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Lee Weight and Generalized Lee Weight for Codes Over $\mathbb{Z}_{2^n}$

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Abstract

Let $\mathbb{Z}_m$ be the ring of integers modulo m in which $m = 2^n$ for arbitrary n. In this paper, we will obtain a relationship between $wt_L(x), wt_L(y)$ and $wt_L(x + y)$ for any $x, y \in \mathbb{Z}_m$. Let $d_r^L(C)$ denote the r-th generalized Lee weight for code C in which C is a linear code of length n over $\mathbb{Z}_4$. Also, suppose that C_1 and C_2 are two codes over $\mathbb{Z}_4$ and C denotes the $(u, u + v)$-construction of them. In this paper, we will obtain an upper bound for $d_r^L(C)$ for all r, $1 \leq r \leq rank(C)$. In addition, we will obtain $d_1^L(C)$ in terms of $d_1^L(C_1)$ and $d_1^L(C_2)$.

Keywords: Linear code, Hamming weight, Lee weight, Generalized Lee weight, $(u, u + v)$- Construction of codes.

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1. Introduction

Consider Z_m as the code alphabet. The Lee weight of an integer i, for $0 \leq i \leq m$, denoted by $wt_L(i)$, is defined as $wt_L(i) = \min\{i, m - i\}$. For $m = 4$, namely in Z_4, we have $wt_L(0) = 0, wt_L(1) = wt_L(3) = 1$ and $wt_L(2) = 2$. The Lee metric on Z_m^n is defined by

$$wt_L(a) = \Sigma_{i=1}^n wt_L(a_i),$$

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where, the sum is taken over N_0, the set of non-negative integers. Also, Lee distance is defined as $d_L(x,y) = wt_L(x-y)$. For more information, see [1].
The concept of generalized Hamming weight (GHW) introduced by V. K. Wei in [2]. After Wei, several authors worked on this topic, see [3, 4]. Moreover, generalized Lee weight (GLW) for codes over Z_4 introduced by B. Hove in [5] for the first time. He showed that there is a relationship between GHW and GLW. After him, several authors studied this concept, see [6, 7].
A code of length n over Z_4 is a subset of the free module Z_4^n and it is called linear if it is a Z_4- submodule of Z_4^n. Let C be a linear code of length n over Z_4 and let $M(C)$ be the $|C| \times n$ array of all codewords in C. Each arbitrary column of $M(C)$, say c, corresponds to the following three cases:
i) c contains only 0,
ii) c contains 0 and 2 equally often,
iii) c contains all elements of $\mathbb{Z}_4$ equally often.

We define the Lee support weight of these columns as 0, 2 and 1, respectively. Also, we define the Lee support weight of code C, denoted by $wt_L(C)$, as the sum of the Lee support weights of all columns of $M(C)$. As an example, let $C = \{(0,0,0),(2,1,2),(0,3,2),(0,2,0),(2,3,2),(2,0,2),(0,1,0),(2,2,0)\}$. Hence we have

$$M(C) = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 2 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \\ 2 & 3 & 2 \\ 2 & 0 & 2 \\ 0 & 1 & 0 \\ 2 & 2 & 0 \end{pmatrix}.$$

If c_i be the i-th column of $M(C)$, then we have $wt_L(c_1) = 2$, $wt_L(c_2) = 1$ and $wt_L(c_3) = 2$. Hence we obtain that $wt_L(C) = 2+1+2 = 5$. For code C with one generator, say x, we have $wt_L(C) = wt_L(x)$.
Note that in Z_8, we cannot present the similar definition for codes. As an example, let $x = (1,3,5)$, so we have $wt_L(x) = 1+3+3 = 7$. For calculating the Lee weight for submodule $C = \langle x \rangle$, we have

$$M(C) = \begin{pmatrix} 1 & 3 & 5 \\ 2 & 6 & 2 \\ 3 & 1 & 7 \\ 4 & 4 & 4 \\ 5 & 7 & 1 \\ 6 & 2 & 4 \\ 7 & 5 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

The above matrix shows that we cannot recognize which column is made by 1 or 3 or 5. In other words, let c_i denote the i-th column of $M(C)$. If we define $wt_L(c_1) = wt_L(1), wt_L(c_2) = wt_L(3)$ and $wt_L(c_3) = wt_L(5)$, as defined for codes over Z_4, then c_1, c_2 and c_3 have different Lee weights but they are the same (each of them contains all elements of Z_8). It is a contradiction.

