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On Hyper CI-algebras

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Abstract

The concept of (*proper*) *hyper CI-algebras*, as a generalization of CI-algebras and hyper BE-algebras, is defined and some properties are presented. Also, the concepts of a *hyper filter* and a *weak hyper filter* over hyper CI-algebras are discussed. Moreover, the notion of a *commutative* hyper CI-algebra is described. Also, we find the number of commutative hyper CI-algebras of order less than 3.

Keywords: (Hyper) BE/CI-algebra, (Weak) hyper filter, Commutative.

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1. Introduction

CI-algebra, defined by Meng [1], is an algebra $(X; \star, 1)$ of type $(2, 0)$ that satisfies:
(CI$_1$) $x \star (y \star z) = y \star (x \star z)$,
(CI$_2$) $x \star x = 1$,
(CI$_3$) $1 \star x = x$, for all $x, y, z \in X$.
The CI-algebra $(X; \star, 1)$ that satisfies the axiom (BE) $x \star 1 = 1$ is called a BE-algebra [2]. It is known that the hyper logical algebras have many applications to sciences. The hyper BE-algebras were defined in [3] as follows:

An algebra $(H; \diamond, 1)$, where H is a nonempty set, $\diamond : H \times H \to \mathcal{P}^*(H)$ a hyper operation and 1 a constant in H, is said to be a hyper BE-algebra, if it satisfies:
(HBE$_1$) $x \diamond (y \diamond z) = y \diamond (x \diamond z)$,
(HBE$_2$) $1 \in x \diamond x$ and $1 \in x \diamond 1$,

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(HBE$_3$) $x \in 1 \diamond x$,
(HBE$_4$) $1 \in 1 \diamond x$ implies $x = 1$, for all $x, y, z \in H$.
Then Rezaei et al. investigated commutative hyper BE-algebras [4]. The notion of good hyper BE-algebras was introduced by Chen et al. [5]. Iranmanesh et al. studied various types of Hv-BE-algebras and discussed on them [6]. Beja et al. presented the notion of the doubt intuitionistic fuzzy hyper filters of hyper BE-algebras and their related properties [7]. Cheng et al. introduced state operators on hyper BE-algebras, state hyper filters and generated state hyper filters and obtained some important results [8]. Recently, Naghibi et al. introduced Hv-BE-algebra as a generalization of a (hyper) BE-algebra [9].

This study presents the concept of a (*commutative*) *hyper CI-algebra*, investigates some relations between hyper BE-algebra, and studies several properties of the hyper BE-algebra. Moreover, (*weak*) *hyper filters* are considered in a hyper CI-algebra.

2. On hyper CI-algebras

In this section, as a generalization of a CI-algebra and a hyper BE-algebra, we define the notion of hyper CI-algebra and investigate some results. In the following, H is a nonempty set, $\mathcal{P}(H)$ denotes the power set of H and $\mathcal{P}^*(H) = \mathcal{P}(H) - \{\emptyset\}$.

Definition 2.1. An algebra $\mathcal{H} = (H; \diamond, 1)$, where $\diamond : H \times H \to \mathcal{P}^*(H)$ is a hyper operation and 1 is a constant, is called a *hyper CI-algebra*, if for all $x, y, z \in H$, it satisfies:
(HCI$_1$) $x \diamond (y \diamond z) = y \diamond (x \diamond z)$,
(HCI$_2$) $1 \in x \diamond x$,
(HCI$_3$) $x \in 1 \diamond x$.

The relation "$\preceq$" is defined by

$$x \preceq y \iff 1 \in x \diamond y.$$

Let G and K be two non-empty subsets of H. Define $G \diamond K = \bigcup_{a \in G, b \in K} a \diamond b$ and

$$G \preceq K \iff \text{there exist } a \in G \text{ and } b \in K \ s.t. \ a \preceq b.$$

We refer to the hyper CI-algebra $\mathcal{H} = (H; \diamond, 1)$ by $\mathcal{H}$.

$\mathcal{H}$ is said to be *good* if for any $x, y, z \in H$, $x \preceq y$ imply $z \diamond x \preceq z \diamond y$.
The following examples show that (HCI$_1$), (HCI$_2$) and (HCI$_3$) are independent.

Example 2.2. (i) Let $A \neq \emptyset$, α a constant element in A, $s, t \in A$, and let define "$\diamond$" on X by

$$s \diamond t = \begin{cases} \{\alpha, t\}; & \text{if } s = \alpha \\ A; & \text{otherwise.} \end{cases}$$

Then $(A; \diamond, \alpha)$ is a hyper CI-algebra.

(ii) Consider the algebra $(H_1 = \{1, \alpha, \beta, \gamma\}; \diamond_1, 1)$, where "$\diamond_1$" is definded by Table 1. Then $(H_1; \diamond_1, 1)$ is a hyper CI-algebra, which is not good. Since $1 \preceq \beta$,

Table 1: Hyper CI-algebra $(H_1; \diamond_1, 1)$

$\diamond_1$	1	α	β	γ
1	$\{1\}$	$\{\alpha\}$	$\{1, \beta\}$	$\{\alpha, \gamma\}$
α	$\{\alpha\}$	$\{1, \alpha\}$	$\{\alpha, \gamma\}$	H_1
β	$\{1\}$	$\{\alpha\}$	$\{1\}$	$\{1, \alpha\}$
γ	$\{1\}$	$\{1\}$	$\{\alpha\}$	$\{1, \alpha\}$

but $1 = \gamma \diamond_1 1 \not\preceq \gamma \diamond_1 \beta = \alpha$.

