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Laplacian Coefficients of a Forest in Terms of the Number of Closed Walks in the Forest and its Line Graph

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Abstract

In this paper, we deal with calculating the laplacian coefficients of a finite simple graph G with the Laplacian polynomial $\psi(G,\lambda) = \sum_{k=0}^{n}(-1)^{n-k}c_k\lambda^k$. We also explore the relationship between the number of closed walks in a graph and a series of its line graphs with the Laplacian coefficients. Our objective is to find a way to determine the Laplacian coefficients using the number of closed walks in a graph and its line graph. Specifically, we have derived the Laplacian coefficients c_{n-k} of a forest F (where $1 \leq k \leq 6$) in terms of the number of closed walks in F and its line graph.

Keywords: Forest, Laplacian coefficient, Closed walk.

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1. Definitions and notations

A *simple undirected graph* is a pair $G = (V, E)$ consisting of a set $V = V(G)$ of vertices and a set $E = E(G)$ of 2−element subsets of V. The elements of E are called *edges* and the number of elements in V is called the *order* of G.

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The notations $n(G)$ and $m(G)$ denote the number of vertices and edges of G, respectively. There are two other graph notations worth mentioning now. The first one is $\deg_G(v)$ which is the number of edges in G with one end point v and the second one is $\deg_G(e)$ which is defined as the degree of vertex e in the line graph of G. Obviously, $deg_G(e) = deg_G(u) + deg_G(v) - 2$, where u and v are the end points of edge e.

We use the notation uvw to denote the path of length two such that vertices u and w have degree one and the vertex v has degree two. In a similar way, we use the notation $uvwx$ to denote a path of length three.

A graph G is said to be *connected* if for arbitrary vertices x and y in V, there exists a sequence $x = x_0, x_1, \ldots, x_r = y$ of vertices such that $x_i x_{i+1} \in E$, $0 \leq i \leq r-1$. The *distance* between two vertices u and v in a connected graph G, $d_G(u, v)$, is defined as the length of a shortest path connecting these vertices and the sum of such numbers is called the *Wiener index* of G, denoted by $W(G)$ [1]. The hyper-Wiener index is a generalization of the Wiener index. It was introduced for trees by Randić in 1993 [2] and for a general graph by Klein et al. in [3]. This topological index is defined as $WW(G) = \frac{1}{2}\sum_{u,v \in V(G)}(d(u,v) + d^2(u,v))$.

A subgraph H of a graph G is a graph with a vertex set $V(H)$ and an edge set $E(H)$, such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. We write $H \leqslant G$ to denote that H is subgraph of G. If $Z \subseteq V$, then the *induced subgraph* $G[Z]$ is the graph with vertex set Z and the edge set $\{uv \in E \mid \{u, v\} \subseteq Z\}$.

In 1972, Gutman and Trinajstić [4] introduced the first degree-based graph invariant applicable in chemistry. This invariant is the first Zagreb index and is defined by the formula $M_1^2(G) = \sum_{v \in V} \deg_G(v)^2$. The second Zagreb index $M_2^1(G) = \sum_{uv \in E} \deg_G(u) \deg_G(v)$ was introduced by Gutman et al. [5] three years later in 1975. The complete history of these graph invariants together with the most important mathematical results about them are reported in [6–8].

The *forgotten index* of G is another variant of the Zagreb group indices defined as $M_1^3(G) = \sum_{v \in V} \deg_G(v)^3 = \sum_{uv \in E}[\deg_G(u)^2 + \deg_G(v)^2]$ [9]. It can be seen that $M_1^\alpha(G) = \sum_{u \in V} \deg_G(u)^\alpha$, $\mathbb{R} \ni \alpha \neq 0, 1$ is the general form of the first Zagreb index. Zhang and Zhang [10] obtained the extremal values of the general Zagreb index in the class of all unicyclic graphs. Milićević et al. [11], reformulated the first and second Zagreb indices in terms of the edge-degrees instead of the vertex-degrees. These invariants were defined the first and second reformulated Zagreb indices defined as $EM_1(G) = \sum_{e \cap f \neq \emptyset}[\deg_G(e) + \deg_G(f)] = \sum_{e \in E} \deg_G(e)^2$ and $EM_2(G) = \sum_{e \cap f \neq \emptyset} \deg_G(e) \deg_G(f)$, respectively.

