Mathematics Interdisciplinary Research **9** (4) (2024) 373 – 383

Original Scientific Paper

Reformulated Zagreb Indices of Trees

Nasrin Dehgardi⋆

Abstract

Zagreb indices were reformulated in terms of the edge degrees instead of the vertex degrees. For a graph G, the first and second reformulated Zagreb indices are defined respectively as:

$$EM_1(G) = \sum_{\varepsilon \in E(G)} d^2(\varepsilon), \qquad EM_2(G) = \sum_{\varepsilon, \varepsilon' \in E(G),\, \varepsilon \sim \varepsilon'} d(\varepsilon)\, d(\varepsilon'),$$

where $d(\varepsilon)$ and $d(\varepsilon')$ denote the degree of the edges ε and ε' respectively, and $\varepsilon \sim \varepsilon'$ means that the edges ε and ε' are adjacent. In this paper, we obtain sharp lower bounds on the first and second reformulated Zagreb indices with a given number of vertices and maximum degree. Furthermore, we will determine the extremal trees that achieve these lower bounds.

Keywords: Tree, Zagreb indices, First and second reformulated Zagreb indices.

2020 Mathematics Subject Classification: 05C05, 05C07, 05C35.

How to cite this article
N. Dehgardi, Reformulated Zagreb indices of trees, *Math. Interdisc. Res.* **9** (4) (2024) 373-383.

1. Introduction

Consider a graph $G = (V(G), E(G))$. Then we put $N_G(\omega) = \{\vartheta \in V(G) \mid \omega\vartheta \in E(G)\}$ and $d_G(\omega) = d(\omega) = |N_G(\omega)|$ be the open neighborhood and the degree of the vertex ω of G respectively. The maximum degree of the graph is denoted by

⋆Corresponding author (E-mail: n.dehgardi@sirjantech.ac.ir)
Academic Editor: Saeid Alikhani
Received 27 May 2024, Accepted 16 August 2024
DOI: 10.22052/MIR.2024.254966.1464

Δ. Two edges ε and ε' are called adjacent if they are distinct and have a common terminal vertex. The degree of an edge $\varepsilon = \omega\vartheta$ in G is denoted by $d_G(\varepsilon) = d(\varepsilon)$ and $d(\varepsilon) = d(\omega) + d(\vartheta) - 2$.

Zagreb indices were first introduced in [1, 2] and most used molecular structure-descriptors. These invariants are defined as:

$$M_1(G) = \sum_{\omega \in V(G)} d^2(\omega),$$

and

$$M_2(G) = \sum_{\omega\vartheta \in E(G)} d(\omega)\, d(\vartheta),$$

respectively. Properties of the Zagreb indices and variants of these indices may be found in [3–9].

Zagreb indices were reformulated by Miličević et al. [10] as:

$$EM_1(G) = \sum_{\varepsilon \in E(G)} d^2(\varepsilon),$$

and

$$EM_2(G) = \sum_{\varepsilon, \varepsilon' \in E(G),\, \varepsilon \sim \varepsilon'} d(\varepsilon)\, d(\varepsilon'),$$

where $d(\varepsilon)$ denotes the degree of the edge ε, and $\varepsilon \sim \varepsilon'$ means that the edges ε and ε' are adjacent.

Some properties of EM_1 and EM_2 are presented in [11–13]. In [14], Ghalavand et al. discussed the maximum and minimum values of trees for reformulated Zagreb indices. Ji et al. [15] investigated the extermal trees, unicyclic and bicyclic graphs with given order for the first reformulated Zagreb index. Reformulated Zagreb indices have recently attracted the attention of many mathematicians and computer scientists, see [16–19].

In this paper, we obtain lower bounds on the reformulated Zagreb indices and determine the extremal trees that achieve these bounds.