(iii) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_2, 1)$, where "$\diamond_2$" is definded by Table 2. Then it is good.

Table 2: Hyper CI-algebra $(H_2; \diamond_2, 1)$

$\diamond_2$	1	α	β
1	$\{1\}$	$\{\alpha\}$	$\{\beta\}$
α	$\{\beta\}$	H_2	$\{\alpha, \beta\}$
β	$\{\beta\}$	H_2	H_2

(iv) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_3, 1)$, where "$\diamond_3$" is definded by Table 3. Then $(H_2; \diamond_3, 1)$ satisfies (HCI_1) and (HCI_3). Since $\alpha \not\preceq \alpha$ and $\beta \not\preceq \beta$. Thus,

Table 3: Hyper CI-algebra $(H_2; \diamond_3, 1)$

$\diamond_3$	1	α	β
1	$\{1\}$	$\{1, \alpha\}$	$\{\alpha, \beta\}$
α	$\{1, \alpha\}$	$\{\alpha, \beta\}$	$\{1, \beta\}$
β	$\{1\}$	$\{\alpha\}$	$\{\beta\}$

(HCI_2) does not hold.

(v) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_4, 1)$, where "$\diamond_4$" is definded by Table 4. Then $(H_2; \diamond_4, 1)$ satisfies (HCI_2) and (HCI_3). Since

$$\{\alpha, \beta\} = \alpha \diamond_4 \{1, \beta\} = \alpha \diamond_4 (\beta \diamond_4 1) \neq \beta \diamond_4 (\alpha \diamond_4 1) = \beta \diamond_4 \alpha = \{\alpha\},$$

(HCI_1) is not valid.

Table 4: Hyper CI-algebra $(H_2; \diamond_4, 1)$

$\diamond_4$	1	α	β
1	$\{1\}$	$\{\alpha\}$	$\{\beta\}$
α	$\{\alpha\}$	$\{1, \beta\}$	$\{\beta\}$
β	$\{1, \beta\}$	$\{\alpha\}$	$\{1, \alpha\}$

Table 5: Hyper CI-algebra $(H_2; \diamond_5, 1)$

$\diamond_5$	1	α	β
1	$\{1\}$	$\{1, \alpha\}$	$\{\alpha\}$
α	$\{1, \alpha\}$	$\{1, \alpha\}$	$\{1, \alpha\}$
β	$\{1\}$	$\{1, \alpha\}$	$\{1, \alpha\}$

(vi) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_5, 1)$, where "$\diamond_5$" is definded by Table 5. Then $(H_2; \diamond_5, 1)$ satisfies (HCI_3) and (HCI_1). Since $\alpha \notin \{\beta\} = 1 \diamond_5 \beta$, ($\mathrm{HCI}_3$) does not hold.

Notice that every CI-algebra and hyper BE-algebra are hyper CI-algebra, but the converse is not true.

Example 2.3. In Example 2.2 (iii), (HBE_4) is not valid, since $\alpha \not\preceq 1$.

$\mathcal{H}$ is *proper* if it is not a hyper BE-algebra.

Proposition 2.4. *Let $(H; \diamond, 1)$ be a hyper BE-algebra, where $g \diamond H = H$, for all $g \in H$, and let $\alpha \notin H$. Define the hyper operation $\to$ on H by*

$$r \to k = \begin{cases} r \diamond k; & \text{if } r, k \in H, \\ \alpha; & \text{if } r = \alpha \text{ and } k \neq \alpha, \\ \alpha; & \text{if } r \neq \alpha \text{ and } k = \alpha, \\ H; & \text{if } r = k = \alpha, \end{cases}$$

for all $r, k \in H$. Then $(H \cup \{\alpha\}; \to, 1)$ is a hyper CI-algebra.

Proof. It is enough to prove (HCI_1). Assume $r, k \in H$, since
$\alpha \to (r \to k) = \alpha = r \to \alpha = r \to (\alpha \to k)$,
$r \to (k \to \alpha) = r \to \alpha = \alpha = k \to \alpha = k \to (r \to \alpha)$,
$\alpha \to (r \to \alpha) = \alpha \to \alpha = H = r \diamond H = r \to H = r \to (\alpha \to \alpha)$. Thus, ($\mathrm{HCI}_1$) is valid. The verification of (HCI_2) and (HCI_3) is trivial. This shows that $(H \cup \{\alpha\}; \to, 1)$ is a proper hyper CI-algebra, since $1 \notin \alpha \to 1 = \alpha$. $\square$

Proposition 2.5. *If $k \preceq r$, then $k \preceq (k \diamond r) \diamond k$ and $r \preceq (k \diamond r) \diamond r$.*

Proof. Suppose $k \preceq r$, then $1 \in g \diamond r$. Using (HBE$_2$) and (HBE$_3$), we get

$$1 \in 1 \diamond 1 \subseteq 1 \diamond (k \diamond k) \subseteq (k \diamond r) \diamond (k \diamond k) = k \diamond ((k \diamond r) \diamond k).$$

Therefore, $k \preceq (k \diamond r) \diamond k$. Similarly, the second part is valid. □

Theorem 2.6. Let G, K and F are nonempty subsets of H. Then

(i) $G \diamond (K \diamond F) = K \diamond (G \diamond F)$,

(ii) $G \preceq G$,

(iii) $G \subseteq 1 \diamond G$,

(iv) $a \in G$ implies $a \preceq G$,

(v) $k \preceq g \diamond h$ implies $g \preceq k \diamond h$,

(vi) $k \in g \diamond h$ implies $g \preceq k \diamond h$, for all $g, h, k \in H$.