A $\{0, 1\}$-matrix is a matrix whose entries consist only of the numbers 0 and 1. Suppose G is a graph with vertex set $V = \{u_1, \ldots, u_n\}$. The adjacency matrix of G is a $\{0, 1\}$-matrix $A(G) = (a_{ij})$ in which $a_{ij} = 1$ if and only if $u_i u_j \in E$. It is clear that A is a real symmetric matrix of order n and so all of its eigenvalues are real. The matrices $D(G) = [d_{ij}]$ and $L(G) = D(G) - A(G)$ in which $d_{ii} = \deg(u_i)$ and $d_{ij} = 0, i \neq j$, are called the diagonal and Laplacian matrices of G, respectively. It is well-known that all eigenvalues of $L(G)$ are non-negative real numbers with

0 as the smallest eigenvalue.

The Laplacian polynomial of a graph G is one of the most important polynomials associated to a graph. If G is a graph, then the Laplacian polynomial of G is the characteristic polynomial of $L(G)$. The roots of this polynomial are called the Laplacian eigenvalues of G. Suppose $\psi(G,x) = \det(xI_n - L) = \sum_{k=0}^{n}(-1)^{n-k}c_k x^k$ denotes the Laplacian polynomial of G. Since the coefficients of the Laplacian polynomial have graph theoretical meaning, some authors took into account the coefficients of this polynomial. For those interested in the latest developments on the Laplacian polynomial and its coefficients, we recommend exploring the following publications and their references: [12–22].

Let f be a topological index and G be a graph. For simplifying our arguments, we usually write f as $f(G)$.

Theorem 1.1. *Suppose G is a graph. The following statements hold:*

1. (Merris [19] and Mohar [20]) $c_0(G) = 0$, $c_1(G) = n\tau(G)$, $c_n(G) = 1$ *and* $c_{n-1}(G) = 2m$, *where* $\tau(G)$ *is the number of spanning trees of* G,
2. (Yan and Yeh [22]) $c_2(G) = W(G)$, *when G is a tree,*
3. (Gutman [18]) $c_3(G) = WW(G)$, *when G is a tree,*
4. (Oliveira et al. [21]) $c_{n-2}(G) = \frac{1}{2}[4m^2 - 2m - M_1^2]$ *and* $c_{n-3}(G) = \frac{1}{3!}[4m^2(2m - 3) - 6M_1^2 m + 6M_1^2 + 2M_1^3 - 12t(G)]$, *where $t(G)$ is the number of triangles in G.*

Suppose λ and ξ are two arbitrary real numbers. We now define two invariants that are useful in simplifying formulas in our results. These are:

$$\alpha_{\lambda,\xi}(G) = \sum_{uv\in E}\Big[\deg_G(u)^{\lambda}\deg_G(v)^{\xi} + \deg_G(u)^{\xi}\deg_G(v)^{\lambda}\Big],$$
$$M_2^{\lambda}(G) = \sum_{uv\in E}\Big(\deg_G(u)\deg_G(v)\Big)^{\lambda}.$$

Note that the second Zagreb index is just the case of $\lambda = 1$ in M_2^{λ}. Let G and H be graphs. Set $\mathcal{S}_H(G) = \{X \mid X \leqslant G \text{ and } X \cong H\}$. In [12, 15–17] we proved the following theorems for the coefficients $c_{n-4}(G)$, $c_{n-5}(G)$, and $c_{n-6}(G)$, when G is a forest, respectively.

