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On w-Neat Rings

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Abstract

In this paper, a novel generalization of the neat ring known as w-neat ring is investigated. Let R be a ring, R is called cleaned poorly (weakly clean), if for every $x \in R$, we have $x = u + e$ or $x = u - e$, where $u \in \mathrm{U}(R)$ and $e \in \mathrm{Id}(R)$. In particular, if all homomorphic images of R are considered cleaned poorly, then R is said to be w-neat. We present some properties of w-neat rings.

Keywords: Clean ring, Cleaned poorly ring, w-Neat ring.

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1. Introduction

Assume that R is a commutative ring that has an identity. If for every $x \in R$ with $x = u + e$ where $u \in \mathrm{U}(R)$ and $e \in \mathrm{Id}(R)$, then R is clean [1]. Every clean ring is considered an exchange ring [1]. Also, if all proper homomorphic images are clean, then R is neat [2]. For every $x \in R$, $x = u + e$ or $x = u - e$ where $u \in \mathrm{U}(R)$ and $e \in \mathrm{Id}(R)$, then R is cleaned poorly (weakly clean) [3–6]. In [3] it is shown that all homomorphic images on cleaned poorly ring is again cleaned poorly. So a w-neat ring is defined. If all proper homomorphic images of R is cleaned poorly, then R is a w-neat. We will obtain some properties of w-neat rings.

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2. Main results

Since all the homomorphic images of a cleaned poorly ring is cleaned poorly, a w-neat ring is defined as follows.

Definition 2.1. *Assume that R is a ring. Then R is called w-neat if every proper homomorphic image is a cleaned poorly ring.*

It is clear that all neat ring is w-neat. However by the next example the converse is not generally holds.

Example 2.2. Let $R = \mathbb{Z}_{(3)} \cap \mathbb{Z}_{(5)} = \{x/s \mid x, s \in \mathbb{Z},\ s \neq 0\ , 3 \nmid s,\ 5 \nmid s\}$. Then by [3], R is a cleaned poorly ring. Since all the homomorphic image on a cleaned poorly ring is again cleaned poorly, R is a w-neat ring. But, R is not considered as clean because a indecomposable ring is considered local, by [7, Theorem 3]. Thus, R is not a neat ring.

Lemma 2.3. Let I be an ideal of R. Then, R/I is a w-neat ring.

Proof. It is straightforward. □

Let $P_0 \subset P_1 \subset \cdots \subset P_n$ be a chain of prime ideals of length n. Then the supremum of all chains of prime ideals length in R is Krull dimension of R. The Krull dimension of a R ring is indicated by $dim(R)$ [8].

Lemma 2.4. Assume that R is a domain of $dim(R) = 1$. Then, R is w-neat.

Proof. Given that R is a $dim(R) = 1$ domain, the Krull dimension of all homomorphic image of R is equal to zero. Hence every homomorphic image in R is considered cleaned poorly, by [7, Corollary 11]. Therefore, R is w-neat. □

Corollary 2.5. Every PID is a w-neat ring.

Proof. According to Lemma 2.4, it is obtained. □

The following exapmle shows that every w-neat ring is generally not to be a cleaned poorly ring.

Example 2.6. Assume A is a field and $R = A[x, y]$. Hence $R/Ry \cong A[x]$ is considered not cleaned poorly, by [3, Theorem 1.9]. So R is not w-neat. Therefore, $A[x]$ is w-neat by Lemma 2.4 which is not weakly clean.

Lemma 2.7. Suppose that R is a w-neat ring considered not cleaned poorly. Then R is reduced.

Proof. Assume that R is a w-neat ring which is considered not cleaned poorly and $Nil(R) \neq 0$. Because R is a w-neat ring, $R/Nil(R)$ is cleaned poorly. So by [3, Theorem 1.9] R is cleaned poorly, which is impossible. Therefore, $Nil(R) = 0$. □

Theorem 2.8. With a ring R, the sentences below are the same:

(1) R is a w-neat ring.

(2) The ring R/xR is cleaned poorly for every $0 \neq x \in R$.

(3) If $\{P_\lambda\}_{\lambda\in\Lambda}$ is a family of nonzero prime ideals of R and $Q = \bigcap_{\lambda\in\Lambda} P_\lambda \neq 0$, then R/Q is considered cleaned poorly.

(4) The ring R/xR is w-neat for every $x \in R$.

(5) R/Q is a cleaned poorly ring for all nonzero semiprime ideal Q of R.

Proof. Similar to [2, Proposition 2.1]. □

Proposition 2.9. If $R = A \oplus B$ for a few A and B ideal of R so that either A or B is not clean, then R is w-neat only when R is cleaned poorly.

Proof. Assume that there are nonzero ideals A and B of R so that $R = A \oplus M$. Let R be a w-neat ring. Then $B \cong R/A$ and $A \cong R/B$ are cleaned poorly, and thus R is a product directly from cleaned poorly rings. Therefore by [3, Theorem 1.7], R cleaned poorly. Conversely, is clear. □

Assume that M is an R-module and R is a ring. If all the family of cosets attaining limited intersection property with nonempty intersection, then M is an R-module that is compact linearly. It is clear that a homomorphic image of an R-module is compact linearly [8]. If R is a linearly compact R-module, R is said to be maximal. Artinian rings are maximal. If R/A is a R-module that is linearly compact for all nonzero ideal A of R, then R is said to be almost maximal [8, 9].

