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Some Properties of Uniformly Harmonically Convex and Uniformly Harmonically Quasi-Convex Functions

Mehdi Dehghanian[⋆], *Yamin Sayyari* and *Arash Ziaee*

Abstract

In this note, we investigate the concepts of uniformly harmonically convexity and uniformly harmonically quasi-convexity, and establish some remarkable properties and basic examples related to these concepts. In addition, we will present methods that allow the construction of η-uniformly harmonically convex and η-uniformly harmonically quasi-convex functions.

Keywords: Harmonically convex, Uniformly harmonically convex, Uniformly harmonically quasi-convex.

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1. Introduction

In past years, the concept of convexity has been expanded in several directions. The notion of uniform convexity functions was first presented by Clarkson in 1936 [1]. Also, the concept of uniformly convex and uniformly quasi-convex functions was studied by Vladimirov et al. in 1978 [2, 3].

⋆Corresponding author (E-mail: mdehghanian@sirjantech.ac.ir)
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In 1983, Zalinescu [4] characterized the uniformly convex functions and gived some examples of such functions. To study more properties of uniformly convex functions, refer to [5, 6]. The theory of uniformly convexity is useful in information theory [7, 8].
Let I be an interval in $\mathbb{R}$. A real value function ψ is called convex on I if

$$\psi\left(tu+(1-t)v\right) \leq t\psi(u)+(1-t)\psi(v),$$

for every $t \in [0,1]$ and every $u, v \in I$.
An important class of convex functions, called harmonic convex functions, has many applications in analysis, mean theory, statistics, optimization and entropy (see [9–14]).
Let $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval. A function $\psi : I \to \mathbb{R}$ will be called harmonically convex if

$$\psi\left(\frac{uv}{tu+(1-t)v}\right) \leq (1-t)\psi(u)+t\psi(v), \tag{1}$$

for every $t \in [0,1]$ and every $u, v \in I$ (see [15]).
In this article, we investigate uniformly harmonically convex functions, and uniformly harmonically quasi-convex functions, and study some basic properties of these types of functions. Moreover, we give some examples of such functions.

2. Uniformly harmonically convex function

In this section, we will introduce the concept of uniformly harmonically convex functions and then study some of their properties.

Definition 2.1. Let $\eta > 0$ and $I \subseteq \mathbb{R}\backslash\{0\}$ be an interval. A function $\psi : I \to \mathbb{R}$ is called η-uniformly harmonically convex (η-UHC) if there is $\delta > 0$ such that

$$\psi\left(\frac{2uv}{u+v}\right) \leq \frac{\psi(u)+\psi(v)}{2} - \delta,$$

for all $u, v \in I$ with $\left|\frac{1}{u} - \frac{1}{v}\right| \geq \eta$.
The function ψ is called to be uniformly harmonically convex (UHC) if it is η-UHC for all $\eta > 0$.

Let us point out the concept of modulus of uniform harmonicall convexity.

$$\delta_{\psi}(\eta) = \inf\left\{\frac{\psi(u)+\psi(v)}{2} - \psi\left(\frac{2uv}{u+v}\right) : u, v \in I, \left|\frac{1}{u} - \frac{1}{v}\right| \geq \eta\right\}.$$

Example 2.2. Let $\eta > 0$ and $I = [a, b] \subseteq \mathbb{R}\backslash\{0\}$. Then $\psi(u) = |u|$ is η-UHC with $\delta = \frac{\eta^2\alpha^4}{4\beta}$ where $\alpha = \min\{|a|, |b|\}$ and $\beta = \max\{|a|, |b|\}$.

Let $\varphi : I \to \mathbb{R}$ be a function and ψ be a real value function such that $\frac{uv}{v-u} \in \mathrm{Dom}(\psi)$ for all $u, v \in I$ with $u \neq v$. We define infimal harmonically convolution $\psi \star \varphi : I \to \mathbb{R}$ by

$$(\psi \star \varphi)(u) = \inf \left\{ \psi\left(\frac{uv}{v-u}\right) + \varphi(v) : u \neq v, v \in I \right\}.$$

Theorem 2.3. *Let ψ, φ be two harmonically convex functions with ψ is η_1-UHC and φ is η_2-UHC for $\eta_1, \eta_2 > 0$. Then $\psi \star \varphi$ is $\eta_1 + \eta_2$-UHC by modulus* $\min\{\delta_\psi(\eta_1), \delta_\varphi(\eta_2)\}$.