Proof. (i) Assume $x \in G \diamond (K \diamond F)$. Then there exist $a \in G$, $b \in K$ and $c \in F$ s.t. $x = a \diamond (b \diamond c)$. Using (HCI$_1$), we get $x = b \diamond (a \diamond c)$, and hence, $x \in K \diamond (G \diamond F)$. Thus, $G \diamond (K \diamond F) \subseteq K \diamond (G \diamond F)$. Similarly, $K \diamond (G \diamond F) \subseteq G \diamond (K \diamond F)$. Therefore, (i) is valid.

(ii) Since $G \neq \emptyset$, there is $a \in G$. Using (HCI$_2$), $1 \in a \diamond a$, for all $a \in G$. Hence, $G \preceq G$.

(iii) Let $a \in G$. By (HCI$_3$), $a \in 1 \diamond a \subseteq 1 \diamond G$. Thus, (iii) holds.

(iv) Using (HCI$_2$), the proof is obvious.

(v) Let $k \preceq g \diamond h$, then there exists $s \in g \diamond h$ s.t. $1 \in k \diamond s$. So, $1 \in k \diamond s \subseteq k \diamond (g \diamond h) = g \diamond (k \diamond h)$, by using (HCI$_1$). It shows that $g \preceq k \diamond g$.

(vi) Suppose $k \in g \diamond h$. By (HCI$_2$) and (HCI$_1$), we have

$$1 \in k \diamond k \subseteq k \diamond (g \diamond h) = g \diamond (k \diamond h).$$

Thus, $g \preceq k \diamond h$. □

Proposition 2.7. *The following hold:*

(i) $1 \diamond r \preceq r$,

(ii) $r \preceq 1 \diamond r$,

(iii) $r \preceq 1 \diamond (1 \diamond (\cdots (1 \diamond r) \cdots))$,

(iv) $t \preceq (t \diamond r) \diamond r$, *for all* $r, t \in H$.

Proof. We prove just (iii), as other parts are easily proven. By (HCI_2), (HCI_1) and Theorem 2.6 (iii), we get

$$\begin{aligned} 1 \in 1 \diamond 1 \subseteq 1 \diamond (r \diamond r) \subseteq 1 \diamond (1 \diamond (r \diamond r)) &\subseteq 1 \diamond (1 \diamond (\cdots (1 \diamond (r \diamond r)) \cdots)) \\ &= 1 \diamond (1 \diamond (\cdots (r \diamond (1 \diamond r)) \cdots)) \\ &\vdots \\ &= r \diamond (1 \diamond (1 \diamond (\cdots (1 \diamond r) \cdots))). \end{aligned}$$

This complete the proof. □

Theorem 2.8. There exist 16 hyper CI-algebras of order 2 that are not isomorphic.

Proof. Set $\mathcal{H} = (\{1, \alpha\}; \diamond, 1)$. By ($HCI_2$), $1 \diamond 1 = \{1\}$ or $1 \diamond 1 = \{1, \alpha\}$, and also $\alpha \diamond \alpha = \{1\}$ or $\alpha \diamond \alpha = \{1, \alpha\}$. Also, by ($HCI_3$), $1 \diamond \alpha = \{\alpha\}$ or $1 \diamond \alpha = \{1, \alpha\}$. Also, by ($HCI_1$), we have

(1) $1 \diamond (\alpha \diamond 1) = \alpha \diamond (1 \diamond 1)$,

(2) $1 \diamond (\alpha \diamond \alpha) = \alpha \diamond (1 \diamond \alpha)$.

Case 1: Let $1 \diamond 1 = \{1\}$ and $1 \diamond \alpha = \{\alpha\}$. Hence, (1) and (2) hold and we get Tables 6-11.

Case 2: Let $1 \diamond 1 = \{1\}$ and $1 \diamond \alpha = \{1, \alpha\}$. Then according to (1), $\alpha \diamond 1 \neq \{\alpha\}$. Hence, $\alpha \diamond 1 = \{1\}$ or $\alpha \diamond 1 = \{1, \alpha\}$. If $\alpha \diamond 1 = \{1\}$, we get Tables 12 and 13. If $\alpha \diamond 1 = \{1, \alpha\}$ by (2) we imply $\alpha \diamond \alpha = \{1, \alpha\}$ and we get Table 14.

Case 3: Let $1 \diamond 1 = \{1, \alpha\}$ and $1 \diamond \alpha = \{\alpha\}$. Then by (1) and (2), $\alpha \diamond 1 \neq \{\alpha\}$ and $\alpha \diamond \alpha = \{1, \alpha\}$. Hence, $\alpha \diamond 1 = \{1\}$ or $\alpha \diamond 1 = \{1, \alpha\}$ and we get Tables 15 and 16.

Case 4: Let $1 \diamond 1 = \{1, \alpha\}$ and $1 \diamond \alpha = \{1, \alpha\}$. Then

(i) $\alpha \diamond \alpha = \{1\}$. Then by (1), $\alpha \diamond 1 \neq 1$. Hence, $\alpha \diamond 1 = \{\alpha\}$ or $\alpha \diamond 1 = \{1, \alpha\}$ and we get Tables 17 and 18.

