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# Exploring the Algebraic Properties of Gyrosemigroups and a Characterization of Gyrosemigroups of Order 2

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**Abstract**

A significant development in the field of gyrogroups was the introduction of the space of all relativistically admissible velocities, which brought gyrogroups into the mainstream. A group has various generalizations, one of which is the notion of gyrogroups. Moreover, for any pair $(a, b)$ in this structure, there exists an automorphism $gyr[a, b]$ that fulfills left associativity and left loop property. The motivation behind this study is to generalize gyrogroups and semigroups, which has led to the introduction of gyrosemigroups. Accordingly, in this paper, some classes of gyrosemigroups are presented. Also, all gyrosemigroups of order 2 are characterized. Furthermore, the gyrosemigroups with an identity or a zero are studied.

Keywords: Semigroup, Groupoid, Gyrosemigroup, Gyrogroup.

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## 1. Introduction

In 1988 Ungar expressed the Lorentz group parametrically in terms of relativistically admissible velocities and orientations in a paper entitled *"Thomas rotation*

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*and the parametrization of the Lorentz transformation group"* [1]. The group structure of the resulting parametric realization of the Lorentz group, along with Einstein velocity addition law, enabled Ungar to discover the group-like structure and it is commonly referred to as a gyrogroup [1, 2]. The research has demonstrated that gyrogroups possess a ubiquitous computational function that transcends the domain of Lorentz groups [3], as pointed out by Chatelin and others [4, page 523].

Clifford algebras have been utilized in previous researches to enhance comprehension of gyrogroups in diverse investigations, including those in the field of [5–9]. Suksumran and Ungar's work on the generalization of gyrogroups to bi-gyrogroups is a significant contribution to the field of mathematics [10]. Recently, Ashrafi et al. characterized gyrogroups up to order 31 and obtained new gyrogroups[11, 12].

A semigroup is a mathematical structure consisting of a set and an associative binary operation. Semigroups are a generalization of groups. Semigroups have various applications in different areas of mathematics and beyond [13–15]. Here are a few examples: Algebraic Structures, Automata Theory, Geometry and Dynamical Systems, Cryptography and Optimization. Gyrogroups also have several interesting applications, although they might not be as widely known or studied as semigroups. A gyrogroup is a mathematical structure that generalizes the concept of a group by relaxing the requirement of associativity. Here are a few areas where gyrogroups find applications: Relativity Theory, Hyperbolic Geometry, Quantum Mechanics and Discrete Mathematics. In some fields such as physics, geometry, and combinatorial mathematics, both semigroups and gyrogroups have applications. This can provide researchers with ideas for finding applications of gyrosemigroups in physics, geometry and combinatorial mathematics. Furthermore, since gyrosemigroups are an extension of semigroups and gyrogroups, understanding the structure of gyrosemigroups helps us better understand these two structures.

Now we introduce a structure that encompasses both the notion of semigroups and that of gyrogroups and call it a gyrosemigroup. Several examples and properties of this structure have been examined. Moreover, we characterize gyrosemigroups of order two up to gyroisomorphism. Finally, the conditions required to obtain a gyrosemigroup with an identity or a zero from another gyrosemigroup will be discussed.

Let us start with the definition of a gyrogroup.

**Definition 1.1.** (Gyrogroup) A groupoid $(G, \oplus)$, where $\oplus$ is a binary action, is called a gyrogroup provided the following axioms are satisfied:

(1) Left associativity law holds, it means for any two elements $u$ and $v$ in $\mathfrak{G}$ there exists a gyroautomorphism $gyr[u, v]$ such that for all $u \in \mathfrak{G}$:

$$u \oplus (v \oplus w) = (u \oplus v) \oplus gyr[u, v](w),$$

(2) For every $u, v \in \mathfrak{G}$, $gyr[u \oplus v, v] = gyr[u, v]$.

(3) Left identity law holds, that means there is an element $0 \in \mathfrak{G}$, such that $0 \oplus u = u$, for all $u \in \mathfrak{G}$.

(4) For every $u \in \mathfrak{G}$, there exists $v \in \mathfrak{G}$ such that $v \oplus u = 0$.