2. Results

A *pendant vertex* is a vertex of degree 1. A tree with exactly one vertex of degree greater than 2 is called a *spider*. A vertex of highest degree in a spider T is called the center of T. A *leg* of a spider is a path from its center to a pendant vertex.

In this section, T denotes a tree with a vertex η where $d_T(\eta) = \Delta$ and $N_T(\eta) = \{\eta_1, \ldots, \eta_\Delta\}$. For positive integers n and Δ, let $\mathcal{T}_{n,\Delta}$ be the set of all trees with n vertices and maximum degree Δ.

Proposition 2.1. *Let $T \in \mathcal{T}_{n,\Delta}$ have a vertex ϑ of degree more than two in maximum distance from η. Then there is a tree $T' \in \mathcal{T}_{n,\Delta}$ such that $EM_1(T') < EM_1(T)$ and $EM_2(T') < EM_2(T)$.*

Proof. Let $d_T(\vartheta) = \gamma \geq 3$ and let $N_T(\vartheta) = \{\vartheta_1, \ldots, \vartheta_\gamma\}$ where ϑ_γ lies on the path from η to ϑ in T. By our assumption, we have $d_T(\vartheta_i) \in \{1, 2\}$ for $1 \leq i \leq \gamma - 1$. Consider the following cases.

Case 1. ϑ is adjacent to at least two pendant vertices such as ϑ_1 and ϑ_2. If $T' = (T - \{\vartheta\vartheta_1\}) \cup \{\vartheta_1\vartheta_2\}$, then

$$\begin{aligned}\lambda_1 &= \sum_{i=3}^{\gamma} d_T^2(\vartheta\vartheta_i) - \sum_{i=3}^{\gamma} d_{T'}^2(\vartheta\vartheta_i) = \sum_{i=3}^{\gamma} [d_T^2(\vartheta\vartheta_i) - (d_T(\vartheta\vartheta_i) - 1)^2] \\ &= \sum_{i=3}^{\gamma} [2\, d_T(\vartheta\vartheta_i) - 1] > 0,\end{aligned}$$

and

$$\begin{aligned}\lambda_2 = &\sum_{3\leq i<j\leq\gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - \sum_{3\leq i<j\leq\gamma} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\vartheta\vartheta_j) \\ &+ \sum_{3\leq i\leq\gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_2) - \sum_{3\leq i\leq\gamma} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\vartheta\vartheta_2) \\ &+ \sum_{i=3}^{\gamma} \sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} d_T(\vartheta\vartheta_i)\, d_T(\omega\vartheta_i) - \sum_{i=3}^{\gamma} \sum_{\omega\in N_{T'}(\vartheta_i)-\{\vartheta\}} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\omega\vartheta_i) \\ = &\sum_{3\leq i<j\leq\gamma} [d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - (d_T(\vartheta\vartheta_i) - 1)\,(d_T(\vartheta\vartheta_j) - 1)] \\ &+ \sum_{3\leq i\leq\gamma} [d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_2) - d_T(\vartheta\vartheta_2)\,(d_T(\vartheta\vartheta_i) - 1)] \\ &+ \sum_{i=3}^{\gamma} \sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} [d_T(\vartheta\vartheta_i)\, d_T(\omega\vartheta_i) - d_T(\omega\vartheta_i)\,(d_T(\vartheta\vartheta_i) - 1)] \\ = &\sum_{3\leq i<j\leq\gamma} [d_T(\vartheta\vartheta_i) + d_T(\vartheta\vartheta_j) - 1] + \sum_{3\leq i\leq\gamma} d_T(\vartheta\vartheta_2) \\ &+ \sum_{i=3}^{\gamma} \sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} d_T(\omega\vartheta_i) > 0.\end{aligned}$$