Proof. Assume that $u_1, u_2 \in \mathrm{Dom}(\psi \star \varphi)$ with $\left|\frac{1}{u_1} - \frac{1}{u_2}\right| \geq \eta_1 + \eta_2$ and $\zeta > 0$. So, we can find $v_1, v_2 \in \mathrm{Dom}(\varphi)$ such that

$$\psi\left(\frac{u_1 v_1}{v_1 - u_1}\right) + \varphi(v_1) \leq (\psi \star \varphi)(u_1) + \zeta,$$
$$\psi\left(\frac{u_2 v_2}{v_2 - u_2}\right) + \varphi(v_2) \leq (\psi \star \varphi)(u_2) + \zeta.$$

Also, we obtain

$$\left|\left(\frac{1}{u_1} - \frac{1}{v_1}\right) - \left(\frac{1}{u_2} - \frac{1}{v_2}\right)\right| + \left|\frac{1}{v_1} - \frac{1}{v_2}\right| \geq \left|\frac{1}{u_1} - \frac{1}{u_2}\right| \geq \eta_1 + \eta_2.$$

Thus, either

$$\left|\left(\frac{1}{u_1} - \frac{1}{v_1}\right) - \left(\frac{1}{u_2} - \frac{1}{v_2}\right)\right| \geq \eta_1, \tag{2}$$

or

$$\left|\frac{1}{v_1} - \frac{1}{v_2}\right| \geq \eta_2, \tag{3}$$

holds. If (2) holds, then we have

$$\begin{aligned}
(\psi \star \varphi)\left(\frac{2u_1u_2}{u_1 + u_2}\right) &\leq \psi\left(\frac{2u_1u_2v_1v_2}{u_1v_1(v_2 - u_2) + u_2v_2(v_1 - u_1)}\right) + \varphi\left(\frac{2v_1v_2}{v_1 + v_2}\right) \\
&\leq \frac{\psi\left(\frac{u_1v_1}{v_1-u_1}\right) + \psi\left(\frac{u_2v_2}{v_2-u_2}\right)}{2} - \delta_\psi(\eta_1) + \frac{\varphi(v_1) + \varphi(v_2)}{2} \\
&\leq \frac{(\psi \star \varphi)(u_1) + (\psi \star \varphi)(u_2)}{2} - \delta_\psi(\eta_1) + \zeta.
\end{aligned}$$

Similarly, if (3) holds, then we get

$$(\psi \star \varphi)\left(\frac{2u_1u_2}{u_1 + u_2}\right) \leq \frac{(\psi \star \varphi)(u_1) + (\psi \star \varphi)(u_2)}{2} - \delta_\varphi(\eta_2) + \zeta,$$

which implies the statement as $\zeta > 0$ was arbitrary. Also

$$\delta_{\psi\star\varphi}(\eta_1+\eta_2)=\min\{\delta_\psi(\eta_1),\delta_\varphi(\eta_2)\}.$$

$\square$

Theorem 2.4. *Let ψ be a harmonically convex function and $\eta > 0$. Then*

$$p\psi(v)+(1-p)\psi(u)-\psi\left(\frac{uv}{pu+(1-p)v}\right)\geq 2\delta_\psi(\eta)\min\{p,1-p\},$$

where $p\in[0,1], u,v\in Dom(\psi)$ and $\left|\frac{1}{u}-\frac{1}{v}\right|\geq\eta$.

Proof. First, without loss of generality, we can suppose that $p\in[0,\frac{1}{2}]$ so $p=\min\{p,1-p\}$. Now, note that

$$\frac{uv}{pu+(1-p)v}=\frac{1}{2p\frac{u+v}{2uv}+(1-2p)\frac{1}{u}}.$$

Since ψ is a harmonically convex functions, we have

$$\begin{aligned}\psi\left(\frac{uv}{pu+(1-p)v}\right)&\leq 2p\psi\left(\frac{2uv}{u+v}\right)+(1-2p)\psi(u)\\&\leq(1-2p)\psi(u)+2p\left(\frac{\psi(u)+\psi(v)}{2}-\delta_\psi(\eta)\right)\\&=(1-p)\psi(u)+p\psi(v)-2p\delta_\psi(\eta).\end{aligned}$$

$\square$

The gauge of uniform harmonically convexity for a harmonically convex function is

$$\Gamma_\psi(\eta)=\inf\left\{\frac{p\psi(v)+(1-p)\psi(u)-\psi\left(\frac{uv}{pu+(1-p)v}\right)}{p(1-p)}:0\leq p\leq 1,\left|\frac{1}{u}-\frac{1}{v}\right|\geq\eta\right\}.$$

Corollary 2.5. *Let ψ be a harmonically convex function on I and $\eta > 0$. Then*

$$2\delta_\psi(\eta)\leq\Gamma_\psi(\eta)\leq 4\delta_\psi(\eta).$$

3. Uniformly harmonically quasi-convex function

In this section, we introduce the concept of uniformly harmonically quasi-convex functions on $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$ and investigate some of its properties.

