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Original Scientific Paper

On New Definitions Related to Golden Ratio

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Abstract

New definitions employing the golden ratio as the characteristic parameter are proposed. The definitions are classified into two categories: Geometrical and Physical properties. In the first category, the golden ratio tree is defined, and its properties are discussed through theorems. Then, decaying and growing type golden ratio spirals are proposed and discussed. The equation producing the golden ratio heart in the analytical two-dimensional space is given. Regarding the second category, the golden ratio ball is defined with respect to collisions with the ground and the collision coefficient is determined. Golden ratio damping is another new definition in which the dimensionless damped parameter is determined in terms of the golden ratio. Theorems are posed and proven regarding the properties of the definitions. Numerical solutions in the form of plots are given when necessary.

Keywords: Fibonacci sequence, Tree, Elastic collision, Damped systems, Spirals, Heart.

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1. Introduction

The ratio of the successive terms converges to the golden ratio as the number of terms increases in a Fibonacci sequence. The golden ratio is encountered in nature

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frequently [1]. It is used in the arts frequently to produce more aesthetic geometries. The golden ratio penetrated into our daily lives, for example, in fashion designs of bags, scarves, and small accessories [2]. To produce more beautiful and harmonic forms, the golden ratio and the Fibonacci sequence have been successfully implemented in the design of Ladies' clothing [3]. The Fibonacci sequence and the golden ratio are employed in textile design [4]. The beauty of the golden spiral to Fibonacci spiral was compared from the perception point of view of attendees of the experiment. The curvature of the golden spiral changes continuously, whereas that of the Fibonacci spiral is discontinuous and %80 of the participants preferred the golden spiral [5]. Various spirals including, the golden and Fibonacci spirals, are discussed in detail [6]. With inspiration from the discrete relation of Fibonacci sequences, a continuous differential equation has been derived that produces special spirals [7]. Golden fractal trees are mathematically investigated in detail [8].

Due to the importance and wide usage of the golden ratio, in this work, new definitions are proposed related to this ratio. The definitions are classified under two general categories: 1) Geometrical definitions, 2) Physical property definitions. Under the former category, a golden ratio tree is proposed first. It is somewhat different from the ones proposed in the literature [8, 9] in the sense that the trunk and branches of the tree possess diameters, not represented by one-dimensional lines. It is shown that while the total length of all branches is infinity, the volume of the tree is finite. The total distance travelled by a liquid drop starting from the trunk and selecting one of the branches at each bifurcation point is also finite. Golden ratio spirals are defined next. The spirals may be growing (the distance to the origin increases with counterclockwise rotation) or decaying (the distance to the origin decreases with counterclockwise rotation). For a detailed analysis of various types of spirals, including more general types than proposed here, see [6]. The curvature of the spiral is continuously increasing/decreasing, which makes it more aesthetic compared to the traditional Fibonacci spiral, which possesses constant curvatures at each 90^0 of rotation [5]. Finally, an algebraic equation of a single parameter, being the golden ratio is presented to draw the two-dimensional heart. There are numerous heart equations resembling the shape of the heart, which can be traced in an internet search. The one-parameter golden ratio heart equation is, however, unique to this work and was not reported before. Under the second category, the golden ratio ball is defined first. Using the definition, the elastic coefficient, the total distance and the time until the ball reaches its static position after bouncing indefinitely from free fall are given. For linear oscillatory systems, golden ratio damping is proposed and the dimensionless damping ratio is expressed in terms of the golden ratio based on the definition. It is shown that the system is lightly damped. While similar definitions, if not exactly the same, can be traced in the literature for the first category, the second category definitions are original, and similar definitions do not exist in the literature.

Table 1: The Fibonacci Sequence.

F_0	F_1	F_2	F_3	F_4	F_5	F_6	F_7	F_8	F_9	F_{10}	F_{11}	F_{12}	F_{13}
0	1	1	2	3	5	8	13	21	34	55	89	144	233

2. Preliminaries

The preliminary knowledge about the Fibonacci sequence and its properties is given in this section [10]. The Fibonacci sequence is calculated by the difference equation

$$F_{j+2} = F_{j+1} + F_j \ , \qquad j = 0, 1, 2 \ldots, \tag{1}$$

with F_0=0 and F_1=1 and the first 14 terms are listed in Table 1. In the sequence, each term is the sum of the previous two terms.