## 2. Gyrosemigroup

In this section, we explore and analyze the notion of gyrosemigroup. A groupoid which satisfies the first and second axioms of Definition 1.1 is called a gyrosemigroup.

**Definition 2.2.** (Gyrosemigroup) A groupoid $(G, \oplus)$ is a gyrosemigroup if its binary operation satisfies the following conditions.

(1) For any $a, b, c \in G$ there exists a unique element $gyr[a, b]c \in G$ such that the binary operation obeys the left associativity law

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c.$$

(2) The map $gyr[a, b] : G \to G$ given by $c \to gyr[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

$$gyr[a, b] \in Aut(G, \oplus).$$

(3) The gyroautomorphism $gyr[a, b]$ generated by any $a, b \in G$ possesses the left reduction property

$$gyr[a, b] = gyr[a \oplus b, b].$$

The automorphism $gyr[a, b]$ of $G$ is called the gyroautomorphism, or the gyration of $G$ generated by $a, b \in G$. The operator $gyr : G \times G \to Aut(G, \oplus)$ is called the gyrator of $G$.

We recall that a gyrosemigroup $(G, \oplus)$ is trivial if $gyr[a, b]$ is the identity function, i.e. $gyr[a, b]x = x$, for every $x \in G$. Now, the question that arises is whether there is a non-obvious gyrosemigroup of any order. The answer in gyrogroups is negative. The first non-trivial gyrogroup is of order eight [11]. We show that the answer is positive in gyrosemigroups.

**Lemma 2.3.** *Let $S$ be a non-empty set . We define $\oplus : S \times S \to S$ by $a \oplus b = a$, for all $a, b \in S$. Then for every $a, b, c \in S$ and for every gyroautomorphism of $gyr[a, b] \in Aut(S, \oplus)$ we have:*

*(1) $gyr[a \oplus b, b] = gyr[a, b]$.*

*(2) $a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c$.*

*Proof.* For every $a, b \in S$ and $gyr[a, b] \in Aut(S, \oplus)$ we have $a \oplus b = a$. Therefore,

(1) $gyr[a \oplus b, b] = gyr[a, b]$.

(2) $a \oplus (b \oplus c) = a \oplus b = a$ and $(a \oplus b) \oplus gyr[a, b]c = a \oplus b = a$.

$\square$

Now by Lemma 2.3, we obtain the next theorem:

**Theorem 2.4.** *If $(S, \oplus)$ be a groupoid with the following Cayley table, then $(S, \oplus)$ is a gyrosemigroup for every $gyr[a, b] \in Aut(S, \oplus)$.*

*Table 1: Non-trivial gyrosemigroup of order n.*

| $\cdot$ | 1 | 2 | $\ldots$ | $n$ |
|---|---|---|---|---|
| 1 | 1 | 1 | $\ldots$ | 1 |
| 2 | 2 | 2 | $\ldots$ | 2 |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | |
| $n$ | $n$ | $n$ | $\ldots$ | $n$ |

Therefore for every $n \in \mathbb{N}$, there exist $(n!)^{n^2}$ gyrosemigroup and a number of these gyrosemigroups are gyroisomorphic.

There exists a groupoid $(S, \oplus)$ such that for every $gyr[a, b] \in Aut(S, \oplus)$, we can not obtain a non-trivial gyrosemigroup.

**Theorem 2.5.** *Suppose that $(S, \oplus)$ is a groupoid with the following Cayley table.*

*Table 2: Gyrosemigroup of order n.*

| $\cdot$ | 1 | 2 | $\ldots$ | $n$ |
|---|---|---|---|---|
| 1 | 1 | 2 | $\ldots$ | $n$ |
| 2 | 1 | 2 | $\ldots$ | $n$ |
| $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | |
| $n$ | 1 | 2 | $\ldots$ | $n$ |

*Then $(S, \oplus)$ is a gyrosemigroup if and only if for every $u, v \in S$, $gyr[u, v]$ is the identity function.*