Definition 3.1. A function $\psi: I\to\overline{\mathbb{R}}$ is said to be

1. harmonically quasi-convex (HQC), if
$$\psi\left(\frac{uv}{pu+(1-p)v}\right) \leq \max\{\psi(u), \psi(v)\},$$
for all nonzero $u, v \in I$ with $u \neq -v$ and $p \in [0,1]$.
2. η-uniformly harmonically quasi-convex (η-UHQC), if for a given $\eta > 0$, there is some $\delta > 0$ such that
$$\psi\left(\frac{2uv}{u+v}\right) \leq \max\{\psi(u), \psi(v)\} - \delta,$$
for all nonzero $u, v \in I$ with $u \neq -v$ and $\left|\frac{1}{u} - \frac{1}{v}\right| \geq \eta$.
3. uniformly harmonically quasi-convex (UHQC) if it is η-UHQC convex for all $\eta > 0$.

Example 3.2. $\psi(u) = \frac{1}{u}$ is a bounded η-UHQC function with modulus $\delta = \frac{\eta}{2}$ on $[a, \infty)$ for all $a > 0$.

Proposition 3.3. *Let $\eta > 0$ and $\psi \geq 0$ be a η-UHQC function with modulus δ and $\inf\{\psi(u) : u \in I\} > \delta$. Then ψ^2 is η-UHQC with modulus δ^2.*

Proposition 3.4. *Let $\eta > 0$. Assume that $\psi \geq 0$ is a harmonically convex and η-UHQC function. Then ψ^2 is η-UHC.*

Proof. Since $\psi \geq 0$ and is harmonically convex,
$$\frac{\psi(u)+\psi(v)}{2} \geq \psi\left(\frac{2uv}{u+v}\right) \geq 0,$$
and therefore
$$\begin{aligned}&\left(\frac{\psi(u)+\psi(v)}{2} - \psi\left(\frac{2uv}{u+v}\right)\right)^2 + \left(\frac{\psi(u)-\psi(v)}{2}\right)^2 \\ &\leq \frac{\psi^2(u)+\psi^2(v)}{2} - \psi^2\left(\frac{2uv}{u+v}\right). \end{aligned} \tag{4}$$
Suppose that $\left|\frac{1}{u} - \frac{1}{v}\right| \geq \eta$ and $\delta > 0$ as in the definition of η-UHQC.
We consider two following cases:
Case 1: If $|\psi(u) - \psi(v)| > \delta$, then from (4), we have
$$\frac{\psi^2(u)+\psi^2(v)}{2} - \psi^2\left(\frac{2uv}{u+v}\right) \geq \frac{\delta^2}{4}.$$
Case2: If $|\psi(u) - \psi(v)| \leq \delta$, then
$$\psi\left(\frac{2uv}{u+v}\right) \leq \max\{\psi(u), \psi(v)\} - \delta \leq \frac{\psi(u)+\psi(v)}{2} - \frac{\delta}{2}.$$

Therefore by using (4), we obtain

$$\frac{\psi^2(u)+\psi^2(v)}{2}-\psi^2\left(\frac{2uv}{u+v}\right)\geq\frac{\delta^2}{4}.$$

□

Theorem 3.5. *Let $\eta>0$ and $\psi:I\to\mathbb{R}$ be a η- UHQC function with modulus $\delta>0$ and ψ is bounded from below and $\beta:=\inf\{\psi(u):u\in I\}$. In addition, if $\varphi:\left[\frac{\beta}{\delta},\infty\right)\to\mathbb{R}_{\geq 0}$ be an increasing function which satisfies $\varphi(z+1)=3\varphi(z)$, then $\varphi\circ\mathcal{J}_\delta\circ\psi$ is η-UHC, where $\mathcal{J}_\delta(u)=\frac{u}{\delta}$.*

