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On Eccentric Adjacency Index of Graphs and Trees

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Abstract

Let $G = (V(G), E(G))$ be a simple and connected graph. The distance between any two vertices x and y, denoted by $d_G(x, y)$, is defined as the length of a shortest path connecting x and y in G. The degree of a vertex x in G, denoted by $\deg_G(x)$, is defined as the number of vertices in G of distance one from x. The eccentric adjacency index (briefly EAI) of a connected graph G is defined as

$$\xi^{ad}(G) = \sum_{u \in V(G)} \mathbf{S}_G(u)\varepsilon_G(u)^{-1},$$

where $\mathbf{S}_G(u) = \sum_{\substack{v \in V(G) \\ d_G(u,v)=1}} \deg_G(v)$ and $\varepsilon_G(u) = \max\{d_G(u, v) \mid v \in V(G)\}$. In this article, we aim to obtain all extremal graphs based on the value of EAI among all simple and connected graphs, all trees, and all trees with perfect matching.

Keywords: Eccentricity, Tree, Eccentric adjacency index, Perfect matching.

2020 Mathematics Subject Classification: 05C12, 05C35, 92E10.

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1. Introduction

In this article, all graphs are finite, simple and connected. Let G be any finite, simple and connected graph with the vertex set $V(G)$ and the edge set $E(G)$. For two vertices u and v in $V(G)$ their distance $d_G(u,v)$ is defined as the length of a shortest path connecting u and v in G. The degree $\deg_G(u)$ of the vertex u in G is defined as the number of neighbors of u in G, i.e., $\deg_G(u) = |\{x \in V(G) \mid d_G(u,x) = 1\}|$. If $\deg_G(u) = 1$, then the vertex u is called *pendant*. The eccentricity $\varepsilon_G(u)$ of the vertex u of G is the distance from u to any vertex farthest away from it in G, i.e., $\varepsilon_G(u) = \max_{v\in V(G)} d_G(u,v)$. The *diameter* $D(G)$ and the *radius* $R(G)$ of G is defined as

$$D(G) = \max_{u\in V(G)} \varepsilon_G(u), \quad R(G) = \min_{u\in V(G)} \varepsilon_G(u).$$

An induced path P in G of length $D(G)$ is called *diametral*. A vertex with minimum eccentricity in G is called *central*. A set of edges $M \subseteq E(G)$ is called a *matching* provided for any two edges $e = xy$ and $f = uv$ in M, $\{x,y\} \cap \{u,v\} = \emptyset$. A matching M is called *perfect* if every vertex of G is incident to exactly one edge from M. One can see that a tree has a perfect matching only if it is of even order.

We denote by S_n, P_n, and K_n, respectively, the star graph, the path graph, and the complete graph of order n. A connected graph with no cycles is called a *tree*. One can see that a tree T has only one central vertex if $D(T)$ is even, and two central vertices if $D(T)$ is odd. A vertex in a tree is called *branching* if its degree is at least three. A tree with exactly one branching vertex is called a *starlike*. We denote a starlike tree with maximum degree Δ by $T(l_1, l_2, \ldots, l_\Delta)$ such that $T(l_1, l_2, \ldots, l_\Delta) - v = P_{l_1} \cup P_{l_2} \cdots \cup P_{l_\Delta}$, where v is the vertex of degree Δ, and $l_1, l_2, \cdots, l_\Delta$ are positive integers. For example, the starlike tree $T(2,2,2,1)$ is depicted in Figure 1. A tree T is called a *spanning tree* of the connected graph G if $V(T) = V(G)$ and $E(T) \subseteq E(G)$.

Chemical graph theory is a field of chemistry that uses graph theory to study and analyze chemical structures. Graph theory is a mathematical framework used to study relationships between objects, and it has proven to be useful in analyzing molecular structures. By representing atoms and bonds as vertices and edges, respectively, a molecular structure can be analyzed using various graph theoretic tools and techniques. One application of chemical graph theory is in the study of biological activity of molecules. The biological activity of a molecule depends on its structure and how it interacts with biological targets such as proteins or enzymes. Chemical graph theory can be used to analyze the structure of molecules and to identify patterns that correlate with specific biological activities. For example, researchers may use chemical graph theory to analyze the structures of compounds with known biological activity, and then develop models to predict the biological activity of new compounds based on their structures. This can help to accelerate the drug discovery process by identifying promising compounds more quickly and efficiently. Overall, chemical graph theory has become an important tool for studying the structure and properties of molecules, and its applications extend to a wide range of fields, including drug discovery, materials science, and environmental chemistry.

A map *Top* from graphs into real numbers is called a *topological index*, if $G \simeq H$ implies that $Top(G) = Top(H)$. Topological indices are graph invariants which are useful tools for predicting and understanding the properties and behavior of molecules. Eccentricity-based topological indices have important applications in network analysis, including in social network analysis, transportation network analysis, and biological network analysis. Several eccentricity based topological indices have been proposed and/or used in QSAR and QSPR studies. The *eccentric connectivity index* was defined in [1] as $\xi^c(G) = \sum_{u\in V(G)} \deg_G(u)\varepsilon_G(u)$. The *connective eccentricity index* which is a modification of the eccentric connectivity index was defined in [2] as $C^\xi(G) = \sum_{u\in V(G)} \frac{\deg_G(u)}{\varepsilon_G(u)}$. The *eccentric adjacency index* (briefly EAI) was proposed in [3] as $\xi^{ad}(G) = \sum_{u\in V(G)} \frac{\mathbf{S}_G(u)}{\varepsilon_G(u)}$, where $\mathbf{S}_G(u) = \sum_{\substack{v\in V(G)\\ d_G(u,v)=1}} \deg_G(v)$. This index was also defined independently by Ediz in 2011 under the name *Ediz eccentric connectivity index* [4]. The *eccentric distance sum* of a connected graph G was defined in [5] as $\xi^d(G) = \sum_{u\in V(G)} \deg_G(u)D_G(u)$, where $D_G(v)$ is the sum of distances between u and other vertices in G. The *augmented eccentric connectivity index* of a connected graph G was introduced in [6] as $\xi^{ac}(G) = \sum_{u\in V(G)} \frac{M_G(u)}{\varepsilon_G(u)}$, where $M_G(u) = \prod_{\substack{v\in V(G)\\ d_G(u,v)=1}} \deg_G(v)$. Mathematical properties of these indices were studied in [7–19]. Although, EAI was introduced much earlier than augmented eccentric connectivity index, but it received less attention from the mathematicians and so its mathematical properties have not been studied much. In this article, inspired from [15], we aim to fill this gap by characterizing all extremal graphs based on EAI among all simple and connected graphs, all trees, and all trees with perfect matching.

