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A Gauge Theory for Extra Dimension Detecting by Point Particle

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Abstract

From the viewpoint of "extra dimension detecting," the phenomenon of the transition of the free point particle into 3d space is investigated. In this way, we formulate the problem using the second-class constrained system. To investigate it using a gauge theoretical approach, we use two methods to convert its two second-class constraints to first-class ones. In symplectic embedding, we construct a pair of scaler and vector gauge potentials, which can be interpreted as interactions for detecting extra dimensions. A Wess-Zumino variable appears as a new coordinate in potentials, and the particle's mass plays the role of a globally conserved charge related to the constructed gauge theory for extra dimensions.

Keywords: Gauge, Extra dimension, Second-class, Symplectic embedding.

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1. Introduction

From the classical point of view, it seems that the gauge symmetry of a system prevents us from reaching a unique solution. However, such an idea is overly simplistic and there are more advantages for a gauge theory. Indeed, the gauge system or more generally the existence of any symmetry for classical and quantum

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theories, helps in selecting the answer that is closest to physical reality among a plethora of answers. As an example, one may consider the theory of fundamental quantum fields, which is usually assumed to have gauge symmetry. The existence of this symmetry helps to select answers from a set of the answers obtained through gauge fixing that has fewer singularities and irrational infinities.

In the theory of constrained systems, the presence of first-class constraints is a sign of the presence of gauge symmetry in that system [1], while the second-class constraints refer to the presence of additional degrees of freedom in the system [2]. We know that the system that exists in a subspace of 3d space (1, 2, and 3d subspaces) is the second class. The constraint analysis of such a system leads to the elimination of its additional degrees of freedom. Therefore, we can examine such a system in a smaller space. In the context of constrained systems, there are methods for converting all or a part of the additional degrees of freedom into gauge degrees of freedom. There are several methods for performing this conversion. Two of which are gauge unfixing and more famous the symplectic embedding of the phase space.

In this study, we focus on the method for constructing the embedded quantum mechanics [3], but it is not a fantastic task, merely. The quantum mechanics resulting from the classical embedded theories have been investigated and applied in various aspects. Examining the scattering of particles present in Riemannian manifolds [4] or the movement of particles on curved surfaces whose curvature is regulated by the potentials caused by the geometry of space [5–7] or physical reality of the sample [8] are examples of their application.

So, in this paper, we look at the problem of analyzing extra dimensions by a zero-dimensional object in such a perspective. In Section 2, we consider a zero-dimensional particle that can be a fundamental particle or a quantum dot (QD). This particle can be driven to the world outside of its zero-dimensional world by an external factor. We will show how to investigate such a problem in the context of the second-class classical systems. Then, in Section 3, we convert the obtained second-class system into the first-class tantamount, using the two methods of gauge unfixing and symplectic embedding of the phase space in order to investigate this phenomenon from the standpoint of gauge symmetry. In continuation, we discuss the limitations of the gauge unfixing method as well as the advantages of the symplectic embedding method. Finally, in the symplectic embedding method, we provide expressions and potentials that can cause particles to transport to extra dimensions.

2. The point particle transition to the 3d

As previously stated, QDs are the best choice for a zero-dimensional system. We assuming the QD, to point out that the event could happen due to both internal and external effects. Moreover, we consider a general quantum particle that evolves from the point where it resides to a wider space.

Assuming that the particle is transferred to a 3d space by an external factor such as the gravitational effects which is caused by gravitational waves or extra dimensions. The coordinates of 3d space are thought to be Cartesian. As a result, the configuration space of the particle will be (x, y, z). This motion in 3d space is clearly from the viewpoint of a 3d observer, and a zero-dimensional observer will not see such a motion. From the perspective of the zero-dimensional observer, we write the particle's Hamiltonian as follows:

$$H_c^0 = \frac{\vec{p}^2}{2m} + E_0. \tag{1}$$

Because the mass of the particle is infinite from the viewpoint of the zero-dimensional observer, it perceives the Hamiltonian as a constant, that is, $H_c^0 = E_0$. But a 3d observer finds the finite value for the first term of the above Hamiltonian. As expected, given that the zero-dimensional observer has no observables on which to write a Hamiltonian. In the next step of our modeling, we examine the problem from the perspective of a 3d observer. So, the configuration space is $\mathbb{R}^3$. However, the particle does not have access to all of 3d space, because we assume that the motion of the QD is limited to the surface $\varphi(x_i)$. Therefore, $\varphi(x_i) = 0$ is the on-shell configuration space of the particle, and the entire of $\mathbb{R}^3$ is its off-shell configuration space. We incorporated to the primary constraint as:

$$\phi = \varphi(x_i), \tag{2}$$

in the formalism of constrained systems. One may imagine the $\varphi(x_i)$ as a coordinate and the ϕ as an identity in formalism. According to the primary constraint introduced, the total Hamiltonian of the particle can be written as follows:

$$H_T = \frac{p^2}{2m} + \lambda\phi. \tag{3}$$

It is clear that the phase space of the particle is a subset of $\mathbb{R}^6$. Since the particle does not have access to the entire configuration space, it also does not have access to the entire phase space. But, it does not imply that the particle's phase space or on shell space is $M' = \mathbb{R}^3|_\phi \otimes \mathbb{R}^3$. The $(\vec{x}, \vec{p})$ are the phase space coordinates of the particle, or in a creative nomenclature "fundamental unobservables for the zero-dimensional observer". The $\mathbb{R}^6$, is the off shell phase space of the particle. To find the on shell phase space, we need to check the compatibility of the primary constraint (2) and the total Hamiltonian (3). In this way, the secondary constraint is extracted as follows:

$$\{\phi, H_T\} = 0 \rightarrow \psi = \vec{p}.\vec{\nabla}\varphi. \tag{4}$$

The expression we obtained for the secondary constraint can be interpreted physically and geometrically as the particle's momentum must always be parallel to

The meaning of $\mathbb{R}^3|_\phi$ is the limitation of the space $\mathbb{R}^3$ by using ϕ constraint.

the surface $\varphi(x_i)$. In other words, there is no momentum that throws the particle out of the $\varphi(x_i)$. In this manner, the particle's reduced phase space is obtained as follows:

$$M = (\mathbb{R}^3|_\phi \otimes \mathbb{R}^3)|_\psi. \tag{5}$$

Now, according to the primary constraint (2) and the secondary constraint (4), the compatibility between them can be checked as:

$$\{\phi, \psi\} = |\vec{\nabla}\varphi|^2. \tag{6}$$

The system is clearly a first class system for the surface with a single point $|\vec{\nabla}\varphi|^2 = 0$, but otherwise it is a second class system. If we remove them from our set of cases, we have a second class system so we can conclude that the number of particle's degrees of freedom remain zero.

Now, using two constraints (2) and (4), we can derive the canonical conjugates $(\vec{x}_I, \vec{p}_I)$ as follows:

$$x_I = f(x_i), \quad i \neq I, \qquad p_I = \tfrac{1}{\partial_I \varphi} p_J \partial_J \varphi_J, \quad J \neq I. \tag{7}$$

Since the configuration space coordinates x_Is are not independent of each other, the momentum of the particles in different directions is not independent of each other. For example, the Ith component is obtained based on the other components of $\vec{p}$. Now, using the canonical conjugates (7), we can obtain a subspace of the configuration space $\mathbb{R}^3$ in which the particle is present.

$$\partial_I \varphi + \partial_J f(x_i) \partial_J \varphi = 0. \tag{8}$$

In this way, we found a gauge orbit in $\mathbb{R}^6$ space for the classical states of the particle. In the following section, we convert this second-class system into a gauge model and investigate the issue through the perspective of gauge symmetry. Since the particle has no propagating degrees of freedom, we can conclude that there is no observability to analyze, by itself. But in both classical and quantum states we, as an extra-dimensional observer, concentrate on mass and energy scalars.

3. Gauging the second-class system

As the previous section shows, we are dealing with a second-class system in this problem. Usually, the examination of a second-class system is full of ambiguity. It will be revealed if we do the process of (7) and (8) up to the last constraint. Several works deal with obtaining gauge invariant theories. Gauge invariant theories are extremely important because a theory must be a gauge theory in order to be quantized, and the quantization of second-class systems is much more difficult than the quantization of first-class theories [9, 10]. Therefore, we have decided to investigate the problem of touching extra dimensions by the zero-dimensional particle from the point of view of gauge symmetric systems.

There are several methods for converting a second-class system into a gauge system. By expanding the phase space or the so-called embedding, some of these methods can provide gauge symmetry to a system with second-class constraints. Among them, the famous BFT embedding [11–14], does work for systems with a constant matrix of Poisson brackets of the constraints. The Faddeev-Jackiw [15, 16] and symplectic embedding [15, 17] work for more general systems. In another category of these methods, unlike the previous ones, the second-class system becomes a gauge system without expanding the phase space. Since the number of second-class constraints is always even, in this method, half of the constraints are considered as gauge symmetry generators and the other half as gauge fixing. In the following, we will convert our model into a first-class system using methods from both of the preceding categories, namely the gauge unfixing (GU) method and symplectic embedding.