Therefore,

$$\begin{aligned}EM_1(T) - EM_1(T') &= \lambda_1 + d_T^2(\vartheta\vartheta_1) + d_T^2(\vartheta\vartheta_2) - d_{T'}^2(\vartheta_2\vartheta_1) - d_{T'}^2(\vartheta\vartheta_2) \\ &= \lambda_1 + 2(\gamma - 1)^2 - (\gamma - 1)^2 - 1 \\ &= \lambda_1 + (\gamma - 1)^2 - 1 > 0,\end{aligned}$$

and

$$\begin{aligned}EM_2(T) - EM_2(T') =& \lambda_2 + d_T(\vartheta\vartheta_1)\, d_T(\vartheta\vartheta_2) + \sum_{3\le i\le\gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_1)\\ &- d_{T'}(\vartheta_2\vartheta_1)\, d_{T'}(\vartheta\vartheta_2)\\ =& \lambda_2 + (\gamma-1)^2 + (\gamma-1)\Big(\sum_{3\le i\le\gamma} d_T(\vartheta\vartheta_i)\Big) - (\gamma-1) > 0.\end{aligned}$$

Case 2. ϑ is adjacent to exactly one pendant vertex. Let ϑ_1 be a pendant vertex and $\vartheta\alpha_1\alpha_2\ldots\alpha_l$ be a path in T for $l \ge 2$ and $\vartheta_2 = \alpha_1$. Assume that $T' = (T - \{\vartheta\vartheta_1\}) \cup \{\vartheta_1\alpha_l\}$. Then

$$\begin{aligned}\lambda_1 =& \sum_{i=2}^{\gamma} d_T^2(\vartheta\vartheta_i) - \sum_{i=2}^{\gamma} d_{T'}^2(\vartheta\vartheta_i) = \sum_{i=2}^{\gamma}[d_T^2(\vartheta\vartheta_i) - (d_T(\vartheta\vartheta_i)-1)^2]\\ =& \sum_{i=2}^{\gamma}[2\, d_T(\vartheta\vartheta_i) - 1] > 0.\end{aligned}$$

If $l = 2$, then

$$\begin{aligned}\lambda_2 =& \sum_{2\le i<j\le\gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - \sum_{2\le i<j\le\gamma} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\vartheta\vartheta_j)\\ &+ \sum_{i=3}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} d_T(\vartheta\vartheta_i)\, d_T(\omega\vartheta_i) - \sum_{i=3}^{\gamma}\sum_{\omega\in N_{T'}(\vartheta_i)-\{\vartheta\}} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\omega\vartheta_i)\\ =& \sum_{2\le i<j\le\gamma}[d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - (d_T(\vartheta\vartheta_i)-1)\,(d_T(\vartheta\vartheta_j)-1)]\\ &+ \sum_{i=3}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}}[d_T(\vartheta\vartheta_i)\, d_T(\omega\vartheta_i) - d_T(\omega\vartheta_i)\,(d_T(\vartheta\vartheta_i)-1)]\\ =& \sum_{2\le i<j\le\gamma}[d_T(\vartheta\vartheta_i) + d_T(\vartheta\vartheta_j) - 1] + \sum_{i=3}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} d_T(\omega\vartheta_i) > 0,\end{aligned}$$

and if $l \ge 3$, then

$$\begin{aligned}\lambda_3 =& \sum_{2\le i<j\le\gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - \sum_{2\le i<j\le\gamma} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\vartheta\vartheta_j)\\ &+ \sum_{i=2}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}} d_T(\vartheta\vartheta_i)\, d_T(\omega\vartheta_i) - \sum_{i=2}^{\gamma}\sum_{\omega\in N_{T'}(\vartheta_i)-\{\vartheta\}} d_{T'}(\vartheta\vartheta_i)\, d_{T'}(\omega\vartheta_i)\\ =& \sum_{2\le i<j\le\gamma}[d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_j) - (d_T(\vartheta\vartheta_i)-1)\,(d_T(\vartheta\vartheta_j)-1)]\end{aligned}$$

$$+\sum_{i=2}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}}[d_T(\vartheta\vartheta_i)\,d_T(\omega\vartheta_i)-d_T(\omega\vartheta_i)\,(d_T(\vartheta\vartheta_i)-1)])$$
$$=\sum_{2\leq i<j\leq\gamma}[d_T(\vartheta\vartheta_i)+d_T(\vartheta\vartheta_j)-1]+\sum_{i=2}^{\gamma}\sum_{\omega\in N_T(\vartheta_i)-\{\vartheta\}}d_T(\omega\vartheta_i)>0.$$