In Proposition 1.1, the exact values of EAI are provided for some graphs which will be later proven as extremal cases. The first three parts of this proposition were proved in [4], the proof of the last one is straightforward.

Proposition 1.1. *For paths P_n on $n \geq 4$ vertices, the star S_n on $n \geq 3$ vertices, the starlikes $T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)$ on $n \geq 6$ and the complete graphs K_n on $n \geq 2$ vertices the following hold:*

1. $\xi^{ad}(P_n) = \dfrac{6}{n-2} + 4\left(\dfrac{1}{n-1} + 2H_{n-3} - H_{\lfloor \frac{n}{2} \rfloor -1} - H_{\lfloor \frac{n-1}{2} \rfloor}\right)$, *where H_k represents the kth harmonic number, defined as the sum of the reciprocals of the first k positive integers.*
2. $\xi^{ad}(S_n) = \dfrac{(n-1)^2 + 2(n-1)}{2}$,
3. $\xi^{ad}(K_n) = n(n-1)^2$,

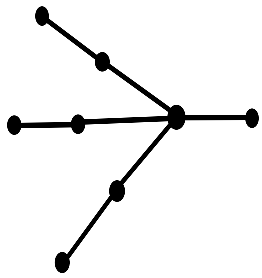

Figure 1: The starlike tree $T(2,2,2,1)$ of order eight and diameter four

4. $\xi^{ad}\big(T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)\big) = \dfrac{n^2+11n-16}{12}$.

The rest of this article is organized as follows. In Section 2, we prove that among all trees of order n, P_n is the unique minimal tree and S_n is the unique maximal tree based on the value of EAI. In Section 3, we prove that among all trees with perfect matching of order n, the path P_n is the unique minimal tree and the starlike tree $T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)$ is the unique maximal tree based on the value of EAI.

2. Extremal trees

In this section, we aim to obtain trees with minimum and maximum value of EAI. We will begin with the minimum case by using a graph transformation that increases the diameter but reduces the value of EAI. This transformation was inspired from that of J. Sedlar for augmented eccentric connectivity index [15], however this is more straightforward and requires fewer computations to obtain the desired inequality.

Transformation A ([15]). Suppose that $T \neq P_n$ is a tree of order n. Let us choose a diametral path $P = v_0v_1\cdots v_D$ in T in which the first branching vertex is the farthest from v_0. Depending on the structure of P, select a vertex u as follows:

(A1) If the vertex v_1 is branching, but v_2 is not, set $u = v_1$,

(A2) If both v_1 and v_2 are branching, set $u = v_2$,

(A3) If the vertex v_1 is not branching, set $u = v_i$, where v_i, $i \neq 1$ is the first branching vertex on P.

Suppose that $x_1,\ldots,x_t$ are the neighbors of u not belonging to P ($t = \deg_T(u)-2$). We create a new tree T^* by deleting the edges $ux_1,\ldots,ux_t$ and adding the edges $v_0x_1,\ldots,v_0x_t$, i.e., $T^* = \big(G - \{ux_1,\ldots,ux_t\}\big) + \{v_0x_1,\ldots,v_0x_t\}$, see Figure 2.

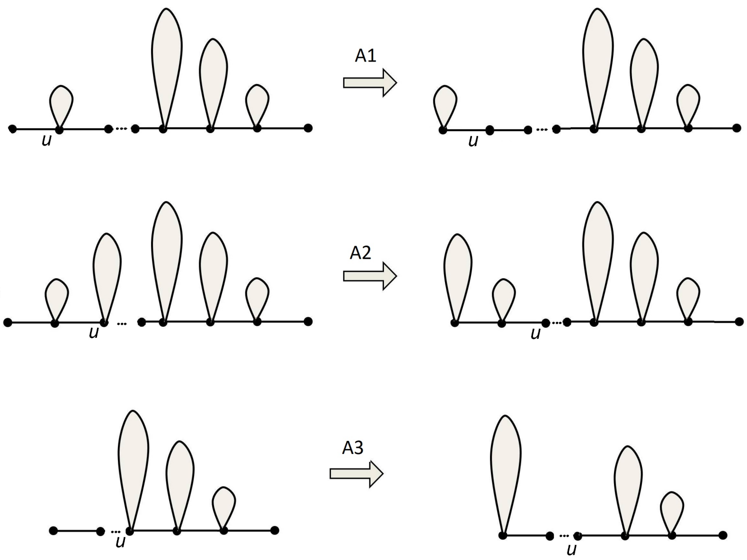

Figure 2: Transformations A1, A2 and A3.

Transformation A is a method for reducing the eccentric adjacency index (EAI) of a given tree T. It involves selecting a diametral path in T and modifying the tree by deleting certain edges and adding new edges. The specific modifications depend on whether certain vertices on the path are branching or not. Let us show that Transformation A decreases the value of EAI.

