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Global Dominator Chromatic Number of Certain Graphs

Hadi Nouri Samani, Saeid Alikhani⋆ *and Nima Ghanbari*

Abstract

For a graph $G = (V, E)$ and a vertex subset $D \subseteq V$, a vertex $v \in V$ is called a dominator of D if v is adjacent to every vertex in D, and an anti-dominator of D if v is not adjacent to any vertex in D. Given a coloring $C = \{V_1, V_2, \ldots, V_k\}$ of G, a color class V_i is a dominating color class (resp. an anti dominating color class) for a vertex v if v dominates all vertices in V_i (resp. v dominates no vertex in V_i). A coloring C is a global dominator coloring if each vertex in G has both a dominating and an anti-dominating color class. The global dominator chromatic number, denoted by $\chi_{gd}(G)$, is the minimum number of colors required for a global dominator coloring of G. In this paper, we investigate the global dominator chromatic number for various classes of graphs.

Keywords: Global domination, Global dominator coloring, Corona, Cactus, Cubic.

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1. Introduction

Let G be an undirected graph without loops or multiple edges. For a vertex v in G, the vertex neighborhood of v, denoted by $N(v)$, is the set of all vertices

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adjacent to v. The vertex and its neighbors, denoted by $N[v]$, form the closed neighborhood of v. Given a subset D of vertices in G, the vertex neighborhood of D, denoted by $N(D)$, is the union of the neighborhoods of all vertices in D and the closed neighborhood of D, denoted by $N[D]$, is the union of D and its vertex neighborhood.

A subset D of vertices is called a dominating set of G if every vertex in G is either in D or adjacent to a vertex in D. The minimum size of the dominating set of G is denoted by $\gamma(G)$. A vertex subset D of G is said to be a global dominating set of G, if D is a dominating set of both G and its complement $\overline{G}$. The smallest possible size of a global dominating set in G is the global domination number, denoted by $\gamma_{g(G)}$ or simply γ_g. Several concepts blending domination and coloring have been explored, including dominator coloring and total dominator coloring. The concept of dominator coloring was introduced and studied by Gera, Horton and Rasmussen [1]. In a dominator coloring, every vertex must dominate all vertices of at least one color class. A total dominator coloring of an isolated free graph G, is a proper coloring of the vertices of G in which every vertex of the graph is adjacent to every vertex of some color class. The dominator chromatic (total dominator chromatic) number $\chi_{d(G)}(\chi_d^{t(G)})$ of G, is the minimum number of colors among all dominator coloring (total dominator coloring) of G (see [2, 3]).

A vertex v is a dominator of a set D if it is adjacent to every vertex in D and v is an anti-dominator of D if it is not adjacent to any vertex in D. Consider a coloring C of G that partitions its vertex set into k color classes $V_1, V_2, \ldots, V_k$.

A color class V_i is termed a dominating color class or an anti-dominating color class for a vertex v if v dominates all vertices in V_i or none, respectively. A coloring C is a global dominator coloring if each vertex in G is assigned a color class that it both dominates and anti-dominates. The smallest number of colors needed to achieve a global dominator coloring of G is the global dominator chromatic number, denoted by $\chi_{gd}(G)$. They introduced this parameter in [4]. Hamid and Rajeswari calculated the global dominator chromatic number for several standard graph classes, including paths, cycles, complete multipartite graphs, and the Petersen graph. They identified graphs G with global dominator chromatic numbers 2 and 3, and for connected bipartite graphs G of order n with at least 4 vertices, they established the sharp bounds $4 \leq \chi_{gd}(G) \leq \lfloor \frac{n}{2} \rfloor + 2$. Rangarajan and Kalarkop, in [5], determined the structure of trees T with at least 6 vertices and $\chi_{gd}(T) = \lfloor \frac{n}{2} \rfloor + 2$, and also provided examples of graph families with global dominator chromatic number 4.

Since a vertex v is always adjacent to itself, v dominates the singleton set $\{v\}$ but does not anti-dominate it. Hence a graph G does not admit a global dominator coloring when $\Delta(G) = n - 1$. For example the friendship graph F_n which is the join of K_1 and nK_2 does not admit a global dominator coloring.

Askari, Mojdeh and Nazari in [6] have initiated a study on total global dominator coloring and studied the complexity of total global dominator coloring. Also they obtained some bounds in terms of order, chromatic number and domination parameters for the total global dominator chromatic number and classified the total global dominator coloring of trees.

This paper extends the investigation of the global dominator chromatic number (gdc number) by determining its value for specific graph classes. Section 2 focuses on the gdc number of corona products of certain graphs and gdc number of grid graphs (Cartesian products of two paths). In Section 3, we delve into cactus chains and compute their gdc numbers. Finally, Section 4 explores the gdc number of cubic graphs of order at most 10.