Theorem 1.2. *([12, 15]). Let G be a graph with n vertices and m edges. Then* $c_{n-4}(G) = \frac{1}{4!}\Big[4m\big(4m^3 - 12m^2 + 51m - 6M_1^2 m - 33M_1^2 + 4M_1^3 + 3\big) + 3M_1^2\big(17M_1^2 - 20\big) + 72M_1^3 - 54M_1^4 - 24M_2^1\Big] - 16\sum_{\{u,v\}\subset V(G)}\binom{\deg_G(u)}{2}\binom{\deg_G(v)}{2} = \frac{1}{4!}\Big[4m\big(4m^3 - 12m^2 + 3m - 6M_1^2 m + 15M_1^2 + 4M_1^3 + 3\big) + 3\big(M_1^2 - 2\big)^2 - 24M_1^3 - 6M_1^4 - 24M_2^1 - 12\Big]$.

Theorem 1.3. *([16]). Let G be a graph with n vertices and m edges. Then* $c_{n-5}(G) = \frac{1}{5!}\Big[2m\big(16m^4 - 80m^3 + 60m^2 - 40M_1^2m^2 + 60m + 180M_1^2m + 40M_1^3m + 15(M_1^2)^2 - 120M_1^2 - 140M_1^3 - 30M_1^4 - 120M_2^1\big) - 20M_1^2\big(3M_1^2 + M_1^3 + 6\big) + 120M_1^3 + 120M_1^4 + 24M_1^5 + 240M_2^1 + 120\alpha_{1,2}\Big]$.

Theorem 1.4. *([17]). Let G be a graph with n vertices and m edges. Then* $c_{n-6}(G) = \frac{1}{6!}\Big[64m^6 - 480m^5 + 720m^4 + 600m^3 - 360m^2 - 480m - 720M_1^5 - 2160\alpha_{1,2} - 720\alpha_{1,3} + 540(M_1^2)^2 - 2340m^2M_1^2 + 2160mM_1^3 - 1080M_1^4 + 720M_2^1 - 120M_1^6 - 240M_1^2M_1^3m - 720M_2^2 + 600M_1^2M_1^3 + 1680M_1^2m^3 - 810(M_1^2)^2m - 1920M_1^3m^2 + 1620M_1^4m + 3600M_2^1m - 720\Theta_2 - 1260mM_1^2 + 720M_1^2 + 480M_1^3 - 240m^4M_1^2 + 320m^3M_1^3 + 360M_1^2M_2^1 + 90M_1^2M_1^4 + 180(M_1^2)^2m^2 - 360M_1^4m^2 - 1440M_2^1m^2 + 288M_1^5m + 1440\alpha_{1,2}m + 40(M_1^3)^2 - 15(M_1^2)^3\Big]$,
where $\Theta_2(G) = \sum_{uvw\in\mathcal{S}_{P_3}(G)} \deg_G(u)\deg_G(w)$.

2. Laplacian coefficients and the number of closed walks

Let G be a graph. A *walk* in G is a sequence $W : v_{i_0}e_{i_1}v_{i_1}e_{i_2}v_{i_2}\ldots e_{i_k}v_{i_k}$ of vertices and edges of G in such a way that for each j, $0 \le j \le k-1$, v_{i_j} and $v_{i_{j+1}}$ are end points of the edge $e_{i_{j+1}}$ in G. The walk is said to be *closed* if it begins and ends at the same vertex. The number of edges of a walk is called the *length* of the walk. The number of closed walks of a given length k, is denoted by $\mathcal{W}_k(G)$. It is easy to see that, in each graph G, $\mathcal{W}_1(G) = 0$, $\mathcal{W}_2(G) = 2m(G)$ and $\mathcal{W}_3(G) = 6t(G)$, where $t(G)$ is the number of triangles in G.

The line graph of a given graph G is another graph $L_1(G)$ that represents the adjacencies between edges of G. This graph is constructed in this way: any edge in G will be a vertex in $L_1(G)$ and for two edges in G with a common vertex, make an edge between their corresponding vertices in $L_1(G)$. For integer k, $k \ge 2$, we define: $L_k(G) = L_1(L_{k-1}(G))$ and $L_0(G) = G$.

