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# Approximation of a Leading Coefficient in an Inverse Heat Conduction Problem via the Ritz Method

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**Abstract**

This paper presents a numerical approach for reconstructing the leading coefficient in an inverse heat conduction problem (IHCP). We consider a one-dimensional heat equation with known input data, including the initial condition, a supplementary temperature measurement at the final time, and two integral observations. By incorporating the terminal condition, the unknown spatially dependent coefficient is eliminated, reducing the problem to a nonclassical parabolic equation. The unknown temperature distribution and its derivatives are approximated and applied to the modified governing equation, which is then discretized using operational matrices of differentiation. To ensure stable derivative estimation, the method is coupled with a regularization technique. A least squares scheme is employed to formulate a nonlinear system of algebraic equations, which is solved using Newton's method. The reliability of the proposed solution is demonstrated through several numerical examples.

Keywords: Least squares technique, Inverse heat equation, Leading coefficient.

2020 Mathematics Subject Classification: 35R30, 65N20, 65N35.

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## 1. Introduction

The present work studies the numerical identification of space-dependent coefficient $a(z)$ [1, 2] and the temperature $H(z,t)$ satisfying the following parabolic equation

$$H_t(z,t) = a(z)H_{zz}(z,t), \quad (z,t) \in \Omega, \tag{1}$$

coupled with the following initial-boundary conditions

$$H(z,0) = \phi(z), \quad 0 \le z \le L, \tag{2}$$

$$\int_0^L \chi_1(z)H(z,t)dz = g_1(t), \quad \int_0^L \chi_2(z)H(z,t)dz = g_2(t), \qquad 0 \le t \le T, \tag{3}$$

$$H(z,T) = \psi(z), \qquad 0 \le z \le L, \tag{4}$$

where $\Omega = \{(x,t)|\ 0 \le x \le L,\ 0 \le t \le T\}$ signifies a bounded domain within $\mathbb{R}^2$ and the space and time variables are denoted by $z$ and $t$, respectively. This model describes the heat conduction procedure in a given medium $[0, L]$ and $a(z)$ can be interpreted as the heat capacity or thermal conductivity/diffusivity [3–5]. The functions $H(z,t)$ and $a(z)$ can also be referred to as the piezometric head in groundwater flow or the pressure in porous media [5]. Concentrating on the heat conduction process with the time span $T$, the initial temperature of the medium is given by $\phi(z)$ and the additional temperature measurement at the final time is available, although it may be imbued with a degree of noise, implying that the inverse problem is not overdetermined as the function $\psi(z)$ is used to reconstruct the unknown coefficient $a(z)$. Despite what is seen in the study of direct or inverse problems in the standard form, namely that Dirichlet or Neumann boundary conditions are given at least at one of the boundaries of the problem, we assume that two averages, in space, of the temperature given by $g_1(t)$ and $g_2(t)$ are accessible as energy over-specifications [6–8]. The kernel functions $\chi_1(z)$ and $\chi_2(z)$, which are supposed to be positive on the interval $[0, L]$, can affect the solvability of the problem and certainly challenge numerical techniques to obtain accurate solutions.

It is assumed that the input data of the problem fulfills the following consistency conditions:

$$\int_0^L \chi_1(z)\phi(z)dz = g_1(0), \quad \int_0^L \chi_2(z)\phi(z)dz = g_2(0), \tag{5}$$

$$\int_0^L \chi_1(z)\psi(z)dz = g_1(T), \quad \int_0^L \chi_2(z)\psi(z)dz = g_2(T), \tag{6}$$

and they are sufficiently smooth to ensure the uniqueness of the solution.

Solving inverse coefficient problems is important in understanding physical and engineering phenomena such as heat transfer in biological tissues, groundwater flow guidance, and oil recovery [3, 9, 10]. Achieving accurate and reliable numerical solutions for this class of inverse problems has always been the focus of researchers. However, the retrieval of the time-dependent diffusion coefficient has received more attention. Analytical investigations into the well-posedness of the problem of finding the thermal coefficient $a(t)$ in one-dimensional heat equation with standard and nonlocal boundary conditions were presented in [5, 11–15]. Various numerical approaches such as the finite difference method (FDM) [14, 16], the Ritz-Galerkin technique [17, 18], the pseudospectral Legendre method [19], the Chebyshev cardinal functions scheme [20], the finite element method (FEM) [21], and the Adomian decomposition method (ADM) [22] were applied for the numerical identification of the time-dependent diffusion coefficient $a(t)$ in IHCPs.

The recovery of the space-dependent coefficient $a(z)$ in Equation (1) or in its most practical form given by

$$H_t(z,t) = \frac{\partial}{\partial z}\left(a(z)\frac{\partial H(z,t)}{\partial z}\right) + f(z,t), \tag{7}$$

has attracted much attention recently, both in terms of analysis of the existence and uniqueness of the solution and from the computational point of view. Regarding theoretical research, we refer to [2, 23] where the existence and uniqueness conditions of the solution of finding space-wise dependent diffusion coefficient in parabolic equation (1) are discussed by means of the Schauder fixed-point theorem and maximum when the initial and Dirichlet boundary conditions are applied to the governing equation and the extra condition is given at the final time. As the summary of numerical techniques existing in the literature, the interested reader is referred to the backward Euler and Crank-Nicolson procedure presented in [2], the iterative fixed point projection method proposed in [4], the polynomial regression technique applied in [24], the combination of FEM with an iterative procedure employed in [25], the combination of adjoint problem approach with conjugate gradient scheme utilized in [26, 27] and the Lagrange-Hermite interpolation method discussed in [28]. An operational matrix approach was proposed in [29] to identify the unknown coefficient $a(z)$, where the governing equation (7) was considered, and the boundary conditions and extra overdetermination were different from Equations (1)-(4). In addition, the extra condition was not satisfied accurately.

In this paper, through an innovative technique, we find the approximate solutions of unknown functions which accurately satisfy all the initial and boundary conditions of the problem (1)-(4) and this is the main incentive for applying this approach since despite the collocation techniques, none of the initial and boundary conditions are approximated. Thus, the computational cost is significantly reduced. The technique is easy to implement and incorporates an appropriate regularization technique to ensure the robustness of the scheme.