Lemma 2.1. *Suppose that $T \neq P_n$ is a tree of order n. If T^* is the tree of order n obtained from $T \neq P_n$ applying Transformation A, then*

$$\xi^{ad}(T) > \xi^{ad}(T^*).$$

Proof. It is easy to verify that Transformation A increases the degree of v_0 and decreases that of u, i.e.,

$$\deg_{T^*}(v_0) = \deg_T(v_0) + t = 1 + t, \qquad \deg_{T^*}(u) = \deg_T(u) - t.$$

The eccentricity of vertices either increases or remains unchanged. For example, $\varepsilon_{T^*}(v_0) = \varepsilon_T(v_0)$, $\varepsilon_T(u) \leq \varepsilon_{T^*}(u)$. Let us distinguish the following three cases: **Case (1)** If $D = 2$, then $T = S_n$, $n \geq 4$. Therefore, we apply Transformation A1. It follows that

$$\begin{aligned}
\xi^{ad}(T)-\xi^{ad}(T^*) &= \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)}-\frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\\
&+\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)}-\sum_{i=1}^{t}\frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)}\\
&=\frac{t+2}{2}-\frac{t+2}{2}+\frac{t+2}{1}-\frac{t+2}{2}+\frac{t+2}{2}-\frac{2}{3}\\
&+\sum_{i=1}^{t}\frac{t+2}{2}-\sum_{i=1}^{t}\frac{t+1}{3}>0.
\end{aligned}$$

Case (2) For $D=3$, we shall distinguish the following two subcases:

Case (2-1) If v_1 is a branching vertex, then v_2 must also be a branching vertex. Therefore, we must apply Transformation A2. It follows that

$$\begin{aligned}
\xi^{ad}(T)-\xi^{ad}(T^*) &= \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\\
&+\frac{\mathbf{S}_T(v_3)}{\varepsilon_T(v_3)}-\frac{\mathbf{S}_{T^*}(v_3)}{\varepsilon_{T^*}(v_3)}+\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)}-\sum_{i=1}^{t}\frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)}\\
&=\frac{\deg_T(v_1)}{3}-\frac{t+\deg_T(v_1)}{3}+\frac{t+\deg_T(v_1)+1}{2}-\frac{\deg_T(v_1)+1}{3}\\
&+\frac{t+2}{3}-\frac{2}{4}+\sum_{i=1}^{t}\frac{t+2}{3}-\sum_{i=1}^{t}\frac{t+1}{4}>0.
\end{aligned}$$

Case (2-2) If v_1 is not branching, then v_2 must be a branching vertex since $T\neq P_4$. Therefore, we apply Transformation A3. It follows that

$$\begin{aligned}
\xi^{ad}(T)-\xi^{ad}(T^*) &= \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)}-\frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\\
&+\sum_{i=1}^{t}\frac{\mathbf{S}_T(w_i)}{\varepsilon_T(w_i)}-\sum_{i=1}^{t}\frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)}\\
&=\frac{2}{3}-\frac{t+2}{3}+\frac{t+3}{2}-\frac{3}{3}+\frac{t+2}{3}-\frac{2}{4}+\sum_{i=1}^{t}\frac{t+2}{3}-\sum_{i=1}^{t}\frac{t+1}{4}>0.
\end{aligned}$$

Case (3) For $D\geq 4$, we distinguish the following four subcases:

Case (3-1) The vertex v_1 is a branching vertex, but v_2 is not. Therefore, we apply Transformation A1. It follows that

$$
\begin{aligned}
\xi^{ad}(T)-\xi^{ad}(T^*) >\ & \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)}-\frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\\
&+\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)}-\sum_{i=1}^{t}\frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)}\\
&=\frac{t+2}{D}-\frac{t+2}{D}+\frac{t+3}{D-1}-\frac{t+3}{D-1}+\frac{\deg_T(v_3)+t+2}{D-2}\\
&-\frac{\deg_T(v_3)+2}{D-2}+\sum_{i=1}^{t}\frac{t+2}{D}-\sum_{i=1}^{t}\frac{t+1}{D+1}>0.
\end{aligned}
$$

Case (3-2) Both v_1 and v_2 are branching vertices. Therefore, we apply Transformation A2. It follows that

$$
\begin{aligned}
\xi^{ad}(T)-\xi^{ad}(T^*) >\ & \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)}-\frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\\
&+\frac{\mathbf{S}_T(v_3)}{\varepsilon_T(v_3)}-\frac{\mathbf{S}_{T^*}(v_3)}{\varepsilon_{T^*}(v_3)}+\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)}-\sum_{i=1}^{t}\frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)}\\
&=\frac{\deg_T(v_1)}{D}-\frac{\left(\sum_{j=1}^{t}\deg_T(x_j)\right)+\deg_T(v_1)}{D}+\frac{\deg_T(v_1)+t+1}{D-1}\\
&-\frac{\deg_T(v_1)+t+1}{D-1}+\frac{\left(\sum_{j=1}^{t}\deg_T(x_j)\right)+\deg_T(v_1)+\deg_T(v_3)}{D-2}\\
&-\frac{\deg_T(v_1)+\deg_T(v_3)}{D-2}+\frac{t+2+\deg_T(v_4)}{D-3}-\frac{2+\deg_T(v_4)}{D-3}\\
&+\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)}{D-1}-\sum_{i=1}^{t}\frac{\mathbf{S}_T(x_i)-1}{D+1}>0.
\end{aligned}
$$