2. Gdc number of certain graphs

In this section, we determine the value of χ_{gd} for some graphs. We need the following results:

Lemma 2.1. *([4]).*

(i) For the path graph P_n with $n \geq 4$ vertices, the global dominator chromatic number, $\chi_{gd}(P_n)$, is given by:

$$\chi_{gd}(P_n) = \begin{cases} \lceil \frac{n}{3} \rceil + 1, & \text{if } n = 7, \\ \lceil \frac{n}{3} \rceil + 2, & \text{otherwise.} \end{cases}$$

(ii) For the cycle graph C_n with $n \geq 4$ vertices, the minimum number of colors needed for a global dominator coloring is $\chi_{gd}(C_n) = \lceil \frac{n}{3} \rceil + 2$.

Theorem 2.2. *([7]). For triangle-free graphs G, the domination number $\gamma(G)$ and the global domination number $\gamma_g(G)$ are related by the following inequality:*

$$\gamma(G) \leq \gamma_g(G) \leq \gamma(G) + 1.$$

Theorem 2.3. *([8]). The global domination number of a tree T exceeds its domination number by one if and only if T is either a star or a diameter-4 tree obtained by joining the centers of several stars (each with at least two leaves) to a common vertex.*

Theorem 2.4. *([4]). For any tree T, the value of χ_{gd} is either $\gamma_g(T) + 1$ or $\gamma_g(T) + 2$.*

The corona product $G \circ H$ of graphs G and H is constructed by taking a single copy of G and $|V(G)|$ copies of H, then connecting each vertex of G to every vertex in its corresponding copy of H. The following theorem gives the global dominator chromatic number of corona of P_n and C_n with $\overline{K_i}$, i.e., $P_n \circ \overline{K_i}$ and $C_n \circ \overline{K_i}$.

Theorem 2.5. *(i) For every $n \geq 4$, $\chi_{gd}(P_n \circ \overline{K_i}) = n + 1$.*

(ii) For every $n \geq 3$, $\chi_{gd}(C_n \circ \overline{K_i}) = n + 1$.

Proof. (i) Suppose that $V(P_n) = \{1, 2, ..., n\}$ and L is the set of all leaves of $P_n \circ \overline{K_i}$. It is easy to see that $\{L\} \cup \{\{1\}, \{2\}, ..., \{n\}\}$ is a global dominator color class with minimum size. Therefore, we have the result.

(ii) It is similar to the proof of (i).

□

By Theorem 2.4, trees can be classified into two types: Class 1: Trees where the global dominator chromatic number is one more than the global domination number.

Class 2: Trees where the global dominator chromatic number is two more than the global domination number.

Determining the class of a given tree is a nontrivial task (see [4]). By Theorem 2.5 we have the following corollary:

Corollary 2.6. *The graphs $P_n \circ \overline{K_i}$ are trees of Class 1.*

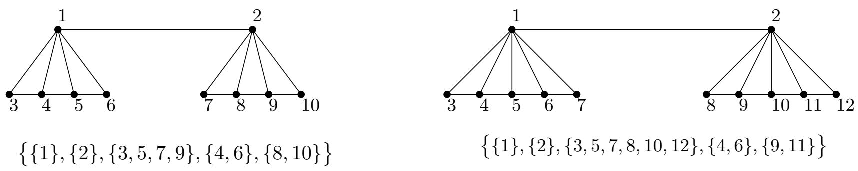

Figure 1: Global dominator coloring of $P_2 \circ P_4$ and $P_2 \circ P_5$.

The following theorem gives the global dominator chromatic number of corona of some graphs:

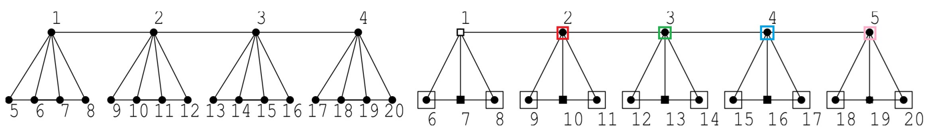

Figure 2: Global dominator coloring of $P_4 \circ P_4$ and $P_5 \circ P_3$.

Theorem 2.7. *(i) For every $n \geq 4$, $\chi_{gd}(P_2 \circ P_n) = \chi_{gd}(P_3 \circ P_n) = 5$.*

(ii) For every $n \geq 4$, and $m \geq 2$, $\chi_{gd}(P_n \circ P_m) = n + 2$.