The rest of the paper is as follows: In Section 2, we discuss the solution procedure for solving the inverse problem. In Section 3, we describe the results of the numerical simulation. Section 4 includes a conclusion.

## 2. Solution procedure

In this section, we provide approximations based on the Legendre polynomials.

First, by applying (4) in (1) we get

$$a(z) = \frac{H_t(z,T)}{\psi''(z)}, \tag{8}$$

provided $\psi''(z) \neq 0,\ z \in [0,L]$. Therefore the governing equation (1) is modified as the following nonclassical parabolic equation

$$H_t(z,t) = \frac{H_t(z,T)}{\psi''(z)} H_{zz}(z,t), \quad (z,t) \in \Omega. \tag{9}$$

Taking the following relation into account

$$\int_0^z \int_0^y H^*_{ss}(s,t)dsdy = H^*(z,t) - H^*(0,t) - zH^*_z(0,t),$$

and defining

$$A(t) := H^*(0,t),\ B(t) = H^*_z(0,t),\ Q(z,t,H^*_{zz}) := \int_0^z \int_0^y H^*_{ss}(s,t)dsdy,$$

we conclude

$$H^*(z,t) = Q(z,t,H^*_{zz}) + A(t) + zB(t). \tag{10}$$

Multiplying Equation (10) by $\chi_1(z)$ and integrating with respect to $z$ over $[0,L]$ and taking into account relation (3) per $H^*(z,t)$, we achieve

$$g_1(t) = \int_0^L \chi_1(z)Q(z,t,H^*_{zz})dz + A(t)\underbrace{\int_0^L \chi_1(z)dz}_{:=y_1} + B(t)\underbrace{\int_0^L z\chi_1(z)dz}_{:=y_2}. \tag{11}$$

Doing the same calculations for $\chi_2(z)$ results

$$g_2(t) = \int_0^L \chi_2(z)Q(z,t,H^*_{zz})dz + A(t)\underbrace{\int_0^L \chi_2(z)dz}_{:=y_3} + B(t)\underbrace{\int_0^L z\chi_2(z)dz}_{:=y_4}. \tag{12}$$

Equivalently

$$\begin{pmatrix} G_1(t) \\ G_2(t) \end{pmatrix} = \begin{pmatrix} y_1 & y_2 \\ y_3 & y_4 \end{pmatrix} \begin{pmatrix} A(t) \\ B(t) \end{pmatrix}, \tag{13}$$

where

$$G_1(t) = g_1(t) - \int_0^L \chi_1(z)Q(z,t,H^*_{zz})dz, \quad g_2(t) - \int_0^L \chi_2(z)Q(z,t,H^*_{zz})dz. \tag{14}$$

Therefore, if $\Delta := y_1y_4 - y_2y_3 \neq 0$, we get

$$A(t) = \frac{y_4G_1(t) - y_2G_2(t)}{y_1y_4 - y_2y_3}, \quad B(t) = \frac{-y_3G_1(t) + y_1G_2(t)}{y_1y_4 - y_2y_3}. \tag{15}$$

Accordingly

$$H^*(z,t) = Q(z,t,H^*_{zz}) + \frac{(y_4 - zy_3)G_1(t) + (zy_1 - y_2)G_2(t)}{\Delta}. \tag{16}$$

It can be seen that

$$\int_0^L \chi_1(z)H^*(z,t)dz = g_1(t), \quad \int_0^L \chi_2(z)H^*(z,t)dz = g_2(t). \tag{17}$$

Next, we denote

$$\tilde{H}(z,t) := \phi(z) + \frac{t}{T}(\psi(z) - \phi(z)), \tag{18}$$

$$H(z,t) := \tilde{H}(z,t) + H^*(z,t) - \left(H^*(z,0) + \frac{t}{T}(H^*(z,T) - H^*(z,0))\right). \tag{19}$$

Taking into account (5)-(6) and paying attention to (16)-(19) it is obvious that

$$H(z,0) = \phi(z),\ H(z,T) = \psi(z),\ \int_0^L \chi_1(z)H(z,t)dz = g_1(t), \tag{20}$$

$$\int_0^L \chi_2(z)H(z,t)dz = g_2(t).$$

We use Equation (19) to construct the approximation of $H(z,t)$. In this direction, the functions $Q(z,t,H^*_{zz})$ and $H^*(z,t)$ are approximated.

Let $P^\top(z) = [p_0(z),...,p_N(z)]$ and $S(t) = [q_0(t),...,q_{N'}(t)]^\top$ be the vectors including the shifted Legendre polynomials [30–33] $p_i(z)$ and $q_j(t)$, defined over the intervals $[0,L]$ and $[0,T]$, respectively. Characterizing the Legendre polynomials of degree $k$ by $\chi_k(s),\ s \in [-1,1]$, then, the shifted Legendre polynomials $p_k(z)$ and $q_k(t)$ are given by $p_k(z) = \chi_k(\frac{2z}{L} - 1)$ and $q_k(t) = \chi_k(\frac{2t}{T} - 1)$. We introduce the approximation of $H^*_{zz}(z,t)$ as:

$$H^*_{zz}(z,t) \simeq \overline{H^*}_{zz}(z,t) = P^\top(z)KS(t),$$

including an $(N+1)\times(N'+1)$ matrix of unknown coefficient as:

$$K = \begin{pmatrix} k_{00} & \cdots & k_{0N'} \\ \vdots & & \vdots \\ k_{N0} & \cdots & k_{NN'} \end{pmatrix}. \tag{21}$$

Defining $P_{\#}^{\top}(z) = [\int_0^z \int_0^y p_0(s)dsdy, ..., \int_0^z \int_0^y p_N(s)dsdy]$, we get the approximation of $Q(z,t,H_{zz}^*)$ as:

$$Q(z,t,H_{zz}^*) \simeq Q(z,t,\overline{H}_{zz}^*) = P_{\#}^{\top}(z)KS(t). \tag{22}$$