3.1 Gauging by unfixing

Mitra and Rajaraman proposed the GU formalism [18], and Vytheeswaran continued the work [19]. Then researchers improved it and the improved GU consists of redefining the phase space variables by making them first-class [20]. In their approach, when the Poisson bracket between the constraints is a constant value, this method can be used to convert a second-class system to a first-class one. According to relation (6), this value for our model is equal to $|\vec{\nabla}\varphi|^2$, which can be have fixed valued or variable depending on the surface on which the zero-dimensional particle is limited on it. As a result, we can choose this surface so that the value of $|\vec{\nabla}\varphi|^2$ is a non-zero constant value for them. This non-zero value is called as Δ, which is equal to $\{\phi, \psi\}$.

The GU method is based on selecting one of the two second class constraints as the generator of gauge symmetry. We use the primary constraint $\phi = \varphi(x_i)$ as the gauge symmetry generator in this case. So, $\tilde{\phi}$ is defined in this case as follows:

$$\tilde{\phi} = \frac{1}{\Delta}\phi. \tag{9}$$

Now $\tilde{\phi}$ and ψ are canonical conjugates. In this case, the gauge invariant Hamiltonian is constructed by a projection as follows [9]:

$$P(H_c) = (\tilde{H}_c)_{GU} = H_c - \psi\{H_c, \tilde{\phi}\} + \frac{1}{2}\psi^2\{\tilde{\phi}, \{\tilde{\phi}, H_c\}\} - ..., \tag{10}$$

which is simplified by vanishing the larger powers, say after $O(\psi^2)$. So, in this short path we derive,

$$H_{GU} = \frac{\vec{p}^2}{2m} + \frac{(\vec{p}.\vec{\nabla}\varphi)^2}{(\vec{\nabla}\varphi)^2 m}. \tag{11}$$

These calculations can be repeated for specific surfaces with $|\vec{\nabla}\varphi| = \texttt{constant}$, which relates to minimal surfaces in Riemannian geometry. With the limitation for primary constraint systems, the disadvantage of this method is that the particle lives in a subset of $\mathbb{R}^4$ instead of our desired $\mathbb{R}^6$. It is because of that, we want to present a gauge potential for entry of the particle into the new 3d universe. For this reason, we choose another way for gauging. In particulars, we focus on the method that added dimension to the problem.

3.2 Gauging by symplectic embedding

We focused on gauging system using the GU method in the previous section. We see that the elected method is used in systems with a constant Poisson bracket between its constraints. It limited our options for the primary constraint. In this section, we present a method that avoids the limitations of the previous method while also being effective for Δ variables. In fact, we embed the second class system in a larger phase space, for the conversion process.

The symplectic approach also has infinite stages, in general. But, practically its levels truncate for realistic physical systems. In it, we must use the first-order Lagrangian, so every second order Lagrangian must be converted to the first-order by expanding its phase space. We use Wess-Zumino (WZ) variables, as suggested by Faddeev [15], to expand the phase space and begin the symplectic embedding process [21]. The WZ variables have been used by many researchers, specially in super-symmetrization. However, share the same conceptual foundation and adhere to the Dirac framework.

The symplectic tensor is the main object formed by this method. If this tensor is singular, the model has symmetry. Otherwise, the work should be repeated until the tensor is singular. We calculate the Hamiltonian of our model using an approach based on Faddeev's proposal [14, 17, 22, 23]. We introduce the first-order Lagrangian of our using the zeroth iterative symplectic tensor, $L^{(0)} = A^{(0)}_\alpha \xi^{(0)}_\alpha - v^{(0)}$, which $v^{(n)}$ is an iterative potential, as follows:

$$L^{(0)} = \dot{\vec{x}}.\vec{p} - \frac{\vec{p}^2}{2m} - \lambda_1\phi_1. \tag{12}$$

The symplectic variables $\xi^{(0)}_\alpha$ and one form canonical momenta $A^{(0)}_\alpha$ can be obtained using Lagrangian (12) in this case

$$A^{(0)}_\alpha = (p_\nu, 0_\nu, 0), \qquad \xi^{(0)}_\alpha = (x_\mu, p_\mu, \lambda_1). \tag{13}$$

According to the definition of symplectic tensor, $f^{(0)}_{\alpha\beta} = \frac{\partial A^{(0)}_\beta}{\partial \xi^{(0)}_\alpha} - \frac{\partial A^{(0)}_\alpha}{\partial \xi^{(0)}_\beta}$, the tensor

Hereafter, the domain for the values of indexes can be understood from zero-order symplectic variable and how it extended in the next steps.