Proof. (i) Consider $P_2 \circ P_n$. Let $u, u_1, u_2, ..., u_n$ be the first vertex of P_2 and the vertices of the first path and $v, v_1, v_2, ..., v_n$ be the second vertex of P_2 and the vertices of the second path. Assume that c is a color function on $P_2 \circ P_n$, such that $c(u) = 1, c(v) = 2, c(u_{2i-1}) = c(v_{2i-1}) = 3, c(u_{2i}) = 4, c(v_{2i}) = 5$ for $1 \leq i \leq n/2$. Now the class with color 1 dominates the color classes 2, 4 and anti-dominates the color class 5, the class with color 2 dominates the color classes 1, 5 and anti-dominate color class 4. Some vertices with color 3 dominate the color class 1 and anti-dominates color class 2, and some vertices with color 3 dominate the color class 2 and the anti-dominates the color class 1. The class with color 4 dominates the color class 1 and anti-dominates color class 2 and the class with color 5 dominates the color class 2 and anti-dominates color class 1. Therefore $\chi_{gd}(P_2 \circ P_n) = 5$ for $n \geq 4$. See Figure 1.

(ii) Let the vertices of the graph $P_n \circ P_m$ be $\{1, ..., n, n+1, ..., n+nm\}$, as shown in Figure 2. The global dominator coloring class of $P_n \circ P_m$

$$\big\{\{1\}, \{2\}, ..., \{n\}, \{n+1, n+3, ..., \}, \{n+2, n+4, ...\}\big\},$$

has the minimum size. This class has the minimum size, because with the smaller size, some vertices does not have a dominating or/and an anti-dominating color class. Therefore, we have the result.

□

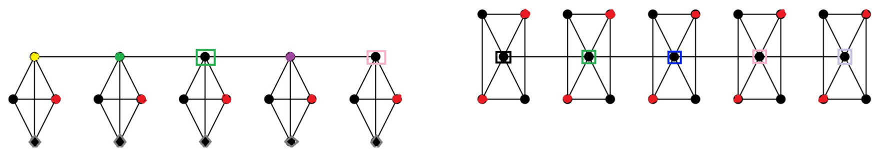

Figure 3: Global dominator coloring of $P_5 \circ C_3$ and $P_5 \circ C_4$.

Below, we obtain, we obtain the global dominator chromatic number of $P_2 \circ C_n$ for even n.

Theorem 2.8. *For even n, $\chi_{gd}(P_2 \circ C_n) = 5$.*

Proof. Consider $P_2 \circ C_n$. Let $v, v_1, v_2, ..., v_{2k}$ be the vertices of the first cycle and the first vertex of P_2 and $u, u_1, u_2, ..., u_{2k}$ be the vertices of the second cycle and the second vertex of P_2. Assume that c is a color function on $P_2 \circ C_n$, such that $c(v) = 1, c(v_{2i}) = 2$, $c(v_{2i-1}) = 3$ for $1 \leq i \leq k$, also $c(u) = 4$, $c(u_{2i}) = 2, c(u_{2i-1}) = 5$ for $1 \leq i \leq k$. Now, vertex v dominates color class 4 and anti-dominates color class 5. The vertices v_{2i} and v_{2i-1} dominate color class 1 and anti-dominate color class 5. Vertex u dominates color class 1 and anti-dominates color class 3. The vertices u_{2i} and u_{2i-1} dominate color class 4 and anti-dominate color class 3. Therefore $\chi_{gd}(P_2 \circ C_n) = 5$ for even n. □

Theorem 2.9. *(i) $\chi_{gd}(P_2 \circ C_3) = 5$ and for every odd $n \geq 5$, $\chi_{gd}(P_2 \circ C_n) = 6$.*

(ii) *For even* $n \geq 3$, $\chi_{gd}(P_3 \circ C_n) = 6$, *and for odd* $n \neq 3$, $\chi_{gd}(P_3 \circ C_n) = 7$.

(iii) *For every* $n \geq 4$ *and* $m \geq 3$,

$$\chi_{gd}(P_n \circ C_m) = \begin{cases} n+2, & \text{if } m \text{ is even,} \\ n+3, & \text{if } m \text{ is odd.} \end{cases}$$

Proof. (i) Consider $P_2 \circ C_n$ for odd n. Let $u, u_1, u_2, ..., u_{2k+1}$ be the vertices of the first cycle and the first vertex of P_2 and $v, v_1, v_2, ..., v_{2k+1}$ be the vertices of the second cycle and the second vertex of P_2. Assume that c is a color function on $P_2 \circ C_n$, such that $c(u) = 1, c(v) = 2, c(u_{2k+1}) = 3, c(v_{2k+1}) = 4, c(\{u_{2i-1}, v_{2i}\}) = 5$, $c(\{u_{2i}, v_{2i-1}\}) = 6$ for $1 \leq i \leq k$.