(ii) $\alpha \diamond \alpha = \{1, \alpha\}$. Then (1) and (2) are valid and we get Tables 19-21.

It is easy to check that the resulting tables are not isomorphic. Therefore, there exist 16 hyper CI-algebras of order 2 in the form of Tables 6-21 below, that are not isomorphic.

Table 6: Hyper CI-algebra $(H; \diamond_6, 1)$

$\diamond_6$	1	α
1	$\{1\}$	$\{\alpha\}$
α	$\{1\}$	$\{1\}$

□

Table 7: Hyper CI-algebra $(H;\diamond_7,1)$

$\diamond_7$	1	α
1	$\{1\}$	$\{\alpha\}$
α	$\{1\}$	H

Table 8: Hyper CI-algebra $(H;\diamond_8,1)$

$\diamond_8$	1	α
1	$\{1\}$	$\{\alpha\}$
α	$\{\alpha\}$	$\{1\}$

Theorem 2.9. Let $(K;\diamond,1_K)$ and $(G;\bullet,1_G)$ be two hyper CI-algebras. Then $(K\times G;\odot,1_K\times 1_G)$ is a hyper CI-algebra, where $K\times G$ is Cartesian product K and G, and $\odot$ is defined by $(k_1,g_1)\odot(k_2,g_2)=(k_1\diamond k_2,g_1\bullet g_2)$, for all $k_1,k_2\in K$ and $g_1,g_2\in G$.

Definition 2.10. Let $g\in H$. $\mathcal{H}$ is said to satisfy the

- *diagonal* (briefly, D-hyper), if $g\diamond g=\{1\}$,
- *row* (briefly, R-hyper), if $1\diamond g=\{g\}$.

Theorem 2.11. Let $g\in H$ and $\mathcal{H}$ satisfy D-hyper. If $g\diamond 1=\{g\}$, then $\mathcal{H}$ satisfy R-hyper.

Proof. It is enough to prove, $1\diamond g=\{g\}$. Assume $h\in 1\diamond g$, then

$$h\in 1\diamond g=1\diamond(g\diamond 1)=g\diamond(1\diamond 1)=g\diamond 1=\{g\}.$$

Therefore, $\mathcal{H}$ satisfy R-hyper. □

3. Hyper filters of hyper CI-algebras

In this section, we describe the concepts of weak hyper filter and hyper filter of $\mathcal{H}$ and prove some properties in this respect.

Definition 3.1. Let $F\subseteq H$ and $1\in F$. Then F is said to be

(i) a *hyper filter* of $\mathcal{H}$ if $x\in F$ and $x\diamond y\approx F$ imply $y\in F$,

(ii) a *weak hyper filter* of $\mathcal{H}$ if $x\in F$ and $x\diamond y\subseteq F$ imply $y\in F$,
for all $x,y\in H$, where $x\diamond y\approx F$ means that $x\diamond y\cap F\neq\emptyset$.

Denote the set of all (weak) hyper filters of $\mathcal{H}$ by $\mathcal{HF}(\mathcal{H})$ (resp. $\mathcal{WHF}(\mathcal{H})$).

Table 9: Hyper CI-algebra $(H; \diamond_9, 1)$

$\diamond_9$	1	α
1	$\{1\}$	$\{\alpha\}$
α	$\{\alpha\}$	H

Table 10: Hyper CI-algebra $(H; \diamond_{10}, 1)$

$\diamond_{10}$	1	α
1	$\{1\}$	$\{\alpha\}$
α	H	$\{1\}$

Example 3.2. (i) In Example 2.2 (iii), $F = \{1, \beta\} \in \mathcal{WHF}(\mathcal{H})$.

(ii) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_{22}, 1)$ where, "$\diamond_{22}$" is definded by Table 22. Then $F = \{1, \alpha\} \in \mathcal{HF}(\mathcal{H})$.

Proposition 3.3. *Let $F \in \mathcal{HF}(\mathcal{H})$, then*

(i) $F \in \mathcal{WHF}(\mathcal{H})$,

(ii) *if $x \in F$ and $x \diamond A \approx F$ imply $A \approx F$, for $x \in H$ and $A \subseteq H$.*

Example 3.4. In Example 2.2 (iii), $\{1, \alpha\} \in \mathcal{WHF}(\mathcal{H})$, but $\{1, \alpha\} \in \mathcal{HF}(\mathcal{H})$, since $\{\alpha, \beta\} = \alpha \diamond_2 \beta \approx \{1, \alpha\}$ and $\alpha \in \{1, \alpha\}$, but $\beta \notin \{1, \alpha\}$.

Theorem 3.5. Let $F \subseteq H$ and $1 \in F$, then

(i) $F \in \mathcal{HF}(\mathcal{H})$ if and only if $F \preceq g \diamond h$ and $g \in F$ implies $h \in F$, for all $g, h \in H$.

(ii) if $g \diamond h \preceq F$ and $g \in F$ implies $h \in F$, for all $g, h \in H$, then $F \in \mathcal{HF}(\mathcal{H})$.

Proof. (i) Assume $F \in \mathcal{HF}(\mathcal{H})$, $F \preceq g \diamond h$ and $g \in F$, then there exist $x \in F$ and $y \in g \diamond h$ s.t. $x \preceq y$. It follows that $1 \in x \diamond y$. Thus, $1 \in x \diamond y \cap F \neq \emptyset$. Hence $h \in F$. The converse implication is obvious.