Therefore,

$$\begin{aligned}EM_1(T)-EM_1(T') =&\lambda_1+d_T^2(\vartheta\vartheta_1)+d_T^2(\alpha_{l-1}\alpha_l)-d_{T'}^2(\vartheta_1\alpha_l)-d_{T'}^2(\alpha_{l-1}\alpha_l)\\ =&\lambda_1+(\gamma-1)^2+1-1-4\\ =&\lambda_1+(\gamma-1)^2-4>0.\end{aligned}$$

If $l=2$, then

$$\begin{aligned}EM_2(T)-EM_2(T') =&\lambda_2+\sum_{2\leq i\leq\gamma}d_T(\vartheta\vartheta_i)\,d_T(\vartheta\vartheta_1)+d_T(\vartheta\vartheta_2)\,d_T(\vartheta_2\alpha_2)\\ &-d_{T'}(\vartheta_1\alpha_2)\,d_{T'}(\vartheta_2\alpha_2)-d_{T'}(\vartheta_2\alpha_2)\,d_{T'}(\vartheta\vartheta_2)\\ =&\lambda_2+(\gamma-1)\Big(\sum_{2\leq i\leq\gamma}d_T(\vartheta\vartheta_i)\Big)+\gamma-2-2(\gamma-1)\\ >&\lambda_2+2(\gamma-1)-\gamma>0,\end{aligned}$$

and if $l\geq 3$, then

$$\begin{aligned}EM_2(T)-EM_2(T') =&\lambda_3+\sum_{2\leq i\leq\gamma}d_T(\vartheta\vartheta_i)\,d_T(\vartheta\vartheta_1)+d_T(\alpha_{l-1}\alpha_l)\,d_T(\alpha_{l-1}\alpha_{l-2})\\ &-d_{T'}(\vartheta_1\alpha_l)\,d_{T'}(\alpha_{l-1}\alpha_l)-d_{T'}(\alpha_{l-1}\alpha_l)\,d_{T'}(\alpha_{l-1}\alpha_{l-2})\\ =&\lambda_3+(\gamma-1)\Big(\sum_{2\leq i\leq\gamma}d_T(\vartheta\vartheta_i)\Big)+2-2-4\\ >&\lambda_3+2(\gamma-1)-4>0.\end{aligned}$$

Case 3. ϑ is not adjacent to any pendant vertex. Let $\vartheta\beta_1\ldots\beta_t$, $\vartheta\alpha_1\ldots\alpha_l$ be two paths in T such that $l,t\geq 2$, $\vartheta_1=\beta_1$ and $\vartheta_2=\alpha_1$. Assume that $T'=(T-\{\vartheta\vartheta_1\})\cup\{\vartheta_1\alpha_l\}$ and consider λ_1, λ_2 and λ_3 defined in Case 2. Then

$$\begin{aligned}EM_1(T)-EM_1(T') =&\lambda_1+d_T^2(\vartheta\vartheta_1)+d_T^2(\alpha_{l-1}\alpha_l)-d_{T'}^2(\vartheta_1\alpha_l)-d_{T'}^2(\alpha_{l-1}\alpha_l)\\ =&\lambda_1+\gamma^2+1-4-4\\ =&\lambda_1+\gamma^2-7>0.\end{aligned}$$

