Mathematics Interdisciplinary Research **8** (4) (2023) 327 – 335

Original Scientific Paper

Bogomolov Multiplier and Isoclinism of Lie Rings

Zeinab Araghi Rostami, Mohsen Parvizi⋆ and Peyman Niroomand

Abstract

In the present paper it is shown that Bogomolov multipliers of isoclinic Lie rings are isomorphic. Also, we show that isoclinic finite Lie rings have isoclinic CP covers. Finally, it is proved that if CE_1 and CE_2 are central extensions which are isoclinic, then CE_2 is a CP extension if CE_1 is so.

Keywords: Isoclinism, Curly exterior product, $\tilde{B}_0$-pairing, Bogomolov multiplier, CP extension.

2020 Mathematics Subject Classification: 17B60, 17B99, 20F40.

How to cite this article
Z. A. Rostami, M. Parvizi and P. Niroomand, Bogomolov multiplier and isoclinism of Lie rings, *Math. Interdisc. Res.* **8** (4) (2023) 327-335.

1. Introduction

The Bogomolov multiplier which is a group-theoretical invariant, was introduced as an obstruction to Noether's problem. The latter is an important problem in invariant theory that was discussed by Emmy Noether in [1]. Let W be a faithful representation of a finite group over a field L. Then G acts naturally upon the field $L(W)$, the rational functions. The problem of rationality or Noether's problem asks about rationality of G-invariant functions $L(W)^G$ over a field L. Saltman in [2] presented a new method that was the application of the unramified cohomology group $H^2_{nr}(\mathbb{C}(W)^G, \mathbb{Q}/\mathbb{Z})$ as an obstruction. Then in the case where $L = \mathbb{C}$, some examples of finite p-groups of order p^9 with negative answer to Noether's problem is given by Saltman himself. In 1988, Bogomolov [3] showed that the unramified cohomology group $H^2_{nr}(\mathbb{C}(W)^H, \mathbb{Q}/\mathbb{Z})$ is canonically isomorphic to

⋆Corresponding author (E-mail: parvizi@um.ac.ir)
Academic Editor: Mohammad Ali Iranmanesh
Received 22 October 2020, Accepted 20 July 2022
DOI: 10.22052/MIR.2022.240335.1252

$$B_0(G) = \bigcap \ker\{res_G^B : H^2(G, \mathbb{Q}/\mathbb{Z}) \to H^2(B, \mathbb{Q}/\mathbb{Z})\},$$

where res_G^B is the usual cohomological restriction maps and B is an abelian subgroup of G. The group $B_0(G)$ was named by Kunyavskii in [4] the *Bogomolov multiplier* of G and it is actually a subgroup of the Schur multiplier $\mathcal{M}(G) = H^2(G, \mathbb{Q}/\mathbb{Z})$. Negating examples to Noether's problem over the complex numbers, can be come up when the Bogomolov multiplier is non trivial. But the computition of the Bogomolov multiplier of a group is not an easy matter. Recently, Moravec in [5] presented homological version of the Bogomolov multiplier using a notion of the non abelian exterior square between groups. He proved that for a finite group G, there is a natural isomorphism between $B_0(G)$ and $\mathrm{Hom}(\tilde{B}_0(G), \mathbb{Q}/\mathbb{Z})$. Also the group $\tilde{B}_0(G)$ can be viewed as a section of $G \wedge G$, the non abelian exterior square of a group G. Among other results he proved that $\tilde{B}_0(G)$ is isomorphic to the quotient group $\mathcal{M}(G)/\mathcal{M}_0(G)$, in which, $\mathcal{M}(G)$ the Schur multipluer of the group G, is considered as the kernel of the so called commutator homomorphism $G \wedge G \to [G, G]$ defined as $x \wedge y \to [x, y]$, and $\mathcal{M}_0(G)$ is generated by all $x \wedge y$ for which $[x, y] = 1$. In finite case, there is a natural isomorphism between $\tilde{B}_0(G)$ and $B_0(G)$. With this interpretation, all really nontrivial nonuniversal commutator relations are gathered into an abelian group that is called the Bogomolov multiplier. In addition, the Bogomolov multiplier is related to the probability of commuting two random elements of a group, by Moravec's method. Also this method introduces the important role of Bogomolov multiplier in central extensions of groups which preserves commutativity, which are famous in K-theory. (See [6, 7] for more information). Furthermore, Moravec showed that isoclinic groups have isomorphic Bogomolov multipliers, CP covers of isoclinic groups are isoclinic and if ce_1 and ce_2 are central extensions with ce_1 a CP extension, then e_2 is a CP extension provided that the extensions are both isoclinic. (for more information see [7]). Here, we want to prove similar results for Lie rings. This should be seen as a continuation of our recent work [8, 9], where we developed the analogous theory of commutativity preserving exterior product, Bogomolov multiplier, CP central extension, and CP cover for finite dimensional Lie algebras over a field and finite Lie rings.