of the model will be as follows:

$$f^{(0)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & 0_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} \\ 0_{1\nu} & 0_{1\nu} & 0_{1\times 1} \end{pmatrix}. \tag{14}$$

At now it is a singular two-form by the zero mode,

$$n^{(0)}_{\alpha} = (0_\mu, 0_\mu, 1). \tag{15}$$

So, the first-order potential is derived as $v^{(0)} = \frac{\vec{p}^2}{2m} + \lambda_1\phi_1$. For the first-order Lagrangian, we find,

$$L^{(1)} = \dot{\vec{x}}.\vec{p} - \dot{\lambda}_1\phi_1 - \frac{\vec{p}^2}{2m}. \tag{16}$$

In new stage, the symplectic variables and one-form momentum become,

$$A^{(1)}_{\alpha} = (p_\nu, 0_\nu, \phi_1), \qquad \xi^{(1)}_{\alpha} = (x_\mu, p_\mu, \lambda_1). \tag{17}$$

Therefore, the symplectic two form are equal to

$$f^{(1)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & \partial_\mu\phi \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} \\ -\partial_\nu\phi & 0_{1\nu} & 0_{1\times 1} \end{pmatrix}. \tag{18}$$

Still the tensor is singular and therefore its zero mode is obtained as $n^{(1)}_{\alpha} = (0_\mu, \partial_\mu\phi, 1)$. It produces the secondary constraint of $\phi_2 = \psi$. We will have the first order potential in this stage, $v^{(1)} = \frac{\vec{p}^2}{2m}$. Now, the second iterative Lagrangian found as follows:

$$L^{(2)} = \dot{\vec{x}}.\vec{p} - \dot{\lambda}_1\phi_1 - \dot{\lambda}_2\psi - \frac{\vec{p}^2}{2m}. \tag{19}$$

Once again, we will have the symplectic variables $\xi^{(2)}_{\alpha}$ and one-form momentum $A^{(2)}_{\alpha}$ as:

$$A^{(2)}_{\alpha} = (p_\mu, 0_\nu, \phi_1, \psi), \qquad \xi^{(2)}_{\alpha} = (x_\mu, p_\mu, \lambda_1, \lambda_2). \tag{20}$$

In this case, the final symplectic two form are obtained.

$$f^{(2)}_{\alpha\beta} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & \partial_\mu\phi & p_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} & \frac{1}{2}\partial_\mu\phi \\ -\partial_\nu\phi & 0_{1\nu} & 0_{1\nu} & 0_{\mu 1} \\ -p_{1\nu} & -\frac{1}{2}\partial_\nu\phi & 0_{1\nu} & 0_{1\times 1} \end{pmatrix}. \tag{21}$$

It is nonsingular, which results in no generation of a new constraint. The process of generating new constraints ends here, but the embedding process begins.

For embed-ness process, first the unknown function dependent on the main phase space and WZ variables is introduced as follows:

$$G(x_\mu, p_\mu, \sigma) = \sum_{n=0}^{\infty} g^{(n)}(x_\mu, p_\mu, \sigma). \tag{22}$$

So, the first-order Lagrangian is modified to

$$\tilde{L}^{(1)} = \dot{\vec{x}}.\vec{p} + \dot{\lambda}\phi - \frac{\vec{p}^2}{2m} + G(x_\mu, p_\mu, \sigma). \tag{23}$$

From the above Lagrangian, we extract the symplectic variables and one-form momenta

$$\tilde{A}_\alpha^{(1)} = (p_\nu, 0_\nu, \phi, 0), \qquad \tilde{\xi}_\alpha^{(1)} = (x_\mu, p_\mu, \lambda, \sigma). \tag{24}$$

As a result, the symplectic tensor is specified.

$$\tilde{f}_{\alpha\beta}^{(1)} = \begin{pmatrix} 0_{\mu\nu} & -\delta_{\mu\nu} & \partial_\mu\phi & 0_{\mu 1} \\ \delta_{\mu\nu} & 0_{\mu\nu} & 0_{\mu 1} & 0_{\mu 1} \\ -\partial_\nu\phi & 0_{1\nu} & 0_{1\times 1} & 0_{1\times 1} \\ 0_{1\nu} & 0_{1\nu} & 0_{1\times 1} & 0_{1\times 1} \end{pmatrix}. \tag{25}$$

This tensor is clearly singular by following zero modes.

$$\tilde{n}_\alpha^{1(1)} = (0_\alpha, 1), \qquad \tilde{n}_\alpha^{2(1)} = (n_\alpha^{(1)}, 0). \tag{26}$$