Theorem 2.1. (*See* [23, Theorem 1.9]) *Let G be a graph with adjacency matrix A and let k be a positive integer. Then* $\operatorname{tr} A^k = \mathcal{W}_k(G)$.

The complete, star and cycle graphs on n vertices are denoted by K_n, S_n and C_n, respectively. Suppose $V(S_5) = \{v_1, v_2, v_3, v_4, v_5\}$ and $E(S_5) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5\}$. The graph S_5^{2e} is constructed from the graph S_5 by adding two edges v_2v_3 and v_4v_5.

Lemma 2.2. *Let G be a graph. Then*

1. $|\mathcal{S}_{P_3}(G)| = m(L_1(G))$,

2. $M_1^2(G) = 2\Big(m(G) + m(L_1(G))\Big)$,

3. $M_1^3(G) = 2\big[m(G) + 3m(L_1(G)) + 3t(L_1(G)) - 3t(G)\big]$.

Proof. 1. Suppose that $e_1, e_2 \in E(G) = V(L_1(G))$. By definition of line graph, $e_1e_2 \in E(L_1(G))$ if and only if e_1 and e_2 have a common vertex. This proves that $|\mathcal{S}_{P_3}(G)| = m(L_1(G))$.

2. Choose a vertex v in G. The number of subgraphs of G isomorphic to P_3 and middle vertex v is equal to $\binom{\deg_G(v)}{2}$. Hence $|\mathcal{S}_{P_3}(G)| = \sum_{v\in V(G)} \binom{\deg_G(v)}{2}$ $= \frac{1}{2}M_1^2(G) - m(G)$. By the case (1), $M_1^2(G) = 2(m(G) + m(L_1(G)))$, as desired.

3. Suppose that $e_1, e_2, e_3 \in E(G) = V(L_1(G))$. By definition, $L_1(G)[\{e_1, e_2, e_3\}]$ $\cong C_3$ if and only if e_1, e_2 and e_3 construct a cycle of length 3 or the star graph S_4. Therefore, $t(L_1(G)) = \sum_{v\in V(G)} \binom{\deg_G(v)}{3} + t(G) = \frac{1}{6}(M_1^3(G) - 3M_1^2(G) + 4m) + t(G)$. We now apply Lemma 2.2(2), to show that $t(L_1(G)) = \frac{1}{6}(M_1^3(G) - 2m(G) - 6m(L_1(G))) + t(G)$. Hence the result follows.

□

Let $C_k : v_1v_2 \dots v_kv_1$ be the cycle graph on k vertices. The graph $C_k[1^{l_1}, 2^{l_2}, \dots, k^{l_k}]$ is constructed from C_k by adding l_i pendant edges, $1 \le i \le k$, to the vertex v_i. For simplicity, if $l_i = 0$, for some i, then we omit i^0 in our notation.

Lemma 2.3. *Let G be a forest with $m(G)$ edges. Then*
(i) $M_1^4(G) = \mathcal{W}_4(L_1(G)) + 2m(G) + 12m(L_1(G)) + 36t(L_1(G)) - 4m(L_2(G))$,
(ii) $M_1^5(G) = \mathcal{W}_5(L_1(G)) + 5M_1^4(G) - 5M_1^3(G) - 15M_1^2(G) + 12m(G) - 5\alpha_{1,2}(G) + 30M_2^1(G)$,
(iii) $M_1^6(G) = \mathcal{W}_6(L_1(G)) - 56m(L_1(G)) + 6M_1^5(G) - 6\alpha_{1,3}(G) - 6M_2^2(G) - 60m(G) - 9M_1^3(G) - 9M_1^4(G) + 61M_1^2(G) - 102M_2^1(G) - 12m(L_2(G)) + 42\alpha_{1,2}(G) - 6\Theta_2(G) - 6t(L_1(G))$.

Proof. Let H be an arbitrary graph.