2. The Bogomolov multiplier and CP covers of Lie rings

In this section we introduce the notion of Bogomolov multiplier and a kind of cover which preserves commutativity and is called CP cover of Lie rings.

Definition 2.1. A Lie ring is an abelian group L together with a $\mathbb{Z}$-bilinear map $[.,.] : L \times L \to L$ defined on it, which is called the Lie bracket with the following conditions:

- $[a, a] = 0$,
- $[a, [b, c]] + [c, [a, b] + [b, [c, a]] = 0$ and $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$ (the Jacobi identity),
  for all $a, b, c \in L$.

The product ($[a, b]$) is called the commutator of a and b.

Note that we can define a Lie ring as a $\mathbb{Z}$-Lie algebra (see [10]). Also, for a given positive integer k, every Lie algebra over $\mathbb{Z}/p^k\mathbb{Z}$ is called a p-Lie ring (see [11]).

Definition 2.2. Assume that L is a Lie ring and $L \cong F/R$ be a free presentation for it . A function $h : L \times L \to L$, which is bilinear is called a Lie-$\tilde{B_0}$-pairing, provided that

(i) $h(\lambda a, b) = h(a, \lambda b) = \lambda h(a, b)$,

(ii) $h(a + a', b) = h(a, b) + h(a', b)$,

(iii) $h(a, b + b') = h(a, b) + h(a, b')$,

(iv) $h([a, a'], b) = h(a, [a', b]) - h(a', [a, b])$,

(v) $h(a, [b, b']) = h([b', a], b) - h([b, a], b')$,

(vi) $h([b, a], [a', b']) = -[h(a, b), h(a', b')]$,

(vii) If $[a, b'] = 0$, then $h(a, b') = 0$,

for all $\lambda \in \mathbb{Z}$ and $a, a', b, b' \in L$.

Definition 2.3. Let L be a Lie ring with a free presentation $L \cong F/R$. The curly exterior square $L \curlywedge L$ is the Lie ring generated by the symbols $m \curlywedge n$ subject to the following list of relations

(i) $\lambda(a \curlywedge b) = \lambda a \curlywedge b = a \curlywedge \lambda b$,

(ii) $(a + a') \curlywedge b = a \curlywedge b + a' \curlywedge b$,

(iii) $a \curlywedge (b + b') = a \curlywedge b + a \curlywedge b'$,

(iv) $[a, a'] \curlywedge b = a \curlywedge [a', b] - a' \curlywedge [a, b]$,

(v) $a \curlywedge [b, b'] = [b', a] \curlywedge b - [b, a] \curlywedge b'$,

(vi) $[(a \curlywedge b), (a' \curlywedge b')] = -[b, a] \curlywedge [a', b']$,

(vii) If $[a, b] = 0$, then $a \curlywedge b = 0$, for all $\lambda \in \mathbb{Z}$ and $a, a', b, b' \in L$.

It is shown by Ellis in [12] that, the kernel of the map $\kappa : L \wedge L \to L^2$ defined by $a \wedge a' \longmapsto [a, a']$ is actually the Schur multiplier of L. On the other hand, $\mathcal{M}_0(L) \leq \mathcal{M}(L) = \ker \kappa$, where $\mathcal{M}_0(L) =< a \wedge b \mid [a, b] = 1 >$. So there is a homomorphism $\tilde{\kappa} : L \wedge L/\mathcal{M}_0(L) \to L^2$ given by $a \wedge a' + \mathcal{M}_0(L) \longmapsto [a, a']$ and $\ker \tilde{\kappa} \cong \mathcal{M}(L)/\mathcal{M}_0(L)$. Similar to groups, we denote $\mathcal{M}(L)/\mathcal{M}_0(L)$ by $\tilde{B_0}(L)$, and we call it the Bogomolov multiplier for the Lie ring L. Also similar to [8, Theorem 3.6], $L \curlywedge L \cong L \wedge L/\mathcal{M}_0(L)$. So, one may consider the following exact sequence of Lie algebras $0 \to \tilde{B_0}(L) \to L \curlywedge L \to L^2 \to 0$.

