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Strong Edge Criticality for Cardinality-Redundance in Graphs

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Abstract

A vertex v in a graph G is said to be over-dominated by a subset S of vertices of G if $|N[v] \cap S| \geq 2$. The cardinality–redundance of S, which we denote by $CR(S)$, is the number of vertices of G that are over-dominated by S. The cardinality–redundance number of a graph G, which we denote by $CR(G)$, is the minimum among all cardinality–redundances $CR(S)$ taken over all dominating sets S. In this paper, we study those graphs whose cardinality–redundance decreases by two upon the removal of any arbitrary edge. We refer to such graphs as cardinality–redundance strong edge critical graphs. We give a general characterization for all cardinality–redundance strong edge critical graphs, and then focus on the cardinality–redundance strong edge critical graphs having cardinality–redundance 2, and present several characterizations for these graphs.

Keywords: Dominating set, Cardinality-Redundance, Critical.

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1. Introduction

For notation and terminology not given here, the reader can consult [1, 2]. Here, we consider undirected and simple graphs $G = (V(G), E(G))$, where the vertex set is $V(G)$ and the edge set is $E(G)$. The *order* of a graph G is given by $n =$

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$n(G) = |V(G)|$. For a vertex v, the *open neighborhood* $N(v) = N_G(v)$ of v is the set of vertices that are adjacent to v, and the *closed neighborhood* $N[v]$ is $N(v) \cup \{v\}$. For any subset $S \subseteq V(G)$, denote $N(A) = N_G(A) = \cup_{v \in A} N(v)$ and $N[A] = \cup_{v \in A} N[v]$. The *degree* of v is the cardinality of $N(v)$, denoted by $\deg(v)$. A graph G is a *bipartite graph* if the vertex set can be partitioned into two sets X and Y such that any edge of G has one end-point in X and the other end-point in Y. A *complete bipartite* graph is a bipartite graph such that any vertex of each partite set is adjacent to all vertices of the other partite set. A complete bipartite graph with partite sets of cardinalities m and n is denoted by $K_{m,n}$. We refer to $K_{1,n-1}$ as the star of order n. We denote by P_n, C_n and K_n the path, the cycle and the complete graph of order n, respectively. A set $S \subseteq V(G)$ of vertices in a graph is called a *dominating set* if $N[S] = V(G)$. The *domination number* of a graph G, which we denote by $\gamma(G)$, is the minimum cardinality of a dominating set among all dominating sets of G. A set $S \subseteq V(G)$ of vertices in a graph is referred as an *independent set* if no pair of vertices of S are adjacent.

The concept of cardinality–redundance in graphs was introduced in [3]. Given a set S and a vertex v, the vertex v is said to be *over-dominated* by S if $|N[v] \cap S| \geq 2$. The *cardinality–redundance* of S, which we denote by $CR(S)$ (or $CR_G(S)$ to refer it to G), is the number of vertices of G that are over-dominated by S. The minimum of $CR(S)$ taken over all the dominating sets S is called the cardinality–redundance number of G, and is denoted by $CR(G)$. A dominating set S with $CR_G(S) = CR(G)$ is called a $CR(G)$-set. The size of a minimum $CR(G)$–set is denoted by $\gamma_{CR}(G)$ and is called *dominating cardinality –redundance number* of G, that is, $\gamma_{CR}(G) = \min\{|S| : S \text{ is a } CR(G)\text{–set}\}$. Any $CR(G)$–set with minimum cardinality is referred as a γ_{CR}-set. To see more references on the concept of cardinality–redundance consult [4, 5].

A fundamental question for various graph operations is the concept of criticality which has been already considered for several domination parameters such as domination, total domination, global domination, secure domination and Roman domination by several authors, (see, for example [6–17]).

In this paper, we consider this concept for the cardinality–redundance in graphs upon edge removal. The paper is organized as follows. In Section 2, we determine the cardinality–redundance number in some classes of graphs, including paths, cycles, and complete bipartite graphs. In Section 3, we introduce cardinality–redundance strong edge critical graphs and give a general characterization for all cardinality–redundance strong edge critical graphs. We then focus on the cardinality–redundance strong edge critical graphs having cardinality–redundance 2, and present several characterizations for these graphs.

