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Approximate Solution of Magnetic Boundary Value Problems in Geometrically Complex Domains Using the Method of Integral Equation Systems

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Abstract

This paper provides a rigorous theoretical justification for the collocation method applied to a system of integral equations arising in magnetic boundary value problems governed by the vector Helmholtz equation. At appropriately selected collocation points, the system of integral equations is transformed into a system of algebraic equations, for which the existence and uniqueness of solutions are rigorously established. The convergence of the algebraic solutions to the exact solution of the integral equations is proven, and the convergence rate of the method is analytically derived. Furthermore, within the framework of the proposed approach, explicitly constructed sequences are rigorously proven to converge to the exact solution of the considered magnetic boundary value problems, thereby providing a systematic and reliable approximation scheme.

Keywords: Magnetic boundary value problems, Vector potentials, Vector Helmholtz equation, Systems of integral equations.

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1. Introduction and problem statement

One of the classical approaches to solving magnetic boundary value problems for the vector Helmholtz equation is to reduce the problem to a system of second-kind integral equations. Such systems generally admit closed-form solutions only in exceptional cases. Therefore, the development of approximate methods with rigorous theoretical justification is of primary importance.

The following function classes are introduced on the domain $G \subset R^3$. Let $C(G)$ denote the space of all continuous functions on G equipped with the norm $\|\varphi\|_\infty = \max\limits_{x \in G} |\varphi(x)|$; $H_\alpha(G)$ denote the space of all continuous functions on G satisfying a Hölder condition with exponent $\alpha \in (0, 1]$; and $\Re(G)$ denote the space of vector-valued functions $E = (E_1, E_2, E_3)$ that are twice continuously differentiable on G and continuous on $\bar{G}$, with $divE \in C(\bar{G})$ and $rotE \in C^3(\bar{G})$, where $C^3(\bar{G}) = C(\bar{G}) \times C(\bar{G}) \times C(\bar{G})$.

Let $D \subset R^3$ be a bounded domain with a C^2-smooth boundary Ω, and let $n(x) = (n_1(x), n_2(x), n_3(x))$ denote the outward unit normal at point $x \in \Omega$. Consider the following boundary value problems for the vector Helmholtz equation:

Interior magnetic boundary value problem. Determine a vector field $E \in \Re(D)$ satisfying the vector Helmholtz equation

$$\Delta E + k^2 E = 0 \text{ in } D,$$

and the boundary condition

$$[[rotE,\, n],\, n] = f, \quad (n,\, E) = g \text{ on } \Omega,$$

where Δ is the Laplace operator, k is the wave number such that $Imk \geq 0$, $[a, b]$ denotes the vector cross product of vectors a and b, and (a, b) denotes the scalar (dot) product of vectors a and b. Here, $g \in H_\alpha(\Omega)$ is a given function, and f is a given tangential field, both satisfying a Hölder condition with exponent $0 < \alpha \leq 1$, i.e.,

$$f \in H^3_{\perp,\alpha}(\Omega) = \left\{ f \in H^3_\alpha(\Omega) \,|\, (n(x),\, f(x)) = 0,\, \forall x \in \Omega \right\},$$

where $H^3_\alpha(\Omega) = H_\alpha(\Omega) \times H_\alpha(\Omega) \times H_\alpha(\Omega)$.

Exterior magnetic boundary value problem. Determine a vector field $E \in \Re\left(R^3 \backslash \bar{D}\right)$ satisfying the vector Helmholtz equation in $R^3 \backslash \bar{D}$, the radiation condition

$$\left[rotE,\, \frac{x}{|x|} \right] + \frac{x}{|x|} divE - ikE = o\left(\frac{1}{|x|} \right),$$

uniformly in all directions $x/|x|$, and the boundary condition

$$[[rotE,\, n],\, n] = f, \quad (n,\, E) = g \text{ on } \Omega,$$

where the functions f and g have the same meaning as in the interior problem.

Let

$$\Phi_k(x,\, y) = \frac{\exp(ik\,|x-y|)}{4\pi\,|x-y|}, \quad x, y \in R^3, \quad x \neq y,$$

be the fundamental solution of the Helmholtz equation. As shown in [[1], Theorem 4.22], the vector field

$$E(x) = \int_{\Omega} \Phi_k(x, y) [n(y), \mu(y)] \, d\Omega_y + grad \int_{\Omega} \Phi_k(x, y) \lambda(y) \, d\Omega_y, \quad x \in D,$$

is a solution to the interior magnetic boundary value problem if the vector function $\mu = (\mu_1, \mu_2, \mu_3) \in H^3_{\perp,\alpha}(\Omega)$ and the scalar function $\lambda \in H_\alpha(\Omega)$ satisfy the system of integral equations

$$\mu(x) + 2 \int_{\Omega} [n(x), [n(x), rot_x \{\Phi_k(x, y) [n(y), \mu(y)]\}]] \, d\Omega_y = 2f(x),$$

$$\lambda(x) + 2 \int_{\Omega} \Phi_k(x, y) (n(x), [n(y), \mu(y)]) \, d\Omega_y + \quad (1)$$

$$+2 \int_{\Omega} \frac{\partial \Phi_k(x, y)}{\partial n(x)} \lambda(y) \, d\Omega_y = 2g(x), \quad x \in \Omega.$$