Moreover, by denoting

$$P_{\chi_r}^{\top}(z) = \int_0^z \chi_r(s)P_{\#}^{\top}(s)ds, \quad r \in \{1,2\},$$

we achieve

$$G_1(t) \simeq \overline{G_1}(t) = g_1(t) - P_{\chi_1}^{\top}(L)KS(t),\ G_2(t) \simeq \overline{G_2}(t) = g_2(t) - P_{\chi_2}^{\top}(L)KS(t), \tag{23}$$

hence

$$H^*(z,t) \simeq \overline{H^*}(z,t) = P_{\#}^{\top}(z)KS(t) + \frac{(y_4 - zy_3)\overline{G_1}(t) + (-y_2 + zy_1)\overline{G_2}(t)}{y_1y_4 - y_2y_3}. \tag{24}$$

Therefore, from (19) and (24) we get the approximation of the temperature $H(z,t)$ as follows:

$$\overline{H}(z,t) = \tilde{H}(z,t) + \overline{H^*}(z,t) - \left(\overline{H^*}(z,0) + \frac{t}{T}(\overline{H^*}(z,T) - \overline{H^*}(z,0))\right). \tag{25}$$

Recalling that the exact solution $H(x,t)$ satisfies the following residual function

$$R(z,t,H) := H_t(z,t) - \frac{H_t(z,T)}{\psi''(z)}H_{zz}(z,t) = 0, \quad (z,t) \in \Omega, \tag{26}$$

we aim to discretize this equation, after deriving the approximations of $H_t(z,t)$ and $H_{zz}(z,t)$, from (25) as follows:

$$\begin{aligned} H_t(z,t) \simeq \overline{H}_t(z,t) = P_{\#}^{\top}(z)KDS(t) + \frac{(y_4 - zy_3)\overline{G_{10}}(t) + (-y_2 + zy_1)\overline{G_{20}}(t)}{y_1y_4 - y_2y_3} \\ + \frac{1}{T}\left(\psi(z) - \phi(z) - \overline{H^*}(z,T) + \overline{H^*}(z,0)\right), \end{aligned} \tag{27}$$

$$\begin{aligned} H_{zz}(z,t) \simeq \overline{H}_{zz}(z,t) &= \phi''(z) + \frac{t}{T}(\psi''(z) - \phi''(z)) \\ &+ P^{\top}(z)K\left\{S(t) - S(0) + \frac{t(S(0) - S(T))}{T}\right\}, \end{aligned} \tag{28}$$

where

$$\overline{G_{10}}(t) = g_1'(t) - P_{\chi_1}^{\top}(L)KDS(t),\ \overline{G_{20}}(t) = g_2'(t) - P_{\chi_2}^{\top}(L)KDS(t),$$

and $D$ is the operational matrix [34] of differentiation of the basis functions $\psi_j(t)$ which satisfies the following equation

$$\frac{d^k}{dt^k}S(t) = D^k S(t), \quad , \ k \in \{1, ..., N'\}. \tag{29}$$

Following relation can be employed to get the elements of $D$ [35]

$$q_k^{'}(t) = \frac{2\sqrt{2k+1}}{T}\sum_{j=0}^{r^*}\sqrt{2k-4j-1}q_{k-2j-1}(t), \quad r^* \in \{\mathbb{Z}|\ 2r^*+1 \leq k\}. \tag{30}$$

We substitute the approximations $\overline{H_t}(z,t)$ and $\overline{H}_{zz}(z,t)$ presented by (27) and (28) in the residual function (26) to get

$$R(z,t,\overline{H}) = \overline{H}_t(z,t) - \frac{\overline{H}_t(z,T)}{\psi''(z)}\overline{H}_{zz}(z,t). \tag{31}$$

Although a system of algebraic equations in terms of the coefficients $\{k_{ij}\},\ i = 0, ..., N,\ j = 0, ..., N'$, can be obtained through the collocation equations as follows

$$R(z_i, t_j, \overline{H}) = 0,\ z_i \in [0, L],\ t_j \in [0, T], \tag{32}$$

in this paper we create it within the least squares problem framework, that is we calculate the following functional

$$J_R[\overline{H}] = \int_0^L \int_0^T R^2(z,t,\overline{H})dtdz, \tag{33}$$

and then utilize the necessary conditions for optimization in Equation (33) as follows:

$$\frac{\partial J_R[\overline{H}]}{\partial k_{ij}} = \frac{\partial}{\partial k_{ij}}\int_0^L \int_0^T R^2(z,t,\overline{H})dtdz = 0, \quad i = 0, ..., N,\ j = 0, ..., N'. \tag{34}$$

Finally, a nonlinear system of equations in terms of elements $k_{ij}$ is produced from (32) or (34) which is solved by Newton's iteration method.

## 2.1 On the convergence of the method

Considering the following integro-differential equation

$$H(z,t) = \phi(z) + \int_0^t \frac{H_s(z,T)}{\psi''(z)}H_{zz}(z,s)ds, \quad (z,t) \in \Omega_1, \tag{35}$$

which is equivalent to Equation (1), we present the convergence of the method on the domain $\Omega_1 = [0,1] \times [0,1]$ and assume that the boundary conditions (3) are homogeneous, i.e. $g_1(t) = g_2(t) = 0$. In this regard, we use the Banach spaces $F(\Omega_1) = \{h : \Omega_1 \to \mathbb{R}|\ h_t,\ h_{zz},\ h_{zzt} \in C(\Omega_1)\}$ and

$$F_0(\Omega_1) = \left\{ h(z,t) \in F(\Omega_1)|\ \int_0^1 \chi_1(z)h(z,t)dz = \int_0^1 \chi_2(z)h(z,t)dz = 0 \right\},$$

equipped with the following norm

$$\|h\|' = \|h\|_\infty + \|h_t\|_\infty + \|h_z\|_\infty + \|h_{zz}\|_\infty + \|h_{zzt}\|_\infty.$$

Considering $h_{zz}(z,t),\ \chi_1(z),\ \chi_2(z) \in C(\Omega_1)$, the Weierstrass approximation theorem implies that there exist the sequences of polynomials $p_{m,n}(z,t),\ q_m^{[1]}(z),\ q_m^{[2]}(z)$ such that

$$\left( \|p_{m,n}(z,t) - h_{zz}(z,t)\|_\infty, \|q_m^{[1]}(z) - \chi_1(z)\|_\infty, \|q_m^{[2]}(z) - \chi_2(z)\|_\infty \right) \to \left(0,0,0\right), \tag{36}$$
$$as\ m,n \to \infty.$$