The first-order correction term in σ can be obtained as:

$$g^{(1)}(x_\mu, p_\mu, \sigma) = \sigma\psi. \tag{27}$$

In this case, the Lagrangian will be

$$\tilde{L}^{(1)} = \dot{\vec{x}}.\vec{p} + \dot{\lambda}\phi - \frac{\vec{p}^2}{2m} + \sigma\psi. \tag{28}$$

Still, it is not gauge invariant because the mode (26) describes a new constraint. In this instance we find

$$\tilde{n}_\alpha^{(1)} \frac{\partial v^{(1)}}{\partial \tilde{\xi}_\alpha^{(1)}} = \frac{1}{2}(\vec{\nabla}\varphi)^2\sigma. \tag{29}$$

So, the next term dependent on the σ is obtained

$$g^{(n)} = -\frac{1}{2}(\vec{\nabla}\varphi)^2\sigma^2. \tag{30}$$

Thus, for the first order Lagrangian, we will have

$$L^{(1)} = \dot{\vec{x}}.\vec{p} + \dot{\lambda}\phi - \frac{\vec{p}^2}{2m} + \sigma\psi - \frac{1}{2}(\vec{\nabla}\varphi)^2\sigma^2. \tag{31}$$

The modes (26) do not introduce any new constraints. As a result, the correction terms $g^{(n)}$ with $n \geq 3$ are zero, and the Lagrangian (31) is invariant. From Lagrangian to Hamiltonian, a Legendre transformation can be used [21]. The gauge Hamiltonian in which the constraints are no longer of the second-class is obtained in this case as:

$$\tilde{H}_c = \frac{\vec{p}^2}{2m} - \lambda\phi - (\vec{p}.\vec{\nabla}\phi)\sigma + \frac{1}{2}(\vec{\nabla}\varphi)^2\sigma^2, \tag{32}$$

where the σ denotes the WZ variable. The gauged phase space will be as shown in $\xi = (\vec{x}, \vec{p}, \lambda, \sigma)$. As is well known, we created the first-class system by embedding the second-class system in a larger phase space. This first-class system is more suitable for quantization. The potential terms that lead to the particle's entry into the 3d space can be extracted using Hamiltonian (32), as follows:

$$V = \lambda\phi + (\vec{p}.\vec{\nabla}\phi)\sigma - \frac{1}{2}(\vec{\nabla}\varphi)^2\sigma^2. \tag{33}$$

If we ask to write the Hamiltonian (32) as the total Hamiltonian, in minimal coupling way, it is

$$H_T = \frac{1}{2m}(\vec{p} - \frac{e}{c}\vec{A}(\vec{x}, \sigma))^2 + eV(|\vec{x}|). \tag{34}$$

In such factorization desired, the $V(x)$ and the $\vec{A}(\vec{x}, \sigma)$ represent the scalar and vector potentials for sending the zero-dimensional particle into the real 3d world, respectively. The e is a conserved charge related to the extra dimension force. The vector and scalar potentials are extracted by comparing two Hamiltonian (32) and (34) as:

$$\tfrac{e}{c}\vec{A} = \sqrt{m}\sigma\vec{\nabla}\phi, \qquad eV(|\vec{x}|) = -\lambda\phi(\vec{x}). \tag{35}$$

The numerical coefficients underlying the Hamiltonian's third and fourth terms, which are related to the second-class constraints, can be adjusted so that the charge of the attraction factor to the extra dimensions becomes $\sqrt{m}$. This means that the particle was not sent to the extra dimension as a result of a new force, but rather as a result of a gravitational effect. Note that, the σ is a new coordinate of the system under review not a new characteristic of the particle. From now on we can study the problem with a dimension-magnetic glance.

4 Conclusion

We investigated the phenomenon of a zero-dimensional quantum particle transited to a space with dimensions greater than zero. The symplectic embedding method was employed in this study. By using the mentioned method, we convert the study in a gauge interaction study. In this way, we found a set of vector and scalar potentials that cause this transition. We also saw that the shape of these two obtained potentials depends on the geometry and shape of the surface that the particle has moved to it. Incorporate to the 2-dimensional surface, which is described by two independent coordinates, that the particle transits on it, a Wess-Zumino variable appeared as the third coordinate. The obtained coordinate and primary first-class constraint lead to constructing the vector gauge potential and vanishing the scalar potential. So, the phenomenon analyzed from this point of view is a fully gauge interaction. It is concluded that the effect of extra dimension detection by the point particle results in its quantum wave function as a geometric quantum phase, suggesting that one can look at the problem by the Berry phase approach.

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