Similarly, the vector field (see [[1], Theorem 4.22])

$$E(x) = \int_{\Omega} \Phi_k(x, y) [n(y), \mu(y)] \, d\Omega_y + grad \int_{\Omega} \Phi_k(x, y) \lambda(y) \, d\Omega_y, \quad x \in R^3 \backslash \bar{D},$$

is a solution to the exterior magnetic boundary value problem if $\mu \in H^3_{\perp,\alpha}(\Omega)$ an $\lambda \in H_\alpha(\Omega)$ solve the system of integral equations

$$\mu(x) - 2 \int_{\Omega} [n(x), [n(x), rot_x \{\Phi_k(x, y) [n(y), \mu(y)]\}]] \, d\Omega_y = -2f(x),$$

$$\lambda(x) - 2 \int_{\Omega} \Phi_k(x, y) (n(x), [n(y), \mu(y)]) \, d\Omega_y - \quad (2)$$

$$-2 \int_{\Omega} \frac{\partial \Phi_k(x, y)}{\partial n(x)} \lambda(y) \, d\Omega_y = -2g(x), \quad x \in \Omega.$$

It should be emphasized that in [2], an approximate solution of an integral equation of the fourth kind was investigated as a mixed system of Volterra integral equations of the first and second kind with a constant delay, and in [3], a numerical algorithm based on Hermite polynomials for solving the Cauchy singular integral equation in the general form is presented. Approximate methods for solving systems of integral equations arising in conjugate boundary value problems for the Helmholtz equation in both two- and three-dimensional spaces were studied in [4] and [5–8], respectively. In [9], solution methods for systems of integral equations related to conjugation problems for Maxwell's equations were examined. The Cauchy problem for systems of nonlinear Volterra-type integral equations was considered in [10], while in [11], numerical methods were proposed for solving systems of hypersingular integral equations associated with a certain class of boundary value problems for the Helmholtz equation. Despite numerous studies on approximate solutions to various boundary value problems using the method of integral equation systems, approximate solutions to electric boundary value problems for the vector Helmholtz equation using the integral equation systems (1) and (2) have not yet been investigated. The present paper aims to address this gap.

2. Justification of the collocation method for the system of integral equations (1) and (2)

It should be emphasized that the counterexample constructed by A.M. Lyapunov (see [[12], Chapter II, paragraph 7]) demonstrates that, in general, the derivative of a single-layer potential with a continuous density does not exist. However, by employing the identity

$$(n(y), [n(y), \mu(y)]) = 0, \quad \forall y \in \Omega,$$

we obtain (see [[1], proof. Theorem 2.26])

$$[n(x), rot_x\{\Phi_k(x, y)[n(y), \mu(y)]\}] =$$
$$= (n(x) - n(y), [n(y), \mu(y)])\, grad_x \Phi_k(x, y) - [n(y), \mu(y)] \frac{\partial \Phi_k(x, y)}{\partial n(x)}.$$

It is then straightforward to compute that

$$\int_\Omega [n(x), [n(x), rot_x\{\Phi_k(x, y)[n(y), \mu(y)]\}]]\, d\Omega_y =$$

$$= e_1 \left(\int_\Omega \left((n_2(x) - n_2(y))\, n_3(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} \right) + \right.\right.$$
$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) +$$
$$\left. + (n_2(x) n_2(y) + n_3(x) n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) \mu_1(y)\, d\Omega_y +$$
$$+ \int_\Omega \left((n_1(x) - n_1(y))\, n_3(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) + \right.$$

$$+\left(n_3(x)-n_3(y)\right)n_1(y)\left(n_2(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}-n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_2}\right)-$$

$$-n_2(x)\,n_1(y)\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\mu_2(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}\Bigg(\left(n_1(x)-n_1(y)\right)n_2(y)\left(n_2(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}-n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_2}\right)+$$

$$+\left(n_2(x)-n_2(y)\right)n_1(y)\left(n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_2}-n_2(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}\right)-$$

$$-n_3(x)\,n_1(y)\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\mu_3(y)\,d\Omega_y\Bigg)+$$

$$+e_2\Bigg(\int\limits_{\Omega}\Bigg(\left(n_2(x)-n_2(y)\right)n_3(y)\left(n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}\right)+$$

$$+\left(n_3(x)-n_3(y)\right)n_2(y)\left(n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}-n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)-$$

$$-n_1(x)\,n_2(y)\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\mu_1(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}\Bigg(\left(n_1(x)-n_1(y)\right)n_3(y)\left(n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}-n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)+$$

$$+\left(n_3(x)-n_3(y)\right)n_1(y)\left(n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}\right)+$$

$$+\left(n_1(x)\,n_1(y)+n_3(x)\,n_3(y)\right)\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\mu_2(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}\Bigg(\left(n_1(x)-n_1(y)\right)n_2(y)\left(n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}\right)+$$

$$+\left(n_2(x)-n_2(y)\right)n_1(y)\left(n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_3}-n_3(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)-$$

$$-n_3(x)\,n_2(y)\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\mu_3(y)\,d\Omega_y\Bigg)+$$

$$+e_3\Bigg(\int\limits_{\Omega}\Bigg(\left(n_2(x)-n_2(y)\right)n_3(y)\left(n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_2}-n_2(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)+$$

$$+\left(n_3(x)-n_3(y)\right)n_2(y)\left(n_2(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\frac{\partial\Phi_k(x,\,y)}{\partial x_2}\right)-$$

$$-n_1(x)\,n_3(y)\,\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\,\mu_1(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}\Bigg((n_1(x)-n_1(y))\,n_3(y)\left(n_2(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_2}\right)+$$

$$+(n_3(x)-n_3(y))\,n_1(y)\left(n_1(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_2}-n_2(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)-$$

$$-n_2(x)\,n_3(y)\,\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\,\mu_2(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}\Bigg((n_1(x)-n_1(y))\,n_2(y)\left(n_1(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_2}-n_2(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_1}\right)+$$