Case (3-3) The vertex v_2 is the closest branching vertex to v_0 in P_D. Therefore, we apply Transformation A3. It follows that

$$
\begin{aligned}
&\xi^{ad}(T)-\xi^{ad}(T^*)\\
&>\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)}-\frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)}+\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)}-\frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}+\sum_{j=1}^{t}\frac{\mathbf{S}_T(x_j)}{\varepsilon_T(x_j)}-\sum_{j=1}^{t}\frac{\mathbf{S}_{T^*}(x_j)}{\varepsilon_{T^*}(x_j)}\\
&\geq\frac{2}{D}-\frac{\left(\sum_{j=1}^{t}\deg_T(x_j)\right)+2}{D}+\frac{\left(\sum_{j=1}^{t}\deg_T(x_j)\right)+\deg_T(v_3)+2}{D-2}\\
&-\frac{\deg_T(v_3)+2}{D-2}+\sum_{j=1}^{t}\frac{\mathbf{S}_T(x_j)}{D-1}-\sum_{j=1}^{t}\frac{\mathbf{S}_T(x_j)-1}{D}>0.
\end{aligned}
$$

Case (3-4) For some $i \geqslant 3$, v_i is the closest branching vertex to v_0 in P_D. Therefore, we apply Transformation A3. It follows that

$$\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*) \\
&> \frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)} + \frac{\mathbf{S}_T(v_{i-1})}{\varepsilon_T(v_{i-1})} - \frac{\mathbf{S}_{T^*}(v_{i-1})}{\varepsilon_{T^*}(v_{i-1})} \\
&+ \frac{\mathbf{S}_T(v_i)}{\varepsilon_T(v_i)} - \frac{\mathbf{S}_{T^*}(v_i)}{\varepsilon_{T^*}(v_i)} + \sum_{j=1}^{t} \frac{\mathbf{S}_T(x_j)}{\varepsilon_T(x_j)} - \sum_{j=1}^{t} \frac{\mathbf{S}_{T^*}(x_j)}{\varepsilon_{T^*}(x_j)} \\
&= \frac{2}{D} - \frac{\left(\sum_{j=1}^{t} \deg_T(x_j)\right) + 2}{D} + \frac{3}{D-1} - \frac{t+3}{D-1} + \frac{t+4}{\varepsilon_T(v_{i-1})} \\
&- \frac{4}{\varepsilon_{T^*}(v_{i-1})} + \frac{\left(\sum_{j=1}^{t} \deg_T(x_j)\right) + \deg_T(v_{i+1}) + 2}{\varepsilon_T(v_i)} \\
&- \frac{\deg_T(v_{i+1}) + 2}{\varepsilon_{T^*}(v_i)} + \sum_{j=1}^{t} \frac{\mathbf{S}_T(x_j)}{\varepsilon_T(x_j)} - \sum_{j=1}^{t} \frac{\mathbf{S}_T(x_j) - 1}{D} > 0.
\end{aligned}$$

□

It is clear that Transformation A increases the diameter of trees containing at least one branching vertex. Applying this transformation consecutively, by Lemma 2.1 we conclude that among trees of order n, P_n is the unique tree with the minimum value of EAI.

Corollary 2.2. *Let $T \neq P_n$ be a tree of order n. Then $\xi^{ad}(T) > \xi^{ad}(P_n)$.*

Now by Proposition 1.1 and Corollary 2.2, we summarize our results as follows:

Theorem 2.3. *Let T be a tree of order n. Then*

$$\xi^{ad}(T) \geq \frac{6}{n-2} + 4\left(\frac{1}{n-1} + 2H_{n-3} - H_{\lfloor \frac{n}{2} \rfloor - 1} - H_{\lfloor \frac{n-1}{2} \rfloor}\right). \tag{1}$$

Equality holds in Equation (1) if and only if $T = P_n$.

Remark 1. In [4] Ediz proved the inequality of Theorem 2.3 in a different way, however, he did not determine the equality case.

Let us find trees maximizing the value of EAI. To do so, we need to introduce a transformation which increases the value of EAI of trees.

Transformation B. Let T be a tree with $D(T) \geqslant 4$. Denote by u a central vertex of T. Let w be a non-pendant and non-central vertex adjacent to u and $x_1, \ldots, x_t$ be the non-central neighbors of w. We denote by T^*, the tree obtained from T by deleting the edges $wx_1, \ldots, wx_t$ and adding the edges $ux_1, \ldots, ux_t$, i.e., $T^* = \left(G - \{wx_1, \ldots, wx_t\}\right) + \{ux_1, \ldots, ux_t\}$.

Let us show that Transformation B increases the value of EAI of trees.

Lemma 2.4. *Suppose that T is a tree with $D(T) \geqslant 4$. Let T^* be the tree obtained from T applying Transformation B. Then $\xi^{ad}(T) < \xi^{ad}(T^*)$.*

Proof. We choose w in a way which w is a non-pendant and non-central vertex adjacent to the central vertex u. Let y be the other neighbor of the central vertex u ($y \neq w$). Also, assume that for each $1 \leq i \leq t$, x_i is a non-central vertex adjacent to w. It follows that