Let M be an R-module. If every family of cosets with the finite intersection property has nonempty intersection, then M is called a linearly compact R-module. It is clear that a homomorphic image of a linearly compact R-module is linearly compact [8]. If R is a linearly compact R-module, then R is said to be maximal. It is clear that Artinian rings are maximal. If R/A is a linearly compact R-module for every nonzero ideal A of R, then R is said to be almost maximal [8, 9].

Theorem 2.10 (Zelinsky)**.** With R as a maximal ring, then $R = R_1 \times \cdots \times R_n$ so that all $R_i (1 \leq i \leq n)$ is considered a local ring.

Corollary 2.11. If R is a maximal ring, then R is cleaned poorly. Moreover, if R is an almost maximal ring, then R is w-neat.

Proof. By Theorem 2.10, $R = R_1 \times \cdots \times R_n$. Thus, every $R_i (1 \leq i \leq n)$ is a local ring. Since every local ring is cleaned poorly, by [7, Proposition 2], every maximal ring is cleaned poorly and every almost maximal ring is w-neat. □

It is known that if all prime ideal of a ring R is limited to a maximal ideal that is unique, then R is a pm-ring [10].
Assume R is a ring. If all elements in R are limited to a finite number of maximal ideals and every proper homomorphic image of R is a pm-ring, then R is h-local

[8]. Also, if each limited generated ideal of R is principal, then R is a Bezout ring [8].
A ring R is said to be a torch ring if it meets:

(1) R is not local.

(2) There exists one minimal prime ideal P of R which is unique where is not zero and the R-submodule creates a chain.

(3) R/P is an h-local domain.

(4) R is almost locally maximal Bezout ring.

To study the examples of a torch ring, see [8].

Theorem 2.12. If R is a commutative torch ring so that $\mathrm{Id}(R) = \{1\}$ and $2 \in \mathrm{U}(R)$, then R is never w-neat.

Proof. Assume P is minimal unique prime ideal of a torch ring R. Suppose that R is w-neat. Hence R/P is a cleaned poorly ring which $\mathrm{Id}(R/P) = \{1 + P\}$ and $2 + P \in \mathrm{U}(R/P)$. Therefore by [3, Theorem 1.6], R is a local ring, which is a contradiction. □

Assume R a ring and all finitely generated R-module $M \cong \oplus K_i$ such that every K_i is a cyclic R-module. Then R is considered an FGC ring [11].

Theorem 2.13 (Brandal)**.** A ring R is an FGC-ring if and only if $R = R_1 \times \cdots \times R_n$ such that, among the following sentences, one is true.

(1) Every $R_i (1 \leq i \leq n)$ is a maximal valuation ring.

(2) Every $R_i (1 \leq i \leq n)$ is a almost maximal Bezout domain.

(3) Every $R_i (1 \leq i \leq n)$ is a torch ring.

Proof. According to [8, Theorem 9.1], it is obtained. □

Theorem 2.14. Assume that R is a commutative FGC-ring where $\mathrm{Id}(R) = \{1\}$ and $2 \in \mathrm{U}(R)$. Therefore, R is cleaned poorly if and only if $R = R_1 \times \cdots \times R_n$ so that each $R_i (1 \leq i \leq n)$ is a local ring. In particular each $R_i (1 \leq i \leq n)$ are almost maximal valuation ring.

Proof. Assume that $R = R_1 \times \cdots \times R_n$ so that every $R_i (1 \leq i \leq n)$ is considered a local ring. Thus, R is considered a ring, which is cleaned poorly. Moreover, assume that R is a cleaned poorly FGC-ring. Because R is FGC, $R = R_1 \times \cdots \times R_n$ so that $R_i (1 \leq i \leq n)$ is a ring introduced in Theorem 2.13. Since R is cleaned poorly, each $R_i (1 \leq i \leq n)$ is cleaned poorly. Based on Theorem 2.12, there is no torch ring in $R_i (1 \leq i \leq n)$ and thus each $R_i (1 \leq i \leq n)$ is a maximal Bezout domain or maximal valuation ring. By Theorem 2.10, a maximal valuation ring and a cleaned poorly domain is local. Since every local Bezout domain is a valuation domain, R is a finite direct product of almost maximal valuation rings. □

Lemma 2.15. Assume that R is an FGC-ring. Then R is w-neat if and only if R is either or an almost maximal Bezout domain that is not local or cleaned poorly.

Proof. Suppose that R is a w-neat FGC-ring. By Proposition 2.9, R is a cleaned poorly ring. Conversely, assume that R is w-neat that such that R is not cleaned poorly . Thus R is not local and so R is indecomposable. Now, R can be an almost maximal Bezout domain or a maximal valuation ring. However, it may not be a maximal ring as it means it is cleaned poorly. Therefore, R is a non-local almost maximal Bezout domain. □

Corollary 2.16. Every FGC-domain is w-neat.

Conflicts of Interest. The author declare that she has no conflicts of interest regarding the publication of this article.

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