2. Preliminary results

In this section, we determine the cardinality–redundance number in the classes of paths, cycles, and complete bipartite graphs.

Proposition 2.1. *For the path P_n, $CR(P_n) = 0$.*

Proof. Let $V(P_n) = \{v_1, v_2, ..., v_n\}$. The result is clear for $n = 1, 2$. Thus, assume that $n \geq 3$. It is straightforward to see that $S_0 = \{v_{3k+2} : k = 0, 1, ..., \frac{n-3}{3}\}$ (for $n \equiv 0 \pmod 3$), $S_1 = \{v_{3k+1} : k = 0, 1, ..., \frac{n-1}{3}\}$ (for $n \equiv 1 \pmod 3$), and $S_2 = \{v_{3k+1} : k = 0, 1, ..., \frac{n-2}{3}\}$ (for $n \equiv 2 \pmod 3$) are the dominating sets for P_n with $CR(S_i) = 0$ for $i = 0, 1, 2$. Thus the result follows. □

Proposition 2.2. *For the cycle C_n, $CR(C_n) = 0$ if $n \equiv 0 \pmod 3$, 1 if $n \equiv 2 \pmod 3$ and 2 if $n \equiv 1 \pmod 3$.*

Proof. Let $V(C_n) = \{v_1, v_2, ..., v_n\}$. First, assume that $n \equiv 0 \pmod 3$. It is easy to see that $S = \{v_{3k+2}, 0 \leq k \leq \frac{n-3}{3}\}$ is a dominating set for C_n with $CR(S) = 0$. Next assume that $n \equiv 2 \pmod 3$. Then $S = \{v_{3k+1}, 0 \leq k \leq \frac{n-2}{3}\}$ is a dominating set for C_n with $CR(S) = 1$. Thus $CR(C_n) \leq 1$. Assume that there is a dominating set D for C_n with $CR(D) = 0$. Let $D = \{v_{i_1}, ..., v_{i_{|D|}}\}$, where $i_j < i_{j+1}$ for $j = 1, ..., |D| - 1$. Then $d(v_{i_1}, v_{i_{|D|}}) = d(v_{i_j}, v_{i_{j+1}}) = 3$ for $j = 1, ..., |D| - 1$. Then $n = |N[D]| = 3|D| \equiv 0 \pmod 3$, a contradiction. Thus, $CR(C_n) = 1$.

Now assume that $n \equiv 1 \pmod 3$. Then $S = \{v_{3k+1}, 0 \leq k \leq \frac{n-1}{3}\}$ is a dominating set for C_n with $CR(S) = 2$. Thus $CR(C_n) \leq 2$. Assume that there is a dominating set D for C_n with $CR(D) = 0$. Let $D = \{v_{i_1}, ..., v_{i_{|D|}}\}$, where $i_j < i_{j+1}$ for $j = 1, ..., |D| - 1$. Then $d(v_{i_1}, v_{i_{|D|}}) = d(v_{i_j}, v_{i_{j+1}}) = 3$ for $j = 1, ..., |D| - 1$. Then $n = |N[D]| = 3|D| \equiv 0 \pmod 3$, a contradiction. Next assume that there is a dominating set D for C_n with $CR(D) = 1$. Let $D = \{v_{i_1}, ..., v_{i_{|D|}}\}$, where $i_j < i_{j+1}$ for $j = 1, ..., |D| - 1$. Then there are precisely one integer t such that $d(v_{i_t}, v_{i_{t+1}}) = 2 \pmod{|D|}$ and $d(v_{i_j}, v_{i_{j+1}}) = 3$ for $j \in \{1, ..., |D|\} - \{t\}$, where the addition is in modulo $|D|$. Then $n = |N[D]| = 3(|D| + 1) - 1$, a contradiction. We deduce that $CR(C_n) = 2$. □

It is well-known that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ (see [1]). Also, for any graph G, $\gamma(G) \leq \gamma_{CR}(G)$ by the definition of cardinality –redundance number of G. It is now straightforward to see that the dominating sets S given in the proof of Proposition 2.2 are $\gamma_{CR}(C_n)$-set. Thus, we obtain the following.