In the following, $K(F)$ and F^2 stand for $\{[x, y] \mid x, y \in F\}$ and $[F, F]$, respectively.

Theorem 2.4. *Let L be a Lie ring. If $L \cong F/R$ is a presentation for L, then $\tilde{B_0}(L) \cong (R \cap F^2)/< K(F) \cap R >$.*

Proof. From [12], $L \wedge L \cong F^2/[R, F]$ and $L^2 \cong F^2/(R \cap F^2)$. Moreover $\ker \tilde{\kappa} = \mathcal{M}(L) \cong (R \cap F^2)/[R, F]$ and $\mathcal{M}_0(L)$ can be considered as the subalgebra of $F/[R, F]$ whose generators are all the commutators in $F/[R, F]$ that lie in $\mathcal{M}(L)$. So we have the following isomorphism for $\mathcal{M}_0(L)$

$$< K(\frac{F}{[R,F]}) \cap \frac{R}{[R,F]} >\cong \frac{< K(F) \cap R > +[R,F]}{[R,F]} \cong \frac{< K(F) \cap R >}{[R,F]}.$$

□

In classical group theory we have a kind of central extensions of groups which preserve commutativity. Here we have such extensions and the Bogomolov multiplier is a key tool in studying these kinds of group extensions, and it is unique up to isomorphism. Also, in finite groups theory, commutativity preserving covers was first studied by Jezernik and Moravec [7]. Unlike Bogomolov multiplier, CP covers are not unique in general. Recently in [9], we defined CP cover for finite Lie rings and finite dimensional Lie algebras and proved that for finite dimensional Lie algebras, all CP covers are isomorphic.

Definition 2.5. Let L, M, and C be Lie rings and $0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$, be an exact sequence. It is called a commutativity preserving extension or CP extension of M by L, if the lifts of each commuting pairs of elements of L in C, commute. Furthermore if the kernel is central, it is named to be a central CP extension.

Proposition 2.6. *([8]). Let $ec : 0 \to M \xrightarrow{\chi} C \xrightarrow{\pi} L \to 0$ be a central extension. Then ec is a CP extension if and only if $\chi(M) \cap K(C) = 0$.*

Definition 2.7. Let C and M be finite Lie rings. The pair of Lie rings (C, M) is a commutativity preserving defining pair or CP defining pair for L, if

(i) $L \cong \frac{C}{M}$,

(ii) $M \subseteq Z(C) \cap C^2$,

(iii) $M \cap K(C) = 0$.

Lemma 2.8. *For a finite Lie ring, if C is the first term in a CP defining pair for L, then $|C|$ is bounded.*

Proof. Put $|C| = n$. We know $|C/Z(C)| \leq |C/M| = |L| = n$. By using Lemma 7 in [13] and Proposition 4.2 in [8], C^2 is finite and we have the following exact sequence

$$\tilde{B_0}(C) \xrightarrow{f} \tilde{B_0}(L) \xrightarrow{g} M \to 0.$$

So, $\tilde{B_0}(L)/\mathrm{Imf} \cong M$ and $|M| \leq |\tilde{B_0}(L)|$. Therefore

$$|C| = |L||M| \leq |L||\tilde{B_0}(L)| = n|\tilde{B_0}(L)|.$$

Hence $|C|$ is bounded. □

Now, we have the following definitions:

Definition 2.9. Let C and M be finite Lie rings. The pair (C, N) of Lie rings is called a maximal commutativity preserving defining pair or maximal CP defining pair for L, provided that the order of C is maximal.

Definition 2.10. If (C, M) is a maximal CP defining pair, we call the Lie ring C a commutativity preserving cover or CP cover for L.