$$+(n_2(x)-n_2(y))\,n_1(y)\left(n_2(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_1}-n_1(x)\,\frac{\partial\Phi_k(x,\,y)}{\partial x_2}\right)+$$

$$+(n_1(x)\,n_1(y)+n_2(x)\,n_2(y))\,\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\Bigg)\,\mu_3(y)\,d\Omega_y\Bigg),$$

and

$$\int\limits_{\Omega}\Phi_k(x,\,y)\,(n(x)\,,[n(y)\,,\mu(y)])\,d\Omega_y+\int\limits_{\Omega}\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\lambda(y)\,d\Omega_y=$$

$$=\int\limits_{\Omega}(\,n_2(x)\,n_3(y)-n_3(x)\,n_2(y))\,\Phi_k(x,\,y)\,\mu_1(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}(\,n_3(x)\,n_1(y)-n_1(x)\,n_3(y))\,\Phi_k(x,\,y)\,\mu_2(y)\,d\Omega_y+$$

$$+\int\limits_{\Omega}(\,n_1(x)\,n_2(y)-n_2(x)\,n_1(y))\,\Phi_k(x,\,y)\,\mu_3(y)\,d\Omega_y+\int\limits_{\Omega}\frac{\partial\Phi_k(x,\,y)}{\partial n(x)}\lambda(y)\,d\Omega_y,$$

where $x=(x_1,x_2,x_3)\in\Omega$, $e_1=(1,0,0)$, $e_2=(0,1,0)$ and $e_3=(0,0,1)$. Therefore, the system of integral Equations (1) can be rewritten in operator form as

$$(I+K)\,u=2v, \tag{3}$$

and the system of integral Equations (2) as

$$(I-K)\,u=-2v, \tag{4}$$

here $u=(\mu_1,\mu_2,\mu_3,\lambda)^{\mathrm{T}}$ is the unknown vector function, $v=(f_1,f_2,f_3,g)^{\mathrm{T}}$ is a given vector function, the notation "a^{T}" denotes the transpose of the vector a, I is the identity operator and

$$K = \begin{pmatrix} K_{11} & K_{12} & K_{13} & K_{14} \\ K_{21} & K_{22} & K_{23} & K_{24} \\ K_{31} & K_{32} & K_{33} & K_{34} \\ K_{41} & K_{42} & K_{43} & K_{44} \end{pmatrix},$$

where

$$(K_{11}\mu_1)(x) = 2\int_{\Omega} \left((n_2(x) - n_2(y))\, n_3(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2}\right) + \right.$$
$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3}\right) +$$
$$\left. + (n_2(x)\, n_2(y) + n_3(x)\, n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) \mu_1(y)\, d\Omega_y,$$

$$(K_{12}\mu_2)(x) = 2\int_{\Omega} \left((n_1(x) - n_1(y))\, n_3(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3}\right) + \right.$$
$$+ (n_3(x) - n_3(y))\, n_1(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2}\right) -$$
$$\left. - n_2(x)\, n_1(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) \mu_2(y)\, d\Omega_y,$$

$$(K_{13}\mu_3)(x) = 2\int_{\Omega} \left((n_1(x) - n_1(y))\, n_2(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2}\right) + \right.$$
$$+ (n_2(x) - n_2(y))\, n_1(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3}\right) -$$
$$\left. - n_3(x)\, n_1(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) \mu_3(y)\, d\Omega_y,$$

$$(K_{14}\lambda)(x) = 0,$$

$$(K_{21}\mu_1)(x) = 2\int_{\Omega} \left((n_2(x) - n_2(y))\, n_3(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x, y)}{\partial x_3}\right) + \right.$$
$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_1(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1}\right) -$$
$$\left. - n_1(x)\, n_2(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)}\right) \mu_1(y)\, d\Omega_y,$$

$$(K_{22}\mu_2)(x) = 2\int_{\Omega} \left((n_1(x) - n_1(y))\, n_3(y) \left(n_1(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1}\right) + \right.$$

$$+\left(n_3\left(x\right)-n_3\left(y\right)\right)n_1\left(y\right)\left(n_3\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_3}\right)+$$

$$+\left(n_1\left(x\right)n_1\left(y\right)+n_3\left(x\right)n_3\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\Bigg)\mu_2\left(y\right)d\Omega_y,$$

$$\left(K_{23}\mu_3\right)\left(x\right)=2\int\limits_{\Omega}\Bigg(\left(n_1\left(x\right)-n_1\left(y\right)\right)n_2\left(y\right)\left(n_3\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_3}\right)+$$

$$+\left(n_2\left(x\right)-n_2\left(y\right)\right)n_1\left(y\right)\left(n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_3}-n_3\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}\right)-$$

$$-n_3\left(x\right)n_2\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\Bigg)\mu_3\left(y\right)d\Omega_y,$$

$$\left(K_{24}\lambda\right)\left(x\right)=0,$$

$$\left(K_{31}\mu_1\right)\left(x\right)=2\int\limits_{\Omega}\Bigg(\left(n_2\left(x\right)-n_2\left(y\right)\right)n_3\left(y\right)\left(n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}-n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}\right)+$$

$$+\left(n_3\left(x\right)-n_3\left(y\right)\right)n_2\left(y\right)\left(n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}\right)-$$

$$-n_1\left(x\right)n_3\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\Bigg)\mu_1\left(y\right)d\Omega_y,$$

$$\left(K_{32}\mu_2\right)\left(x\right)=2\int\limits_{\Omega}\Bigg(\left(n_1\left(x\right)-n_1\left(y\right)\right)n_3\left(y\right)\left(n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}\right)+$$