(*i*) It can be easily seen that

$$\mathcal{W}_4(H) = 2m(H) + 4|\mathcal{S}_{P_3}(H)| + 8|\mathcal{S}_{C_4}(H)|. \tag{1}$$

Since G is a forest, $|\mathcal{S}_{C_4}(L_1(G))| = 3|\mathcal{S}_{K_4}(L_1(G))| = 3\sum_{v\in V(G)} \binom{\deg_G(v)}{4} = \frac{3}{24}\left(M_1^4(G) - 6M_1^3(G) + 11M_1^2(G) - 12m(G)\right)$, and by Lemma 2.2(2,3),

$$|\mathcal{S}_{C_4}(L_1(G))| = \frac{3}{24}\left(M_1^4(G) - 2m(G) - 14m(L_1(G)) - 36t(L_1(G))\right). \tag{2}$$

Also, by Lemma 2.2 (1), $|\mathcal{S}_{P_3}(L_1(G))| = m(L_2(G))$. We now apply Equations (1) and (2) to deduce that $M_1^4(G) = \mathcal{W}_4(L_1(G)) + 2m(G) + 12m(L_1(G))$ $+ 36t(L_1(G)) - 4m(L_2(G))$, as desired.

(ii) By an easy calculation, one can see that

$$\mathcal{W}_5(H) = 30t(H) + 10|\mathcal{S}_{C_5}(H)| + 10|\mathcal{S}_{C_3[1^1]}(L_1(G))|. \tag{3}$$

Since G is a forest, $|\mathcal{S}_{C_5}(L_1(G))| = 12|\mathcal{S}_{K_5}(L_1(G))| = 12\sum_{v\in V(G)}\binom{\deg_G(v)}{5}$ $= \frac{12}{120}(M_1^5(G) - 10M_1^4(G) + 35M_1^3(G) - 50M_1^2(G) + 48m(G))$ and $|\mathcal{S}_{C_3[1^1]}(L_1(G))| = \sum_{uv\in E(G)}[\binom{\deg_G(u)-1}{2}(\deg_G(u) + \deg_G(v) - 4) + \binom{\deg_G(v)-1}{2}(\deg_G(u) + \deg_G(v) - 4)] = \frac{1}{2}M_1^4(G) + \frac{1}{2}\alpha_{1,2}(G) - \frac{7}{2}M_1^3(G) - 3M_2^1(G) + 8M_1^2(G) - 8m(G)$. Now the result follows from Equation (3).

(iii) By some easy calculations, one can see that

$$\begin{aligned}\mathcal{W}_6(H) = {} & 2m(H) + 12|\mathcal{S}_{P_3}(H)| + 6|\mathcal{S}_{P_4}(H)| + 12|\mathcal{S}_{S_4}(H)| + 24t(H) + \\ & 48|\mathcal{S}_{C_4}(H)| + 36|\mathcal{S}_{K_4-e}(H)| + 12|\mathcal{S}_{C_4[1^1]}(H)| + 12|\mathcal{S}_{C_6}(H)| \\ & + 24|\mathcal{S}_{S_4^{2e}}(H)|.\end{aligned} \tag{4}$$

We now assume that $x = uvw \in \mathcal{S}_{P_3}(G) = E(L_1(G))$. Then the number of paths constructed from three edges in $L_1(G)$ with x as its middle edge can be computed via $(\deg_G(u) + \deg_G(v) - 3)(\deg_G(v) + \deg_G(w) - 3) - t_x(L_1(G))$, where $t_x(L_1(G))$ denotes the number of triangles constructed on the edge x of $L_1(G)$. Therefore,

$$\begin{aligned} & |\mathcal{S}_{P_4}(L_1(G))| \\ = {} & \sum_{uvw\in\mathcal{S}_{P_3}(G)} [(\deg_G(u) + \deg_G(v) - 3)(\deg_G(v) + \deg_G(w) - 3) - t_x(L_1(G))] \\ = {} & \sum_{uv\in E(G)} \deg_G(u)\deg_G(v)(\deg_G(u) + \deg_G(v) - 2) \\ - {} & 3\sum_{uv\in E(G)} [\deg_G(u)(\deg_G(v) - 1) + (\deg_G(u) - 1)\deg_G(v)] + \Theta_2(G) \\ - {} & 3t(L_1(G)) + 9m(L_1(G)) + \sum_{v\in V(G)} \binom{\deg_G(v)}{2}(\deg_G(v)^2 - 6\deg_G(v)). \end{aligned}$$