Proposition 2.3. *For a cycle C_n, $\gamma_{CR}(C_n) = \lceil \frac{n}{3} \rceil$.*

Proposition 2.4. *For the complete bipartite graph $K_{m,n}$ with $m, n \geq 2$, $CR(K_{m,n}) = 2$.*

Proof. Let $G = K_{m,n}$, where $m, n \geq 2$. Let $V(G) = A \cup B$, where $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_n\}$ are partite sets of G. Clearly $S = \{a_1, b_1\}$ is a dominating set for G with $CR(S) = 2$. Thus, $CR(G) \leq 2$. Assume that there is a dominating set D for G with $CR(D) < 2$. Let $D = \{v_1, v_2, ..., v_k\}$. Clearly D is an independent set and so, without lost of generality, assume that $D \subseteq A$. Since $|D| \geq 2$ and $|B| \geq 2$ we obtain that there are at least two vertices in B that are over-dominated by D, contradicting the fact that $CR(G) = 0$. □

Observe that $\gamma_{CR}(K_{m,n}) \geq \gamma(K_{m,n}) = 2$. Now, following the proof of Proposition 2.4, we obtain the following.

Proposition 2.5. $\gamma_{CR}(K_{m,n}) = 2$.

3. Criticality upon edge removal

We begin with the following.

Proposition 3.1. *For an arbitrary edge uv of a graph G, $CR(G) - 2 \leq CR(G - uv) \leq CR(G) + \Delta(G) - 1$.*

Proof. Let $G = (V(G), E(G))$ be a graph and uv be an arbitrary edge of G, and let S be a $\gamma_{CR}(G-uv)$-set. If $S \cap \{u, v\} = \emptyset$ then S is a dominating set for G and it is evident that $CR_G(S) = CR_{G-uv}(S)$. Consequently, $CR(G) \leq CR_G(S) = CR_{G-uv}(S)$. Thus, assume that $S \cap \{u, v\} \neq \emptyset$. Without loss of generality, assume that $v \in S$. Assume that $u \notin S$. Clearly $N(u) \cap S \neq \emptyset$. If $|N(u) \cap S| \geq 2$, then $CR_G(S) = CR_{G-uv}(S)$, and thus $CR(G) \leq CR_G(S) = CR_{G-uv}(S)$. Thus, assume that $|N(u) \cap S| \geq 1$. Then $CR_G(S) = CR_{G-uv}(S)+1$, and thus $CR(G) \leq CR_G(S) + 1 = CR_{G-uv}(S) + 1$. Next, assume that $u \in S$. If $N(u) \cap N(v) \cap S \neq \emptyset$, then $CR_G(S) = CR_{G-uv}(S)$ and so $CR(G) \leq CR_G(S) = CR_{G-uv}(S)$, and if $N(u) \cap N(v) \cap S = \emptyset$ then $CR_G(S) = CR_{G-uv}(S) + 2$, and thus $CR(G) \leq CR_G(S) + 2 = CR_{G-uv}(S) + 2$. Thus, the lower bound follows.

Now let S be a $\gamma_{CR}(G)$-set. If $\{u, v\} \cap S = \emptyset$, then $CR_{G-uv}(S) = CR_G(S)$, and so $CR(G - uv) \leq CR_G(S) = CR(G)$. Thus assume that $\{u, v\} \cap S \neq \emptyset$. Without loss of generality, assume that $v \in S$. Assume that $u \notin S$. If $N_G(u) \cap S \neq \{v\}$, then $CR_{G-uv}(S) = CR_G(S)$, and so $CR(G - uv) \leq CR_G(S) = CR(G)$. Thus assume that $N_G(u) \cap S = \{v\}$. Let $S' = S \cup \{u\}$. Then $CR_{G-uv}(S') = CR_G(S') + \deg(u) - 1 \leq CR_G(S') + \Delta(G) - 1$. Thus, $CR(G - uv') \leq CR(G) + \Delta(G) - 1$. □

We define a graph G *cardinality redundance edge-critical* (or just CR-edge critical) if $CR(G - uv) < CR(G)$ for any edge uv. It is *cardinality redundance strong edge-critical* (or just CR-strong edge critical) if $CR(G - uv) = CR(G) - 2$ for any edge uv, and *cardinality redundance weak edge-critical* (or just CR-weak edge critical) if $CR(G - uv) = CR(G) - 1$ for any edge uv. If G is a CR-strong (weak) edge critical graph and $CR(G) = k$ then we call G a k-CR-strong (weak) edge critical. In this paper, we focus on the CR-strong edge critical graphs. The following is a direct consequence of Propositions 2.1 to 2.3 which shows that for any integer $k \geq 2$, there is a CR-strong edge critical graph G with $\gamma_{CR}(G) = k$.