Paying attention to Equation (16), we define

$$p^*_{m,n}(z,t) = Q^\#_{m,n}(z,t) + \frac{(y_4 - zy_3)G_{1mn}(t) + (zy_1 - y_2)G_{2mn}(t)}{\Delta}, \tag{37}$$

where

$$Q^\#_{m,n}(z,t) = \int_0^z \int_0^y p_{m,n}(s,t)dsdy, \quad G_{kmn}(t) = -\int_0^1 q_m^{[k]}(z)Q^\#_{m,n}(z,t)dz,\ k \in \{1,2\}.$$

Supposing $\Gamma = \sup_{[0,1]}\{|\chi_1(z)|, |\chi_2(z)|\}$, from (36) and (37) we have:

$$|p^*_{m,n}(z,t) - H^*(z,t)| = \Bigg| \int_0^z \int_0^y p_{m,n}(s,t) - h_{ss}(s,t)dsdy \tag{38}$$
$$+\frac{(y_4 - zy_3)\Big(G_{1mn}(t) - G_1(t)\Big) + (zy_1 - y_2)\Big(G_{2mn}(t) - G_2(t)\Big)}{\Delta}\Bigg|$$
$$\le |p_{m,n}(z,t) - h_{zz}(z,t)| + \frac{1}{\Delta}\Big((|y_4|+|y_3|)|G_{1mn}(t) - G_1(t)| + (|y_1|+|y_2|)|G_{2mn}(t) - G_2(t)|\Big)$$
$$\le \|p_{m,n}(z,t) - h_{zz}(z,t)\|_\infty \left(1 + \frac{\Gamma}{\Delta}(|y_1| + |y_2| + |y_4| + |y_3|)\right).$$

Hence, $\|p^*_{m,n}(z,t) - H^*(z,t)\|_\infty \to 0$ as $m,n \to \infty$. Doing simple calculations results:

$$\left| \frac{\partial p^*_{m,n}(z,t)}{\partial z} - \frac{\partial H^*(z,t)}{\partial z} \right| \le \|p_{m,n}(z,t) - h_{zz}(z,t)\|_\infty \left(1 + \frac{\Gamma}{\Delta}(|y_1| + |y_3|)\right), \tag{39}$$

$$\left|\frac{\partial p^*_{m,n}(z,t)}{\partial t}-\frac{\partial H^*(z,t)}{\partial t}\right|\leq \|\frac{\partial p_{m,n}(z,t)}{\partial t}-h_{zzt}(z,t)\|_\infty\left(1+\frac{\Gamma}{\Delta}(|y_1|+|y_2|+|y_4|+|y_3|)\right), \tag{40}$$

$$\left|\frac{\partial^3 p^*_{m,n}(z,t)}{\partial t\partial z^2}-\frac{\partial^3 H^*(z,t)}{\partial t\partial z^2}\right|\leq \|\frac{\partial p_{m,n}(z,t)}{\partial t}-h_{zzt}(z,t)\|_\infty. \tag{41}$$

Therefore, we conclude $\|p^*_{m,n}(z,t)-H^*(z,t)\|' \to 0$ as $m,n\to\infty$ and taking into account

$$\int_0^1 q^{[1]}_m(z)p^*_{m,n}(z,t)=\int_0^1 q^{[2]}_m(z)p^*_{m,n}(z,t)=0,$$

we claim the following lemma.

**Lemma 2.1.** *the polynomials of space $F_0(\Omega_1)$ are dense in that space.*

Next, we dwell on Equation (35) and consider the functional $J_{R_*}:\left(F(\Omega_1),\|.\|'\right)\to\mathbb{R}$ as:

$$J_{R_*}(H)=\int_0^1\int_0^1 R^2_*(z,t,H)dtdz, \tag{42}$$

with $R_*(z,t,H)=H(z,t)-\phi(z)-\int_0^t \frac{H_s(z,T)}{\psi''(z)}H_{zz}(z,s)ds$.

**Lemma 2.2.** *$J_{\bar{R}_*}$ is continuous on the Banach space $\left(F(\Omega_1),\|.\|'\right)$.*

*Proof.* Let $h^{(1)}\in F(\Omega_1)$ and

$$\mho=\Omega_1\times[-\kappa-\tau,\kappa+\tau]^5,\ \kappa=\max\{\|h^{(1)}\|_\infty,\|h^{(1)}_t\|_\infty,\|h^{(1)}_z\|_\infty,\|h^{(1)}_{zz}\|_\infty,\|h^{(1)}_{zzt}\|_\infty\},\ \tau>0.$$

Denoting $e_{h^{(1)}}=(z,t,h^{(1)},h^{(1)}_t,h^{(1)}_z,h^{(1)}_{zz},h^{(1)}_{zzt})\in\mho$ and considering $h^{(2)}\in F(\Omega_1)$ subject to $\|h^{(2)}-h^{(1)}\|'<\rho$, it can be seen that for small enough value of $\rho$ we have:

$$e_{h^{(2)}}=(z,t,h^{(2)},h^{(2)}_t,h^{(2)}_z,h^{(2)}_{zz},h^{(2)}_{zzt})\in\mho.$$

Furthermore, $R_*$ is continuous on the compact set $\mho$ with respect to its arguments, thus it is uniformly continuous on $\mho$. Therefore, given $\epsilon>0$, if $\rho>0$ is sufficiently small such that $\|e_{h^{(2)}}-e_{h^{(1)}}\|_\infty<\rho$, we have $|R_*(z,t,h^{(2)})-R_*(z,t,h^{(1)})|<\epsilon$ and $|J_{R_*}(h^{(2)})-J_{R_*}(h^{(1)})|<\epsilon$. □

**Theorem 2.3.** *Let $\lambda_{mn}$ be the minimum of the functional $J_{R_*}(H)$ on*

$$\Lambda^{m,n}(\Omega_1)=F_0(\Omega_1)\cap Span\{p_i(z)q_j(t)|\quad i=0,...m,\ j=0,...,n\}.$$

*Then,* $\lim_{m,n\to\infty}\lambda_{mn}=0$.