$$\begin{aligned}
\xi^{ad}(T) - \xi^{ad}(T^*) &< \frac{\mathbf{S}_T(u)}{\varepsilon_T(u)} - \frac{\mathbf{S}_{T^*}(u)}{\varepsilon_{T^*}(u)} + (\frac{\mathbf{S}_T(y)}{\varepsilon_T(y)} - \frac{\mathbf{S}_{T^*}(y)}{\varepsilon_{T^*}(y)}) + \frac{\mathbf{S}_T(w)}{\varepsilon_T(w)} - \frac{\mathbf{S}_{T^*}(w)}{\varepsilon_{T^*}(w)} \\
&+ \sum_{i=1}^{n} \frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)} - \sum_{i=1}^{n} \frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)} \\
&< \frac{\mathbf{S}_T(u)}{R(T)} - \frac{\mathbf{S}_T(u) + (\sum_{i=1}^{t} \deg(x_i)) - t}{R(T)} - \frac{t}{R(T)} \\
&+ \frac{(\sum_{i=1}^{t} \deg(x_i)) + \deg(u)}{R(T)+1} - \frac{t + \deg(u)}{R(T)+1} \\
&+ \frac{\sum_{i=1}^{t} \mathbf{S}_T(x_i)}{R(T)+2} - \frac{(\sum_{i=1}^{t} \mathbf{S}_T(x_i)) + \deg(u) - 1}{R(T)+1} \\
&= -\frac{\sum_{i=1}^{t} \deg(x_i)}{R(T)} + \frac{\left(\sum_{i=1}^{t} \deg(x_i)\right) - t}{R(T)+1} + \frac{\sum_{i=1}^{t} \mathbf{S}_T(x_i)}{R(T)+2} \\
&- \frac{\sum_{i=1}^{t} \mathbf{S}_T(x_i) + \deg(u) - 1}{R(T)+1} < 0,
\end{aligned}$$

proving the result.

□

Applying Transformation B, consecutively, we obtain a tree of diameter three, but such a tree is not maximal based on the value of EAI.

Lemma 2.5. *Suppose that T is a tree of order n with $D(T) = 3$ with the central vertices u and v. Let $x_1, \ldots, x_t$ and $y_1, \ldots, y_l$ be the non-central neighbors of u and v, respectively. Construct the tree T^* from T by deleting the edges $ux_1, \ldots, ux_t$ and adding the edges $vx_1, \ldots, vx_t$, i.e., $T^* = \big(G - \{ux_1, \ldots, ux_t\}\big) + \{vx_1, \ldots, vx_t\}$. Then T^* is nothing but the star graph S_n. Moreover, $\xi^{ad}(T) < \xi^{ad}(S_n)$.*

Proof. The first part of the lemma is clear. Let us verify the second part:

$$\begin{aligned}
\xi^{ad}(T) - \xi^{ad}(T^*) &= \frac{\mathbf{S}_T(u)}{\varepsilon_T(u)} - \frac{\mathbf{S}_{T^*}(u)}{\varepsilon_{T^*}(u)} + \frac{\mathbf{S}_T(v)}{\varepsilon_T(v)} - \frac{\mathbf{S}_{T^*}(v)}{\varepsilon_{T^*}(v)} + \sum_{i=1}^{t} \frac{\mathbf{S}_T(x_i)}{\varepsilon_T(x_i)} \\
&- \sum_{i=1}^{t} \frac{\mathbf{S}_{T^*}(x_i)}{\varepsilon_{T^*}(x_i)} + \sum_{i=1}^{l} \frac{\mathbf{S}_T(y_i)}{\varepsilon_T(y_i)} - \sum_{i=1}^{l} \frac{\mathbf{S}_{T^*}(y_i)}{\varepsilon_{T^*}(y_i)} \\
&= \frac{t+l+1}{2} - \frac{t+l+1}{2} + \frac{t+l+1}{2} - \frac{t+l+1}{1} \\
&+ \sum_{i=1}^{t} \frac{t+1}{3} - \sum_{i=1}^{t} \frac{t+l+1}{2} + \sum_{i=1}^{l} \frac{l+1}{3} - \sum_{i=1}^{l} \frac{t+l+1}{2} < 0.
\end{aligned}$$

□

Corollary 2.6. *Among trees of order $n \geq 3$, the star graph S_n is the unique maximal tree based on the value of* EAI.

Proof. Let T be a tree of order $n \geq 3$ and diameter D. The result is clear for $D = 2$, since S_n is the unique tree of diameter 2. If $D = 3$, the result follows from Lemma 2.5. If $D \geq 4$, applying Transformation B on T, consecutively, we obtain a tree of diameter 3, say T^*. By Lemma 2.4, $\xi^{ad}(T) < \xi^{ad}(T^*)$. Moreover, it follows from Lemma 2.5 that $\xi^{ad}(T) < \xi^{ad}(T^*) < \xi^{ad}(S_n)$, concluding the result. □

Now by Proposition 1.1 and Corollary 2.6, we have:

Theorem 2.7. *Let T be a tree of order n. Then*

$$\xi^{ad}(T) \leq \frac{(n-1)^2 + 2(n-1)}{2}. \tag{2}$$

Equality holds in Equation (2) if and only if $T = S_n$.

Now we utilize the previous results to obtain graphs maximizing and minimizing the value of EAI. First we recall the following trivial fact:

Lemma 2.8. *Let G be a connected graph with an arbitrary edge uv. Let G^* be the connected graph obtained from G by deleting the edge uv. Then for each $x \in \{u, v\}$, we have:*

$$\varepsilon_G(x) \leq \varepsilon_{G^*}(x), \quad \deg_{G^*}(x) = \deg_G(x) - 1.$$

Note that the contribution of every vertex to the value of EAI in the complete graph K_n is the maximum possible. By Lemma 2.8, deleting any edge will decrease the value of EAI. It follows that for a connected graph G of order n, we have $\xi^{ad}(G) \leq n(n-1)^2$, with equality if and only if $G = K_n$. Therefore, we obtain the unique maximal graph with respect to the value of EAI. Now we shall obtain the minimal one.

Proposition 2.9. *For a connected graph G of order n, we have*

$$\xi^{ad}(G) \geq \frac{6}{n-2} + 4\left(\frac{1}{n-1} + 2H_{n-3} - H_{\lfloor \frac{n}{2} \rfloor -1} - H_{\lfloor \frac{n-1}{2} \rfloor}\right). \tag{3}$$

Equality holds in Equation (3) if and only if $G = P_n$.