Now we simplify the last summation to deduce that

$$\begin{aligned}|\mathcal{S}_{P_4}(L_1(G))| = {} & \Theta_2(G) - 3t(L_1(G)) + 9m(L_1(G)) + \alpha_{1,2}(G) - 8M_2^1(G) \\ & + 6M_1^2(G) + \frac{1}{2}M_1^4(G) - \frac{7}{2}M_1^3(G).\end{aligned} \tag{5}$$

Suppose that $e = uv \in E(G) = V(L_1(G))$. The number of stars isomorphic to S_4 in $L_1(G)$ with e as its center is computed by $\binom{\deg_G(u)+\deg_G(v)-2}{3}$ and so

$$\begin{aligned}|\mathcal{S}_{S_4}(L_1(G))| = {} & \frac{1}{6}M_1^4(G) + \frac{1}{2}\alpha_{1,2}(G) - \frac{3}{2}M_1^3(G) \\ & - 3M_2^1(G) + \frac{13}{3}M_1^2(G) - 4m(G).\end{aligned} \tag{6}$$

By the proof of Case (1), we have

$$|\mathcal{S}_{C_4}(L_1(G))| = \frac{3}{24}\Big(M_1^4(G) - 6M_1^3(G) + 11M_1^2(G) - 12m(G)\Big). \quad (7)$$

Furthermore, it can be seen that

$$|\mathcal{S}_{K_4-e}(L_1(G))| = 2|\mathcal{S}_{C_4}(L_1(G))|. \quad (8)$$

On the other hand, by definition of complete graphs,

$$|\mathcal{S}_{C_n}(K_n)| = \frac{1}{2}(n-1)!. \quad (9)$$

Note that four edges in G give an induced subgraph of $L_1(G)$ isomorphic to K_4 if and only if those edges has a common vertex. Thus,

$$\begin{aligned}
|\mathcal{S}_{C_4[1^1]}(L_1(G))| = 3 \sum_{e=uv\in E(G)} \Big[&\binom{\deg_G(u)-1}{3}(\deg_G(u)+\deg_G(v)-5) \\
&+ \binom{\deg_G(v)-1}{3}(\deg_G(u)+\deg_G(v)-5)\Big] \quad (10) \\
&= \frac{1}{2}M_1^5(G) + \frac{1}{2}\alpha_{1,3}(G) - \frac{11}{2}M_1^4(G) - 3\alpha_{1,2}(G) \\
&+ \frac{41}{2}M_1^3(G) - \frac{67}{2}M_1^2(G) + 11M_2^1(G) + 30m(G). \quad (11)
\end{aligned}$$

Furthermore, six edges of G give a cycle in $L_1(G)$ if and only if those edges have a common vertex. We now apply Equation (9) to deduce that

$$\begin{aligned}
|\mathcal{S}_{C_6}(L_1(G))| = 60|\mathcal{S}_{K_6}(L_1(G))| &= 60 \sum_{v\in V(G)} \binom{\deg_G(v)}{6} \\
&= \frac{1}{12}M_1^6(G) - \frac{5}{4}M_1^5(G) + \frac{85}{12}M_1^4(G) - \frac{75}{4}M_1^3(G) \\
&+ \frac{137}{6}M_1^2(G) - 20m(G). \quad (12)
\end{aligned}$$