(ii) By Proposition 3.3 (i), it is enough to prove $F \in \mathcal{HF}(\mathcal{H})$. Let $g, h \in H$ s.t. $t \in g \diamond h \cap F$ and $g \in F$. This shows that $1 \in t \diamond t \subseteq (g \diamond h) \diamond F$ and therefore, $g \diamond h \preceq F$. By assumption $h \in F$. Therefore, $F \in \mathcal{HF}(\mathcal{H})$. $\square$

Let $g, h \in H$. Define

$$\tilde{U}(g, h) = \{t \in H |\ t = 1 \text{ } or \text{ } g \preceq h \diamond t\} \text{ and } \tilde{U}(g) = \{t \in H |\ t = 1 \text{ } or \text{ } g \preceq t\}.$$

Example 3.6. In Example 3.2 (ii), $\tilde{U}(\alpha) = \{1, \alpha\}$, $\tilde{U}(\beta) = \{1, \beta\}$ and $\tilde{U}(\alpha, \beta) = \{1, \beta\}$.

Table 11: Hyper CI-algebra $(H; \diamond_{11}, 1)$

$\diamond_{11}$	1	α
1	$\{1\}$	$\{\alpha\}$
α	H	H

Table 12: Hyper CI-algebra $(H; \diamond_{12}, 1)$

$\diamond_{12}$	1	α
1	$\{1\}$	H
α	$\{1\}$	$\{1\}$

Proposition 3.7. *The following hold:*

- (i) $1 \in \tilde{U}(g)$,
- (ii) $g \in \tilde{U}(g)$,
- (iii) $g \in \tilde{U}(g, 1)$,
- (iv) $\tilde{U}(g) \subseteq \tilde{U}(1, g)$,
- (v) $\tilde{U}(g, h) = \tilde{U}(h, g)$,
- (vi) *if* $h \preceq 1$, *then* $\tilde{U}(g) \subseteq \tilde{U}(g, h)$,
- (vii) *if* $\tilde{U}(g) \in \mathcal{HF}(\mathcal{H})$, *then* $\tilde{U}(g, h) \subseteq \tilde{U}(g)$; *for all* $g, h \in H$.

Theorem 3.8. Let $\emptyset \neq F \subseteq H$, then $F \in \mathcal{HF}(\mathcal{H})$ if and only if $\tilde{U}(g, h) \subseteq F$, for all $g, h \in F$.

Proof. Assume $F \in \mathcal{HF}(\mathcal{H})$, $g, h \in F$ and $t \in \tilde{U}(g, h)$. Then $t = 1$ or $g \preceq h \diamond t$. If $t = 1$, then $t \in F$. If $g \preceq h \diamond t$, then $1 \in g \diamond (h \diamond t)$ and $g \diamond (h \diamond x) \approx F$. By Proposition 3.3 (ii) and $g \in F$, $h \diamond t \approx F$. As a result, since $F \in \mathcal{HF}(\mathcal{H})$ and $h \in F$, $t \in F$, we obtain $\tilde{U}(g, h) \subseteq F$.

Now, let $\tilde{U}(g, h) \subseteq F$ for all $g, h \in F$. Let $t \diamond s \approx F$ and $t \in F$. Hence, there exists $z \in (t \diamond s) \cap F$. Thus, $1 \in z \diamond z \subseteq z \diamond (t \diamond s) \subseteq \tilde{U}(z, t) \subseteq F$, and consequently $F \in \mathcal{HF}(\mathcal{H})$. □

Theorem 3.9. Let $F \in \mathcal{HF}(\mathcal{H})$, then $F = \bigcup_{g,h \in F} \tilde{U}(g, h)$.

Table 13: Hyper CI-algebra $(H; \diamond_{13}, 1)$

$\diamond_{13}$	1	α
1	$\{1\}$	H
α	$\{1\}$	H

Table 14: Hyper CI-algebra $(H; \diamond_{14}, 1)$

$\diamond_{14}$	1	α
1	$\{1\}$	H
α	H	H

Proof. By Theorem 3.8, $\tilde{U}(g,h) \subseteq F$ for all $g,h \in H$. Then $\bigcup_{g,h\in F} \tilde{U}(g,h) \subseteq F$. Now, let $t \in F$. By Proposition 3.7 (i) and (ii), $t \in \tilde{U}(t) \subseteq \tilde{U}(1,t) \subseteq \bigcup_{g,h\in F} \tilde{U}(g,h)$. Hence, $F \subseteq \bigcup_{g,h\in F} \tilde{U}(g,h)$. Thus, $F = \bigcup_{g,h\in F} \tilde{U}(g,h)$. □

Theorem 3.10. Let $F \in \mathcal{HF}(\mathcal{H})$, then $F = \bigcup_{g\in F} \tilde{U}(g)$.

Proof. Let $t \in \bigcup_{g\in F} \tilde{U}(g)$, then $t \in \tilde{U}(g)$. This shows that $g \preceq t$ and $1 \in g \diamond t$. Hence $a \diamond t \cap F \neq \emptyset$. Thus, $t \in F$ and it implies $\tilde{U}(g) \subseteq F$. □

Proposition 3.11. *(i) If $\mathcal{H}$ satisfy D-hyper, then $\{1\}$ is closed,*

(ii) $\{1\}$, $H \in \mathcal{HF}(\mathcal{H}) \cap \mathcal{WHF}(\mathcal{H})$.