Proof. If G is a tree, then the result follows from Theorem 2.3. If G is not a tree, then G has a spanning tree, say T. It follows from the definition that T is obtained from G by deleting some of its edges. So by Lemma 2.8, $\xi^{ad}(G) > \xi^{ad}(T)$. Besides, by Theorem 2.3, $\xi^{ad}(T) \geq \xi^{ad}(P_n)$, concluding the result. □

3. Extremal trees with perfect matching

In this section, we assume trees have an even order, since only such trees can have a perfect matching. If a tree has a perfect matching, then the matching is unique, see [20]. Besides, in a tree with perfect matching, each vertex can obviously be adjacent to at most one pendant vertex. Besides, consider a diametral path $P = v_0v_1 \cdots v_D$ in a tree with perfect matching. Note that v_1 and v_{D-1} must be of degree two since each of them are already adjacent to a pendant vertex and can not be adjacent to more vertices. By the same reasoning the neighbors of v_2 outside of P are either of degree two or degree one.

It is worth noting that the path P_n has a perfect matching if n is even. This implies that P_n has the minimum value of EAI among trees of order n with a perfect matching, according to Theorem 2.3. However, characterizing trees with perfect matching and maximum value of EAI is challenging since the star S_n, for $n \geq 3$, does not have a perfect matching. To address this, we propose a transformation that preserves the existence of a perfect matching and increases the value of EAI.

Note that P_2 is the unique tree with perfect matching of diameter one. The unique tree with perfect matching of diameter three is P_4. Besides, one can see that the starlike tree $T(\overbrace{2, \ldots, 2}^{\frac{n}{2}-1}, 1)$ is the unique tree with perfect matching of diameter four. We also know that trees of diameter two has no perfect matching. Hence, we need to find maximal trees based on EAI among trees with perfect matching of diameter $D \geq 4$.

Transformation D. Let T be a tree with $D(T) \geq 4$, and let $P = v_0v_1 \cdots v_D$ be a diametral path in T. Denote by u the central vertex of P with the largest index. Let $x_1, \ldots, x_t$ be all vertices of degree two from $V(T) \setminus \{v_3\}$ being adjacent to v_2. Let us denote by T^* the tree obtained from T by deleting the edges $v_2x_1, \ldots, v_2x_t$ and adding the edges $ux_1, \ldots, ux_t$, see Figure 3.

Lemma 3.1. *Let T be a tree of order n with $D(T) \geq 4$ with perfect matching M. Suppose that T^* is a tree which is obtained from T by applying Transformation D. Then T^* is a tree of order n with perfect matching M and $\xi^{ad}(T) - \xi^{ad}(T^*) < 0$.*

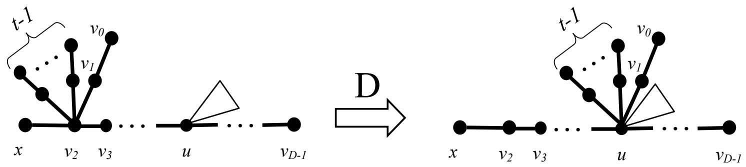

Figure 3: Transformations D

Proof. Since for each $1 \leq i \leq t$, $v_2 x_i \notin M$, we conclude that M is a perfect matching for T^*. Note that v_1 belongs to $\{x_1, \ldots, x_t\}$. Also, assume x as a possible pendant vertex adjacent to v_2. We shall distinguish the following cases:

Case (1) $D = 5$**:**

$$
\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*) \\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big) \\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + (\frac{\mathbf{S}_T(u)}{R(T)} - \frac{\mathbf{S}_{T^*}(u)}{R(T)-1}) + (\frac{\mathbf{S}_T(v_{u+1})}{R(T)+1} - \frac{\mathbf{S}_{T^*}(v_{u+1})}{R(T)}) \\
&< t\Big(\frac{2}{5} - \frac{2}{4} + \frac{\deg_T(v_2)+1}{4} - \frac{\deg(u)+t+1}{3}\Big) + (\frac{t}{3}) + \frac{t}{4} - (\frac{t}{2}) - (\frac{t}{3}) < 0.
\end{aligned}
$$

Case (2) $D = 6$**:**

$$
\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*) \\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big) \\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + (\frac{\mathbf{S}_T(u)}{R(T)} - \frac{\mathbf{S}_{T^*}(u)}{R(T)-1}) + (\frac{\mathbf{S}_T(v_{u+1})}{R(T)+1} - \frac{\mathbf{S}_{T^*}(v_{u+1})}{R(T)}) \\
&< t\Big(\frac{2}{6} - \frac{2}{5} + \frac{\deg_T(v_2)+1}{5} - \frac{\deg(u)+t+1}{4}\Big) + (\frac{t}{4}) + \frac{t}{5} - (\frac{2t}{3}) < 0.
\end{aligned}
$$

Case (3) $D = 7$:

$$
\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*)\\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big)\\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + (\frac{\mathbf{S}_T(u)}{R(T)} - \frac{\mathbf{S}_{T^*}(u)}{R(T)-1})\\
&+ (\frac{\mathbf{S}_T(v_{u+1})}{R(T)(T)+1} - \frac{\mathbf{S}_{T^*}(v_{u+1})}{R(T)(T)})\\
&< t\Big(\frac{2}{7} - \frac{2}{6} + \frac{\deg_T(v_2)+1}{6} - \frac{\deg(u)+t+1}{5}\Big) + (\frac{2t}{5}) + \frac{t}{6} - (\frac{2t}{3})\\
&- (\frac{t}{4}) < 0.
\end{aligned}
$$