$$+\left(n_3\left(x\right)-n_3\left(y\right)\right)n_1\left(y\right)\left(n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}-n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}\right)-$$

$$-n_2\left(x\right)n_3\left(y\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\Bigg)\mu_2\left(y\right)d\Omega_y,$$

$$\left(K_{33}\mu_3\right)\left(x\right)=2\int\limits_{\Omega}\Bigg(\left(n_1\left(x\right)-n_1\left(y\right)\right)n_2\left(y\right)\left(n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}-n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}\right)+$$

$$+\left(n_2\left(x\right)-n_2\left(y\right)\right)n_1\left(y\right)\left(n_2\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_1}-n_1\left(x\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial x_2}\right)+$$

$$+\left(n_1\left(x\right)n_1\left(y\right)+n_2\left(x\right)n_2\left(y\right)\right)\frac{\partial\Phi_k\left(x,\,y\right)}{\partial n\left(x\right)}\Bigg)\mu_3\left(y\right)d\Omega_y,$$

$$\left(K_{34}\lambda\right)\left(x\right)=0,$$

$$\left(K_{41}\mu_1\right)\left(x\right)=2\int\limits_{\Omega}\left(n_2\left(x\right)n_3\left(y\right)-n_3\left(x\right)n_2\left(y\right)\right)\Phi_k\left(x,\,y\right)\mu_1\left(y\right)d\Omega_y,$$

$$(K_{42}\mu_2)(x) = 2\int\limits_{\Omega} (n_3(x)\, n_1(y) - n_1(x)\, n_3(y))\, \Phi_k(x, y)\, \mu_2(y)\, d\Omega_y,$$

$$(K_{43}\mu_3)(x) = 2\int\limits_{\Omega} (n_1(x)\, n_2(y) - n_2(x)\, n_1(y))\, \Phi_k(x, y)\, \mu_3(y)\, d\Omega_y,$$

$$(K_{44}\lambda)(x) = 2\int\limits_{\Omega} \frac{\partial \Phi_k(x, y)}{\partial n(x)} \lambda(y)\, d\Omega_y.$$

As can be seen, by taking into account the inequality (see [[13], Chapter V, paragraph 27, section 4])

$$|n(y) - n(x)| \leq M\, |y - x|\, , \forall x, y \in \Omega,$$

it follows that the expressions $(K_{mp}\varphi)(x)$, $x \in \Omega$, $m, p = \overline{1, 4}$, are weakly singular integrals. Therefore, the operators K_{mp}, $m, p = \overline{1, 4}$, are compact in the space $C(\Omega)$. Consequently, the solutions of the integral equation systems (3) and (4) can be studied in the broader function space $C^4(\Omega) = C(\Omega) \times C(\Omega) \times C(\Omega) \times C(\Omega)$, equipped with the norm $\|\rho\|_4 = \max\limits_{m=\overline{1,4}} \|\rho_m\|_\infty$, where $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)$.

We now provide a justification of the collocation method for the systems of integral Equations (3) and (4). To do so, we first partition the surface Ω into "regular" elementary components $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$. A "regular" elementary component is defined as a set of points satisfying the following conditions:

(1) For any $l \in \{1, 2, ..., N\}$, the elementary component Ω_l is closed, and its set of interior points relative to Ω, denoted $\overset{0}{\Omega}_l$, is non-empty. Moreover, $mes\, \overset{0}{\Omega}_l = mes\Omega_l$ and $j \in \{1, 2, ...N\}\, , j \neq l$, $\overset{0}{\Omega}_l \cap \overset{0}{\Omega}_j = \emptyset$ hold;

(2) For any $l \in \{1, 2, ..., N\}$, the elementary component Ω_l is a connected portion of the surface Ω with a continuous boundary;

(3) For any $l \in \{1, 2, ..., N\}$, there exists a so-called reference point

$$x(l) = (x_1(l)\, , x_2(l)\, , x_3(l)) \in \Omega_l,$$

such that:

(3.1) $r_l(N) \sim R_l(N)$ $(r_l(N) \sim R_l(N) \Leftrightarrow C_1 \leq \frac{r_l(N)}{R_l(N)} \leq C_2$, where C_1 and C_2 are positive constants independent of N), where $r_l(N) = \min\limits_{x \in \partial\Omega_l} |x - x(l)|$ and $R_l(N) = \max\limits_{x \in \partial\Omega_l} |x - x(l)|$;

(3.2) $R_l(N) \leq \frac{r_0}{2}$, where r_0 is the radius of the standard sphere for Ω (see [[13], Chapter V, paragraph 27, section 4]);

(3.3) $r_j(N) \sim r_l(N)$, $\forall j \in \{1, 2, ..., N\}$.

From here on we will denote by M positive constants that are different in different inequalities.

It is clear that $r(N) \sim R(N)$ and $\lim\limits_{N\to\infty} r(N) = \lim\limits_{N\to\infty} R(N) = 0$, where $R(N) = \max\limits_{l=\overline{1,\,N}} R_l(N)$ and $r(N) = \min\limits_{l=\overline{1,\,N}} r_l(N)$.

Lemma 2.1. *([14]). When the surface* Ω *is partitioned into "regular" elementary components* $\Omega = \bigcup\limits_{l=1}^{N} \Omega_l$*, the following relation holds:* $R(N) \sim \frac{1}{\sqrt{N}}$.