**Theorem 2.6.** *If $(G, \cdot)$ is an Abelian group, then $(G, \oplus)$ is a gyrosemigroup, where for every $a, b \in G$, $a \oplus b = a \cdot b^{-1}$ and $gyr[a, b]c = c^{-1}$.*

**Theorem 2.7.** *The gyrosemigroup $(G, \oplus)$ in Theorem 2.6 is a gyrogroup if and only if for all $a \in G$, we have $o(a) = 2$, when $o(a)$ is the order or period of the element $a$ in the Abelian group $(G, \cdot)$.*

*Proof.* If for all $a \in G$, $o(a) = 2$, then the result is clear.
Conversely, let $c \in G$ be an identity for $\oplus$. Then for every $a \in G$, $a = a \oplus c = ac^{-1}$ and $a = c \oplus a = ca^{-1}$. So $a^{-1} = (ca^{-1})^{-1} = ac^{-1} = a$ and $o(a) = 2$. $\square$

**Example 2.8.** Let $(\mathbb{Z}_n, +)$ be the cyclic group of order $n$. We define $a \oplus b = a + (-b)$. $(\mathbb{Z}_n, \oplus)$ is not a semigroup, but it is a gyrosemigroup by putting for every $a, b \in G$, $gyr[a, b]c = -c$.

**Theorem 2.9.** *For every $n \in \mathbb{N}$, $n \neq 1$, there exists a groupoid $(G, \oplus)$ such that for every gyrator of $G$ , $(G, \oplus)$ is not a gyrosemigroup.*

*Proof.* If $n > 2$, consider the cyclic group $G = (\mathbb{Z}_n, +)$ of order $n$. Define the operation $\oplus : \mathbb{Z}_n \times \mathbb{Z}_n \to \mathbb{Z}_n$ by $a \oplus b = b - a$. We have $1 \oplus (1 \oplus 0) = -2$ and $(1 \oplus 1) \oplus gyr[1, 1](0) = 0 - (1 - 1) = 0$ , but for $n > 2$ we have $0 \neq -2$ and this is a contradiction. For $n = 2$ we prove it in Theorem 3.24. □

## 3. Gyrosemigroups of order 2

In this section all gyrosemigroups of order 2 are introduced and characterized. Let $G = \{0, 1\}$. If $(G, \oplus)$ is a groupoid, then $Aut(G, \oplus) = \{A\}$ or $Aut(G, \oplus) = \{A, T\}$, where $A$ is the identity automorphism and $T$ is the transposition $(01)$. This means that $T(0) = 1$ and $T(1) = 0$.

**Theorem 3.10.** *There exist 5 non-isomorphic semigroups of order 2 as the following Cayley tables show:*

*Table 3: All semigroups of order 2 up to isomorphism.*

$S_1$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

$S_2$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

$S_3$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

$S_4$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0 | 1 |

$S_5$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

**Theorem 3.11.** *Let $(G, \oplus)$ be a semigroup and for every $a, b \in G$, $gyr[a, b] = A$. Then $(G, \oplus)$ is a gyrosemigroup.*

*Proof.* For every $a, b, c \in G$, we have $a \oplus (b \oplus c) = (a \oplus b) \oplus c = (a \oplus b) \oplus A(c) = (a \oplus b) \oplus gyr[a, b]c$. □

**Theorem 3.12.** *Let $G = \{0, 1\}$. If $0 \oplus 0 = 1 \oplus 1$, then $(G, \oplus)$ is a gyrosemigroup if and only if $(G, \oplus)$ is a semigroup and for every $a, b \in G$, $gyr[a, b] = A$.*

*Proof.* First, assume that $(G, \oplus)$ is a gyrosemigroup. To the contrary, let for some $a, b \in G$, $gyr[a, b] \neq A$. So $gyr[a, b](0) = 1$ and $gyr[a, b](1) = 0$. Let $0 \oplus 0 = 1 \oplus 1 = 0$ then

$$1 = gyr[a, b](0) = gyr[a, b](0 \oplus 0) = gyr[a, b](0) \oplus gyr[a, b](0) = 1 \oplus 1 = 0,$$

and it is a contradiction. In a similar fashion, $0 \oplus 0 = 1 \oplus 1 = 1$ makes a contradiction. Therefore for every $a, b \in G$, $gyr[a, b] = A$. Thus a groupoid of order 2 with $0 \oplus 0 = 1 \oplus 1$ cannot be a gyrosemigroup with any non-trivial gyrator. Moreover, when the gyrator is trivial the concepts of semigroup and gyrosemigroup coincide. The inverse statement is clear by Theorem 3.11 and we are done. □