If $l = 2$, then

$$\begin{aligned}EM_2(T) - EM_2(T') =&\lambda_2 + \sum_{2\leq i\leq \gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_1) + d_T(\vartheta\vartheta_2)\, d_T(\vartheta_2\alpha_2)\\ &- d_{T'}(\vartheta_1\alpha_2)\, d_{T'}(\vartheta_2\alpha_2) - d_{T'}(\vartheta_2\alpha_2)\, d_{T'}(\vartheta\vartheta_2)\\ =&\lambda_2 + \gamma\Big(\sum_{2\leq i\leq \gamma} d_T(\vartheta\vartheta_i)\Big) + \gamma - 4 - 2(\gamma - 1)\\ >&\lambda_2 + \gamma - 2 > 0,\end{aligned}$$

and if $l \geq 3$, then

$$\begin{aligned}EM_2(T) - EM_2(T') =&\lambda_3 + \sum_{2\leq i\leq \gamma} d_T(\vartheta\vartheta_i)\, d_T(\vartheta\vartheta_1) + d_T(\alpha_{l-1}\alpha_l)\, d_T(\alpha_{l-1}\alpha_{l-2})\\ &- d_{T'}(\vartheta_1\alpha_l)\, d_{T'}(\alpha_{l-1}\alpha_l) - d_{T'}(\alpha_{l-1}\alpha_l)\, d_{T'}(\alpha_{l-1}\alpha_{l-2})\\ =&\lambda_3 + \gamma\Big(\sum_{2\leq i\leq \gamma} d_T(\vartheta\vartheta_i)\Big) + 2 - 4 - 4\\ >&\lambda_3 + 2\gamma - 6 > 0.\end{aligned}$$

This completes the proof. □

Proposition 2.2. *Let $T \in \mathcal{T}_{n,\Delta}$ be a spider with $\Delta \geq 3$ and assume T has at least two legs of length more than two. Then there exists a spider $T' \in \mathcal{T}_{n,\Delta}$ such that $EM_1(T') < EM_1(T)$ and $EM_2(T') < EM_2(T)$.*

Proof. Let $d_T(\eta) = \Delta$ and $\eta\beta_1\beta_2\ldots\beta_t$, $\eta\alpha_1\alpha_2\ldots\alpha_l$ be two legs of length more than two such that $\beta_1 = \eta_1$ and $\alpha_1 = \eta_2$. Assume that $T' = (T - \{\beta_1\beta_2\}) \cup \{\beta_2\alpha_l\}$. First we show that $EM_1(T') < EM_1(T)$. If $t = 2$, then

$$\begin{aligned}EM_1(T) - EM_1(T') =&d_T^2(\eta\beta_1) + d_T^2(\beta_1\beta_2) + d_T^2(\alpha_{l-1}\alpha_l)\\ &- d_{T'}^2(\eta\beta_1) - d_{T'}^2(\beta_1\alpha_l) - d_{T'}^2(\alpha_{l-1}\alpha_l)\\ =&\Delta^2 + 1 + 1 - (\Delta - 1)^2 - 1 - 4\\ =&2\Delta - 4 > 0,\end{aligned}$$

and if $t > 2$, then

$$\begin{aligned}EM_1(T) - EM_1(T') =&d_T^2(\eta\beta_1) + d_T^2(\beta_1\beta_2) + d_T^2(\alpha_{l-1}\alpha_l)\\ &- d_{T'}^2(\eta\beta_1) - d_{T'}^2(\beta_1\alpha_l) - d_{T'}^2(\alpha_{l-1}\alpha_l)\\ =&\Delta^2 + 4 + 1 - (\Delta - 1)^2 - 4 - 4\\ =&2\Delta - 4 > 0.\end{aligned}$$

Now, we prove $EM_2(T') < EM_2(T)$. We have the following cases.