Now, vertex u dominates color class 3 and anti-dominates color class 4. The vertices u_{2i} and u_{2i-1} dominate color class 1 and anti-dominate color class 4, the vertex v dominates color class 4 and anti-dominates color class 3, the vertices v_{2i} and v_{2i-1} dominate color class 2 and anti-dominate color class 3. Therefore $\chi_{gd}(P_2 \circ C_n) = 6$ for odd n.

(ii) Proof is similar to proof of (i).

(iii) As we see in Figure 3, to obtain a global dominator color class with minimum size, we put any n vertices of P_n in the class of size one and for the cycle C_m with even m we have two classes of size $m/2$ and for odd m, we have three classes. This coloring has the minimum size. Therefore, we have the result. □

Using similar methods as those used in the proof of Theorem 2.9, we have the following result.

Theorem 2.10. *(i)*

$$\chi_{gd}(C_3 \circ C_m) = \begin{cases} 6, & \text{if } m = 3,\ m \text{ is even,} \\ 7, & \text{if } m \text{ is odd.} \end{cases}$$

(ii) *For every* $n \geq 4$ *and* $m \geq 3$,

$$\chi_{gd}(C_n \circ C_m) = \begin{cases} n+2, & \text{if } m \text{ is even,} \\ n+3, & \text{if } m \text{ is odd.} \end{cases}$$

Here, we compute the global dominator chromatic number of the cartesian product of two paths, i.e., $P_n \Box P_m$. First we state some examples:

Example 2.11. (i) As we can see in Figure 4 for the graph $P_2\Box P_8$, the global dominator color class $\big\{\{1\},\{4\},\{7\},\{10\},\{13\},\{16\},\{2,5,8,11,14\},$ $\{3,6,9,12,15\}\big\}$ has the minimum size, and so $\chi_{gd}(P_2\Box P_8)=8$.

(ii) We see in Figure 4, that for the graph $P_4\Box P_8$, the global dominator color class $\big\{\{1\},\{4\},\{7\},\{11\},\{14\},\{18\},\{21\},\{24\},\{26\},\{29\},\{32\},$ $\{3,6,9,12,15,17,20,23,27,30\},\{2,5,8,10,13,16,19,22,25,28,31\}\big\}$ has the minimum size, and so $\chi_{gd}(P_4\Box P_8)=13$.

(iii) For the graph $P_3\Box P_6$, the global dominator color $\big\{\{1\},\{4\},\{7\},\{10\},\{14\},\{17\},$ $\{2,5,9,12,15,18\},\{3,6,8,11,13,16\}\big\}$ has the minimum size, and so $\chi_{gd}(P_3\Box P_6)=8$ (see Figure 4).

(iv) It is easy to see that $\chi_{gd}(P_5\Box P_6)=12$ (see Figure 4).

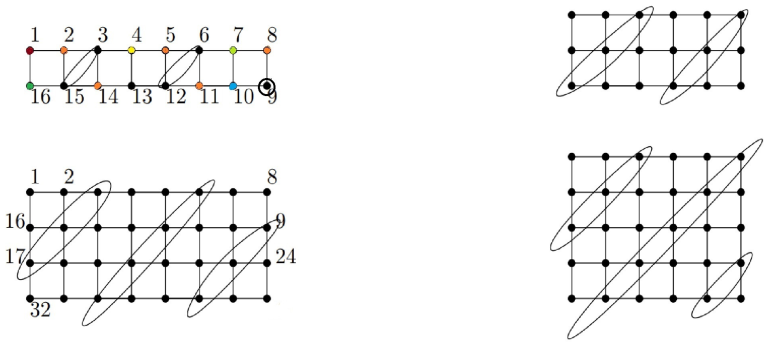

Figure 4: Global dominator coloring of some grid graphs.

The following theorem gives an upper bound for $\chi_{gd}(P_n\Box P_m)$ based on the order of grid, i.e., $|V|=mn$. This upper bounds is sharp for many m and n.

Theorem 2.12. *(i)* $\chi_{gd}(P_2\Box P_4)=4=\left\lceil\frac{nm}{3}\right\rceil+1$.

(ii) For any $n,m\geq 2$, we have $\chi_{gd}(P_n\Box P_m)\leq\left\lceil\frac{nm}{3}\right\rceil+2$.

Proof. (i) Suppose that u_i and v_i $(1\leq i\leq 4)$ are the vertices of the first and the second copy of P_4, respectively in the graph $P_2\Box P_4$. The color function c with $c(u_i)=i$ and $c(v_1)=2, c(v_2)=1, c(v_3)=4, c(v_4)=3$ is a global dominator coloring of this graph with minimum size. Therefore $\chi_{gd}(P_2\Box P_4)=4$.