Suppose $f = uv \in E(G) = V(L_1(G))$. The number of subgraphs of $L_1(G)$ isomorphic to S_5^{2e} with the property that f is a vertex of degree 4 can be obtained from $\binom{\deg_G(u)-1}{2}\binom{\deg_G(v)-1}{2} + 3\binom{\deg_G(u)-1}{4} + 3\binom{\deg_G(v)-1}{4}$. Therefore,

$$\begin{aligned}
|\mathcal{S}_{S_5^{2e}}(L_1(G))| = \frac{1}{4}M_2^2(G) - \frac{3}{4}\alpha_{1,2}(G) + \frac{39}{8}M_1^3(G) + \frac{9}{4}M_2^1(G) \\
- \frac{31}{4}M_1^2(G) + 7m(G) + \frac{1}{8}M_1^5(G) - \frac{5}{4}M_1^4(G). \quad (13)
\end{aligned}$$

We now apply Lemma 2.2 and Equations (4), (5), (6), (8), (9), (11), (12) and (13) to complete the proof of this case. Hence the result comes up.

□

Lemma 2.4. *([17]). Let G be a graph with n vertices and m edges. Then* $EM_2(G) = \alpha_{1,2} - 6M_2^1 + \frac{1}{2}M_1^4 - \frac{5}{2}M_1^3 + 6M_1^2 - 4m + \Theta_2$.

Lemma 2.5. *Let G be a graph. Then*

$$\begin{aligned}
M_2^1(G) &= \frac{1}{2}M_1^2(L_1(G)) - \frac{1}{2}M_1^3(G) + 2M_1^2(G) - 2m(G),\\
EM_2(G) &= \frac{1}{2}M_1^2(L_2(G)) - \frac{1}{2}M_1^3(L_1(G)) + 2M_1^2(L_1(G)) - 2m(L_1(G)),\\
\alpha_{1,2}(G) &= \frac{1}{3}M_1^3(L_1(G)) - \frac{1}{3}M_1^4(G) + 2M_1^3(G) + 4M_2^1(G) - 4M_1^2(G) + \frac{8}{3}m(G),\\
\Theta_2(G) &= \frac{1}{2}M_1^2(L_2(G)) - \frac{5}{6}M_1^3(L_1(G)) + 2M_1^2(L_1(G)) - 2m(L_1(G))\\
&\quad - \frac{1}{6}M_1^4(G) + \frac{1}{2}M_1^3(G) + 2M_2^1(G) - 2M_1^2(G) + \frac{4}{3}m(G).
\end{aligned}$$

Proof. Suppose that $e = uv \in E(G)$. By definition of line graph, $\deg_{L_1(G)}(e) = \deg_G(u) + \deg_G(v) - 2$. Hence $M_1^2(L_1(G)) = \sum_{uv\in E(G)}(\deg_G(u) + \deg_G(v) - 2)^2 = M_1^3(G) + 2M_2^1(G) - 4M_1^2(G) + 4m(G)$ which completes the proof of the first equality. The second equality follows from $EM_2(G) = M_2(L_1(G))$ and the first equality.

Next, we prove the third equality. We have $M_1^3(L_1(G)) = \sum_{uv\in E(G)}(\deg_G(u) + \deg_G(v) - 2)^3 = 3\alpha_{1,2}(G) + M_1^4(G) - 6M_1^3(G) - 12M_2^1(G) + 12M_1^2(G) - 8m(G)$, as desired. Finally, by Lemma 2.4, $\Theta_2(G) = EM_2(G) - \alpha_{1,2} + 6M_2^1(G) - \frac{1}{2}M_1^4(G) + \frac{5}{2}M_1^3(G) - 6M_1^2(G) + 4m(G)$. Now, the fourth equality follows from the above three equalities. □

We are now ready to prove the main result of this section.