Proof. Putting $\zeta:=\frac{1}{6}\varphi\left(\frac{\beta}{\delta}\right)$. Let $u,v\in I$ be defined such that $\left|\frac{1}{u}-\frac{1}{v}\right|\geq\eta$. Without loss of generality we may suppose that $\psi(v)\leq\psi(u)$. Thus

$$\varphi\circ\mathcal{J}_\delta\left(\psi\left(\frac{2uv}{u+v}\right)\right)\leq\varphi\circ\mathcal{J}_\delta(\psi(u)-\delta). \tag{5}$$

On the other hand,

$$\varphi\circ\mathcal{J}_\delta(\beta-\delta)\leq\varphi\circ\mathcal{J}_\delta(\psi(u)-\delta)=\frac{1}{3}\varphi\circ\mathcal{J}_\delta(\psi(u)). \tag{6}$$

Also, from

$$4\zeta=\frac{2}{3}\varphi\left(\mathcal{J}_\delta(\beta)\right)\leq\frac{2}{3}\varphi\left(\mathcal{J}_\delta(\psi(u))\right),$$

we have

$$\frac{1}{3}\varphi\left(\mathcal{J}_\delta(\psi(u))\right)\leq\varphi\left(\mathcal{J}_\delta(\psi(u))\right)-4\zeta. \tag{7}$$

Therefore, from (5), (6) and (7), we get

$$\varphi\circ\mathcal{J}_\delta\left(\psi\left(\frac{2uv}{u+v}\right)\right)\leq\varphi\left(\mathcal{J}_\delta(\psi(u))\right)-4\zeta. \tag{8}$$

Since $\varphi\geq 0$,

$$3\varphi\circ\mathcal{J}_\delta\left(\psi\left(\frac{2uv}{u+v}\right)\right)\leq\varphi\left(\mathcal{J}_\delta(\psi(u))\right)+2\varphi\left(\mathcal{J}_\delta(\psi(v))\right). \tag{9}$$

It follows from (8) and (9) that

$$\varphi\circ\mathcal{J}_\delta\left(\psi\left(\frac{2uv}{u+v}\right)\right)\leq\frac{\varphi\left(\mathcal{J}_\delta(\psi(u))\right)+\varphi\left(\mathcal{J}_\delta(\psi(v))\right)}{2}-\zeta,$$

which say $\varphi\circ\mathcal{J}_\delta\circ\psi$ is η-UHC. □

Example 3.6. Let $\eta > 0$ and $\psi : I \to \mathbb{R}$ be a η- UHQC function with modulus $\delta > 0$. Then the function $\Phi(u) = 3^{\left[\frac{\psi(u)}{\delta}\right]}$ is η-UHC where $[\cdot]$ is floor function. (Just put $\varphi(u) = 3^{[u]}$ in Theorem 3.5.)

Example 3.7. Let $\eta > 0$ and $I = [a, b] \subseteq \mathbb{R}\backslash\{0\}$. Then $\psi(u) = |u|$ is η-UHQC with $\delta = \frac{\eta\alpha^3}{2\beta}$ where $\alpha = \min\{|a|, |b|\}$ and $\beta = \max\{|a|, |b|\}$.

Example 3.8. Let $\eta > 0$. Then the function $\Phi(u) = 3^{\left[\frac{2\beta|u|}{\eta\alpha^3}\right]}$ is η-UHC on $[a, b]$ where $\alpha = \min\{|a|, |b|\}$ and $\beta = \max\{|a|, |b|\}$. (The result follows from the Examples 3.6 and 3.7.)

Theorem 3.9. *Let $\eta > 0$ and $\psi : I \to \mathbb{R}$ be a η- UHQC function with modulus $\delta > 0$. In addition, let φ be an increasing function which satisfies $\varphi(z) \leq k\varphi(z+\delta)$ for some $0 < k < 1$ and $\beta := \inf\{\varphi(z) : z \in Dom(\varphi)\} > 0$, then $\varphi \circ \psi$ is η-UHQC with modulus $(1 - k)\beta$.*