In the recent work [9], we proved that for an arbitrary CP defining pair (C, M), the Lie rings M and C are isomorphic to a quotient of the Bogomolov multiplier and CP cover of L, respectively. Also the maximality of the pair (C, M), implies M to be isomorphic to the Bogomolov multiplier of L. In this paper we intend to prove that for a finite Lie ring each two CP covers are isoclinic.

3. Bogomolov multiplier and isoclinism of Lie rings

Hall in [14] presented an equivalence relation on the class of all groups that named isoclinism. In 1994, Moneyhun in [15] defined the same concept for Lie algebras. Now, since Lie rings are $\mathbb{Z}$-Lie algebras, we have similar definitions for Lie rings. Here we want to show Bogomolov multipliers of isoclinic Lie rings are isomorphic.

Definition 3.1. Two Lie rings L_1 and L_2 are isoclinic, if there exist isomorphisms:

$$\alpha : \frac{L_1}{Z(L_1)} \to \frac{L_2}{Z(L_2)}, \qquad \beta : L_1^2 \to L_2^2,$$

such that for all $l_1, l_1' \in L_1$, if $\alpha(l_1 + Z(L_1)) = l_2 + Z(L_2)$ and $\alpha(l_1' + Z(L_1)) = l_2' + Z(L_2)$, then $\beta([l_1, l_1']) = [l_2, l_2']$ where $l_2, l_2' \in L_2$. In this case, the pair (α, β) is called an isoclinism from L_1 to L_2.

Theorem 3.2. *Let L and K be isoclinic Lie rings. Then $\tilde{B_0}(L) \cong \tilde{B_0}(K)$.*

Proof. There are two isomorphisms $\alpha : L/Z(L) \to K/Z(K)$ and $\beta : L^2 \to K^2$ such that for all $l_1, l_2 \in L$ and $k_1, k_2 \in K$, if $\alpha(l_1 + Z(L)) = k_1 + Z(K)$ and $\alpha(l_2 + Z(L)) = k_2 + Z(K)$, then $\beta([l_1, l_2]) = [k_1, k_2]$. Define a map $\Phi : L \times L \to K \curlywedge K$ given by $\Phi(l_1, l_2) = k_1 \curlywedge k_2$. For proving to see that this is well-defined, suppose $\alpha(l_i + Z(L)) = k_i + Z(K) = k'_i + Z(K)$, for $i = 1, 2$. Therefore $k_1 - k'_1, k_2 - k'_2 \in Z(K)$, and we can let $k'_1 = k_1 + z$ and $k'_2 = k_2 + w$ for some $z, w \in Z(K)$. The definition of $K \curlywedge K$ implies that $k'_1 \curlywedge k'_2 = (k_1 + z) \curlywedge (k_2 + w) = k_1 \curlywedge k_2$. So, Φ is well-defined.

Now, let $l, l', s, s' \in L$ and $k, k', k_s, k_{s'} \in K$ such that $\alpha(s + Z(L)) = k_s + Z(K)$, $\alpha(s' + Z(L)) = k_{s'} + Z(K)$, $\alpha(l + Z(L)) = k + Z(K)$, and $\alpha(l' + Z(L)) = k' + Z(K)$. As L and K be isoclinic, then $\beta([l, l']) = [k, k']$ and $\beta([s, s']) = [k_s, k_{s'}]$. Also, we have

$$\Phi(l, [s, s']) = k \curlywedge [k_s, k_{s'}] = ([k_{s'}, k] \curlywedge k_s) - ([k_s, k] \curlywedge k_{s'}) = \Phi([s', l], s) - \Phi([s, l], s'),$$

and

$$\Phi([l, s], [l', s']) = [k, k_s] \curlywedge [k', k_{s'}] = -[k_s \curlywedge k, k_{s'} \curlywedge k'] = -[\Phi(s, l), \Phi(s', l')].$$