Proposition 3.2. *For $n \equiv 1 \pmod 3$, the cycle C_n is CR-strong edge critical.*

In the following we give a general characterization for all CR-strong edge critical graphs.

Theorem 3.3. *A graph G is a CR-strong edge critical if and only if for any edge uv, there is a dominating set S containing both u and v such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G - uv)$-set and a $CR(G)$-set.*

Proof. Let G be a CR-strong edge critical graph and let uv be an edge of G and $G' = G - uv$. Then $CR(G') = CR(G) - 2$. Let S' be a $\gamma_{CR}(G')$-set. If $\{u, v\} \cap S' = \emptyset$, then $CR_G(S') = CR_{G'}(S')$, and so $CR(G) \le CR_G(S') = CR_{G'}(S') = CR(G') = CR(G) - 2$, a contraction. Thus, $\{u, v\} \cap S' \neq \emptyset$. Without loss of generality, assume that $v \in S'$. Assume that $u \notin S'$. Clearly $N_{G'}(u) \cap S' \neq \emptyset$. If $|N_{G'}(u) \cap S'| \ge 3$, then $CR_G(S') = CR_{G'}(S')$, and so $CR(G) \le CR_G(S') = CR_{G'}(S') = CR(G') = CR(G) - 2$, a contraction. Thus assume that $|N_{G'}(u) \cap S'| = 2$. Then $CR_G(S') = CR_{G'}(S') + 1$, and so $CR(G) \le CR_G(S') + 1 = CR_{G'}(S') + 1 = CR(G') + 1 = CR(G) - 1$, a contraction. We deduce that $u \in S'$. Suppose that $N_G(u) \cap S' \neq \{v\}$. If $N_G(v) \cap S' = \{v\}$ then $CR_G(S') = CR_{G'}(S') + 1$, and so $CR(G) \le CR_G(S') + 1 = CR_{G'}(S') + 1 = CR(G') + 1 = CR(G) - 1$, a contraction. Thus, $N_G(v) \cap S' \neq \{v\}$. Then $CR_G(S') = CR_{G'}(S')$, and so $CR(G) \le CR_G(S')$, a contraction. We deduce that $N_G(u) \cap S' = \{v\}$, and likewise $N_G(v) \cap S' = \{u\}$. Consequently, $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$. Note that S' is a $\gamma_{CR}(G - uv)$-set. Since $CR_G(S') = CR_{G'}(S') + 2 = CR(G') + 2 = CR(G)$, S' is a $CR(G)$-set, as desired.

For the conserve, assume that for any edge uv, there is a dominating set S containing both u and v such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G - uv)$-set and a $CR(G)$-set. Let $e = uv$ be an arbitrary edge, and let S be the dominating set satisfying the assumption. By Proposition 3.1, $CR(G) - 2 \le CR(G')$. Thus, we show that $CR(G) - 2 \ge CR(G')$. It is evident that S is a dominating set in G' and $CR_{G'}(S) = CR_G(S) - 2$. Thus, $CR(G') \le CR_{G'}(S) = CR_G(S) - 2$, and thus the result follows. □

The following is an immediate consequence of Theorem 3.3.

Corollary 3.4. *If G is a CR-strong edge critical graph, then*
(1) G has no triangle,
(2) $\delta(G) \ge 2$,
(3) $\gamma_{CR}(G) \le \gamma_{CR}(G - uv)$.

We now focus on the 2-CR-strong edge critical graphs. The following is a general characterization for 2-CR-strong edge critical graphs G with $2 \le \gamma_{CR}(G) \le 3$.