Let us now consider the $4N$ dimensional matrix $K^{4N} = (k_{lj})_{l,j=1}^{4N}$ with elements:

$$k_{ll} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{lj} = 2\left((n_2(x(l)) - n_2(x(j)))\, n_3(x(j)) \times \right.$$
$$\times \left(n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} - n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} \right) +$$
$$+ (n_3(x(l)) - n_3(x(j)))\, n_2(x(j)) \times$$
$$\times \left(n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} - n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} \right) +$$
$$\left. + (n_2(x(l))\, n_2(x(j)) + n_3(x(l))\, n_3(x(j))) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial n(x(l))} \right) mes\Omega_j$$
$$\text{for } l, j = \overline{1,\,N} \text{ and } j \neq l,$$

$$k_{l,N+l} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{l,N+j} = 2\left((n_1(x(l)) - n_1(x(j)))\, n_3(x(j)) \times \right.$$
$$\times \left(n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} - n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} \right) +$$
$$+ (n_3(x(l)) - n_3(x(j)))\, n_1(x(j)) \times$$
$$\times \left(n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} - n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} \right) -$$
$$\left. - n_2(x(l))\, n_1(x(j)) \tfrac{\partial \Phi_k(x(l),\, x(j))}{\partial n(x(l))} \right) mes\Omega_j \text{ for } l, j = \overline{1,\,N} \text{ and } j \neq l;$$

$$k_{l,2N+l} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{l,2N+j} = 2\left((n_1(x(l)) - n_1(x(j)))\, n_2(x(j)) \times \right.$$
$$\times \left(n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} - n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} \right) +$$
$$+ (n_2(x(l)) - n_2(x(j)))\, n_1(x(j)) \times$$
$$\times \left(n_3(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_2} - n_2(x(l)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial x_3} \right) -$$
$$\left. - n_3(x(l))\, n_1(x(j)) \frac{\partial \Phi_k(x(l),\, x(j))}{\partial n(x(l))} \right) mes\Omega_j \; for\, l, j = \overline{1,\,N}\, and\, j \neq l,$$

$$k_{l,3N+j} = 0 \quad \text{for } l, j = \overline{1,\,N},$$

$$k_{N+l,l} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{N+l,j} = 2(n_2\left(x\left(l\right)\right) - n_2\left(x\left(j\right)\right)) \times$$
$$\times \left(n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3}\right) +$$
$$\left(n_3\left(x\left(l\right)\right) - n_3\left(x\left(j\right)\right)\right) n_2\left(x\left(j\right)\right)$$
$$\times \left(n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3} - n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) -$$
$$-n_1\left(x\left(l\right)\right) n_2\left(x\left(j\right)\right) \tfrac{\partial\Phi_k\left(x(l),\, x(j)\right)}{\partial n(x(l))}\Big) mes\Omega_j \text{ for } l, j = \overline{1,\,N} \text{ and } j \neq l,$$
$$k_{N+l,N+l} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{N+l,N+j} = 2\left(\left(n_1\left(x\left(l\right)\right) - n_1\left(x\left(j\right)\right)\right) n_3\left(x\left(j\right)\right) \times\right.$$
$$\times \left(n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3} - n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) +$$
$$+ \left(n_3\left(x\left(l\right)\right) - n_3\left(x\left(j\right)\right)\right) n_1\left(x\left(j\right)\right) \times$$
$$\times \left(n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3}\right) +$$
$$\left. + \left(n_1\left(x\left(l\right)\right) n_1\left(x\left(j\right)\right) + n_3\left(x\left(l\right)\right) n_3\left(x\left(j\right)\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial n\left(x\left(l\right)\right)}\right) mes\Omega_j$$

$$\text{for } l, j = \overline{1,\,N} \text{ and } j \neq l,$$

$$k_{N+l,2N+l} = 0 \text{ for } l = \overline{1,\,N},$$

$$k_{N+l,2N+j} = 2\ \left(\left(n_1\left(x\left(l\right)\right) - n_1\left(x\left(j\right)\right)\right) n_2\left(x\left(j\right)\right) \times\right.$$
$$\times \left(n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3}\right) +$$
$$+ \left(n_2\left(x\left(l\right)\right) - n_2\left(x\left(j\right)\right)\right) n_1\left(x\left(j\right)\right) \times$$
$$\times \left(n_1\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_3} - n_3\left(x\left(l\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) -$$

$$\left. -n_3\left(x\left(l\right)\right) n_2\left(x\left(j\right)\right) \frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial n\left(x\left(l\right)\right)}\right) mes\Omega_j\, for\, l, j = \overline{1,\,N}\, and\, j \neq l;$$

$$k_{N+l,3N+j} = 0 \text{ for } l, j = \overline{1,\,N},$$

$$k_{2N+l,l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{2N+l,j} = 2\left(\left(n_2\left(x\left(l\right)\right) - n_2\left(x\left(j\right)\right)\right) n_3\left(x\left(j\right)\right) \times\right.$$
$$\times\left(n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2} - n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) +$$
$$+\left(n_3\left(x\left(l\right)\right) - n_3\left(x\left(j\right)\right)\right) n_2\left(x\left(j\right)\right) \times$$
$$\times\left(n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2}\right) -$$

$$\left.-n_1\left(x\left(l\right)\right) n_3\left(x\left(j\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial n\left(x\left(l\right)\right)}\right) mes\Omega_j \; for\, l, j = \overline{1,\, N} \, and\, j \neq l;$$