Case (4) $D = 8$:

$$
\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*)\\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big)\\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + \frac{\mathbf{S}_T(v_3) - \mathbf{S}_{T^*}(v_3)}{\varepsilon_T(v_3)} + (\frac{\mathbf{S}_T(u)}{R(T)} - \frac{\mathbf{S}_{T^*}(u)}{R(T)-1})\\
&+ (\frac{\mathbf{S}_T(v_{u+1})}{R(T)+1} - \frac{\mathbf{S}_{T^*}(v_{u+1})}{R(T)})\\
&< t\Big(\frac{2}{8} - \frac{2}{7} + \frac{\deg_T(v_2)+1}{7} - \frac{\deg(u)+t+1}{6}\Big) + (\frac{2t}{6}) + \frac{t}{7} - (\frac{3t}{4}) < 0.
\end{aligned}
$$

Case (5) $9 \leq D$ and D is odd:

$$
\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*)\\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big)\\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + \frac{\mathbf{S}_T(v_3) - \mathbf{S}_{T^*}(v_3)}{\varepsilon_T(v_3)} + \frac{\mathbf{S}_T(u) - \mathbf{S}_{T^*}(u)}{R(T)}\\
&+ \Big(\frac{\mathbf{S}_T(v_{u-1}) - \mathbf{S}_{T^*}(v_{u-1})}{R(T)} + \frac{\mathbf{S}_T(v_{u+1}) - \mathbf{S}_{T^*}(v_{u+1})}{R(T)+1}\Big)\\
&\leq t\Big(\frac{2}{D} - \frac{2}{R(T)+2} + \frac{\deg_T(v_2)+1}{D-1} - \frac{\deg(u)+t+1}{R(T)+1}\Big)\\
&+ (\frac{2t}{D-2}) + \frac{t}{D-1} + \frac{t}{D-3} - \frac{2t}{R(T)} - \Big(\frac{t}{R(T)} + \frac{t}{R(T)+1}\Big) < 0.
\end{aligned}
$$

Case (6) $9 \leq D$ and D is even:

$$\begin{aligned}
&\xi^{ad}(T) - \xi^{ad}(T^*) \\
&\leq t\Big(\frac{\mathbf{S}_T(v_0)}{\varepsilon_T(v_0)} - \frac{\mathbf{S}_{T^*}(v_0)}{\varepsilon_{T^*}(v_0)} + \frac{\mathbf{S}_T(v_1)}{\varepsilon_T(v_1)} - \frac{\mathbf{S}_{T^*}(v_1)}{\varepsilon_{T^*}(v_1)}\Big) + \Big(\frac{\mathbf{S}_T(v_2)}{\varepsilon_T(v_2)} - \frac{\mathbf{S}_{T^*}(v_2)}{\varepsilon_{T^*}(v_2)}\Big) \\
&+ \frac{\mathbf{S}_T(x) - \mathbf{S}_{T^*}(x)}{\varepsilon_T(x)} + \frac{\mathbf{S}_T(v_3) - \mathbf{S}_{T^*}(v_3)}{\varepsilon_T(v_3)} + \frac{\mathbf{S}_T(u) - \mathbf{S}_{T^*}(u)}{R(T)} \\
&+ \Big(\frac{\mathbf{S}_T(v_{u-1}) - \mathbf{S}_{T^*}(v_{u-1})}{R(T)+1} + \frac{\mathbf{S}_T(v_{u+1}) - \mathbf{S}_{T^*}(v_{u+1})}{R(T)+1}\Big) \\
&\leq t\Big(\frac{2}{D} - \frac{2}{R(T)+2} + \frac{\deg_T(v_2)+1}{D-1} - \frac{\deg_T(u)+t+1}{R(T)+1}\Big) \\
&+ (\frac{2t}{D-2}) + \frac{t}{D-1} + \frac{t}{D-3} - \frac{2t}{R(T)} - \frac{2t}{R(T)+1} < 0.
\end{aligned}$$

□

It follows from Lemma 3.1 that the starlike tree $T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)$ maximizes the value of EAI among all trees with perfect matching of order n.

Corollary 3.2. *Suppose that $T \neq T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)$ is a tree of order n. If T has a perfect matching, then*

$$\xi^{ad}(T) < \xi^{ad}\big(T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)\big).$$

Proof. Note that Transformation D decreases the diameter by at least one, and at most two. If we apply consecutively Transformation D on T, then we eventually arrive at a tree with perfect matching of order n and diameter four. Such a tree is nothing but the starlike tree $T(\overbrace{2,\ldots,2}^{\frac{n}{2}-1},1)$. Hence, by Lemma 3.1 we conclude the result. □

Example 3.3. Consider the tree T depicted in Figure 4. Applying Transformation D on T, consecutively, we obtain a sequence of trees T_1, T_2, T_3 and $T_4 = T(\overbrace{2,\ldots,2}^{8},1)$, satisfying the following:

$$\xi^{ad}(T) < \xi^{ad}(T_1) < \xi^{ad}(T_2) < \xi^{ad}(T_3) < \xi^{ad}(T_4).$$

Now by Proposition 1.1 and Corollary 3.2, we obtain a sharp upper bound for the value of EAI among all trees with perfect matching of order n.

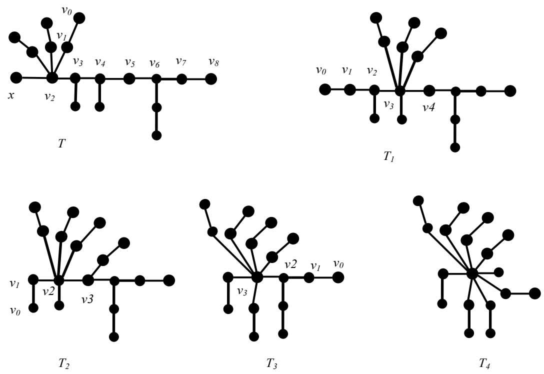

Figure 4: A sequence of trees obtained from T applying Transformation D.