Also, let $l, s \in L$ commute, and let $k, k_s \in K$ be as above, Then $\beta([l, s]) = [k, k_s] = 0$, hence $k \curlywedge k_s = 0$. Thus Φ is $\tilde{B_0}$-pairing. Therefore Φ induces a homomorphism $\Phi^* : L \curlywedge L \to K \curlywedge K$ given by $(l_1 \curlywedge l_2 \longmapsto k_1 \curlywedge k_2)$ for all $l_1, l_2 \in L$. Similarly, there is a homomorphism $\Psi^* : K \curlywedge K \to L \curlywedge L$ given by $k_1 \curlywedge k_2 \longmapsto l_1 \curlywedge l_2$, for all $k_1, k_2 \in K$. We can write $\Psi^* \Phi^* = \Phi^* \Psi^* = 1$, hence Φ^* is an isomorphism.
Let $k^*_L : L \curlywedge L \to L^2$ be given by $(l_1 \curlywedge l_2 \longmapsto [l_1, l_2])$ and let $k^*_K : K \curlywedge K \to K^2$ be given by $(k_1 \curlywedge k_2 \longmapsto [l_1, l_2])$, for all $l_1, l_2 \in L$ and $k_1, k_2 \in K$. Since $\beta k^*_L(l_1 \curlywedge l_2) = \beta([l_1, l_2]) = [k_1, k_2] = k^*_K(\Phi^*(l_1 \curlywedge l_2))$, we have the following diagram which is commutative

$$\begin{array}{ccccccc}
0 \to \tilde{B_0}(L) & \longrightarrow & L \curlywedge L & \xrightarrow{K^*_L} & L^2 \to 0 \\
\downarrow{\scriptstyle\gamma} & & \downarrow{\scriptstyle\Phi^*} & & \downarrow{\scriptstyle\beta} \\
0 \to \tilde{B_0}(K) & \longrightarrow & K \curlywedge K & \xrightarrow{K^*_K} & K^2 \to 0
\end{array}$$

Since β and Φ^* are isomorphisms, so is γ. □

4. CP extension and isoclinism of Lie rings

In this section, similar to groups, we want to study the connection between CP extension and isoclinism and then we will prove that all CP covers of isoclinic Lie rings are isoclinic.

Definition 4.1. Let N_i, C_i, and L_i $(i = 1, 2)$ be Lie rings, and let $ce_1 : 0 \to N_1 \xrightarrow{\chi_1} C_1 \xrightarrow{\pi_1} L_1 \to 0$ and $ce_2 : 0 \to N_2 \xrightarrow{\chi_2} C_2 \xrightarrow{\pi_2} L_2 \to 0$ be two central extensions. Then ce_1 and ce_2 are isoclinic, if there are isomorphisms $\eta : L_1 \to L_2$ and $\xi : C_1^2 \to C_2^2$ such that for all $c_1, c_2 \in C_1$, we have $\xi([c_1, c_2]) = [c_1', c_2']$ where $c_1', c_2' \in C_2$ and $\eta\pi_1(c_i) = \pi_2(c_i'), (i = 1, 2)$. In this case, the pair (η, ξ) is called an isoclinism from ce_1 to ce_2.

Proposition 4.2. *Let $e_i : 0 \to N_i \xrightarrow{\chi_i} C_i \xrightarrow{\pi_i} L_i \to 0$, $i = 1, 2$, be two central extensions which are isoclinic. If e_1 is a CP extension, then so is e_2.*

Proof. Let (x_2, y_2) be a commuting pair of L_2. Since e_1 and e_2 are isoclinic central extensions, there are two isomorphisms $\eta : L_1 \to L_2$ and $\xi : C_1^2 \to C_2^2$. Also, there are $x_1, y_1 \in L_1$ such that $y_2 = \eta(y_1)$, $x_2 = \eta(x_1)$ and $[x_1, y_1] = 0$. Also, there are $c_1, c_1' \in C_1$ such that $x_1 = \pi_1(c_1)$ and $y_1 = \pi_1(c_1')$. Since e_1 is a central CP extension, then $[c_1, c_1'] = 0$. Also there are $c_2, c_2' \in C_2$ such that

$$x_2 = \eta(x_1) = \eta\pi_1(c_1) = \pi_2(c_2) \quad , \quad y_2 = \eta(y_1) = \eta\pi_1(c_1') = \pi_2(c_2').$$

Also, $0 = \xi([c_1, c_1']) = [\xi(c_1), \xi(c_1')] = [c_2, c_2']$. Thus e_2 is a CP extension. □

Note that similar to Definition 3.1, we have the concept of isoclinism for CP defining pairs.