$$k_{2N+l,N+l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{2N+l,N+j} = 2\left(\left(n_1\left(x\left(l\right)\right) - n_1\left(x\left(j\right)\right)\right) n_3\left(x\left(j\right)\right) \times\right.$$
$$\times\left(n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2}\right) +$$
$$+\left(n_3\left(x\left(l\right)\right) - n_3\left(x\left(j\right)\right)\right) n_1\left(x\left(j\right)\right) \times$$
$$\times\left(n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2} - n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) -$$
$$\left.-n_2\left(x\left(l\right)\right) n_3\left(x\left(j\right)\right)\tfrac{\partial\Phi_k\left(x(l),\, x(j)\right)}{\partial n(x(l))}\right) mes\Omega_j \text{ for } l, j = \overline{1,\, N} \text{ and } j \neq l,$$
$$k_{2N+l,2N+l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{2N+l,2N+j} = 2\left(\left(n_1\left(x\left(l\right)\right) - n_1\left(x\left(j\right)\right)\right) n_2\left(x\left(j\right)\right) \times\right.$$

$$\times\left(n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2} - n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1}\right) +$$
$$+\left(n_2\left(x\left(l\right)\right) - n_2\left(x\left(j\right)\right)\right) n_1\left(x\left(j\right)\right) \times$$
$$\times\left(n_2\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_1} - n_1\left(x\left(l\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial x_2}\right) +$$
$$\left.+\left(n_1\left(x\left(l\right)\right) n_1\left(x\left(j\right)\right) + n_2\left(x\left(l\right)\right) n_2\left(x\left(j\right)\right)\right)\frac{\partial\Phi_k\left(x\left(l\right),\, x\left(j\right)\right)}{\partial n\left(x\left(l\right)\right)}\right) mes\Omega_j$$

$$\text{for } l, j = \overline{1,\, N} \text{ and } j \neq l,$$
$$k_{2N+l,3N+j} = 0 \text{ for } l, j = \overline{1,\, N},$$
$$k_{3N+l,l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{3N+l,j} = 2\left(n_2\left(x\left(l\right)\right) n_3\left(x\left(j\right)\right) - n_3\left(x\left(l\right)\right) n_2\left(x\left(j\right)\right)\right)\Phi_k\left(x\left(l\right),\, x\left(j\right)\right) mes\Omega_j$$

$$\text{for } l, j = \overline{1,\, N} \text{ and } j \neq l,$$

$$k_{3N+l,N+l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{3N+l,N+j} = 2\left(n_3\left(x\left(l\right)\right) n_1\left(x\left(j\right)\right) - n_1\left(x\left(l\right)\right) n_3\left(x\left(j\right)\right)\right) \Phi_k\left(x\left(l\right),\, x\left(j\right)\right) mes\Omega_j$$

$$\text{for } l, j = \overline{1,\, N} \text{ and } j \neq l,$$

$$k_{3N+l,2N+l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{3N+l,2N+j} = 2\left(n_1\left(x\left(l\right)\right) n_2\left(x\left(j\right)\right) - n_2\left(x\left(l\right)\right) n_1\left(x\left(j\right)\right)\right) \Phi_k\left(x\left(l\right),\, x\left(j\right)\right) mes\Omega_j$$

$$\text{for } l, j = \overline{1,\, N} \text{ and} j \neq l,$$

$$k_{3N+l,3N+l} = 0 \text{ for } l = \overline{1,\, N},$$

$$k_{3N+l,3N+j} = 2\frac{\partial\Phi_k(x(l),\, x(j))}{\partial n(x(l))} mes\Omega_j \text{ for } l, j = \overline{1,\, N} \text{ and } j \neq l.$$

Proceeding in the same manner as in [15] and taking into account Lemma 2.1, it is not difficult to show that the expressions

$$\left(K_{mp}^{N}\varphi\right)\left(x\left(l\right)\right) = \sum_{j=1}^{N} k_{(m-1)N+l,\,(p-1)N+j}\, \varphi\left(x\left(j\right)\right), \quad m, p = \overline{1,\, 4},$$

evaluated at the reference points $x\left(l\right)$, $l = \overline{1,\, N}$, represent cubature formulas for the integrals $\left(K_{mp}\varphi\right)(x)$, $m, p = \overline{1,\, 4}$, respectively, and moreover,

$$\max_{l=\overline{1,\, N}} \left|\left(K_{mp}\varphi\right)\left(x\left(l\right)\right) - \left(K_{mp}^{N}\varphi\right)\left(x\left(l\right)\right)\right| \leq M\left(\left\|\varphi\right\|_{\infty} N^{-\frac{1}{2}} \ln N + \omega\left(\varphi, N^{-\frac{1}{2}}\right)\right),$$

where $\omega\left(\varphi, \delta\right)$ is the modulus of continuity of the function $\varphi \in C\left(\Omega\right)$, i.e.

$$\omega\left(\varphi, \delta\right) = \max_{\substack{|x-y| \leq \delta \\ x,\, y \in \Omega}} \left|\varphi(x) - \varphi(y)\right|, \quad \delta > 0.$$