The difference equation can be solved by assuming a solution $F_n = r^n$. Inserting into (1) and simplifying yields

$$r^2 - r - 1 = 0. \tag{2}$$

The solution is

$$r_{1,2} = \frac{1 \mp \sqrt{5}}{2}. \tag{3}$$

Hence, the general solution of the difference equation is

$$F_n = c_1 \left(\frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left(\frac{1 - \sqrt{5}}{2} \right)^n. \tag{4}$$

Applying the initial conditions $F_0 = 0$ and $F_1 = 1$, the coefficients are $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$. Hence a direct formula to calculate the terms in a Fibonacci sequence is derived

$$F_n = \frac{1}{\sqrt{5}} \left(\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right). \tag{5}$$

Despite the irrational numbers in the formula, the results are always integers. If one calculates the limit of the ratio as the terms get larger

$$\lim_{n \to \infty} \frac{F_{n+1}}{F_n} = \frac{1 + \sqrt{5}}{2} \ \cong 1.6180339887, \tag{6}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ is called the Golden Ratio. This ratio has long been considered as a measure of aesthetics in arts and architecture. The ratio was identified in the structure of cosmic objects as well as in nature, such as seashells, vegetables, the human body, the orientation of leaves, etc. The n-th power of the Golden ratio can always be expressed as a linear function of itself

$$\varphi^n = F_n \varphi + F_{n-1} \qquad n = 1, 2, 3, \ldots \tag{7}$$

Another important property of the Golden ratio is that

$$\frac{1}{\varphi} = \varphi - 1, \tag{8}$$

as can be verified from Equation (2).

3. Geometrical definitions

Some new geometrical definitions are given in this section. Similar results, although not exactly the same, can be retrieved from the literature.

3.1. Golden ratio tree

Golden ratio tree is defined in two-dimensional space as a tree having trunk with length L and diameter D. The trunk bifurcates into two branches with each of them having length L/φ and diamerer $D/\ \varphi$, where $\varphi = \frac{1+\sqrt{5}}{2} \cong 1.618034$ is the famous golden ratio which is the ratio achieved when $\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$ with F_n representing the n-th Fibonacci number. The bifurcation angle is $\frac{360}{1+\varphi} \cong 137.5$. The tree is obtained by repeating the process up to infinity (Figure 1), and its properties are summarized in the following definition:

Definition 3.1. The golden ratio tree is a slender tree with a specified trunk length and diameter that bifurcates into two branches at the ends at each step up to infinity. The length and diameter are reduced by the golden ratio after each bifurcation with constant bifurcation angles 137.5^0 between the branches

A line version of the tree exists in the literature in which the diameters of all branches are zero, but the bifurcation angles vary [9]. In fact, the tree possesses a fractal geometry, which is a special case of the Mandelbrot fractal trees. Fractal geometries are geometries that repeat themselves on different geometric scales. Any small portion of the proposed golden ratio tree when magnified will resemble the shape of the original tree. Within the specific contraction ratios $\frac{1}{\varphi} = 0.618034$ and angles 137.5^0, the branches are avoiding type that is no branch intersects with each other [8]. Although the analysis was presented in [8] for no thickness in the branches, for sufficiently small slenderness ratios of $\frac{D}{L} \ll 1$, theoretical results of [8] will also apply to our case.

Assuming a liquid drop entering to the trunk from the roots of the tree, when the fluid reaches the far end of one of the branches of the infinite tree, the total distance travelled by the drop would be finite as stated by the following theorem.

Theorem 3.2. *The liquid drop entering the trunk and reaching the far end of one of the branches must travel a finite route of length*

$$L_T = L(1 + \varphi), \tag{9}$$

subject to the condition that it does not split into parts during the travel.