Definition 4.3. Let C_i, N_i and L_i $(i = 1, 2)$ be Lie rings and let (C_1, N_1) and (C_2, N_2) be two CP defining pairs. Then (C_1, N_1) and (C_2, N_2) are isoclinic, if there are isomorphisms $\eta : L_1 \to L_2$ and $\xi : C_1^2 \to C_2^2$ such that for all $c_1, c_2 \in C_1$, we have $\xi([c_1, c_2]) = [c_1', c_2']$ where $c_1', c_2' \in C_2$ and $\eta\pi_1(c_i) = \pi_2(c_i'), (i = 1, 2)$. In this case, the pair (η, ξ) is called an isoclinism from (C_1, N_1) to (C_2, N_2).

Lemma 4.4. *Let (C_1, N_1) and (C_2, N_2) be isoclinic CP defining pairs for L_1 and L_2, respectively, where C_1 and C_2 are finite Lie rings. Then $N_1 \cong N_2$ and $|C_1| = |C_2|$.*

Proof. Since $N_i \subseteq Z(C_i) \cap C_i^2 (i = 1, 2)$ and $\xi(N_1) = N_2$, we have $N_1 \cong N_2$. Also for finite C_i's, $|(C_1/N_1)| = |(C_2/N_2)|$. So $|C_1| = |C_2|$. □

In [9], we proved that all CP covers of a finite dimensional Lie algebra are isomorphic. Now we want to show that all CP covers of a finite Lie ring are isoclinic.

Proposition 4.5. *Let L_1 and L_2 be isoclinic finite Lie rings. Then the CP covers of $L1$ and those of $L2$ are isoclinic.*

Proof. Let C_1 and C_2 be arbitrary CP covers of L_1 and L_2, respectively. Hence we have the exact sequence $0 \to N_1 \xrightarrow{\subseteq} C_1 \xrightarrow{\pi_1} L_1 \to 0$ such that $N_1 \subseteq Z(C_1) \cap C_1^2$, $N_1 \cap K(C_1) = 0$ and also an exact sequence $0 \to N_2 \xrightarrow{\subseteq} C_2 \xrightarrow{\pi_2} L_2 \to 0$ such that $N_2 \subseteq Z(C_2) \cap C_2^2$, $N_2 \cap K(C_2) = 0$. Now, let (α, β) be an isoclinic pair of L_1 and L_2

such that $\alpha : L_2/Z(L_2) \to L_1/Z(L_1)$ and $\beta : L_2^2 \to L_1^2$ are isomorphisms. Here, we want to show that C_1 and C_2 are isoclinic. The maps $\bar{\pi}_1 : C_1/Z(C_1) \to L_1/Z(L_1)$ and $\bar{\pi}_2 : C_2/Z(C_2) \to L_2/Z(L_2)$ that are induced by π_1 and π_2, are in fact isomorphisms because the extensions are central CP covers. So, we can define $\bar{\alpha} : C_2/Z(C_2) \to C_1/Z(C_1)$ given by $c_2 + Z(C_2) \mapsto \bar{\pi}_1^{-1}\alpha\bar{\pi}_2(c_2 + Z(C_2))$. By using curly exterior product in Lie algebra [8], π_1 and π_2 induce two isomorphisms $\pi_1 \curlywedge \pi_1 : C_1^2 \to L_1 \curlywedge L_1$ given by $[x_1, y_1] \mapsto \pi_1(x_1) \curlywedge \pi_1(y_1)$ and $\pi_2 \curlywedge \pi_2 : C_2^2 \to L_2 \curlywedge L_2$ defined by $[x_2, y_2] \mapsto \pi_2(x_2) \curlywedge \pi_2(y_2)$, for $x_1, y_1 \in C_1$ and $x_2, y_2 \in C_2$. Furthermore as in the proof of Theorem 3.2, there exists an isomorphism $\tilde{\alpha} : L_2 \curlywedge L_2 \to L_1 \curlywedge L_1$ which is given by $l_2 \curlywedge l'_2 \mapsto l_1 \curlywedge l'_1$ induced by α, such that $l_1 + Z(L_1) = \alpha(l_2 + Z(L_2))$ and $l'_1 + Z(L_1) = \alpha(l'_2 + Z(L_2))$, for $l_1, l'_1 \in L_1$ and $l_2, l'_2 \in L_2$. Thus we can define an isomorphism $\bar{\beta} : C_2^2 \to C_1^2$ given by $[c_2, c'_2] \mapsto (\pi_1 \curlywedge \pi_1)^{-1}\tilde{\alpha}(\pi_2 \curlywedge \pi_2)([c_2, c'_2])$, for $c_2, c'_2 \in C_2$. Now, as L_1 and L_2 are isoclinic by (α, β), then C_1 and C_2 are isoclinic by $(\bar{\alpha}, \bar{\beta})$. □