- If $t = l = 2$, then

$$\begin{aligned}
EM_2(T) - EM_2(T') &= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i)\, d_T(\eta\eta_1) + d_T(\eta\alpha_1)\, d_T(\alpha_1\alpha_2) \\
&\quad + d_T(\eta\beta_1)\, d_T(\beta_1\beta_2) - \sum_{2\leq i\leq \Delta} d_{T'}(\eta\eta_i)\, d_{T'}(\eta\eta_1) \\
&\quad - d_{T'}(\beta_2\alpha_2)\, d_{T'}(\alpha_1\alpha_2) - d_{T'}(\alpha_1\alpha_2)\, d_{T'}(\eta\alpha_1) \\
&= \sum_{2\leq i\leq \Delta} (\Delta - (\Delta - 1)) d_T(\eta\eta_i) + \Delta + \Delta - 2 - 2\Delta \\
&= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i) - 2 \\
&> d_T(\eta\eta_2) - 2 = \Delta - 2 > 0.
\end{aligned}$$

- If $t = 2$ and $l \geq 3$, then

$$\begin{aligned}
EM_2(T) - EM_2(T') &= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i)\, d_T(\eta\eta_1) + d_T(\eta\beta_1)\, d_T(\beta_1\beta_2) \\
&\quad + d_T(\alpha_l\alpha_{l-1})\, d_T(\alpha_{l-1}\alpha_{l-2}) - \sum_{2\leq i\leq \Delta} d_{T'}(\eta\eta_i)\, d_{T'}(\eta\eta_1) \\
&\quad - d_{T'}(\beta_2\alpha_l)\, d_{T'}(\alpha_l\alpha_{l-1}) - d_{T'}(\alpha_l\alpha_{l-1})\, d_{T'}(\alpha_{l-1}\alpha_{l-2}) \\
&= \sum_{2\leq i\leq \Delta} (\Delta - (\Delta - 1)) d_T(\eta\eta_i) + \Delta + 2 - 2 - 4 \\
&= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i) + \Delta - 4 \\
&> d_T(\eta\eta_2) + \Delta - 4 = 2\Delta - 4 > 0.
\end{aligned}$$

- If $t \geq 3$ and $l \geq 3$, then

$$\begin{aligned}
EM_2(T) - EM_2(T') &= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i)\, d_T(\eta\eta_1) + d_T(\eta\beta_1)\, d_T(\beta_1\beta_2) \\
&\quad + d_T(\alpha_l\alpha_{l-1})\, d_T(\alpha_{l-1}\alpha_{l-2}) - \sum_{2\leq i\leq \Delta} d_{T'}(\eta\eta_i)\, d_{T'}(\eta\eta_1) \\
&\quad - d_{T'}(\beta_2\alpha_l)\, d_{T'}(\alpha_l\alpha_{l-1}) - d_{T'}(\alpha_l\alpha_{l-1})\, d_{T'}(\alpha_{l-1}\alpha_{l-2}) \\
&= \sum_{2\leq i\leq \Delta} (\Delta - (\Delta - 1)) d_T(\eta\eta_i) + \Delta + 4 - 4 - 4 \\
&= \sum_{2\leq i\leq \Delta} d_T(\eta\eta_i) + \Delta - 4 \\
&> d_T(\eta\eta_2) + \Delta - 4 = 2\Delta - 4 > 0.
\end{aligned}$$

This completes the proof. □

Theorem 2.3. *Let $T \in \mathcal{T}_{n,\Delta}$. Then*

$$EM_1(T) = (\Delta - 1)^3 + \Delta^2 + 1,$$

when $\Delta = n - 2$, and

$$EM_1(T) = \Delta(\Delta - 1)^2,$$

when $\Delta = n - 1$. Also

$$EM_1(T) \geq (\Delta - 1)^3 + \Delta^2 + 4(n - \Delta - 2) + 1,$$

when $\Delta < n - 2$, with equality if and only if T is a spider with exactly one leg of length more than two.