Proof. Assume that $u, v \in I$ and $\left|\frac{1}{u} - \frac{1}{v}\right| \geq \eta$. Without loss of generality we may suppose that $\psi(v) \leq \psi(u)$. Therefore

$$\psi\left(\frac{2uv}{u+v}\right) \leq \psi(u) - \delta.$$

Therefore,

$$\begin{aligned}\varphi\left(\psi\left(\frac{2uv}{u+v}\right)\right) &\leq \varphi(\psi(u) - \delta) \leq k\varphi(\psi(u)) \\ &\leq \varphi(\psi(u)) - (1-k)\beta = \max\{\varphi(\psi(u)), \varphi(\psi(v))\} - (1-k)\beta,\end{aligned}$$

hence, $\varphi \circ \psi$ is η-UHQC with modulus $(1 - k)\beta$. □

Example 3.10. Let $\eta > 0, a > 1$ and let $\psi : (0, \infty) \to (0, \infty)$ be a η- UHQC function with modulus $\delta > 0$. Then $a^{\psi(u)}$ is η-UHQC with modulus $1 - a^{-\delta}$. (Setting $\varphi(z) = a^z$ and $k = a^{-\delta}$ in Theorem 3.9.)

Let $\varphi : I \to \overline{\mathbb{R}}$ be a function and ψ be a function such that $\frac{uv}{v-u} \in \mathrm{Dom}(\psi)$ for all $u, v \in I$ with $u \neq v$. We define infimal harmonically quasi convolution $\psi \star_q \varphi : I \to \overline{\mathbb{R}}$ by

$$(\psi \star_q \varphi)(u) = \inf\left\{\max\left\{\psi\left(\frac{uv}{v-u}\right), \varphi(v)\right\} : u \neq v, v \in I\right\}.$$

Example 3.11. Let $\psi(u) = \varphi(u)$ defined by $\frac{1}{u}$ on $(0, \infty)$. Then $(\psi \star_q \varphi)(u) = \frac{1}{2u}$ (see Figure 1).

Theorem 3.12. *Let ψ, φ be two HQC functions. Then $\psi \star_q \varphi$ is HQC.*

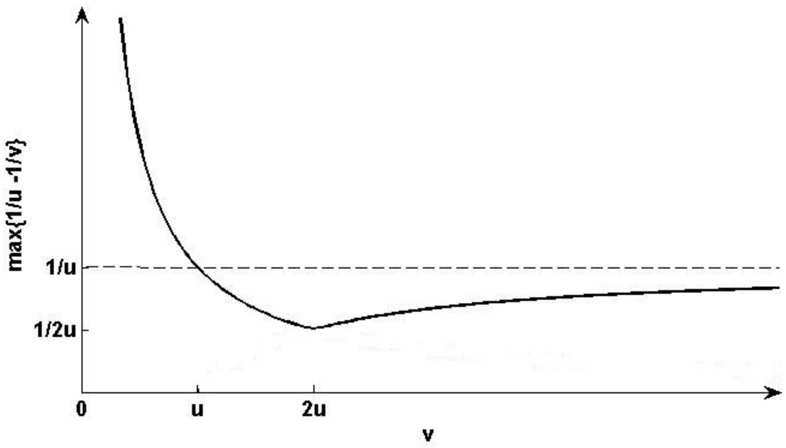

Figure 1: $\max\left\{\frac{1}{u}-\frac{1}{v},\frac{1}{v}\right\}$.

Proof. Assume that $u_1, u_2 \in \mathrm{Dom}(\psi \star_q \varphi)$ and $\zeta > 0$. So, we can find $v_1, v_2 \in \mathrm{Dom}(\varphi)$ such that

$$\max\left\{\psi\left(\frac{u_1v_1}{v_1-u_1}\right),\varphi(v_1)\right\} \le (\psi \star_q \varphi)(u_1)+\zeta,$$
$$\max\left\{\psi\left(\frac{u_2v_2}{v_2-u_2}\right),\varphi(v_2)\right\} \le (\psi \star_q \varphi)(u_2)+\zeta.$$

Hence,

$$\begin{aligned}
&(\psi \star_q \varphi)\left(\frac{2u_1u_2}{u_1+u_2}\right) \le \max\left\{\psi\left(\frac{2u_1u_2v_1v_2}{u_1v_1(v_2-u_2)+u_2v_2(v_1-u_1)}\right),\varphi\left(\frac{2v_1v_2}{v_1+v_2}\right)\right\}\\
&\le \max\left\{\max\left\{\psi\left(\frac{u_1v_1}{v_1-u_1}\right),\psi\left(\frac{u_2v_2}{v_2-u_2}\right)\right\},\max\left\{\varphi(v_1),\varphi(v_2)\right\}\right\}\\
&= \max\left\{\max\left\{\psi\left(\frac{u_1v_1}{v_1-u_1}\right),\varphi(v_1)\right\},\max\left\{\psi\left(\frac{u_2v_2}{v_2-u_2}\right),\varphi(v_2)\right\}\right\}\\
&\le \max\left\{(\psi \star_q \varphi)(u_1),(\psi \star_q \varphi)(u_2)\right\}+\zeta,
\end{aligned}$$

which completes the proof. □

Theorem 3.13. *Let $\eta > 0$ and $\psi : (0,\infty) \to \mathbb{R}$ be a η- UHQC function with modulus δ which is bounded from below. Then $\lim_{u\to 0^+}\psi(u) = +\infty$.*