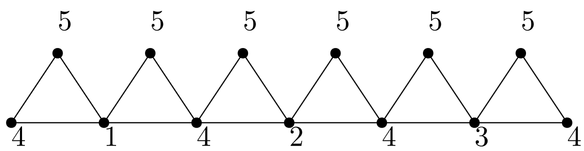

Figure 5: Chain triangular cactus T_6.

(ii) As we can see in Figure 4, if nm is even, then there are two global dominator color classes of sizes $\lceil\frac{nm}{3}\rceil - 1$, $\lceil\frac{nm}{3}\rceil$ and $nm + 1 - 2\lceil\frac{nm}{3}\rceil$ classes of size one. If nm is odd, then there are two global dominator color classes of size $\lceil\frac{nm}{3}\rceil$ and $nm - 2\lceil\frac{nm}{3}\rceil$ of size one. Therefore, we have the result.

□

3. Gdc number of some cactus chains

This section focuses on a specific type of linear polymer known as a cactus chain. Cactus graphs, initially referred to as Husimi trees, were introduced over sixty years ago in the context of cluster integrals within statistical mechanics (see [9–11]). For more details on various graph parameters of cactus graphs, please refer to [12–14]. A cactus graph is a connected graph where no edge is shared by more than one cycle. Consequently, each block of a cactus graph is either a single edge or a cycle. When all blocks of a cactus G are cycles of the same size k, we call it a k-uniform cactus.

In this section, we concentrate on 3-, 4-, and 6-uniform cactus graphs, appear naturally in chemistry. A triangular cactus is a specific type of 3-uniform cactus, where all blocks are triangles.

A triangular cactus is a graph composed solely of triangle blocks. A vertex shared by multiple triangles is termed a cut-vertex. A chain triangular cactus is a specific type of triangular cactus where each triangle has at most two cut-vertices, and each cut-vertex is shared by exactly two triangles.

By replacing triangles with 4-cycles in the definition of a chain triangular cactus, we obtain square cacti. These squares can be classified as ortho-squares (cut-vertices are adjacent) or para-squares (cut-vertices are not adjacent) (see [15]).

We begin by analyzing chain triangular cacti. Figure 5 illustrates an example of such a cactus. The length of a chain triangular cactus is defined as the number of triangles it contains. It is clear that chain triangular cacti with the same length are isomorphic. Consequently, we denote a chain triangular cactus of length n by T_n.

In the following, we determine the global dominator chromatic number of T_n.

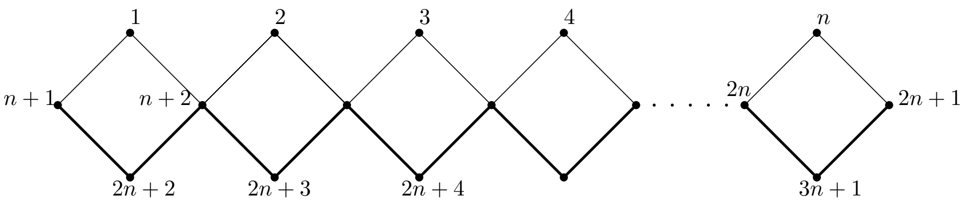

Figure 6: Para-chain square cactus graph Q_n.

Theorem 3.1. *For every $n \geq 3$,*

$$\chi_{gd}(T_n) = \begin{cases} \lceil \frac{n}{2} \rceil + 3, & \text{if } n = 3, 4, \\ \\ \lceil \frac{n}{2} \rceil + 2, & \text{otherwise.} \end{cases}$$

Proof. To have a global dominator coloring of T_n, first we should choose the vertices with the maximum degree (which have degree four in T_n) as single classes, because they dominate more vertices. But since we are looking for the smallest possible size of the global dominator coloring, we must choose them in such a way that they have dominator class and also have anti-dominator class. We consider the path P_{n+1} with vertex set, say $\{v_0, v_1, ..., v_n\}$ (an induced path in T_n whose edges are the bases of triangles, see Figure 5). We select the vertices $v_1, v_3, v_5, ...$ of this path and assign them a color (we have used $\lceil \frac{n}{2} \rceil$ colors so far). Now it is enough to assign one color to all other vertices of the path (i.e., vertices $v_2, v_4, ...$) and another color to the top vertices of triangles. Such a coloring is a a global dominator coloring with minimum size. Therefore we have the result. □

By substituting triangles in the definition of the triangular cactus T_n with 4-cycles, we obtain cacti whose blocks are all 4-cycles. These cacti are referred to as square cacti. Figure 6 illustrates an example of a square cactus chain. The internal squares in a square cactus chain can be connected to their neighbors in two ways:

Ortho-square: The cut-vertices of the square are adjacent.

Para-square: The cut-vertices of the square are not adjacent.

We focus on para-chains, denoted by Q_n, as depicted in Figure 6. The following theorem provides the global dominator chromatic number of Q_n.