Theorem 3.5. *If a graph G is 2-CR-strong edge critical with $2 \le \gamma_{CR}(G) \le 3$, then for any edge uv, there is a dominating set S containing both u and v such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G - uv)$-set and a $\gamma_{CR}(G)$-set.*

Proof. Let G be a 2-CR-strong edge critical graph, and let $e = uv$ be an edge of G. By Theorem 3.3, there is a set S containing both u and v such that $N_G[u] \cap S =$

$N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G-uv)$-set and a $CR(G)$-set. We show that S is a $\gamma_{CR}(G)$-set as well. Clearly, by Corollary 3.4 (3), $\gamma_{CR}(G) \le \gamma_{CR}(G-uv)$. Suppose that $\gamma_{CR}(G) < \gamma_{CR}(G-uv)$. We consider the following cases:

Case 1. Let $\gamma_{CR}(G) = 2$. Then, $\gamma_{CR}(G-uv) \ge 3$. Let $S = \{u, v, z_1, z_2, ..., z_{k-2}\}$, where $k \ge 3$, be a $\gamma_{CR}(G-uv)$-set and let A be a $\gamma_{CR}(G)$-set with $|A| = \gamma_{CR}(G) = 2$. Since S is dominating set for G and G is triangle-free, we find that $A \cap (N[u] \cup N[v]) \neq \emptyset$ and for each $i = 1, 2, ..., k-2$, $A \cap N[z_i] \neq \emptyset$. Thus, we obtain that $k \le 3$, and consequently, $|k| = 3$. Suppose that $A \cap \{u, v\} = \emptyset$. Then $A \cap N(u) \neq \emptyset$ and $A \cap N(v) \neq \emptyset$. Note that $A \cap N[z_1] \neq \emptyset$. Then $|A| \ge 3$, a contradiction. Thus assume, without lost of generally, that $u \in A$. Clearly in this state $A \cap N(z_1) \neq \emptyset$. If $z_1 \in A$, then no vertex of $N(v) - \{u\}$ is dominated by A, a contradiction. Thus, $z_1 \notin A$. Assume that $z'_1, z'_2 \in N(z_1)$, where $z'_1 \in A$. Then z'_2 is not dominated by A, since G is triangle-free.

Case 2. Let $\gamma_{CR}(G) = 3$. Then $\gamma_{CR}(G-uu) \ge 4$. Let $S = \{u, v, z_1, z_2, ..., z_{k-2}\}$, where $k \ge 4$, be a $\gamma_{CR}(G-uv)$-set and A be a $\gamma_{CR}(G)$-set with $|A| = 3$. Since S is dominating set for G and G is triangle-free, we find that $A \cap (N[u] \cup N[v]) \neq \emptyset$ and for each $i = 1, 2, ...z_{k-2}$, $A \cap N[z_i] \neq \emptyset$. Thus, we obtain that $k \le 4$, and consequently, $k = 4$. Observe that $N(z_1) \cap N(z_2) = \emptyset$. If $A \cap \{u, v\} = \emptyset$, then $A \cap N(u) \neq \emptyset$ and $A \cap N(v) \neq \emptyset$, since G is triangle–free. Since, $A \cap N[z_i] \neq \emptyset$ for $i = 1, 2$, we find that $|A| \ge 4$, a contradiction. Thus $A \cap \{u, v\} \neq \emptyset$ and so we find that $|A \cap \{u, v\}| = 1$. Without lost of generality suppose $u \in A$.