In general $\{1\}$ is not closed, see $(H; \diamond_{15}, 1)$.

Proposition 3.12. *If $F_i \in \mathcal{HF}(\mathcal{H})$, for all $i \in \Lambda$, then $\bigcap_{i\in\Lambda} F_i \in \mathcal{HF}(\mathcal{H})$.*

The union of two weak hyper filters of $\mathcal{H}$, may not be a weak hyper filter.

Example 3.13. In Example 2.2 (ii), we have $F_1 = \{1, \alpha\}$, $F_2 = \{1, \beta\} \in \mathcal{WHF}(\mathcal{H})$, but $F_1 \cup F_2 = \{1, \alpha, \beta\} \notin \mathcal{WHF}(\mathcal{H})$, since $y \diamond_1 \gamma = \{1, \alpha\} \subseteq F_1 \cup F_2$, but $\gamma \notin F_1 \cup F_2$.

Given $\emptyset \neq G \subseteq H$, the *hyper filter generated* by G is the least hyper filter of $\mathcal{H}$ which contains G and denote by $[G)$. Obviously, $[1) = \{1\}$ and $[H) = H$.

Table 15: Hyper CI-algebra $(H; \diamond_{15}, 1)$

$\diamond_{15}$	1	α
1	H	$\{\alpha\}$
α	$\{1\}$	H

Table 16: Hyper CI-algebra $(H; \diamond_{16}, 1)$

$\diamond_{16}$	1	α
1	H	$\{\alpha\}$
α	H	H

Theorem 3.14. Let $\mathcal{H}$ be good and $\emptyset \neq G \subseteq H$. Then
$[G) = \{t \in H : t = 1 \text{ } or \text{ } 1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond t)...)), \exists\, r_1, r_2, ..., r_n \in G, n \in \mathbb{N}\}$.

Proof. We take

$$T := \{t \in H : t = 1 \text{ } or \text{ } 1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond t)...)), \exists\, r_1, r_2, ..., r_n \in G, n \in \mathbb{N}\}.$$

It is enough to prove that T is the least hyper filter of $\mathcal{H}$ containing G. Given $r \in G$, $1 \in r \diamond r$, we get $r \in T$. Hence, $G \subseteq T$. Clearly, $1 \in T$, since $1 \in 1 \diamond 1$. Let $a \diamond b \approx T$ and $a \in T$. Also, suppose $z \in a \diamond b \cap T$. If $z = 1$, then $1 \in a \diamond b$ and since $a \in T$, we obtain $b \in T$. If $z \neq 1$, since $z \in T$, then there exist $r_1, r_2, ..., r_n \in G$, s.t.
$1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond z)...))$. Since $z \in a \diamond b$, one has

$$1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond z)...)) \subseteq r_1 \diamond (r_2 \diamond (...(r_n \diamond (a \diamond b))...),$$

and so $1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond (a \diamond b))...)$.
If $a = 1$, since $1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond (1 \diamond b))...)$ and $b \in 1 \diamond b$, we obtain $b \in T$.

If $a \neq 1$ and $a \in T$, then there exist $h_1, h_2, ..., h_m \in G$ s.t.

$$1 \in h_1 \diamond (h_2 \diamond (...(h_m \diamond a)...)).$$

On the other hand, using (HCI$_1$), we get

$$1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond (a \diamond b))...) = a \diamond (r_1 \diamond (r_2 \diamond (...(r_n \diamond b))...).$$

Hence, $a \preceq (r_1 \diamond (r_2 \diamond (...(r_n \diamond b))...)$. Since $\mathcal{H}$ is good, we obtain

$$h_1 \diamond (h_2 \diamond (...(h_m \diamond a)...)) \preceq h_1 \diamond (h_2 \diamond (...(h_m ... \diamond (r_1 \diamond (r_2 \diamond (...(r_n \diamond b)))...).$$

Since, $1 \in h_1 \diamond (h_2 \diamond (...(h_m \diamond a)...))$, we have

$$1 \in h_1 \diamond (h_2 \diamond (...(h_m ... \diamond (r_1 \diamond (r_2 \diamond (...(r_n \diamond b)))...).$$

Table 17: Hyper CI-algebra $(H; \diamond_{17}, 1)$

$\diamond_{17}$	1	α
1	H	H
α	$\{\alpha\}$	$\{1\}$

Table 18: Hyper CI-algebra $(H; \diamond_{18}, 1)$

$\diamond_{18}$	1	α
1	H	H
α	H	$\{1\}$

Therefore, $b \in T$, and so $T \in \mathcal{HF}(\mathcal{H})$.

Now, let $F \in \mathcal{HF}(\mathcal{H})$ and $G \subseteq F$. We show that $T \subseteq F$. Let $t \in T$. If $t = 1$, then $t \in F$. If $t \neq 1$, then there exist $r_1, r_2, ..., r_n \in G$, s.t. $1 \in r_1 \diamond (r_2 \diamond (...(r_n \diamond t)...))$. Then $r_1 \diamond (r_2 \diamond (...(r_n \diamond t)...)) \cap F \neq \emptyset$. Since $r_1 \in G \subseteq F$ and $F \in \mathcal{HF}(\mathcal{H})$, by Proposition 3.3 (ii), we get $r_2 \diamond (...(r_n \diamond t)...) \cap F \neq \emptyset$. Using Proposition 3.3 (ii), n times, we have $x \in F$. Thus, $[G) = T$. □

In Theorem 3.14 the condition good is necessary, see the following example.