*Proof.* By the property of minimum, $\forall\ \epsilon > 0$ , there exists an element $h^{\#}(z,t) \in F_0(\Omega_1)$ such that

$$|J_{R_*}(h^{\#})| < \epsilon. \tag{43}$$

From Lemma 2.2, since $J_{R_*}$ is continuous on $\Big(F(\Omega_1), \|.\|^{'}\Big)$, for small enough $\rho > 0$ subject to $\|h - h^{\#}\| < \rho$, we have

$$|J_{R_*}(h) - J_{R_*}(h^{\#})| < \epsilon. \tag{44}$$

Using Lemma 2.1, it is implied that for large enough values of $m$ and $n$, there exists a sequence of polynomials $r_{m,n}(z,t)$ such that $\|r_{m,n}(z,t) - h^{\#}(z,t)\|^{'} < \rho$. Finally, considering $u^{mn}(z,t) \in \Lambda^{m,n}(\Omega_1)$ such that $J_{R_*}(u^{mn}) = \lambda_{mn}$ and applying (43) and (44) for $u^{mn}(z,t)$ and $r_{m,n}(z,t)$ we get:

$$0 \leq J_{R_*}(u^{mn}) \leq J_{R_*}(r_{m,n}) < 2\epsilon.$$

So,

$$\lim_{m,n\to\infty} \lambda_{mn} = \lim_{m,n\to\infty} J_{R_*}(u^{mn}) = 0.$$

□

## 2.2 Numerical differentiation technique

Assuming $N_1$ as the number of discrete data points $z_k \in [0, L]$ and considering $\psi_\delta(z)$ as the measured data of $\psi(z)$ such that $\max|\psi(z_k) - \psi_\delta(z_k)| \leq \delta$, we utilize the mollification technique proposed by [36] to compute the function $\psi_\delta^{''}(z)$. The procedure is based on the convolution smoothing with a Gaussian mollifier given by $G_\sigma(t) = \frac{\exp^{-\frac{t^2}{\sigma^2}}}{\sigma\sqrt{\pi}}$ subject to the regularization parameter $\sigma > 0$. The main idea to solve the problem under consideration is based on using the convolution formula

$$\Big(G_\sigma * \psi\Big)(z) = \int_{-\infty}^{\infty} G_\sigma(\tau)\psi(z-\tau)d\tau, \tag{45}$$

and paying attention to the following property

$$\int_{-\infty}^{\infty} G_\sigma(\tau)\psi^{''}(z-\tau)d\tau = \int_{-\infty}^{\infty} G_\sigma^{''}(\tau)\psi(z-\tau)d\tau. \tag{46}$$

The mollified derivative is obtained as:

$$\Big(G_\sigma * \psi_\delta^{''}\Big)(z) = \int_{-\infty}^{\infty} G_\sigma^{''}(\tau)\psi_\delta(z-\tau)d\tau. \tag{47}$$

After recovering $\Big(G_\sigma * \psi_\delta^{''}\Big)(z_k)$ from (47), we apply the curve fitting technique to get the approximation $\hat{\psi}_\delta^{''}(z) = \sum_{k=0}^{N_2} \mu_k p_k(z)$. That is we solve the following

Table 1: The results of $\|abs(H)\|_2$, $\|re(H)\|_2$, $\|abs(a)\|_2$, $\|re(a)\|_2$ when different number of parameters $N$ and $N'$ and exact boundary data are applied in Example 3.1.

| $(N, N')$ | $\|abs(H)\|_2$ | $\|re(H)\|_2$ | $\|abs(a)\|_2$ | $\|re(a)\|_2$ | $CPU$ |
|---|---|---|---|---|---|
| $(2, 2)$ | 0.000815674 | 0.000687078 | 0.00178325 | 0.0040825 | 15.96 |
| $(4, 4)$ | $2.69633 \times 10^{-6}$ | $1.93116 \times 10^{-6}$ | 0.000010265 | 0.0000230208 | 48.81 |

system for the elements $\mu_k$

$$\frac{\partial}{\partial \mu_k} \sum_{m=1}^{N_1} \left\{ \hat{\psi}_\delta''(z_m) - \left( G_\sigma * \psi_\delta'' \right)(z_m) \right\}^2 = 0, \quad k = 0, 1, ..., N_2. \tag{48}$$

We select the regularization parameter $\sigma$ such that for given $\eta > 0$, the following inequality is fulfilled

$$\|R(z, t, \overline{H})\|_\infty \leq \eta.$$

## 3. Numerical tests

In this section, we solve three examples. Numerical simulations are implemented by the Mathematica software version 12.3 where routine command such as "FindRoot" is used to solve the nonlinear systems of algebraic equations (32) or (34). Moreover, the functional (33) is calculated by Simpson's rule. The absolute error and relative error functions of $a(z)$, $H(z, t)$ termed by $abs(a)$, $abs(H)$ and $re(a)$, $re(H)$ are used to exhibit the accuracy of the approximate solutions.

**Example 3.1.** We consider the following system of equations [2]

$$H_t(z, t) = a(z) H_{zz}(z, t), \quad (z, t) \in [0, 1] \times [0, 1], \tag{49}$$

$$H(z, 0) = z^2 - z + 1, \quad 0 \leq z \leq 1, \tag{50}$$

$$\int_0^1 \cos(z) H(z, t) dz = (\cos(1) - \sin(1) + 1) e^t, \quad \int_0^1 (1 + z) H(z, t) dz = 1.25 e^t, \quad 0 \leq t \leq 1, \tag{51}$$

$$H(z, 1) = (z^2 - z + 1) e, \qquad 0 \leq z \leq 1, \tag{52}$$

and solve it to estimate the pair solution $(a(z), H(z, t)) = (0.5(z^2 - z + 1), (z^2 - z + 1)e^t)$.