Suppose that $A \cap \{z_1, z_2\} \neq \emptyset$. Since A dominates all vertices in $N(v) \backslash \{u\}$, we find that $A \cap \{z_1, z_2\} = 1$ and the third vertex of A dominates all vertices in $N(v) \backslash \{u\}$. Without loss of generality, assume that $z_1 \in A$. Then $A = \{u, z_1, {z_2}'\}$, where ${z_2}' \in N(z_2)$. But then A does not dominate the vertices of $N(z_2) - \{z'_2\}$, since $N(z_1) \cap N(z_2) = \emptyset$. This ia a contradiction. Thus, $A \cap \{z_1, z_2\} = \emptyset$. Then $A = \{u, z'_1, z'_2\}$, where $z_1 \in N(z_1)$ and $z_2 \in N(z_2)$. Then any vertex in $N(z_2) - \{z'_2\}$ is adjacent to z'_1 and any vertex in $N(z_1) - \{z'_1\}$ is adjacent to z'_2. Now considering the edge $z_1 z'_1$, by Theorem 3.3, there is a dominating set T containing both z_1 and z'_1 such that $N_G[z_1] \cap T = N_G[z'_1] \cap T = \{z_1, z'_1\}$ and T is both a $\gamma_{CR}(G - z_1 z'_1)$-set and a $CR(G)$-set. Since T dominates z_2, we obtain that $T \cap N[z_2] \neq \emptyset$. If $z_2 \in T$, then $N(z_2) \cap N(z'_1) \neq \emptyset$, a contradiction, since $CR(G) = 2$. Thus, $z_2 \notin T$. If $z'_2 \in T$, then $N(z_1) \cap N(z'_2) \neq \emptyset$, a contradiction. Thus, $z'_2 \notin T$. Then T contains a vertex $z'_3 \neq z'_2$ in $N(z_2)$. This is a contradiction, since $z'_3 \in N(z'_1)$. □

We next characterize 2-CR-strong edge critical graphs with $2 \le \gamma_{CR}(G) \le 3$.

Theorem 3.6. *A graph G with $\gamma_{CR}(G) = 2$ is 2-CR-strong edge critical if and only if $G = K_{m,n}$, where $m, n \ge 2$.*

Proof. Let G be a 2-CR-strong edge critical with $\gamma_{CR}(G) = 2$ and let uv be an edge of G. By Corollary 3.4 (2), $|N(u)| \ge 2$ and $|N(v)| \ge 2$. By Theorem 3.5 there is a dominating set S containing both u and v such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G-uv)$-set and a $\gamma_{CR}(G)$-set. Let $N(u) = \{u_1, u_2, ..., u_k\}$ and $N(v) = \{v_1, v_2, ..., v_l\}$. Note that so that $V(G) = \{u, v, u_1, u_2, ..., u_k, v_1, v_2, ..., v_l\}$.

Considering the edge uu_1, by Theorem 3.5, there is a dominating set S' containing both u and u_1 such that $N_G[u] \cap S' = N_G[u_1] \cap S' = \{u, u_1\}$ and S' is both a $\gamma_{CR}(G-uu_1)$-set and a $\gamma_{CR}(G)$-set. Then u_1 dominates all vertices in $N(v)-\{u\}$. Since uu_1 was chosen arbitrary, we deduce that each vertex of $N(u) - \{v\}$ is adjacent to all vertices of $N(v)-\{u\}$, and each vertex of $N(v)-\{u\}$ is adjacent to all vertices of $N(u) - \{v\}$. Consequently, G is a bipartite graph with partite sets $N(u) \cup \{v\}$ and $N(v) \cup \{u\}$ that is $G = K_{|N(u)|+1,|N(v)|+1}$. The converse follows from Propositions 2.4 and 2.5. □

Theorem 3.7. *A graph G with $\gamma_{CR}(G) = 3$ is 2-CR-strong edge critical if and only if $G = C_7$.*

Proof. Let G be a 2-CR-strong edge critical with $\gamma_{CR}(G) = 3$. We proceed with Claim 1.

Claim 1. For any edge uv, there is a dominating set $S = \{u, v, z_{uv}\}$ such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$, S is both a $\gamma_{CR}(G - uv)$-set and a $\gamma_{CR}(G)$-set, and any neighbors of z_{uv} is adjacent to all vertices $N(u) - \{v\}$ or $N(v) - \{u\}$.