**Corollary 3.13.** *Among 128 groupoids of order 2 with* $0 \oplus 0 = 1 \oplus 1$, *there are 124 non-gyrosemigroups and 4 gyrosemigroups.*

**Theorem 3.14.** *Let* $G = \{0, 1\}$. *If* $(G, \oplus)$ *is a commutative groupoid, then* $(G, \oplus)$ *is a gyrosemigroup if and only if* $(G, \oplus)$ *is a semigroup and for every* $a, b \in G$, $gyr[a, b] = A$.

*Proof.* Let for some $a, b \in G$, $gyr[a, b] \neq A$ so $gyr[a, b](0) = 1$ and $gyr[a, b](1) = 0$. Let $0 \oplus 1 = 1 \oplus 0 = 0$, then

$$1 = gyr[a, b](0) = gyr[a, b](0 \oplus 1) = gyr[a, b](0) \oplus gyr[a, b](1) = 1 \oplus 0 = 0,$$

and it is a contradiction. Similarly, $0 \oplus 1 = 1 \oplus 0 = 1$ makes a contradiction. Therefore for every $a, b \in G$, $gyr[a, b] = A$. So, a commutative groupoid of order 2 can not be a gyrosemigroup with any non-trivial gyration. The inverse is clear by Theorem 3.11 and the proof is complete. □

**Corollary 3.15.** *There are 62 commutative non-gyrosemigroup groupoids of order 2 and 2 commutative gyrosemigroups such that* $0 \oplus 0 = 1 \oplus 1$ *dose not hold.*

**Theorem 3.16.** *Let* $a \oplus a = a \oplus b = a$, *for every* $a, b \in G = \{0, 1\}$. *Then,* $(G, \oplus)$ *is a gyrosemigroup, for every gyrator of* $G$.

*Proof.* For every $a, b, c \in G$, we have $a \oplus (b \oplus c) = a$ and $(a \oplus b) \oplus gyr[a, b]c = a \oplus gyr[a, b]c = a$, for every $gyr[a, b] \in Aut(G)$. □

**Corollary 3.17.** *There are 16 gyrosemigroups of order 2 with* $a \oplus a = a \oplus b = a$.

**Theorem 3.18.** *If* $(G, \oplus)$ *is a gyrosemigroup and* $a \oplus b = b'$, *where* $b' = 0$ *for* $b = 1$ *and* $b' = 1$ *for* $b = 0$, *then* $(G, \oplus)$ *is a gyrosemigroup if and only if for every* $a, b \in G$, $gyr[a, b] = T$.

*Proof.* For every $a, b, c \in G$,

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c \text{ and } gyr(a \oplus b, b) = gyr(a, b) \Leftrightarrow gyr[a, b] = T.$$

□

It is noteworthy that the groupoids in the last theorem do not satisfy the conditions of Theorems 3.12, 3.14 and 3.16.

**Corollary 3.19.** *There are 15 non-gyrosemigroup groupoids of order 2 with* $0 \oplus 0 = 1 \oplus 0 = 1$, $1 \oplus 1 = 0 \oplus 1 = 0$ *and one gyrosemigroup with such property.*

**Theorem 3.20.** *If* $0 \oplus 0 = 0, 0 \oplus 1 = 1, 1 \oplus 0 = 0$ *and* $1 \oplus 1 = 1$ *and* $gyr[a, b] \neq A$ *then* $(G, \oplus)$ *is not a gyrosemigroup.*

*Proof.* (1) If $(a, b) = (0, 0)$, then $0 \oplus (0 \oplus 0) = 0$ while $(0 \oplus 0) \oplus gyr[0, 0]0 = 0 \oplus 1 = 1$.