Example 3.15. We take $T = \{t \in H_1 : t = 1 \text{ or } 1 \in 1 \diamond_2 (1 \diamond_2 (...(1 \diamond_2 t)...))\}$, in Example 2.2 (ii). Then $T = \{1, \beta\}$, but $[1) = \{1\} \neq \{1, \beta\} = T$.

Corollary 3.16. *If $\mathcal{H}$ is good and $g \in H$, then*

$$[g) = \{t \in H : t = 1 \text{ or } 1 \in g \diamond (g \diamond (...(g \diamond t)...))\}.$$

Theorem 3.17. Let $\mathcal{H}$ be good, $F \in \mathcal{HF}(\mathcal{H})$ and $g \in H - F$. Then

$$[F \cup \{g\}) = \{t \in H : t = 1 \text{ or } g \diamond (g \diamond (...(g \diamond t)...)) \cap F \neq \emptyset\}.$$

Proof. Similar to [5, Theorem 3. 2]. □

Definition 3.18. Let $\emptyset \neq I \subseteq H$. Then I is said to be a *hyper ideal* of $\mathcal{H}$, if

(I_1) $g \in H$ and $i \in I$, then $g \diamond i \subseteq I$,

(I_2) $g \in H$ and $i, j \in I$, then $(i \diamond (j \diamond g)) \diamond g \subseteq I$.

Example 3.19. (i) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_{23}, 1)$, where "$\diamond_{23}$" is definded by Table 23. Then $I = \{1, \alpha\} \in \mathcal{HI}(\mathcal{H})$.

(ii) Consider Example 3.2 (ii) and take $I = \{1, \beta\}$. Then (I_1) is valid, but does not satisfy (I_2), since $(\beta \diamond_{22} (\beta \diamond_{22} \alpha)) \diamond_{22} = \{1, \alpha\} \not\subseteq I$.

Table 19: Hyper CI-algebra $(H; \diamond_{19}, 1)$

$\diamond_{19}$	1	α
1	H	H
α	$\{1\}$	H

Table 20: Hyper CI-algebra $(H; \diamond_{20}, 1)$

$\diamond_{20}$	1	α
1	H	H
α	$\{\alpha\}$	H

(iii) Consider the algebra $(H_2 = \{1, \alpha, \beta\}; \diamond_{24}, 1)$, where "$\diamond_{24}$" is definded by Table 24. Then $I = \{1, \alpha\}$ satisfies (I_2), but does not satisfy (I_1), since $\beta \diamond_{24} \alpha = \{\beta\} \nsubseteq I$.

Denote the set of all hyper ideals of $\mathcal{H}$ by $\mathcal{HI}(\mathcal{H})$.

Proposition 3.20. *Let $I \in \mathcal{HI}(\mathcal{H})$.*

(*i*) $1 \in I$,

(*ii*) $I \in \mathcal{HF}(\mathcal{H})$.

Proof. (i) Assume $i \in I$. Since $I \in \mathcal{HI}(\mathcal{H})$ and $1 \in i \diamond i \subseteq I$, we get $1 \in I$.
(ii) Suppose $I \in \mathcal{HI}(\mathcal{H})$, then $I \neq \emptyset$. Let $i \in I$, then $1 \in i \diamond i \subseteq I$. Therefore, $1 \in I$. Now, suppose $g, h \in H$ s.t. $g \diamond h \cap I \neq \emptyset$ and $g \in I$. Let $s \in g \diamond h \cap I$. By ($\mathrm{HCI}_3$) and ($\mathrm{I}_2$), we have

$$h \in 1 \diamond h \subseteq (s \diamond s) \diamond h \subseteq (s \diamond (g \diamond h)) \diamond h \subseteq I.$$

Thus, $I \in \mathcal{HF}(\mathcal{H})$. □

Every hyper filter is not a hyper ideal, as the following example shows:

Example 3.21. In Example 3.2 (ii), $F = \{1, \alpha\} \in \mathcal{HF}(\mathcal{H})$, but does not an hyper ideal, since $\beta \diamond_{22} \alpha = \{\beta\} \nsubseteq F$.

4. On commutative hyper CI-algebras

Now, we consider the commutative hyper CI-algebras, and enumerate them of order 2.

Definition 4.1. $\mathcal{H}$ is said to be *commutative* if $(h \diamond g) \diamond g = (g \diamond h) \diamond h$, for all $g, h \in H$.

Table 21: Hyper CI-algebra $(H;\diamond_{21},1)$

$\diamond_{21}$	1	α
1	H	H
α	H	H

Table 22: Hyper CI-algebra $(H_2;\diamond_{22},1)$

$\diamond_{22}$	1	α	β
1	$\{1\}$	$\{1,\alpha\}$	$\{\beta\}$
α	$\{1\}$	$\{1\}$	$\{\beta\}$
β	$\{\beta\}$	$\{\beta\}$	$\{1\}$

Example 4.2. (i) In Example 2.2 (i), $(A;\diamond,c)$ is commutative.
(ii) Consider the algebra $(H_2=\{1,\alpha,\beta\};\diamond_{25},1)$, where "$\diamond_{25}$" is definded by Table 25. Then $(H_2;\diamond_{25},1)$ is commutative.