Theorem 3.2. *For every $n \geq 1$,*

$$\chi_{gd}(Q_n) = \begin{cases} n + 3, & \text{if } n = 1, \\ \\ n + 2, & \text{otherwise.} \end{cases}$$

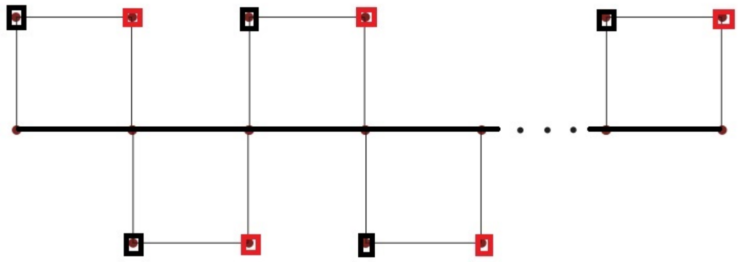

Figure 7: Ortho-chain square cactus graph O_n.

Proof. To construct a global dominator color class with minimum size of Q_n, we consider two induced path P_{2n+1} (one path has vertices $\{n+1, 2n+2, n+2, 2n+3, ..., 3n+1, 2n+1\}$ which is indicated by thicker edges in Figure 6 and another path has vertices $\{n+1, 1, n+2, 2, ..., n, 2n+1\}$). We select the vertices with label $2n+2, 2n+3, ..., 3n+1$ of one of this path and also the vertices with label $1, 2, 3, ..., n$ and assign them a color, say $n+2$ and assign another colors, say, $1, 2, ..., n+1$ to other vertices of P_{2n+1}. This coloring is a global dominator coloring of Q_n with minimum size. Therefore we have the result. □

Now we consider another kind of square cactus chain and compute its global dominator chromatic number (Figure 7). The following theorem gives the global dominator chromatic number of O_n.

Theorem 3.3. *For every $n \geq 2$,*

$$\chi_{gd}(O_n) = \begin{cases} n+2, & \text{if } n = 2, \\ n+3, & \text{otherwise.} \end{cases}$$

Proof. To construct a global dominator color class with minimum size of O_n, we color all $n+1$ vertices on the induced path P_{n+1} (which is indicated by thicker edges in Figure 7) by different colors (so we use $n+1$ colors for P_{n+1}) and color the right corners of the squares by $(n+2)$-th color (red color) and use $(n+3)$-th color (black color) to color the left corners as shown in Figure 7. Therefore, $\chi_{gd}(O_n) = n+3$. □

4. Gdc number of cubic graphs of order at most 10

In this section, we compute the global dominator chromatic number of the cubic graphs of order at most 10. Recently, Alikhani, Golmohammadi and Konstantinova in [16] have studied the coalition numbers of all cubic graphs of order at most 10.

Also Alikhani and Peng have studied the domination polynomials (which is the generating function for the number of dominating sets of a graph) of cubic graphs of order 10 in [17]. As a consequence, they have shown that the Petersen graph is determined uniquely by its domination polynomial.

4.1 Results for the cubic graphs of order 6 and 8

In this subsection, we compute the global dominator chromatic number of the cubic graphs of order 6 and 8. First, we consider the cubic graphs of order 6 that are shown in Figure 4. Observe that for the graph G_1, the global dominator color class

$$\{1,5\},\{2\},\{3\},\{4\},\{6\},$$

is with minimum size and for the graph G_2, the global dominator color class

$$\{1,5\},\{2,4\},\{3\},\{6\},$$

is with minimum size. So we have the following observation:

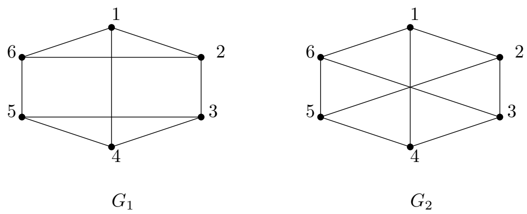

Figure 8: Cubic graphs of order 6.

Observation 4.1. *If G_1 and G_2 are the cubic graphs of order 6 (Figure 8), then $\chi_{gd}(G_1) = 5$ and $\chi_{gd}(G_2) = 4$.*

Here, we obtain the global dominator chromatic number of the cubic graphs of order 8 as shown in Figure 9.

Theorem 4.2. *For the cubic graphs of order 8 in Figure 9,*

(i) $\chi_{gd}(G_1) = \chi_{gd}(G_4) = \chi_{gd}(G_5) = 4$.

(ii) $\chi_{gd}(G_2) = \chi_{gd}(G_3) = \chi_{gd}(G_6) = 5$.