(2) If $(a,b)=(0,1)$, then $0\oplus(1\oplus 0)=0$ while $(0\oplus 1)\oplus gyr[0,1]0=1\oplus 1=1$.

(3) If $(a,b)=(1,0)$, then $1\oplus(0\oplus 1)=1$ while $(1\oplus 0)\oplus gyr[1,0]1=0\oplus 0=0$.

(4) If $(a,b)=(1,1)$, then $1\oplus(1\oplus 0)=0$ while $(1\oplus 1)\oplus gyr[1,1]0=1\oplus 1=1$.

□

**Theorem 3.21.** *If $a\oplus b=b$ for every $a,b\in G=\{0,1\}$, then $(G,\oplus)$ is a gyrosemigroup if for every $a,b\in G$, $gyr[a,b]=A$.*

*Proof.* For every $a,b,c\in G$,

$$c=a\oplus(b\oplus c)=(a\oplus b)\oplus gyr[a,b]c=gyr[a,b]c,$$

Thus,

$$gyr[a,b]=A.$$

□

**Corollary 3.22.** *There are 15 non-gyrosemigroup groupoids of order 2 with $a\oplus b=b$ and one gyrosemigroup with such property.*

**Theorem 3.23.** *If $(G,\oplus)$ is a semigroup of order 2, non-isomorphic to $S_3$, then the gyrator of every gyrosemigroup on $(G,\oplus)$ is the identity function.*

*Proof.* It easy to see that $Aut(G,\oplus)=\{A\}$ so the result follows. □

**Theorem 3.24.** *If $0\oplus 0=1, 0\oplus 1=1, 1\oplus 0=0$ and $1\oplus 1=0$ then $(G,\oplus)$ is not a gyrosemigroup with any gyrator of $G$.*

*Proof.* (1) If $(a,b)=(0,0)$, then $0\oplus(0\oplus 0)=1$ while $(0\oplus 0)\oplus gyr[0,0]0=0$.

(2) If $(a,b)=(0,1)$, then $0\oplus(1\oplus 0)=1$ while $(0\oplus 1)\oplus gyr[0,1]0=0$.

(3) If $(a,b)=(1,0)$, then $1\oplus(0\oplus 1)=0$ while $(1\oplus 0)\oplus gyr[1,0]1=1$.

(4) If $(a,b)=(1,1)$, then $1\oplus(1\oplus 0)=0$ while $(1\oplus 1)\oplus gyr[1,1]0=1$.
So there is no gyrator such that $(G,\oplus)$ is a gyrosemigroup. □

**Corollary 3.25.** *There are 16 non-gyrosemigroup groupoids of order 2 with $0\oplus 0=1, 0\oplus 1=1, 1\oplus 0=0$ and $1\oplus 1=0$.*

**Theorem 3.26.** *There are exactly 24 gyrosemigroups of order 2 called $GS_1, GS_2,\dots,GS_{24}$, as shown in Table 4.*

Table 4: All gyrosemigroups of order 2.

$GS_1$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 0 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_2$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_3$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_4$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $T$ | $A$ |

$GS_5$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $T$ |

$GS_6$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $T$ | $T$ |

$GS_7$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $T$ |

$GS_8$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $T$ |
| 1 | $A$ | $T$ |

$GS_9$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $T$ |
| 1 | $T$ | $A$ |

$GS_{10}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $T$ |
| 1 | $T$ | $T$ |

$GS_{11}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $A$ |
| 1 | $A$ | $A$ |

$GS_{12}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $A$ |
| 1 | $A$ | $T$ |

$GS_{13}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $A$ |
| 1 | $T$ | $A$ |

$GS_{14}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $A$ |
| 1 | $T$ | $T$ |

$GS_{15}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $A$ | $A$ |

$GS_{16}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $A$ | $T$ |

$GS_{17}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $T$ | $A$ |

$GS_{18}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $T$ | $T$ |

$GS_{19}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 0 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_{20}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 0 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_{21}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_{22}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

$GS_{23}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 1 | 0 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $T$ | $T$ |
| 1 | $T$ | $T$ |

$GS_{24}$

| $\cdot$ | 0 | 1 |
|---|---|---|
| 0 | 1 | 1 |
| 1 | 1 | 1 |

| $gyr$ | 0 | 1 |
|---|---|---|
| 0 | $A$ | $A$ |
| 1 | $A$ | $A$ |

Now, the question arises as to whether some of these gyrosemigroups can be isomorphic. To answer the question, we first need to introduce the concept of an isomorphism in gyrosemigroups.