Proposition 4.3. *Let $\mathcal{H}$ be commutative and satisfies* (HBE$_4$). *Then $\mathcal{H}$ is a hyper BE-algebra.*

Proof. Assume $H=\{1,g\}$. It is enough to prove $1\in g\diamond 1$. On the contrary, suppose $1\notin g\diamond 1$. Therefore, $g\diamond 1=g$. On the other hand, by (HCI$_2$), (HCI$_3$) and commutative law, we obtain

$$1\in g\diamond g\subseteq (1\diamond g)\diamond g=(g\diamond 1)\diamond 1=g\diamond 1,$$

which is a contradiction. Therefore, $1\in g\diamond 1$ and so $\mathcal{H}$ is a hyper BE-algebra. □

The Proposition 4.3 is valid for the commutative hyper CI-algebra of order 2, and also (HBE$_4$) is necessary.

Example 4.4. (i) Consider $H=\{1,g,h\}$ and define "$\diamond_{26}$" by Table 26. Then $(H;\diamond_{26},1)$ is commutative of order 3 and satisfies (HBE$_4$), but not a hyper BE-algebra, since $1\not\preceq g$.
(ii) Consider Table 18. Then $(H;\diamond_{18},1)$ is commutative, but does not satisfy (HBE$_4$).

Proposition 4.5. *Let $\mathcal{H}$ satisfy R-hyper and be commutative. Then $g\diamond h=h\diamond g=\{1\}$ implies $g=h$.*

Proposition 4.6. *Let $\mathcal{H}$ be commutative. Then $(g\diamond 1)\preceq 1$, for all $g\in H$.*

Notice that in Proposition 4.6 we can not remove the commutative law.

Table 23: Hyper CI-algebra $(H_2; \diamond_{23}, 1)$

$\diamond_{23}$	1	α	β
1	$\{1\}$	$\{\alpha\}$	$\{\beta\}$
α	$\{1, \alpha\}$	$\{1\}$	$\{\beta\}$
β	$\{1, \alpha\}$	$\{1\}$	$\{1, \alpha\}$

Table 24: Hyper CI-algebra $(H_2; \diamond_{24}, 1)$

$\diamond_{24}$	1	α	β
1	$\{1\}$	$\{\alpha\}$	$\{\beta\}$
α	$\{1, \alpha\}$	$\{1\}$	$\{\beta\}$
β	$\{\beta\}$	$\{\beta\}$	$\{1, \alpha\}$

Example 4.7. In Example 2.2 (iii), $(H; \diamond_2, 1)$ is not commutative, since $H = (1 \diamond_2 x) \diamond_2 x \neq (x \diamond_2 1) \diamond_2 1 = \{y\}$. Also, we have $1 \notin (y \diamond_2 1) \diamond_2 1 = \{y\}$, and $y = (y \diamond_2 1) \not\preceq 1$.

Theorem 4.8. Let $\mathcal{H}$ be commutative. If for all $g \in H$, $g \preceq 1$ then $g \diamond h \preceq (h \diamond k) \diamond (g \diamond k)$, for all $g, h, k \in H$.

Proof. Given $t \in k \diamond h$, we have

$$\begin{aligned} 1 \in t \diamond 1 \subseteq (k \diamond h) \diamond 1 \subseteq (k \diamond h) \diamond [(g \diamond h) \diamond (g \diamond h)] &= (g \diamond h) \diamond [(k \diamond h) \diamond (g \diamond h)] \\ &= (g \diamond h) \diamond [g \diamond ((k \diamond h) \diamond h)] \\ &= (g \diamond h) \diamond [g \diamond ((h \diamond k) \diamond k))] \\ &= (g \diamond h) \diamond [(h \diamond k) \diamond (g \diamond k)]. \end{aligned}$$

It follows that $g \diamond h \preceq (h \diamond k) \diamond (g \diamond k)$. □

Theorem 4.9. There exist 7 commutative hyper CI-algebras of order 2 that are not isomorphic.

Proof. Using Theorem 2.8, there are 16 hyper CI-algebras $(H; \diamond_i, 1)$ of order 2, for $i \in \{6, 7, \cdots, 13\}$. One can see that the hyper operations $\diamond_6$, $\diamond_{11}$, $\diamond_{14}$, $\diamond_{15}$, $\diamond_{16}$, $\diamond_{18}$ and $\diamond_{21}$ satisfy the commutative law. □

5. Conclusions

We have defined the notion of a (*proper*) *hyper CI-algebra*, and presented some results in this respect. Also, (*weak*) *hyper filters* in this structure are studied. It is

Table 25: Hyper CI-algebra $(H_2; \diamond_{25}, 1)$

$\diamond_{25}$	1	α	β
1	$\{1\}$	$\{1, \alpha\}$	$\{\beta\}$
α	H_2	H_2	$\{\beta\}$
β	$\{1, \alpha\}$	H_2	H_2

Table 26: Hyper CI-algebra $(H; \diamond_{26}, 1)$

$\diamond_{26}$	1	α	β
1	$\{1\}$	$\{\alpha\}$	$\{\beta\}$
α	$\{\beta\}$	$\{1, \beta\}$	$\{1\}$
β	$\{1, \beta\}$	$\{\beta\}$	$\{1, \beta\}$

shown that there exist 16 hyper CI-algebra, and 7 commutative hyper CI-algebra of order less than 3, up to isomorphism. In future work, we will investigate among filters of hyper CI-algebras and characterize hyper CI-algebras in cases $|H| = 3$ and 4.

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