Proof. (i) The graphs G_1 and G_4, have the global dominator color class $\{1,3,6\}$, $\{2,5,7\}$, $\{4\}$, $\{8\}$ with the minimum size. Also for the graph G_5, the global dominator color class $\{1,3,5\}$, $\{2,6,8\}$, $\{4\}$, $\{7\}$ has the minimum size. Therefore, we have the result.

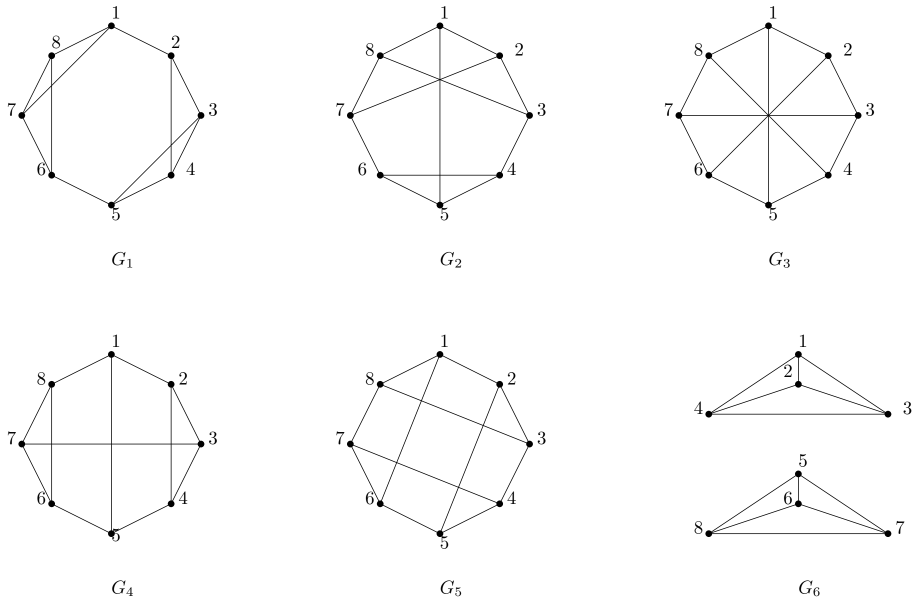

Figure 9: Cubic graphs of order 8.

(ii) The graph G_2 has the global dominator color class $\{1,6\},\{3,5,7\},\{2\},\{4\},\{8\}$ with the minimum size. For the graph G_3, the global dominator color class $\{1,4,7\},\{2,5\},\{3\},\{6\},\{8\}$ has the minimum size and the graph G_6 has the global dominator class $\{1,5\},\{2,6\},\{4,8\},\{3\},\{7\}$. Therefore, we have the result.

□

4.2 Results for cubic graphs of order 10

In this subsection, we compute the global dominator chromatic number of cubic graphs of order at most 10. There are exactly 21 cubic graphs of order 10 given in Figure 10 (see [17]). Note that the graph G_{17} is the Petersen graph.

Theorem 4.3. *For the cubic graphs of order* 10 *in Figure 10,*

(i) If $G \in \{G_i\}$ *for any* $i \in \{1,2,...,21\} \setminus \{10,20,21\}$*, then* $\chi_{gd}(G_i) = 5$.

(ii) $\chi_{gd}(G_{10}) = \chi_{gd}(G_{20}) = \chi_{gd}(G_{21}) = 6$.

Proof. (i) The graphs G_1 and G_2, have the global dominator color class $\{1,4,8\}$, $\{2,5,7,10\},\{3\},\{6\},\{9\}$ with the minimum size. For the graph G_3, the global dominator color class $\{1,5,8\},\{2,4,7,9\},\{3\},\{6\},\{10\}$ has the minimum size. For the graph G_4, the global dominator color class $\{1,4,9\},\{3,5,10\}$, $\{6,8\},\{2\},\{7\}$ has the minimum size.