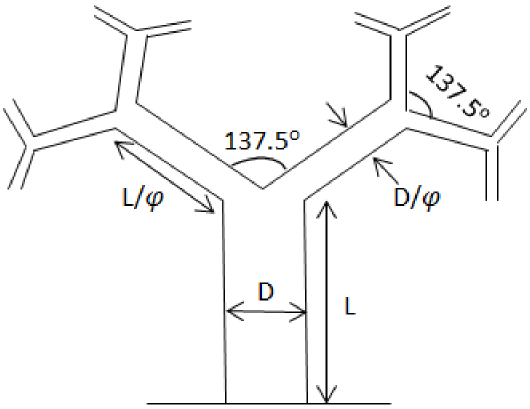

Figure 1: Golden Ratio Tree.

Proof. Add the total lengths travelled by the tiny drop

$$\begin{aligned} L_T &= L + \frac{L}{\varphi} + \frac{L}{\varphi^2} + \cdots = L\left(1 + \frac{1}{\varphi} + \frac{1}{\varphi^2} + \dots\right) \\ &= L\frac{1}{1-\frac{1}{\varphi}} = L\frac{\varphi}{\varphi - 1} = L\varphi^2 = L(\varphi + 1). \end{aligned}$$

In the calculations, the properties $\frac{1}{\varphi} = \varphi - 1$ and $\varphi^2 = \varphi + 1$ given in (7) and (8) are used as can be verified numerically. Therefore, the total distance travelled is approximately $2.618L$ □

Theorem 3.3. *The total length of the tree including the trunk and all branches is infinite*

Proof. Add the total length of the trunk and all branches

$$\begin{aligned} L_T &= L + 2\frac{L}{\varphi} + 2^2\frac{L}{\varphi^2} + \cdots = L\left(1 + \frac{2}{\varphi} + \left(\frac{2}{\varphi}\right)^2 + \dots\right) \\ &= L\lim_{n\to\infty}\frac{1-\left(\frac{2}{\varphi}\right)^{n+1}}{1-\frac{2}{\varphi}} = \infty, \end{aligned}$$

since for the geometric series, the ratio $\frac{2}{\varphi} > 1$ □

Theorem 3.4. *The total volume of the tree including the trunk and all branches* *is finite and equal to*

$$V_T = \frac{\pi D^2}{4} L \frac{2\varphi+1}{2\varphi-1}. \tag{10}$$

Proof. Add the total volume of the trunk and all branches

$$V_T = \frac{\pi D^2}{4}L + \frac{\pi D^2}{4}L\frac{2}{\varphi^3} + \frac{\pi D^2}{4}L\frac{4}{\varphi^6}\cdots = \frac{\pi D^2}{4}L\left(1+\frac{2}{\varphi^3}+\frac{4}{\varphi^6}\ldots\right).$$

The parenthesis is a convergent series since the ratio $\frac{2}{\varphi^3} < 1$. The result is

$$\left(1+\frac{2}{\varphi^3}+\frac{4}{\varphi^6}\ldots\right) = \lim_{n\to\infty}\frac{1-\left(\frac{2}{\varphi^3}\right)^{n+1}}{1-\frac{2}{\varphi^3}} = \frac{\varphi^3}{\varphi^3-2} = \frac{2\varphi+1}{2\varphi-1}.$$

Note that $\varphi^3 = 2\varphi+1$ from (7). Hence the total volume is $V_T = \frac{\pi D^2}{4}L\frac{2\varphi+1}{2\varphi-1}$ or 1.894 times the volume of the trunk.

It is interesting for the golden ratio tree that while the total length is infinite, the total volume is finite. □

3.2. Golden ratio spiral

A slightly different definition from the existing golden spirals in the literature is given here. The spirals are given in polar coordinates and in the exponential function form with a decaying or growing nature.

Definition 3.5. A golden ratio spiral is defined in polar coordinates as

$$r = r_0 e^{\mp\alpha\theta}, \tag{11}$$

where $\alpha = \frac{2}{\pi} ln\varphi$ with $\frac{\pi}{2}$ radians corresponding to the constant step size in which the radial distance grows or diminishes by the golden ratio. Positive sign corresponds to the growing and negative sign to the decaying behaviour.