Theorem 2.6. *Let G be a graph, $A = A(G)$ and $A_1 = A(L_1(G))$. Then*

1. $c_{n-1}(G) = \operatorname{tr} A^2$ *and* $c_{n-2}(G) = \frac{1}{2}\Big[\prod_{i=0}^{1}\big(\operatorname{tr} A^2 - 2i\big) - \operatorname{tr} A_1^2\Big]$,
2. $c_{n-3}(G) = \frac{1}{3!}\Big[\prod_{i=0}^{2}\big(\operatorname{tr} A^2 - 2i\big) - 3\operatorname{tr} A^2 \operatorname{tr} A_1^2 + \operatorname{tr}(12A_1^2 + 2A_1^3) - 4\operatorname{tr} A^3\Big]$,
3. $c_{n-4}(G) = \frac{1}{4!}\Big[\prod_{i=0}^{3}\big(\operatorname{tr} A^2 - 2i\big) - 6(\operatorname{tr} A^2)^2 \operatorname{tr} A_1^2 + \operatorname{tr} A^2 \operatorname{tr}(60A_1^2 + 8A_1^3) - \operatorname{tr}(144A_1^2 + 48A_1^3 + 6A_1^4) + 3(\operatorname{tr} A_1^2)^2\Big]$, *when G is a forest,*
4. $c_{n-5}(G) = \frac{1}{5!}\Big[\prod_{i=0}^{4}\big(\operatorname{tr} A^2 - 2i\big) - 10(\operatorname{tr} A^2)^3 \operatorname{tr} A_1^2 + (\operatorname{tr} A^2)^2 \operatorname{tr}(180A_1^2 + 20A_1^3) - \operatorname{tr} A^2 \operatorname{tr}(1040A_1^2 + 280A_1^3 + 30A_1^4) + 15\operatorname{tr} A^2(\operatorname{tr} A_1^2)^2 - \operatorname{tr} A_1^2 \operatorname{tr}(120A_1^2 + 20A_1^3) + \operatorname{tr}(1920A_1^2 + 960A_1^3 + 240A_1^4 + 24A_1^5)\Big]$, *when G is a forest,*

5. $c_{n-6}(G) = \frac{1}{6!}\Big[\prod_{i=0}^{5}\big(\operatorname{tr}A^2-2i\big)-15(\operatorname{tr}A^2)^4\operatorname{tr}A_1^2+(\operatorname{tr}A^2)^3\operatorname{tr}(420A_1^2+40A_1^3) - (\operatorname{tr}A^2)^2\operatorname{tr}(4260A_1^2+960A_1^3+90A_1^4)+45(\operatorname{tr}A^2\operatorname{tr}A_1^2)^2-810\operatorname{tr}A^2(\operatorname{tr}A_1^2)^2 + \operatorname{tr}A^2\operatorname{tr}(18480A_1^2+7520A_1^3+1620A_1^4+144A_1^5)-15(\operatorname{tr}A_1^2)^3+3600 (\operatorname{tr}A_1^2)^2-28800\operatorname{tr}A_1^2+\operatorname{tr}A_1^2\operatorname{tr}(1200A_1^3+90A_1^4)+40(\operatorname{tr}A_1^3)^2-120\operatorname{tr}A^2 \operatorname{tr}A_1^2\ \operatorname{tr}A_1^3-\operatorname{tr}(19200A_1^3-7200A_1^4-1440A_1^5-120A_1^6)\Big]$, *when G is a forest.*

Proof. The proof is derived from Theorems 1.1 to 1.4 and 2.1 and Lemmas 2.2, 2.3 and 2.5 along with some straightforward calculations. For instance, by applying Theorem 1.1(4), we obtain

$$c_{n-2}(G) = \frac{1}{2}\left[4m(G)^2 - 2m(G) - M_1^2(G)\right].$$

Consequently, employing Lemma 2.2 (2), we deduce that

$$c_{n-2}(G) = \frac{1}{2}\left[4m(G)^2 - 4m(G) - 2m(L_1(G))\right].$$

On the other hand, it is known that $\operatorname{tr}A^2 = 2m(G)$ and $\operatorname{tr}A_1^2 = 2m(L_1(G))$. Therefore, we can conclude that

$$c_{n-2}(G) = \frac{1}{2}\Big[\prod_{i=0}^{1}\big(\operatorname{tr}A^2 - 2i\big) - \operatorname{tr}A_1^2\Big].$$

The proofs for the remaining cases follow a similar pattern, and for brevity, we have omitted their detailed presentation here. □

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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