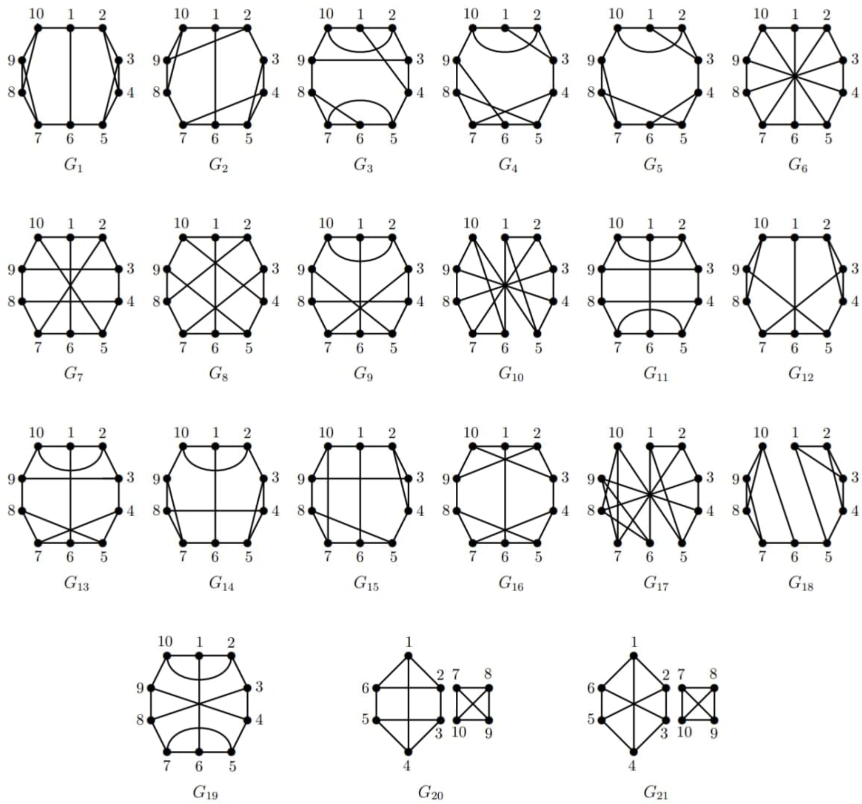

Figure 10: Cubic graphs of order 10.

For the graph G_5, the global dominator color class $\{1, 4, 9\}, \{3, 5, 7, 10\}, \{2\}, \{6\}, \{8\}$ has the minimum size.

For the graph G_6, the global dominator color class $\{1, 5, 7\}, \{2, 4, 8, 10\}, \{3\}, \{6\}, \{9\}$ has the minimum size.

For the graph G_7, the global dominator color class $\{2, 4, 6, 9\}, \{3, 7, 10\}, \{1\}, \{5\}, \{8\}$ has the minimum size.

For the graph G_8, the global dominator color class $\{2, 5, 7, 10\}, \{1, 3, 9\}, \{4\}, \{6\}, \{8\}$ has the minimum size.

For the graph G_9, the global dominator color class $\{1, 4, 9\}, \{3, 6, 8, 10\}, \{2\}, \{5\}, \{7\}$ has the minimum size.

For the graph G_{11}, the global dominator color class $\{1, 3, 5, 8\}, \{2, 7, 9\}, \{4\}, \{6\}, \{10\}$ has the minimum size.

For the graph G_{12}, the global dominator color class $\{1,4,7,9\},\{2,6,8\},\{3\},\{5\},$ $\{10\}$ has the minimum size.

For the graph G_{13}, the global dominator color class $\{3,5,7,10\},\{2,6,9\},\{1\},\{4\},$ $\{8\}$ has the minimum size.

For the graph G_{14}, the global dominator color class $\{1,5,8\},\{2,4,6,9\},\{3\},\{7\},$ $\{10\}$ has the minimum size.

For the graph G_{15}, the global dominator color class $\{1,4,7,9\},\{2,6,8\},\{3\},\{5\},$ $\{10\}$ has the minimum size.

The global dominator color class $\{1,3,5,9\},\{2,6,8\},\{4\},\{7\},\{10\}$ has the minimum size for the graph G_{16}.

The global dominator color class $\{1,7,10\},\{2,4,6,8\},\{3\},\{5\},\{9\}$ has the minimum size for the graph G_{17}.

The global dominator color class $\{1,4,7,10\},\{3,5,8\},\{2\},\{6\},\{9\}$ has the minimum size for the graph G_{18}.

The global dominator color class $\{2,4,6,10\},\{5,7,9\},\{1\},\{3\},\{8\}$ has the minimum size for the graph G_{19}.

(ii) For the graph G_{10}, the global dominator color class $\{1,3,5,7,9\},\{2\},\{4\},\{6\},$ $\{8\},\{10\}$ has the minimum size.

For the graph G_{20}, the global dominator color class $\{1,5,7\},\{2,4,8\},\{3\},\{6\},$ $\{9\},\{10\}$ has the minimum size.

For the graph G_{21}, the global dominator color class $\{1,3,8\},\{2,4,7\},\{5\},\{6\},$ $\{9\},\{10\}$ has the minimum size.

□

5. Conclusion

This paper calculates the global dominator chromatic number (gdc number) for several classes of graphs, including grid graphs (cartesian products of two paths) and corona products of specific graphs. We also determine the gdc number for certain cactus chains and cubic graphs of order at most 10.

Several open problems remain for future research. Some of these are:

(i) Graph Products: What is the gdc number of products of graphs, e.g., cartesian,, corona, join, and lexicographic products?

(ii) Graph Operations: How do vertex and edge operations on a graph impact its global domination number γ_g and global dominator chromatic number $\chi_{gd}(G)$?

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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