**Definition 3.27.** Let $(G, \oplus)$ and $(G', \oplus')$ be two gyrosemigroups. If $f : G \to G'$ is a groupoid homomorphism such that for every $a, b, c \in G$, $gyr'[f(a), f(b)]f(c) = f(gyr[a, b]c)$, then $f$ is called a gyrohomomorphism.

If a gyrohomomorphism $f$ is onto and one to one, then we say that $f$ is a gyroisomorphism. In this case, $G$ and $G'$ are gyroisomorphic and and we write $G \simeq G'$.

By applying the last definition to the gyrosemigroups of Theorem 3.26, we conclude the following result.

**Theorem 3.28.** *There are exactly 15 non-gyroisomorphic gyrosemigroups of order 2.*

*Proof.* By Definition 3.27, one can see that $GS_1 \simeq GS_{24}$, $GS_2 \simeq GS_{21}$, $GS_4 \simeq GS_{11}$, $GS_5 \simeq GS_7$, $GS_6 \simeq GS_{15}$, $GS_8 \simeq GS_{13}$, $GS_{10} \simeq GS_{17}$, $GS_{14} \simeq GS_{16}$, $GS_{20} \simeq GS_{22}$ and the result follows. □

**Remark 1.** The isomorphism between $GS_4$ and $GS_{11}$, shows that the statement "if two gyrosemigroups with the same groupoid are gyroisomorphic, then the gyrators of them are the same" is not necessarily true.

## 4. Gyromonoids and gyrosemigroups with zero

Let $S$ be a semigroup. If there exists an element 1 of $S$ such that for all $s \in S$, $s \oplus 1 = 1 \oplus s = s$, then 1 is called an identity of $S$ and such $S$ is a called monoid.

Assume $(S, \oplus)$ is not a monoid. Then we can add an extra element 1 to the set $S$ to obtain a monoid $(S \cup \{1\}, \oplus)$ with the following data,

(1) for all $s \in S$, $1 \oplus s = s \oplus 1 = s$,

(2) $1 \oplus 1 = 1$,

We denote this monoid by $S^1$.

Let $G$ be a gyrosemigroup. If there exists an element $e$ of $G$ such that for all $a \in G$, $a \oplus e = e \oplus a = a$, we say that $e$ is an identity (element) of $G$ and such $G$ is a gyrosemigroup with identity, or a gyromonoid. Similar to the semigroups, gyrosemigroups have at most one identity.

**Theorem 4.29.** *Let $(G, \oplus)$ be a gyrosemigroup with no identity and for every $a \in G$, $gyr[a, a] = A$, where $A$ is the identity automorphism. Set $G^1 = G \cup \{e\}$ and define $(G^1, \oplus')$ with the following data,*

*(1) $e \oplus' a = a \oplus' e = a$, for all $a \in G$,*

*(2) $e \oplus' e = e$ and $a \oplus' b = a \oplus b$, for all $a, b \in G$,*

*(3) $gyr'[e, a] = gyr'[a, e] = A$, for all $a \in G$,*

*(4) $gyr'[e, e] = A$,*

*(5) $gyr'[a, b]c = gyr[a, b]c$ and $gyr'[a, b]e = e$, for all $a, b, c \in G$.*

*Then $(G^1, \oplus')$ with $gyr'$ becomes a gyrosemigroup with the identity element $e$.*

*Proof.* First, we prove the gyroassociativity law. For all $a, b \in G$ we have

(1) $e \oplus' (a \oplus' b) = a \oplus' b$ and $(e \oplus' a) \oplus' gyr'[e, a]b = a \oplus' b$.

(2) $a \oplus' (e \oplus b) = a \oplus b$ and $(a \oplus' e) \oplus' gyr'[a, e]b = a \oplus' b$.