It can be verified from the definition that at each step size $\frac{\pi}{2}$, the radial distance grows by a factor of φ (positive case) or decays by a factor of $1/\varphi$ (negative case). Sample plots of growing and decaying golden ratio spirals are given in Figures 2 and 3. Although both spirals can be derived from each other, the important issue is the control of the initial value. In a growing spiral, one has control over the minimum radial distance value, whereas in a decaying spiral, one can fix the largest value. In standard Fibonacci spirals, the radial distance is constant within each $\frac{\pi}{2}$ step size. However, a gradual increase/decrease is achieved in this exponential form. By defining the Fibonacci differential equation from the Fibonacci series, a different continuous form of the spiral was also proposed by Pakdemirli (2023). The step size in that definition turns out to be 44.61^0.

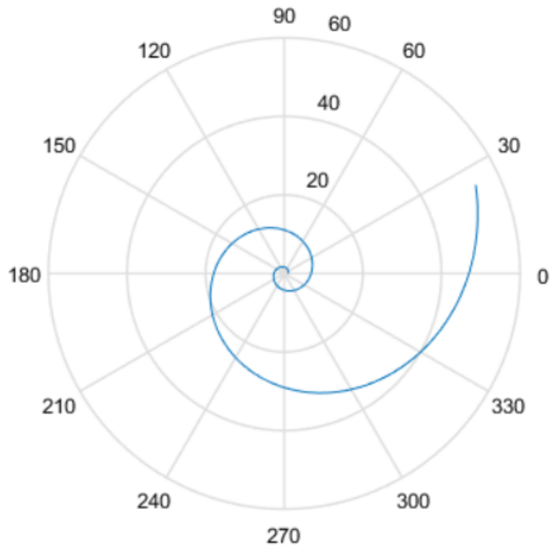

Figure 2: Golden ratio spiral with distance to origin growing with counterclockwise rotation.

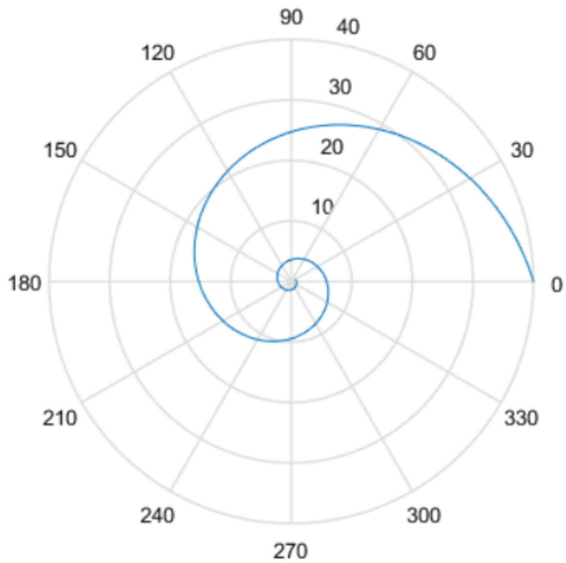

Figure 3: Golden ratio spiral with distance to origin decaying with counterclockwise rotation.

3.3. Golden ratio heart

There are tremendous numbers of different heart equations suggested. An internet search will reveal the different types of such equations describing the shape of a heart in two-dimensional analytical space. A simple and aesthetic heart equation is proposed here for the first time by employing the golden ratio. The definition is as follows:

Definition 3.6. Golden ratio heart is defined with the following Cartesian relationship

$$x^2 + \left(y - (x^2)^{1/\varphi^2}\right)^2 = 1. \tag{12}$$

A plot of the golden ratio heart is given in Figure 4.