Proof. Assume that for every $T \in \mathcal{T}_{n,\Delta}$, $EM_1(T') \leq EM_1(T)$. If $\Delta = 2$, then T' is a path and $EM_1(P_n) = 4n - 10 = (\Delta - 1)^3 + \Delta^2 + 4(n - \Delta - 2) + 1$. If $\Delta \geq 3$, then by Proposition 2.1, T' is a spider. If T' is a star, then $EM_1(T') = \Delta(\Delta - 1)^2$ and if $\Delta = n - 2$, then

$$EM_1(T') = (\Delta - 1)^3 + \Delta^2 + 1.$$

Now let $\Delta < n - 2$. Then by Proposition 2.2, T' has exactly one leg of length more than two. Then

$$EM_1(T') = (\Delta - 1)^3 + \Delta^2 + 4(n - \Delta - 2) + 1.$$

□

The proof of the following theorem uses the arguments provided in the proof of Theorem 2.3.

Theorem 2.4. *Let $T \in \mathcal{T}_{n,\Delta}$. Then*

$$EM_2(T) = \frac{(\Delta - 2)(\Delta - 1)^3}{2} + \Delta(\Delta - 1)^2 + \Delta,$$

when $\Delta = n - 2$, and

$$EM_2(T) = \frac{\Delta(\Delta - 1)^3}{2},$$

when $\Delta = n - 1$. Also

$$EM_2(T) \geq \frac{(\Delta - 2)(\Delta - 1)^3}{2} + \Delta(\Delta - 1)^2 + 4(n - \Delta - 3) + 2\Delta + 2,$$

when $\Delta < n - 2$, with equality if and only if T is a spider with exactly one leg of length more than two.

By definitions of EM_1 and EM_2, we have the following observation.

Observation 2.5. *Let G be a graph. Then for every edge $e \notin E(G)$,*

$$EM_1(G+e) > EM_1(G), \qquad EM_2(G+e) > EM_2(G).$$

By applying Observation 2.5, we obtain the next two Theorems.

Theorem 2.6. *Let G be a simple and connected graph with n vertices and maximum degree Δ. Then*

$$EM_1(G) \geq (\Delta-1)^3 + \Delta^2 + 4(n-\Delta-2) + 1,$$

when $\Delta < n-2$,

$$EM_1(G) \geq (\Delta-1)^3 + \Delta^2 + 1,$$

when $\Delta = n-2$, and

$$EM_1(G) \geq \Delta(\Delta-1)^2,$$

when $\Delta = n-1$. Moreover, the equality holds if and only if G is a spider with at most one leg of length more than two.

Theorem 2.7. *Let G be a simple and connected graph with n vertices and maximum degree Δ. Then*

$$EM_2(G) \geq \frac{(\Delta-2)(\Delta-1)^3}{2} + \Delta(\Delta-1)^2 + 4(n-\Delta-3) + 2\Delta + 2,$$

when $\Delta < n-2$,

$$EM_2(G) \geq \frac{(\Delta-2)(\Delta-1)^3}{2} + \Delta(\Delta-1)^2 + \Delta,$$

when $\Delta = n-2$, and

$$EM_2(G) \geq \frac{\Delta(\Delta-1)^3}{2},$$

when $\Delta = n-1$. Furthermore, the equality holds if and only if G is a spider with at most one leg of length more than two.

Conflicts of Interest. The author declares that she has no conflicts of interest regarding the publication of this article

References

[1] I. Gutman, B. Ruščić, N. Trinajstić and C. F. Wilcox, Graph theory and molecular orbitals, XII. Acyclic polyenes, *J. Chem. Phys.* **62** (1975) 3399 – 3405, https://doi.org/10.1063/1.430994.

[2] I. Gutman and N. Trinajstić, Graph theory and molecular orbitals. Total π-electron energy of alternant hydrocarbons, *Chem. Phys. Lett.* **17** (1972) $535-538$, https://doi.org/10.1016/0009-2614(72)85099-1.