Proof of Claim 1. Let uv be an edge of G. By Theorem 3.5, there is a dominating set S containing both u and v such that $N_G[u] \cap S = N_G[v] \cap S = \{u, v\}$ and S is both a $\gamma_{CR}(G - uv)$-set and a $\gamma_{CR}(G)$-set. Let $S = \{u, v, z_{uv}\}$. Let $z_1 \in N(z_{uv})$. Considering the edge z_1z_{uv}, by Theorem 3.5, there is a dominating set $S' = \{z_1, z_{uv}, y\}$ such that $N_G[z_1] \cap S' = N_G[z_{uv}] \cap S' = \{z_1, z_{uv}\}$ and S' is both a $\gamma_{CR}(G - z_1z_{uv})$-set and a $\gamma_{CR}(G)$-set. Since G is triangle–free, $y \notin N(u) - \{v\}$ and $y \notin N(v) - \{u\}$. Thus $y \in \{u, v\}$. Without lost of generality, assume that $y = u$. Then all vertices in $N(v) - \{u\}$ are dominated by z_1. This completes the proof of Claim 1.

Now let $e = uv$ be an arbitrary edge, and $S = \{u, v, z_{uv}\}$ be the set satisfying the Claim 1. We proceed with Claim 2.

Claim 2. No pair of vertices of $N(z_{uv})$ have a common neighbor in $V(G) - \{z_{uv}\}$.

Proof of Claim 2. Let $z_1, z_2 \in N(z_{uv})$ have a common neighbor $u_1 \neq z_{uv}$. Since G is triangle-free, we may assume, without loss of generality, that $u_1 \in N(u)-\{v\}$. Considering the edge z_1u_1, by Claim 1 there is a dominating set $S' = \{u_1, z_1, y\}$, where $y = z_{u_1z_1}$, such that $N_G[u_1] \cap S' = N_G[z_1] \cap S' = \{u_1, z_1\}$ and S' is both a $\gamma_{CR}(G - z_1u_1)$-set and a $\gamma_{CR}(G)$-set, and any neighbors of y is adjacent to all vertices $N(u_1) - \{z_1\}$ or $N(z_1) - \{u_1\}$. By Claim 1, v is adjacent to either z_{uv} or z_2, a contradiction.

By Claims 1 and 2, we find that $\deg(z_{uv}) = 2$. We next prove that $\deg(u) = 2$. Suppose that $\deg(u) \geq 3$. Let $N(u) - \{v\} = \{u_1, u_2, ..., u_k\}$, where $k \geq 2$. Let $z_1 \in N(z_{uv})$ be adjacent to all vertices of $N(u) - \{v\}$. We consider the edge u_1z_1. By Claim 1 there is a dominating set $S' = \{u_1, z_1, y\}$, where $y = z_{u_1z_1}$, such that $N_G[u_1] \cap S' = N_G[z_1] \cap S' = \{u_1, z_1\}$ and S' is both a $\gamma_{CR}(G - z_1u_1)$-set and a $\gamma_{CR}(G)$-set, and any neighbors of $z_{u_1z_1}$ is adjacent to all vertices $N(z_1) - \{u_1\}$ or $N(u_1) - \{z_1\}$. It is evident that $y \in N(v) - \{u\}$. By Claim 2, $z_2 \in N(y)$.

Then by Claim 1, z_2 is adjacent to all vertices in $N(u_1) - \{z_1\}$ or $N(z_1) - \{u_1\}$, a contradiction, since none of u nor u_2 is adjacent to z_2. We conclude that $\deg(u) = 2$. Similarly, $\deg(v) = 2$. Consequently $G = C_7$.

The converse follows from Propositions 2.2 and 2.3. □

4. Concluding remarks

We believe that the cycle C_n for $n \equiv 1 \pmod 3$ and $K_{m,n}$ for $m, n \geq 2$, are the only 2-CR-strong edge critical graphs with $\gamma_{CR}(G) \geq 2$. We thus propose the following problem.

Problem 1. Is it true that the cycle C_n, where $n \equiv 1 \pmod 3$, and the complete bipartite graphs $K_{m,n}$, where $m, n \geq 2$ are the only 2-CR-strong edge critical graphs G with $\gamma_{CR}(G) \geq 2$?

We also were not able to find a k-CR-strong edge critical graph with $k > 2$, and it seems to us that there is no such graphs. If this is correct then we reach to the following conjecture.

Conjecture 1. A graph G is CR-strong edge critical if and only if (1) $G \cong K_{m,n}$, where $m, n \geq 2$, or (2) $G \cong C_n$, where $n \equiv 1 \pmod 3$.

Conflicts of Interest. The authors declare that they have no conflicts of interest regarding the publication of this article.

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