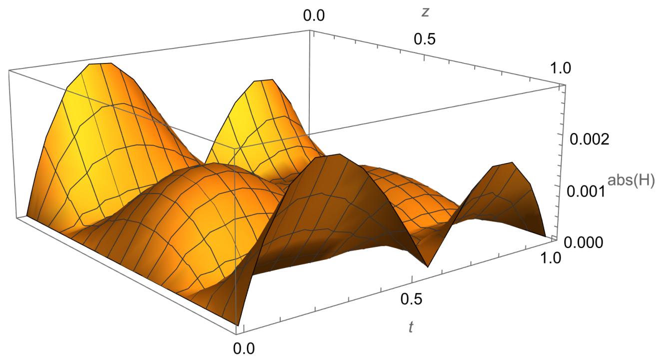


Figure 1: Graph of the absolute error for $H(z,t)$ when accurate boundary data are applied in Example 3.1.

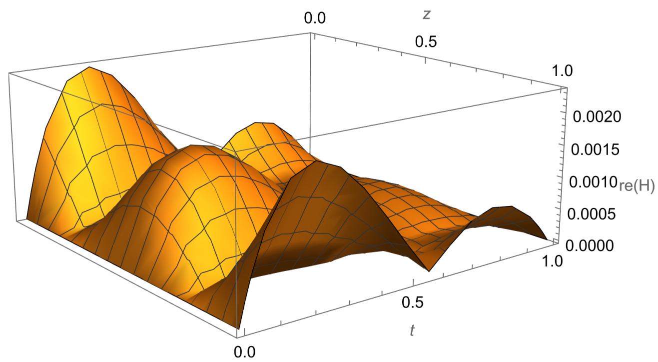


Figure 2: Graph of the relative error for $H(z,t)$ when $N = N' = 2$ and accurate boundary data are applied in Example 3.1.

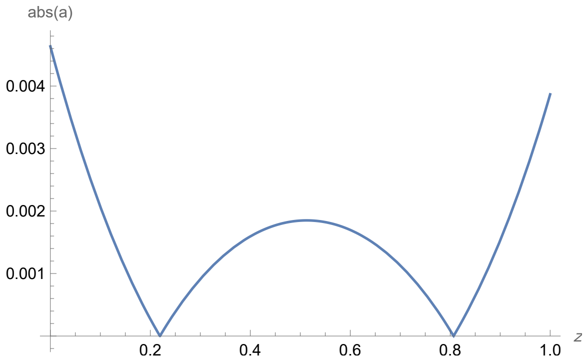


Figure 3: Graph of the absolute error for $a(z)$ when $N = N' = 2$ and accurate boundary data are applied in Example 3.1.

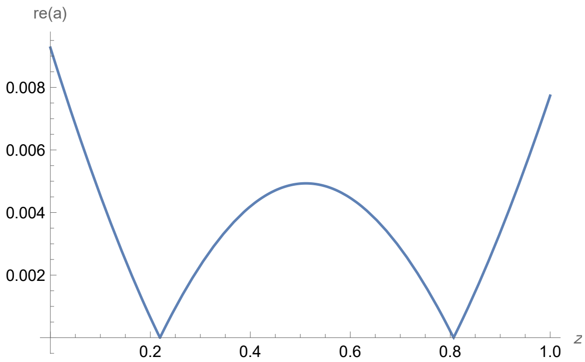


Figure 4: Graph of the relative error for $a(z)$ when $N = N' = 2$ and accurate boundary data are applied in Example 3.1.

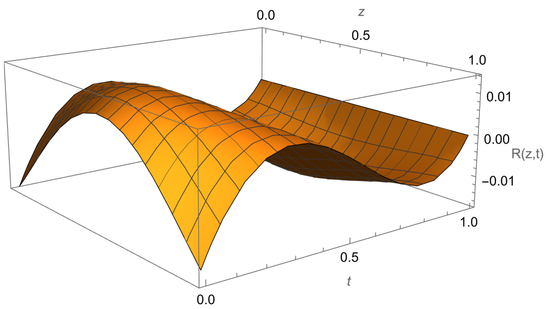


Figure 5: Graph of the residual function $R(z, t, \overline{H})$ when $N = N' = 2$ and accurate boundary data are applied in Example 3.1.

We apply the numerical method of Section 2 in the presence of accurate input data. To ensure the validity of the solution, the quantities

$$y_1 = 6.48176097,\ y_2 = -2.96947585,\ y_3 = -11.6671697,\ y_4 = 6.5450565,$$

are calculated to get $\Delta = 7.77811 \neq 0$. In the case $N = N' = 2$, the results are demonstrated in Figures 1 to 5. The experiment with $N = N' = 4$ is also performed to see the impact of the greater values of $N$ and $N'$, as shown in Table 1. The results obtained in this example indicate the good performance of the method in finding accurate approximate solutions.

**Example 3.2.** Consider the inverse problem presented by equations (1)-(4), defined over the computational domain $\Omega = [0,1] \times [0,1]$ with the following input data [2]:

$$\phi(z) = z^2 + z + z^2e^{2z},\ \psi(z) = (z^2 + z + z^2e^{2z})e,\ \chi_1(z) = 1,\ \chi_2(z) = 1 + z^2, \quad (53)$$

$$g_1(t) = \frac{1}{12}e^t(7 + 3e^2),\ g_2(t) = \frac{1}{60}e^t(17 + 30e^2). \quad (54)$$

To check the validity of our estimations, we obtain

$$y_1 = 9,\ y_2 = -6,\ y_3 = -16,\ y_4 = 12,\ \Delta = 12 \neq 0,$$

and then follow the procedure presented in Section  to approximate the exact solutions

$$H(z,t) = (z^2 + z + z^2e^{2z})e^t, \quad a(z) = \frac{0.5z(1 + z + ze^{2z})}{1 + (1 + 4z + 2z^2)e^{2z}}. \quad (55)$$

First, we assume that $N = N' = 4$ and no perturbation is applied to the boundary condition (4). The outcomes are shown in Figures 6 to 8 and Table 2 to observe the agreement of the approximate solutions with the exact solutions.