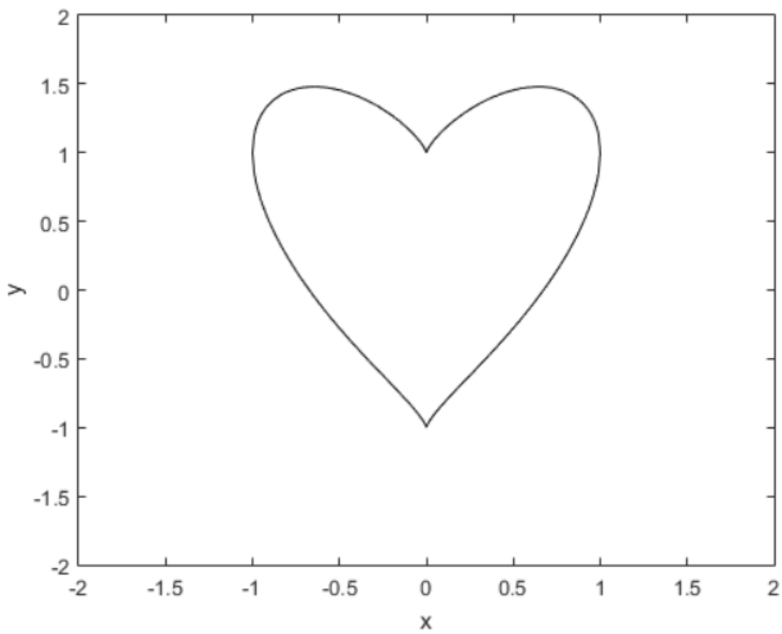

Figure 4: Golden ratio heart.

4. Physical property definitions

In this section, the golden ratio is employed in defining physical properties of the dynamical systems. The definitions are quite original, and similar definitions cannot be traced in the literature.

4.1. Golden ratio ball

The golden ratio ball is defined as follows:

Definition 4.1. If the golden ratio ball is dropped from a reasonable height h, which is not great to the smooth ground, it bounces back to a height h/φ and continues to bounce back at the same ratio of $1/\varphi$ of the previous height indefinitely (Figure 5).

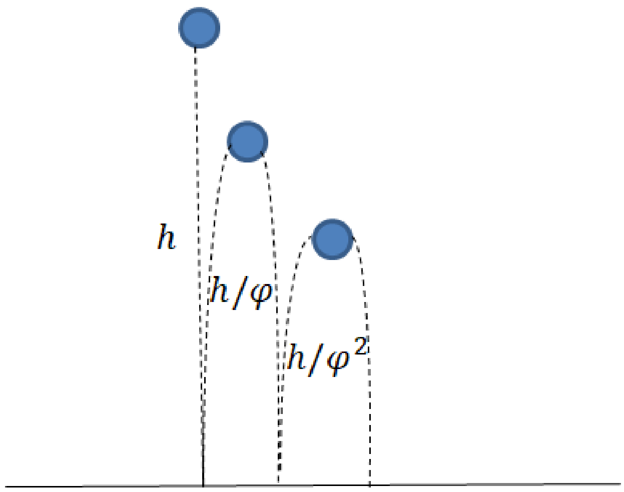

Figure 5: Golden ratio ball.

The following theorem is an immediate consequence of the definition.

Theorem 4.2. *The elastic collision coefficient of a golden ratio ball neglecting air friction is*

$$e = \frac{1}{\sqrt{\varphi}}. \tag{13}$$

Proof. In calculating (13), air friction is neglected. For a free fall, the ball starts from zero velocity and attains its maximum value at the collision with the ground. If the dropped height is not great, the velocities are not high, ensuring that the air friction (drag force) can be neglected. However, for dropping from extremely high positions, the drag force cannot be neglected since it is an increasing function of the velocity (linear or quadratic, depending on the magnitude of velocity). At a critical intermediate height, the gravitational force and the drag force equal each other, maintaining a constant velocity drop in the remaining part of the motion. The elastic collision coefficient can be measured by taking the ratio of final to initial velocities of an object just after and before collision if one of the objects is stationary (infinite mass), such as the smooth ground [11]. Hence the velocity just before collision is $\sqrt{2gh}$ from energy conservation and the velocity just after collision is $\sqrt{2g\frac{h}{\varphi}}$ if air friction (drag force) is neglected. The ratio of the velocities is the elastic coefficient $e = \frac{1}{\sqrt{\varphi}} \cong 0.786$.