(3) $a \oplus' (b \oplus' e) = a \oplus' b$ and $(a \oplus' b) \oplus' gyr'[a, b]e = a \oplus' b$.

(4) $e \oplus' (e \oplus' a) = a$ and $(e \oplus' e) \oplus' gyr'[e, e]a = a$.

(5) $a \oplus' (b \oplus' c) = (a \oplus' b) \oplus' gyr[a, b]c = (a \oplus' b) \oplus' gyr'[a, b]c$.

Now, note that for all $a, b \in G$, we have $gyr'[a, e] = A = gyr'[a \oplus' e, e]$, $gyr'[e, a] = A = gyr'[e \oplus' a, a]$ and $gyr'[a, b] = gyr[a, b] = gyr[a \oplus b, b] = gyr'[a \oplus' b, b]$ and the proof is complete. □

We say that $S$ is a semigroup with zero, if there exists an element $0 \in S$ (it is called zero) such that for all $s \in S$, $s \oplus 0 = 0 \oplus s = 0$. If $S$ has no zero element, then we can add an extra element 0 to the set $S$. Then we define for every $s \in S$, $0 \oplus s = s \oplus 0 = 0$, and $0 \oplus 0 = 0$. Hence $S \cup \{0\}$ becomes a semigroup with zero element 0. We shall consistently use the notation $S^0$ (semigroup obtained from $S$ by adding a zero) with the following definition:

$$S^0 = \begin{cases} S, & if\ S\ has\ a\ zero\ element\ 0, \\ S \cup \{0\}, & otherwise. \end{cases}$$

Let $G$ be a gyrosemigroup. If there exists an element 0 of $G$ such that for all $a \in G$, $a \oplus 0 = 0 \oplus a = 0$, we say that 0 is a zero (element) of $G$ and such $G$ is a gyrosemigroup with a zero.

**Theorem 4.30.** *Let* $(G, \oplus)$ *be a gyrosemigroup with no zero element and for every* $a \in G$, $gyr[a, a] = A$, *where* $A$ *is the identity automorphism. Set* $G^0 = G \cup \{0\}$ *and define* $(G^0, \oplus')$ *with the following data,*

*(1)* $0 \oplus' a = a \oplus' 0 = 0$, *for all* $a \in G$,

*(2)* $0 \oplus 0 = 0$ *and* $a \oplus' b = a \oplus b$, *for all* $a, b \in G$,

*(3)* $gyr'[0, a] = gyr'[a, 0] = A$, *for all* $a \in G$,

*(4)* $gyr'[0, 0] = A$,

*(5)* $gyr'[a, b]c = gyr[a, b]c$ *and* $gyr'[a, b]0 = 0$, *for all* $a, b, c \in G$.

*Then* $(G^0, \oplus')$ *with* $gyr'$ *becomes a gyrosemigroup with the zero element* $0$.

*Proof.* First, we prove the gyroassociativity law.

(1) $0 \oplus (a \oplus' b) = a \oplus' b$ and $(0 \oplus' a) \oplus' gyr'[0, a]b = a \oplus' b$.

(2) $a \oplus' (0 \oplus' b) = a \oplus' b$ and $(a \oplus' 0) \oplus' gyr'[a, 0]b = a \oplus' b$.

(3) $a \oplus' (b \oplus' 0) = a \oplus' b$ and $(a \oplus' b) \oplus' gyr'[a, b]0 = a \oplus' b$.

(4) $0 \oplus' (0 \oplus' a) = a$ and $(0 \oplus' 0) \oplus' gyr'[0, 0]a = a$.

(5) $a \oplus' (b \oplus' c) = (a \oplus' b) \oplus' gyr[a, b]c = (a \oplus' b) \oplus' gyr'[a, b]c$.

Now, note that for every $a, b \in G$, $gyr'[a, 0] = A = gyr'[a \oplus' 0, 0]$, $gyr'[0, a] = A = gyr'[0 \oplus' a, a]$ and $gyr'[a, b] = gyr[a, b] = gyr[a \oplus b, b] = gyr'[a \oplus' b, b]$ and the proof is complete. □

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