Let C be a code of length n over ring Z_4. The rank of C, denoted by $rank(C)$, is defined as the minimum number of generators of C, see [6].

For $1 \leq r \leq rank(C)$, the r-th generalized Lee weight with respect to *rank* (GLWR) for C, denoted by $d_r^L(C)$, is defined as follows:

$$d_r^L(C) = min\{wt_L(D) \mid D \text{ is a } \mathbb{Z}_4 - \text{submodule of C with rank(D)} = r\}.$$

A linear code C of length n and $rank = k$, is called an $[n, k]$ code.

2. The results

In this section, we will derive several properties of Lee weight and GLW for codes over $\mathbb{Z}_m$.

Theorem 2.1. *Let C_i be an $[n, k_i]$ linear code over $\mathbb{Z}_4$, for $i = 1, 2$. Then the $(u, u + v)$- construction of C_1 and C_2 defined by*

$$C = \{(c_1, c_1 + c_2) \mid c_1 \in C_1, c_2 \in C_2\},$$

is a $[2n, k_1 + k_2]$ linear code over $\mathbb{Z}_4$.

Proof. The proof is easy. □

Theorem 2.2. *[6] Let C_1 and C_2 be $[n; k_1, k_2]$ codes over $\mathbb{Z}_4$. Then,*

$$wt_L(C) = \frac{4}{|C|} \Sigma_{x \in C}(wt_L(x) - wt(x)),$$

where $wt(x)$ is the Hamming weight for a vector x.

Lemma 2.3. *For any x in ring $\mathbb{Z}_m$, where m is an arbitrary power of 2, we have*

$$wt_L(x) = \begin{cases} x & 0 \leq x \leq m/2, \\ m - x & x > m/2. \end{cases}$$

Proof. From the definition of Lee weight, we have $wt_L(x) = min\{x, m - x\}$. It is sufficient to investigate x and $m - x$ in all possible cases. We have the following three cases:

i) If $0 \leq x < m/2$, noticing that m, x and Lee weight are integers, we have $m - x > m/2$. So, x is less than $m - x$. Based on this, we obtain $min\{x, m - x\} = x$. Therefore, $wt_L(x) = x$.

ii) If $x = m/2$, then $x = m - x = m/2$. Hence, $min\{x, m - x\} = m/2$ and $wt_L(x) = x$.

iii) If $x > m/2$, then $m - x$ is less than x. Therefore, $min\{x, m - x\} = m - x$. Hence, we have $wt_L(x) = m - x$. □

Theorem 2.4. *For any x and y in ring $\mathbb{Z}_m$, where m is an arbitrary power of 2, we have*

$$wt_L(x) + wt_L(y) \geq wt_L(x + y) \geq wt_L(x) - wt_L(y).$$

Proof. First, we show that $wt_L(x) + wt_L(y) \geq wt_L(x + y)$. It is clear that it is hold when one of x, y and $x + y$ is zero, so we can assume that x, y and $x + y$ are non-zero. The following cases should be considered:

1. Let $1 \leq x, y \leq m/2$. From Lemma 2.3, we have $wt_L(x) = x$ and $wt_L(y) = y$. We have the following subcases:

i) If $x + y \leq m/2$, then we have $wt_L(x + y) = x + y$ by Lemma 2.3. Hence,

$$wt_L(x) + wt_L(y) = wt_L(x + y).$$

ii) If $x + y > m/2$, then we have $wt_L(x + y) = m - x - y$ by Lemma 2.3. Since x, y and Lee weight are integers, we have $x + y \geq m - x - y$. Hence,

$$wt_L(x) + wt_L(y) \geq wt_L(x + y).$$

2. For $1 \leq x \leq m/2$ and $m/2 < y < m$, we have $wt_L(x) = x$ and $wt_L(y) = m - y$. The following cases can be occurred:

i) If $m/2 + 1 \leq x + y < m$, then $wt_L(x + y) = m - x - y$. Now, we have $x + m - y \geq m - x - y$. Based on this inequality,

$$wt_L(x) + wt_L(y) \geq wt_L(x + y).$$

ii) If $m < x + y \leq \frac{3}{2}m$, there exists an integer, say a, in which $x + y = m + a$ and $a \leq m/2$. Hence, $x + y \overset{m}{\equiv} a$ and $wt_L(x + y) = wt_L(a) = a$. Now, we have $x + m - y \geq x + y - m(= a)$. Hence, $wt_L(x) + wt_L(y) \geq wt_L(x + y)$.

3. If $m/2 < x < m$ and $m/2 < y < m$, then $wt_L(x) = m - x$ and $wt_L(y) = m - y$. Since $m+1 \leq x+y \leq 2m-1$, there exists an integer, say a, in which $x+y = m+a$. We have the following two subcases:

i) If $a \leq \frac{m}{2}$, then we have $wt_L(x+y) = a$ and $2m-(m+a) \geq a$. Now, $m-x+m-y$

is greater than or equall to a. This means that $wt_L(x) + wt_L(y) \geq wt_L(x + y)$.

ii) If $a > m/2$, then $wt_L(x + y) = m - a$. So, $2m - (x + y) \geq m - a$. In other words, $m - x + m - y \geq m - a$. Therefore, $wt_L(x) + wt_L(y) \geq wt_L(x + y)$.

Also, by the similar method for $wt_L(x + y)$ and $wt_L(x) - wt_L(y)$, we obtain $wt_L(x + y) \geq wt_L(x) - wt_L(y)$. □

Corollary 2.5. *For any $x, y \in Z_4^n$, we have*

$$wt_L(x) + wt_L(y) \geq wt_L(x + y).$$

Remark 1. It is easy to show that for any x in ring $\mathbb{Z}_m$, we have

$$wt_L((m - 1)x) = wt_L(x).$$

In particular, for any $x = (x_1, x_2, ..., x_n) \in Z_4^n$, we have

$$wt_L(x) = wt_L(3x).$$

The following theorem is similar to the theorem that we have for Hamming weight [8].

Theorem 2.6. *Let C_1 and C_2 be linear codes over $\mathbb{Z}_4$ and $C = \{(c_1, c_1 + c_2) \mid c_1 \in C_1, c_2 \in C_2\}$. Then*

$$d_1^L(C) = min\{2d_1^L(C_1), d_1^L(C_2)\}.$$

Proof. Let $d_1^L(C_1) = wt_L(D_1)$ where $D_1 = \langle x \rangle$ for x in C_1 and let $d_1^L(C_2) = wt_L(D_2)$ where $D_2 = \langle y \rangle$ for y in C_2. We have $d_1^L(C_1) = wt_L(x)$ and $d_1^L(C_2) = wt_L(y)$. Note that $(x, x) \in C$. Let $D = \langle (x, x) \rangle$. Hence, $rank(D) = 1$. We have $wt_L(D) = wt_L(x, x) = 2wt_L(x) = 2d_1^L(C_1)$. Also, $(0, y) \in C$. Now, let $D' = \langle (0, y) \rangle$. So, we obtain $wt_L(D') = wt_L(0, y) = wt_L(y) = d_1^L(C_2)$. Since D and D' satisfy $\{wt_L(H); H \leqslant C, rank(H) = 1\}$ and $min\{wt_L(H); H \leqslant C, rank(H) = 1\} = d_1^L(C_1)$, we have

$$d_1^L(C) \leqslant wt_L(D) = 2d_1^L(C_1),$$
$$d_1^L(C) \leqslant wt_L(D') = d_1^L(C_2).$$