If $e = 1$, there is no energy loss in the collision, and the ball retains its original height, which is called a pure elastic collision. If $e = 0$, the other limiting value, the collision is pure plastic and the ball does not bounce back at all. The golden ratio ball has an elastic-plastic collision property with its elastic behaviour being more pronounced compared to its plastic behaviour. □

Theorem 4.3. *When dropped from an height h, the golden ratio ball travels a total height of*

$$H_T = h(2\varphi + 1). \tag{14}$$

Proof. Writing the heights at each interval and adding up them

$$H_T = h+2\frac{h}{\varphi}+2\frac{h}{\varphi^2}+\cdots = 2h\left(1+\frac{1}{\varphi}+\frac{1}{\varphi^2}+\ldots\right)-h = 2h\lim_{n\to\infty}\frac{1-\left(\frac{1}{\varphi}\right)^{n+1}}{1-\frac{1}{\varphi}}-h,$$

or

$$H_T = 2h\frac{1}{1-\frac{1}{\varphi}} - h = 2h\varphi^2 - h = 2h\left(\varphi+1\right) - h = h(2\varphi+1) \cong 4.236h.$$

□

Theorem 4.4. *When dropped from a height h, the total time elapsed up to static position for a golden ratio ball is*

$$\tau = \sqrt{\frac{2g}{h}}\frac{\sqrt{\varphi}+1}{\sqrt{\varphi}-1}. \tag{15}$$

Proof. The time elapsed for a free falling ball starting with no initial velocity and attaining a velocity v at the collision is v/g or if the height is h, it is $\frac{\sqrt{2gh}}{g} = \sqrt{\frac{2h}{g}}$. Adding all times

$$\tau = \sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h}{g\varphi}} + 2\sqrt{\frac{2h}{g\varphi^2}} + \cdots = 2\sqrt{\frac{2h}{g}}\left(1+\frac{1}{\sqrt{\varphi}}+\frac{1}{\varphi}+\frac{1}{\varphi^{3/2}}\ldots\right) - \sqrt{\frac{2h}{g}},$$

or

$$\tau = 2\sqrt{\frac{2h}{g}}\frac{1}{1-\frac{1}{\sqrt{\varphi}}} - \sqrt{\frac{2h}{g}} = \sqrt{\frac{2h}{g}}\ \left(\frac{2\sqrt{\varphi}}{\sqrt{\varphi}-1} - 1\right) = \sqrt{\frac{2g}{h}}\frac{\sqrt{\varphi}+1}{\sqrt{\varphi}-1}.$$

Since $\frac{\sqrt{\varphi}+1}{\sqrt{\varphi}-1} \cong 8.352$, the total time elapsed is 8.352 times the time to the first collision. □

4.2. Golden ratio damping

In mechanical vibrations, one of the fundamental systems is the free vibration of viscous damped systems. For linear systems, the response is harmonic and at each period, the maximum amplitude becomes a constant fraction of the previous amplitude. The response approaches to zero as time tends to infinity. The following definition is given for the golden ratio damped system:

Definition 4.5. Golden ratio damped system is defined as the viscously damped system where the ratio of the $(n+1)$-th maximum amplitude to the n-th maximum amplitude is $\frac{1}{\varphi}$, i.e., $\frac{x_{n+1}}{x_n} = \frac{1}{\varphi}$.

The dimensionless viscosity coefficient is the most important parameter characterizing such systems. The following theorem gives the viscosity coefficient for golden ratio damped systems.

Theorem 4.6. *For a golden ratio damped system, the dimensionless viscosity coefficient is*

$$\zeta = \frac{ln\varphi}{\sqrt{4\pi^2 + ln^2\varphi}}. \tag{16}$$

Proof. For a viscously damped system, the equation of motion is [12]

$$\frac{d^2x}{dt^2} + 2\zeta\omega\frac{dx}{dt} + \omega^2 x = 0, \tag{17}$$

where t is the time, $x(t)$ is the vibrational response, ζ is the dimensionless damping and ω is the natural frequency of the system. The response is solved from above

$$x\,(t) = ae^{-\zeta\omega t}\cos(\omega\sqrt{1-\zeta^2}t + \gamma), \tag{18}$$

where $ae^{-\zeta\omega t}$ is the decaying amplitude of vibrations and γ is the phase angle. The period of the damped system is