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

References

[1] E. Noether, Gleichungen mit vorgeschriebener Gruppe, *Math. Ann.* **78** (1917) 221 – 229, https://doi.org/10.1007/BF01457099.

[2] D. J. Saltman, Noether's problem over an algebraically closed field, *Invent Math.* **77** (1984) 71 – 84, https://doi.org/10.1007/BF01389135.

[3] F. A. Bogomolov, The Brauer group of quotient spaces by linear group actions, *Math. USSR Izv.* **30** (1988) p. 455, https://doi.org/10.1070/IM1988v030n03ABEH001024.

[4] B. Kunyavskiĭ, The Bogomolov multiplier of finite simple groups, Cohomological and geometric approaches to rationality problems, *Progr. Math.* **282** 209 – 217, Birkhauser Boston, Inc, Boston, MA, 2010.

[5] P. Moravec, Unramified brauer groups of finite and infinite groups, *Amer. J. Math.* **134** (2012) 1679 – 1704.

[6] U. Jezernik and P. Moravec, Universal commutator relations, Bogomolov multipliers, and commuting probability, *J. Algebra* **428** (2015) 1 – 25, https://doi.org/10.1016/j.jalgebra.2014.12.034.

[7] U. Jezernik and P. Moravec, Commutativity preserving extensions of groups, *Proc. R. Soc. Edinb. A: Math.* **148** (2018) 575 – 592, https://doi.org/10.1017/S0308210517000270.

[8] Z. A. Rostami, M. Parvizi and P. Niroomand, The Bogomolov multiplier of Lie algebras, *Hacet. J. Math. Stat.* **49** (2020) 1190 – 1205, https://doi.org/10.15672/hujms.455076.

[9] Z. A. Rostami, M. Parvizi and P. Niroomand, Bogomolov multiplier and lazard correspondence, *Commun Algebra* **48** (2020) 1201 – 1211, https://doi.org/10.1080/00927872.2019.1677694.

[10] D. Groves, *Problem in Lie Rings and Groups*, PhD diss., University of Oxford, 2000.

[11] V. Naik, *Lazard Correspondence up to Isoclinism*, PhD diss., The University of Chicago, 2013.

[12] G. Ellis, Non-abelian exterior products of Lie algebras and an exact sequence in the homology of Lie algebras, *J. Pure Appl. Algebra* **46** (1987) 111 – 115, https://doi.org/10.1016/0022-4049(87)90088-0.

[13] B. Eick, M. Horn and S. Zandi, Schur multipliers and the Lazard correspondence, *Arch. Math.* **99** (2012) 217 – 226, https://doi.org/10.1007/s00013-012-0426-7.

[14] P. Hall, The classification of prime-power groups, *J. für die Reine und Angew. Math.* **182** (1940) 130 – 141, https://doi.org/10.1515/crll.1940.182.130.

[15] K. Moneyhun, Isoclinisms in Lie algebras, *Algebras Groups Geom.* **11** (1994) 9 – 22.

Zeinab Araghi Rostami
Department of Pure Mathematics,
Ferdowsi University of Mashhad,
Mashhad, I. R. Iran
e-mail: araghirostami@gmail.com; zeinabaraghirostami@stu.um.ac.ir

Mohsen Parvizi
Department of Pure Mathematics,
Ferdowsi University of Mashhad,
Mashhad, I. R. Iran
e-mail: parvizi@um.ac.ir

Peyman Niroomand
School of Mathematics and Computer Science,
Damghan University,
Damghan, I. R. Iran
e-mail: niroomand@du.ac.ir; p_niroomand@yahoo.com