Theorem 3.4. *Let T be a tree with perfect matching of order n. Then*

$$\xi^{ad}(T) \leq \frac{n^2 + 11n - 16}{12}. \tag{4}$$

Equality holds in Equation (4) if and only if $T = T(\overbrace{2, \ldots, 2}^{\frac{n}{2}-1}, 1)$.

Conclusion In this article, we prove that among all trees of order n, the path P_n is the unique minimal tree and the star S_n is the unique maximal tree based on the value of EAI. Moreover, we prove that among all trees with perfect matching of order n, the path P_n is the unique minimal tree and the starlike tree $T(\overbrace{2, \ldots, 2}^{\frac{n}{2}-1}, 1)$ is the unique maximal tree based on the value of EAI.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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References

[1] V. Sharma, R. Goswami and A. K. Madan, Eccentric Connectivity Index: a novel highly discriminating topological descriptor for structure-property and structure-activity studies, *J. Chem. Inf. Comput. Sci.* **37** (1997) 273–282.

[2] S. Gupta, M. Singh and A. K. Madan, Connective Eccentricity Index: a Novel Topological Descriptor for Predicting Biological Activity, *J. Mol. Graph. Model.* **18** (2000) 18–25, https://doi.org/10.1016/S1093-3263(00)00027-9.

[3] S. Gupta, M. Singh and A. K. Madan, Predicting Anti-Hiv Activity: Computational Approach Using a Novel Topological Descriptor, *J. Comput. Aided. Mol. Des.* **15** (2001) 671–678, https://doi.org/10.1023/A:1011964003474.

[4] S. Ediz, On The Ediz Eccentric Connectivity Index of a Graph, *Optoelectron. Adv. Mat.* **5** (2011) 1263–1264.

[5] S. Gupta, M. Singh and A. K. Madan, Eccentric Distance Sum: a Novel Graph Invariant for Predicting Biological and Physical Properties, *J. Math. Anal. Appl.* **275** (2002) 386–401, https://doi.org/10.1016/S0022-247X(02)00373-6.

[6] S. Bajaj, S. S. Sambi and A. K. Madan, Topological Models For Prediction of Anti-Hiv Activity of Acylthiocarbamates, *Bioorg. Med. Chem.* **13** (2005) 3263–3268, https://doi.org/10.1016/j.bmc.2005.02.033.

[7] S. Akhter and R. Farooq, The Eccentric Adjacency Index of Unicyclic Graphs and Trees, *Asian-Eur. J. Math.* **13** (2020) 2050028, 16 pp, https://doi.org/10.1142/S179355712050028X.

[8] T. Došlić and M. Saheli, Augmented Eccentric Connectivity Index, *Miskolc Math. Notes* **12** (2011) 149–157, https://doi.org/10.18514/MMN.2011.331.

[9] T. Došlić, M. Saheli and D. Vukičević, Eccentric Connectivity Index: extremal graphs and values, *Iranian J. Math. Chem.* **1** (2010) 45–56, https://doi.org/10.22052/IJMC.2010.5154.

[10] M. Ghorbani, K. Malekjani and A. Khaki, Eccentric Connectivity Index of Some Dendrimer Graphs, *Iranian J. Math. Chem.* **3** (2012) 7–18, https://doi.org/ 10.22052/IJMC.2012.5270.

[11] A. Ilić, G. Yu and L. Feng, On the Eccentric Distance Sum of Graphs, *J. Math. Anal. Appl.* **381** (2011) 590–600, https://doi.org/10.1016/j.jmaa.2011.02.086.

[12] M. A. Malik, Two Degree-Distance Based Topological Descriptors of Some Product Graphs, *Discrete Appl. Math.* **236** (2018) 315–328, https://doi.org/10.1016/j.dam.2017.11.002.

[13] Y. Nacaroğlu, On the Eccentric Adjacency Index of Graphs, *New Trends Math. Sci.* **6** (2018) 128–136, https://doi.org/10.20852/ntmsci.2018.301.

[14] Y. Nacaroğlu and A. D. Maden, On the Eccentric Connectivity Index of Unicyclic Graphs, *Iranian J. Math. Chem.* **9** (2018) 47–56, https://doi.org/10.22052/IJMC.2017.59425.1231.

[15] J. Sedlar, On Augmented Eccentric Connectivity Index of Graphs and Trees, *MATCH Commun. Math. Comput. Chem.* **68** (2012) 325–342.

[16] R. Sharafdini and M. Safazadeh, On Eccentric Adjacency Index of Several Infinite Classes of Fullerenes, *British J. Math. Comput. Sci.* **12** (2016) 1–11, https://doi.org/10.9734/BJMCS/2016/20567.

[17] Z. Yarahmadi, Eccentric Connectivity and Augmented Eccentric Connectivity Indices of N-Branched Phenylacetylenes Nanostar Dendrimers, *Iranian J. Math. Chem.* **1** (2010) 105–110, https://doi.org/10.22052/IJMC.2010.5160.

[18] Z. Yarahmadi, T. Došlić and S. Moradi, Chain Hexagonal Cacti: Extremal with Respect to the Eccentric Connectivity Index, *Iranian J. Math. Chem.* **4** (2013) 123–136, https://doi.org/10.22052/IJMC.2013.5286.

[19] B. Zhou and Z. Du, On Eccentric Connectivity Index, *MATCH Commun. Math. Comput. Chem.* **63** (2010) 181–198.

[20] Y. Hou and J. Li, Bounds On The Largest Eigenvalue of Trees with a Given Size of Matching, *Linear Algebra Appl.* **342** (2002) 203–217, https://doi.org/10.1016/S0024-3795(01)00465-7.

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