Proof. Taking $\beta := \{\inf \psi(u) : u \in I\}$. Choose $u_0 > 0$ that $\psi(u_0) < \beta + \frac{\delta}{2}$.

By induction on n, we show that if $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq 2^n\eta$, then $\psi(u) \geq (n+1)\delta+\beta$ for each $n \in \mathbb{N}$. Suppose that $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq \eta$ and $\psi(u) < \beta+\delta$. Therefore,

$$\psi\left(\frac{2uu_0}{u+u_0}\right) \leq \max\{\psi(u), \psi(u_0)\} - \delta = \psi(u) - \delta < \beta,$$

which is contradiction. Hence $\psi(u) \geq \beta+\delta$. Now, assume that $\psi(u) \geq (k+1)\delta+\beta$ whenever $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq 2^k\eta$.
Let $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq 2^{k+1}\eta$, then $\left|\frac{u+u_0}{2uu_0}-\frac{1}{u_0}\right| \geq 2^k\eta$. It follows that

$$\psi\left(\frac{2uu_0}{u+u_0}\right) \geq (k+1)\delta+\beta.$$

Since $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq 2^{k+1}\eta > \eta$ then $\psi\left(\frac{2uu_0}{u+u_0}\right) \leq \psi(u)-\delta$ and so

$$(k+2)\delta+\beta \leq \phi(u),$$

whenever $\left|\frac{1}{u}-\frac{1}{u_0}\right| \geq 2^{k+1}\eta$. Therefore $\lim_{u\to 0^+}\psi(u) = +\infty$. □

The following corollary is an immediate consequence of Theorem 3.13.

Corollary 3.14. *Every η- UHQC function on $(0,\infty)$ is unbounded.*

Example 3.2 shows that Corollary 3.14 is not necessarily true on $[a,\infty)$ for $a>0$.

Theorem 3.15. *Let $A>0, B\geq 0$ and $\psi:(0,\infty)\to\mathbb{R}$ be a η- UHQC function with modulus δ. Then $\Psi(u)=\psi\left(\frac{u}{A+Bu}\right)$ is $\frac{\eta}{A}$- UHQC function with modulus δ.*

Proof. Suppose that $\eta>0$ and $u,v\in(0,\infty)$ with $|\frac{1}{u}-\frac{1}{v}| \geq \frac{\eta}{A}$. So

$$\begin{aligned}
\Psi\left(\frac{2uv}{u+v}\right) &= \psi\left(\frac{2uv}{A(u+v)+2Buv}\right)\\
&= \psi\left(\frac{2\times\frac{u}{A+Bu}\times\frac{v}{A+Bv}}{\frac{u}{A+Bu}+\frac{v}{A+Bv}}\right)\\
&\leq \max\left\{\psi\left(\frac{u}{A+Bu}\right), \psi\left(\frac{v}{A+Bv}\right)\right\} - \delta,
\end{aligned}$$

whenever $|\frac{A}{u}-\frac{A}{v}| \geq \eta$. Therefore,

$$\Psi\left(\frac{2uv}{u+v}\right) \leq \max\{\Psi(u), \Psi(v)\} - \delta,$$

whenever $|\frac{1}{u}-\frac{1}{v}| \geq \frac{\eta}{A}$. □

Conclusion

In this paper, we investigate the concepts of uniformly harmonically convexity and uniformly harmonically quasi-convexity and establish some remarkable properties. Moreover, we will introduce methods that allow the construction of η-uniformly harmonically convex and η-uniformly harmonically quasi-convex functions.

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Mehdi Dehghanian
Department of Mathematics,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: mdehghanian@sirjantech.ac.ir

Yamin Sayyari
Department of Mathematics,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: y.sayyari@sirjantech.ac.ir

Arash Ziaee
Department of Civil Engineering,
Sirjan University of Technology,
Sirjan, I. R. Iran
e-mail: ziaee@sirjantech.ac.ir