[3] B. Basavanagoud and S. Patil, A note on hyper-Zagreb index of graph operations, *Iranian J. Math. Chem.* **7** (2016) $89-92$, https://doi.org/10.22052/IJMC.2016.12405.

[4] B. Borovićanin, K. C. Das, B. Furtula and I. Gutman, Bounds for Zagreb indices, *MATCH Commun. Math. Comput. Chem.* **78** (2017) $17-100$.

[5] N. Dehgardi, A note on the re-defined third Zagreb index of trees, *Commun. Comb. Optim.* (In press) https://doi.org/ 10.22049/CCO.2023.28868.1757.

[6] N. Dehgardi and T. Došlić, Lower bounds on the general first Zagreb index of graphs with low cyclomatic number, *Discrete Appl. Math.* **345** (2024) $52-61$, https://doi.org/10.1016/j.dam.2023.11.033.

[7] R. Kazemi, A. Behtoei and A. Kohansal, The Zagreb index of bucket recursive trees, *Math. Interdisc. Res.* **5** (2020) $103-111$, https://doi.org/10.22052/MIR.2020.204312.1166.

[8] A. M. Naji, N. D. Soner and I. Gutman, On leap Zagreb indices of graphs, *Commun. Comb. Optim.* **2** (2017) $99-117$, https://doi.org/10.22049/cco.2017.25949.1059.

[9] R. Rasi, S. M. Sheikholeslam and A. Behmaram, Trees with extreme values of second Zagreb index and coindex, *Math. Interdisc. Res.* **4** (2019) $227-238$, https://doi.org/10.22052/MIR.2018.130441.1100.

[10] A. Miličević, S. Nikolić and N. Trinajstić, On reformulated Zagreb indices, *Mol. Divers.* **8** (2004) $393-399$, https://doi.org/10.1023/b:modi.0000047504.14261.2a.

[11] N. De, Some bounds of reformulated Zagreb indices, *Appl. Math. Sci.* **6** (2012) $5005-5012$.

[12] A. Ilić and B. Zhou, On reformulated Zagreb indices, *Discrete Appl. Math.* **160** (2012) $204-209$, https://doi.org/10.1016/j.dam.2011.09.021.

[13] B. Zhou and N. Trinajstić, Some properties of the reformulated Zagreb indices, *J. Math. Chem.* **48** (2010) $714-719$, https://doi.org/10.1007/s10910-010-9704-4.

[14] A. Ghalavand and A. R. Ashrafi, Extremal trees with respect to the first and second reformulated Zagreb index, *Malaya J. Mat.* **5** (2017) $524-530$, https://doi.org/10.26637/mjm503/006.

[15] S. Ji, X. Li and B. Huo, On reformulated Zagreb indices with respect to acyclic, unicyclic and bicyclic graphs, *MATCH Commun. Math. Comput. Chem.* **72** (2014) 723 – 732.

[16] M. K. Jamil and I. Tomescu, First reformulated Zagreb index and some graph operations, *Ars Comb.* **138** (2018) 193 – 209.

[17] S. Ji, Y. Qu and X. Li, The reformulated Zagreb indcies of tricyclic graphs, *Appl. Math. Comput.* **268** (2015) 590 – 595, https://doi.org/10.1016/j.amc.2015.06.058.

[18] E. I. Milovanović, I. Z. Milovanović, E. C. Dolićanin and E. Glogić, A note on the first reformulated Zagreb index, *Appl. Math. Comput.* **273** (2016) 16 – 20, https://doi.org/10.1016/j.amc.2015.09.088.

[19] G. Su, L. Xiong, L. Xu and B. Ma, On the maximum and minimum first reformulated Zagreb index of graphs with connectivity at most k, *Filomat* **25** (2011) 75 – 83.

Nasrin Dehgardi
Department of Mathematics and Computer Science,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: n.dehgardi@sirjantech.ac.ir