Moreover, to incorporate inaccurate boundary data in the computations, we utilize the formula [37–39] $\psi_\delta(z_m) = \psi(z_m) + \delta RandomReal[\{-1,1\}]$ with $\delta = 0.04$ (as the percentage of the noise) and use the technique described in Subsection 2.2 with

$$N_1 = 20,\ N_2 = 6,\ \sigma = 0.02,\ \eta = 0.03,$$

to generate

$$\hat{\psi}''_\delta(z) = 10.9035 + 32.1172z + 70.8554z^2 + 47.0464z^3 + 114.088z^4 - 41.6872z^5 + 53.5672z^6.$$

By utilizing $\hat{\psi}''_\delta(z)$ in the calculations with $N = N' = 4$, we obtain the results illustrated in Figures 9 to 11 indicating a reasonable reaction, proportional to the amount of error in the input data.

Table 2: The results of $re(a)$ and $re(H)$ when $N = N' = 4$ and exact boundary data are applied in Example 3.2. The CPU time per seconds for this experiment is 35.68.

| $(z,t)$ | $re(a)$ | $re(H)$ |
|---|---|---|
| $(0.1, 0.1)$ | 0.00154922 | 0.000177055 |
| $(0.2, 0.2)$ | 0.000229452 | $7.52023 * 10^{-6}$ |
| $(0.3, 0.3)$ | 0.000425337 | 0.000113969 |
| $(0.4, 0.4)$ | 0.000200605 | 0.000047016 |
| $(0.5, 0.5)$ | $7.24903 \times 10^{-6}$ | 0.000010881 |
| $(0.6, 0.6)$ | 0.0000465314 | 0.0000196079 |
| $(0.7, 0.7)$ | 0.0000230428 | 0.0000132521 |
| $(0.8, 0.8)$ | $7.51288 * 10^{-6}$ | $4.07597 * 10^{-6}$ |
| $(0.9, 0.9)$ | 0.0000133318 | $2.3156 * 10^{-6}$ |
| $(1, 1)$ | 0.0000529661 | $2.78403 * 10^{-16}$ |

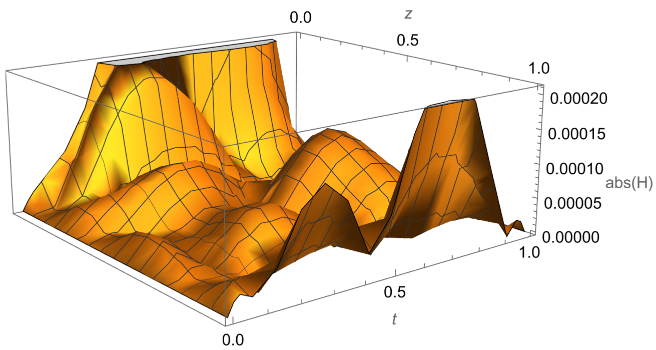


Figure 6: Graph of the absolute error for $H(z,t)$ when accurate boundary data are applied in Example 3.2.

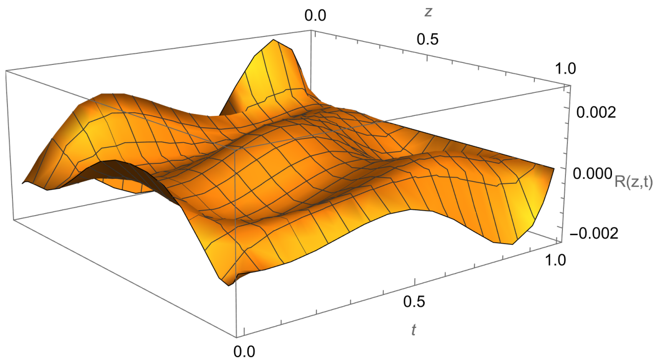


Figure 7: Graph of the residual function $R(z,t,\overline{H})$ when accurate boundary data are applied in Example 3.2.

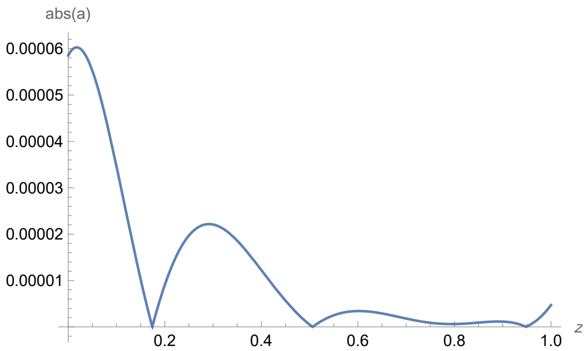


Figure 8: Graph of the absolute error for $a(z)$ when accurate boundary data are applied in Example 3.2.

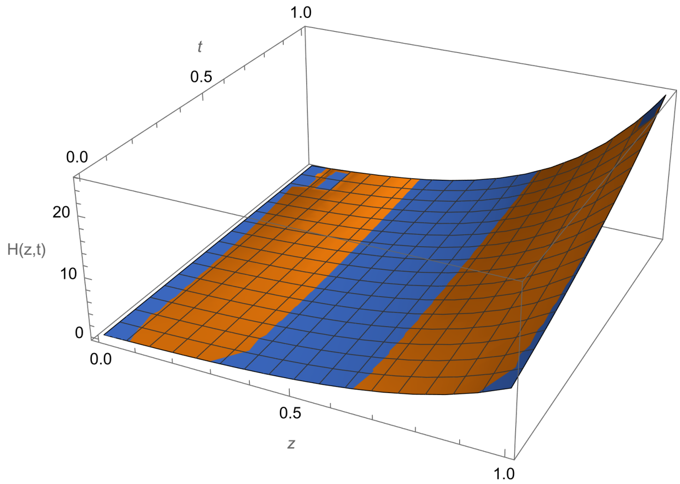


Figure 9: Graphs of the exact and approximate solutions of $H(z,t)$ are pictured when perturbed boundary data are applied in Example 3.2.

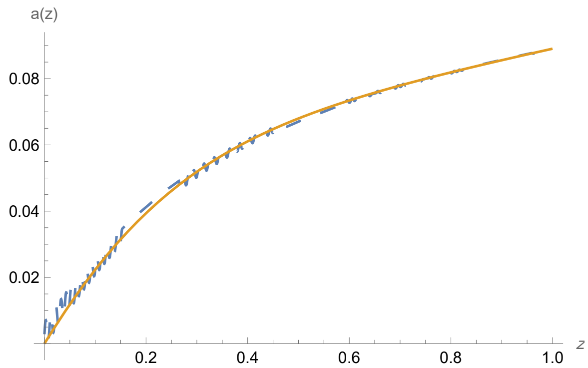


Figure 10: Graphs of the exact (orange) and approximate (blue) solutions for $a(z)$ when perturbed boundary data are applied in Example 3.2.