Let C^{4N} be the space of vectors

$$z^{4N} = \left(z_1^{4N}, z_2^{4N}, ..., z_{4N}^{4N}\right)^{\mathrm{T}}, \quad z_l^{4N} \in C, \quad l = \overline{1,\, 4N},$$

equipped with the norm $\left\|z^{4N}\right\| = \max\limits_{l=\overline{1,\, 4N}} \left|z_l^{4N}\right|$, and let I^{4N} denote the identity operator in the space C^{4N}. Then, if we denote by z_l^{4N}, $l = \overline{1,\, N}$, the approximate values of $\mu_1\left(x\left(l\right)\right)$; by z_{N+l}^{4N}, $l = \overline{1,\, N}$, the approximate values of $\mu_2\left(x\left(l\right)\right)$; by z_{2N+l}^{4N}, $l = \overline{1,\, N}$, the approximate values of $\mu_3\left(x\left(l\right)\right)$; and by z_{3N+l}^{4N}, $l = \overline{1,\, N}$, the approximate values of $\lambda\left(x\left(l\right)\right)$, the systems of integral Equations (3) and (4) are reduced to systems of algebraic equations with respect to $z^{4N} \in C^{4N}$, which we write in the form:

$$\left(I^{4N} + K^{4N}\right) z^{4N} = 2v^{4N}, \tag{5}$$

and

$$\left(I^{4N} - K^{4N}\right) z^{4N} = -2v^{4N}, \tag{6}$$

respectively, where $v^{4N} = p^{4N}v$, and $p^{4N} : C^4(\Omega) \to C^{4N}$ is a bounded linear operator defined by

$$p^{4N}v = (f_1(x(1)), ..., f_1(x(N)), f_2(x(1)), ..., f_2(x(N))$$
$$f_3(x(1)), ..., f_3(x(N)), g(x(1)), ..., g(x(N)))^{\mathrm{T}}.$$

Theorem 2.2. *Let $Imk > 0$ and $v \in C^4(\Omega)$. Then the systems of Equations (3) and (5) have unique solutions $u_* = (\mu_1^*, \mu_2^*, \mu_3^*, \lambda^*)^{\mathrm{T}} \in C^4(\Omega)$ and $w^{4N} \in C^{4N}$, respectively, and the following estimate holds:*

$$\left\| w^{4N} - p^{4N}u_* \right\| \le M\left(\omega\left(v, N^{-\frac{1}{2}}\right) + \|v\|_4 N^{-\frac{1}{2}} \ln N\right).$$

Proof. To prove the theorem, we use G.M. Vainikko's theorem on the convergence of linear operator equations (see [16]). Let us verify the conditions of Theorem 4.2 from [16], using the notations, definitions, and propositions provided therein. It is shown in [[1], formula (4.64)] that if k is not an eigenvalue of the electric or magnetic boundary value problem for the vector Helmholtz equations, then operators $I + K$ and $I - K$ are invertible, i.e. in this case $Ker\ (I \pm K) = \{0\}$. Moreover, [[1], Theorem 4.16] proves that if $Imk > 0$, then the electric and magnetic boundary value problems admit at most one solution, i.e., if $Imk > 0$, then the homogeneous electric and magnetic boundary value problems have only the trivial solution. Hence, if $Imk > 0$, then $Ker\ (I + K) = \{0\}$. It is clear that the operators $I^{4N} + K^{4N}$ are Fredholm operators of index zero. Taking into account the method of partitioning the surface Ω into "regular" elementary components, it follows that for any vector function $\rho = (\rho_1, \rho_2, \rho_3, \rho_4)^{\mathrm{T}} \in C^4(\Omega)$, the identity

$$\lim_{N\to\infty} \left\| p^{4N}\rho \right\| = \lim_{N\to\infty} \max_{m=\overline{1,4}} \left\{ \max_{l=\overline{1,N}} |\rho_m(x(l))| \right\} = \max_{m=\overline{1,4}} \left\{ \max_{x\in\Omega} |\rho_m(x)| \right\} = \|\rho\|_4,$$

holds. Therefore, the operator system $P = \left\{p^{4N}\right\}$ is linking the spaces $C^4(\Omega)$ and C^{4N}. Then $v^{4N} \xrightarrow{P} v$, and by Definition 2.1 in [16], $I^{4N} + K^{4N} \xrightarrow{PP} I + K$. Since, by Definition 3.2 in [16], $I^{4N} \to I$ is stable, it remains, by Proposition 3.5 and Definition 3.3 in [16], to verify the compactness condition. According to Proposition 1.1 from [16], this is equivalent to the existence of a relatively compact sequence $\left\{K_{4N} z^{4N}\right\} \subset C^4(\Omega)$ such that

$$\left\| K^{4N} z^{4N} - p^{4N}\left(K_{4N} z^{4N}\right) \right\| \to 0 \text{ as } N \to \infty,$$

given that $\forall \left\{ z^{4N} \right\}$, $z^{4N} \in C^{4N}$ and $\left\| z^{4N} \right\| \leq M$. As $\left\{ K_{4N} z^{4N} \right\}$, we choose the sequence

$$\left(K_{4N} z^{4N} \right)(x) = \left(\sum_{j=1}^{4N} k_j^{(1)}(x)\, z_j^{4N},\ \sum_{j=1}^{4N} k_j^{(2)}(x)\, z_j^{4N},\ \sum_{j=1}^{4N} k_j^{(3)}(x)\, z_j^{4N},\ \sum_{j=1}^{4N} k_j^{(4)}(x)\, z_j^{4N} \right)^{\mathrm{T}},$$

where

$$k_j^{(1)}(x) = 2\int\limits_{\Omega_j} \left((n_2(x) - n_2(y))\, n_3(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) +$$

$$\left. + (n_2(x)\, n_2(y) + n_3(x)\, n_3(y)) \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) d\Omega_y \; for\; j = \overline{1, N},$$

$$k_j^{(1)}(x) = \int\limits_{\Omega_{j-N}} \left((n_1(x) - n_1(y))\, n_3(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_1(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} \right) -$$

$$\left. - n_2(x)\, n_1(y) \tfrac{\partial \Phi_k(x, y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{N, 2N},$$

$$k_j^{(1)}(x) = 2\int\limits_{\Omega_{j-2N}} \left((n_1(x) - n_1(y))\, n_2(y) \left(n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} \right) + \right.$$