$$\tau = \frac{2\pi}{\omega\sqrt{1-\zeta^2}}. \tag{19}$$

The ratio of the $(n+1)$-th peak to the n-th peak response is then

$$\frac{x_{n+1}}{x_n} = \frac{ae^{-\zeta\omega(n+1)\tau}\cos(\omega\sqrt{1-\zeta^2}(n+1)\tau + \gamma)}{ae^{-\zeta\omega n\tau}\cos(\omega\sqrt{1-\zeta^2}n\tau + \gamma)} = e^{-\zeta\omega\tau}, \tag{20}$$

since cosine terms attain their maximum values of 1 at the peak points. Substituting the period from (19) with successive maximum amplitudes being $\frac{1}{\varphi}$ by definition

$$\frac{1}{\varphi} = \exp\left(-\zeta\frac{2\pi}{\sqrt{1-\zeta^2}}\right), \tag{21}$$

and solving for the damping coefficient, the result is

$$\zeta = \frac{ln\varphi}{\sqrt{4\pi^2 + ln^2\varphi}}. \tag{22}$$

□

When the golden ratio is substituted, the numerical value of the damping coefficient is $\zeta \cong 0.076$ which indicates that the system is lightly damped. A sample plot of the golden ratio damped system is given in Figure 6.

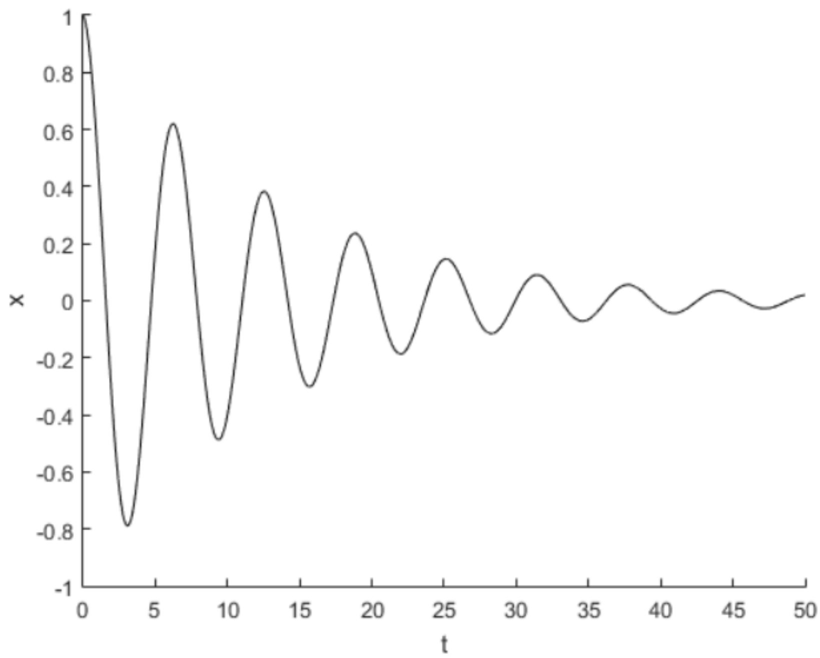

Figure 6: Golden ratio damped system.

5. Concluding remarks

New definitions incorporating golden ratios are proposed in this work. Specific properties of the suggested definitions are worked via theorems and numerical results. This work may open new horizons for developing new definitions related to the golden ratio. Two and three-dimensional new geometric objects with an implementation of the golden ratio can be proposed. The ratio can also be implemented in two and three-dimensional analytical geometry equations. Physical quantities other than those appearing in the worked examples (bouncing ball, damped oscillations) can also be expressed in terms of the golden ratio with appropriate definitions. Applications to nature and real-life problems need further investigation. One wide application topic is the spiral forms observed in nature. There are debates about the match of these forms with the classical Fibonacci spirals, which have a constant radius of curvature for each 90^0. The match might be better if spirals in which the radius of curvature changes continuously, as outlined in this work, can be implemented.

Conflicts of Interest. The author declares that he has no conflicts of interest regarding the publication of this article.

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