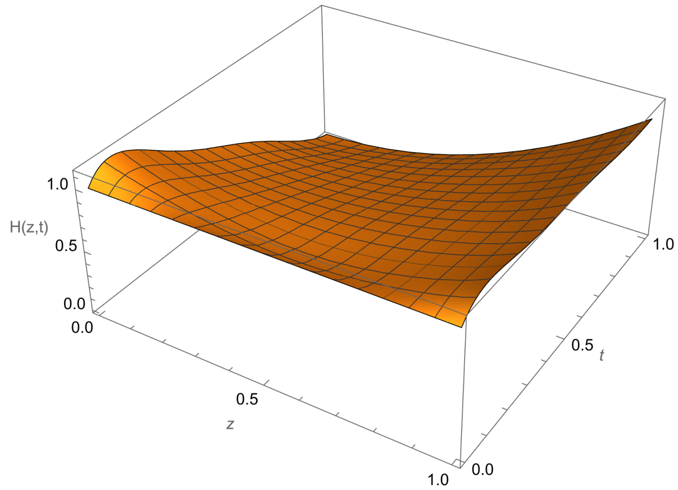


Figure 11: Graph of the approximate solution of $H(z,t)$ in Example 3.3.

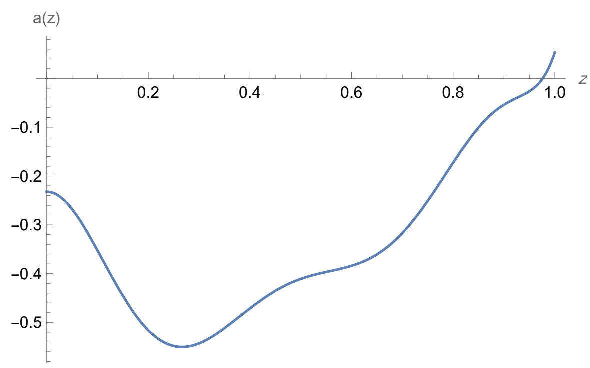


Figure 12: Graph of the approximate solution of $a(z)$ in Example 3.3.

Table 3: The values of approximate solutions for $a(z)$ and $H(z,t)$ at $z_i = t_i = \frac{i}{10}$, i.e. $a(\frac{i}{10})$ and $H(\frac{i}{10}, \frac{i}{10})$ in Example 3.3. The CPU time per seconds for this experiment is 71.86.

| $(z,t)$ | $a(z)$ | $H(z,t)$ |
|---|---|---|
| $(0,0)$ | $-0.232169$ | 1 |
| $(0.1,0.1)$ | $-0.352462$ | 1.02601 |
| $(0.2,0.2)$ | $-0.516134$ | 0.845193 |
| $(0.3,0.3)$ | $-0.542811$ | 0.712643 |
| $(0.4,0.4)$ | $-0.471264$ | 0.636488 |
| $(0.5,0.5)$ | $-0.410839$ | 0.594199 |
| $(0.6,0.6)$ | $-0.383954$ | 0.590887 |
| $(0.7,0.7)$ | $-0.316889$ | 0.6301 |
| $(0.8,0.8)$ | $-0.172387$ | 0.706367 |
| $(0.9,0.9)$ | $-0.0537919$ | 0.824762 |
| $(1,1)$ | $0.0532423$ | 1 |

**Example 3.3.** In this example, we aim to find the approximate solution of the inverse problem (1)-(4) on the computational domain $\Omega = [0,1] \times [0,1]$ with the following input data

$$\phi(z) = 1,\ \psi(z) = z^2,\ g_1(t) = 1 - \frac{2t}{3},\ g_2(t) = \frac{5-3t}{4},\ \chi_1(z) = 1,\ \chi_2(z) = 1 + z^3. \tag{56}$$

The analytical solutions for $H(z,t)$ and $a(z)$ are not available and the input data (56) are selected according to conditions (5)-(6). Simple calculations yield

$$y_1 = \frac{28}{3},\ y_2 = \frac{-20}{3},\ y_3 = \frac{-50}{3},\ y_4 = \frac{40}{3},\ \Delta = \frac{40}{3}.$$

Then, we solve the inverse problem using the proposed method with $N = N' = 6$ and get the approximations depicted in Figures 11 and 12. It should be noted that in this example, the $L^2$-norm of the residual function $R(z, t, \overline{H})$ is equal to 0.0304507. Furthermore, the values of the approximations of $a(z)$ and $H(z, t)$ at discrete points $z_i = t_i = \frac{i}{10}$ are tabulated in Table 3.

## 4. Conclusion

This paper presents a numerical method based on Legendre polynomials for identifying the leading coefficient and temperature distribution in a one-dimensional parabolic equation. The original problem is reformulated as a nonclassical parabolic equation, wherein approximations of the unknown temperature and its derivatives are substituted into the updated governing equation. The resulting residual function is then discretized using the operational matrices of differentiation associated with Legendre polynomial bases.

This approach accurately satisfies all initial and boundary conditions while significantly reducing computational effort, as only a small number of basis functions are required to achieve acceptable solutions. To ensure stability in cases with nonsmooth boundary data, a Gaussian mollifier is incorporated via a mollification scheme. The convergence of the method is formally proven, and numerical simulations are provided to illustrate its effectiveness. Unlike previous studies that focus solely on standard Dirichlet or Neumann boundary conditions, the proposed method effectively handles nonstandard boundary conditions where traditional techniques such as finite difference (FDM) and finite element methods (FEM) often struggle, particularly with integral constraints. In contrast to collocation-based approaches, our technique offers improved accuracy with reduced computational complexity. Moreover, the proposed framework is versatile and can be extended to solve a wide range of direct and inverse problems involving various partial differential equations (PDEs) with complex boundary conditions.

**Conflicts of Interest.** The authors declare that they have no conflicts of interest regarding the publication of this article.

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