Therefore, we obtain

$$d_1^L(C) \leqslant min\{2d_1^L(C_1), d_1^L(C_2)\}. \quad (1)$$

On the other hand, let $d_1^L(C) = wt_L(H)$. So, $rank(H) = 1$ and $H =< (x, x + y) >$ for $x \in C_1$ and $y \in C_2$. Now,

$$wt_L(H) = wt_L(x, x + y) = wt_L(x) + wt_L(x + y).$$

We have the following three cases:
i) If $x = 0, y \neq 0$ then $wt_L(H) = wt_L(y) \geq d_1^L(C_2)$.
ii) If $x \neq 0, y = 0$ then $wt_L(H) = 2wt_L(x) \geq 2d_1^L(C_1)$.
iii) If $x \neq 0, y \neq 0$ then by using Remark 1 and Theorem 2.4, we have

$$wt_L(H) = wt_L(3x) + wt_L(x+y) \geq wt_L(4x+y) = wt_L(y) \geq d_1^L(C_2).$$

Finally,

$$d_1^L(C) \geq min\{2d_1^L(C_1), d_1^L(C_2)\}. \tag{2}$$

By using Equations (1) and (2), the proof is completed. □

Theorem 2.7. *Let C_1 and C_2 be linear codes over $\mathbb{Z}_4$. Let $C = \{(c_1, c_1 + c_2) \mid c_1 \in C_1, c_2 \in C_2\}$. Then*

$$d_r^L(C) \leq min\{2d_r^L(C_1), d_r^L(C_2)\}.$$

Proof. Supppose that $d_r^L(C_1) = wt_L(D_1)$ in which $D_1 = \langle x_1, x_2, ..., x_r\rangle$ and $d_r^L(C_2) = wt_L(D_2)$ in which $D_2 = \langle y_1, y_2, ..., y_r\rangle$.

Let $D'_1 = \langle (x_1, x_1), (x_2, x_2), ..., (x_r, x_r)\rangle$. By using Theorem 2.2, we have

$$\begin{aligned} wt_L(D'_1) &= \frac{4}{|D'_1|}\Sigma_{\alpha_1,\ldots,\alpha_r \in Z_4}[wt_L(\alpha_1(x_1,x_1) + ... + \alpha_r(x_r,x_r)) \\ &- wt(\alpha_1(x_1,x_1) + ... + \alpha_r(x_r,x_r))] \\ &= \frac{4}{|D'_1|}\Sigma 2wt_L(\alpha_1 x_1 + ... + \alpha_r x_r) - 2wt(\alpha_1 x_1 + ... + \alpha_r x_r) \\ &= \frac{2\times 4}{|D_1|}\Sigma_{t\in D_1} wt_L(t) - wt(t) = 2wt_L(D_1) = 2d_r^L(C_1). \end{aligned}$$

Hence, $wt_L(D'_1) = 2d_r^L(C_1)$. By using the above method for $D'_2 = \langle (0, y_1), \ldots, (0, y_r)\rangle$, we have $wt_L(D'_2) = wt_L(D_2) = d_r^L(C_2)$. Since D'_1 and D'_2 are submodule of C of rank r in which satisfy $\{wt_L(H); H \leq C, rank(H) = r\}$ and $min\{wt_L(H); H \leq C, rank(H) = r\} = d_r^L(C)$, we have

$$d_r^L(C) \leq wt_L(D'_1) = 2d_r^L(C_1), \qquad d_r^L(C) \leq wt_L(D'_2) = d_r^L(C_2).$$

Finally, we obtain

$$d_r^L(C) \leq min\{2d_r^L(C_1), d_r^L(C_2)\}.$$

□

Conflicts of Interest. The author declare that she has no conflicts of interest regarding the publication of this article.

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