$$+ (n_2(x) - n_2(y))\, n_1(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) -$$

$$\left. - n_3(x)\, n_1(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) d\Omega_y \; for\; j = \overline{2N, 3N},$$

$$k_j^{(1)}(x) = 0 \text{ for } j = \overline{3N, 4N},$$

$$k_j^{(2)}(x) = 2\int\limits_{\Omega_j} \left((n_2(x) - n_2(y))\, n_3(y) \left(n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_1(x) \frac{\partial \Phi_k(x, y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x, y)}{\partial x_1} \right) -$$

$$\left. - n_1(x)\, n_2(y) \frac{\partial \Phi_k(x, y)}{\partial n(x)} \right) d\Omega_y \; for\; j = \overline{1, N},$$

$$k_j^{(2)}(x) = 2\int\limits_{\Omega_{j-N}} \left((n_1(x) - n_1(y))\, n_3(y) \left(n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_1(y) \left(n_3(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_3} \right) +$$

$$\left. + (n_1(x)\, n_1(y) + n_3(x)\, n_3(y))\, \tfrac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{N,\, 2N},$$

$$k_j^{(2)}(x) = 2\int\limits_{\Omega_{j-2N}} \left((n_1(x) - n_1(y))\, n_2(y) \left(n_3(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_3} \right) + \right.$$

$$+ (n_2(x) - n_2(y))\, n_1(y) \left(n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_3} - n_3(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} \right) -$$

$$\left. - n_3(x)\, n_2(y) \frac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y\, for\, j = \overline{2N,\, 3N},$$

$$k_j^{(2)}(x) = 0 \text{ for } j = \overline{3N,\, 4N},$$

$$k_j^{(3)}(x) = 2\int\limits_{\Omega_j} \left((n_2(x) - n_2(y))\, n_3(y) \left(n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_2(y) \left(n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} \right) -$$

$$\left. - n_1(x)\, n_3(y)\, \tfrac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{1,\, N},$$

$$k_j^{(3)}(x) = 2\int\limits_{\Omega_{j-N}} \left((n_1(x) - n_1(y))\, n_3(y) \left(n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} \right) + \right.$$

$$+ (n_3(x) - n_3(y))\, n_1(y) \left(n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} \right) -$$

$$\left. - n_2(x)\, n_3(y)\, \tfrac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{N,\, 2N},$$

$$k_j^{(3)}(x) = 2\int\limits_{\Omega_{j-2N}} \left((n_1(x) - n_1(y))\, n_2(y) \left(n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} - n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} \right) + \right.$$

$$+ (n_2(x) - n_2(y))\, n_1(y) \left(n_2(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_1} - n_1(x) \frac{\partial \Phi_k(x,\,y)}{\partial x_2} \right) +$$

$$\left. + (n_1(x)\, n_1(y) + n_2(x)\, n_2(y))\, \tfrac{\partial \Phi_k(x,\,y)}{\partial n(x)} \right) d\Omega_y \text{ for } j = \overline{2N,\, 3N};$$

$$k_j^{(3)}(x) = 0 \quad \text{for } j = \overline{3N\,,\,4N},$$
$$k_j^{(4)}(x) = 2 \int\limits_{\Omega_j} \left(n_2(x)\, n_3(y) - n_3(x)\, n_2(y) \right) \Phi_k(x,\,y)\, d\Omega_y \quad \text{for } j = \overline{1,\,N},$$
$$k_j^{(4)}(x) = 2 \int\limits_{\Omega_{j-N}} \left(n_3(x)\, n_1(y) - n_1(x)\, n_3(y) \right) \Phi_k(x,\,y)\, d\Omega_y \quad \text{for } j = \overline{N,\,2N},$$
$$k_j^{(4)}(x) = 2 \int\limits_{\Omega_{j-2N}} \left(n_1(x)\, n_2(y) - n_2(x)\, n_1(y) \right) \Phi_k(x,\,y)\, d\Omega_y \quad \text{for } j = \overline{2N,\,3N},$$
$$k_j^{(4)}(x) = 2 \int\limits_{\Omega_{j-3N}} \frac{\partial \Phi_k(x,y)}{\partial n(x)} d\Omega_y \text{ for } j = \overline{3N\,,\,4N}.$$

As can be seen, the expressions $k_j^{(m)}(x)$, $j = \overline{1,\,4N}$, $m = \overline{1,\,4}$, are weakly singular integrals. Therefore,

$$\left| \sum_{j=1}^{4N} k_j^{(m)}(x)\; z_j^{4N} \right| \le M \left\| z^{4N} \right\|, \quad \forall\, x \in \Omega, \quad m = \overline{1,\,4}.$$

Moreover, following the approach in [17], it is not difficult to show that

$$\left| \sum_{j=1}^{4N} k_j^{(m)}(x')\; z_j^{4N} - \sum_{j=1}^{4N} k_j^{(m)}(x'')\; z_j^{4N} \right| \le$$

$$\le M \left\| z^{4N} \right\| \left| x' - x'' \right| \left| \ln\left| x' - x'' \right| \right|, \quad \forall\, x',\, x'' \in \Omega, \quad m = \overline{1,\,4}.$$

Hence,

$$\left| \left(K_{4N}\, z^{4N} \right)(x) \right| \le M \left\| z^{4N} \right\|, \quad \forall\, x \in \Omega,$$

and

$$\left| \left(K_{4N}\, z^{4N} \right)(x') - \left(K_{4N}\, z^{4N} \right)(x'') \right| \le M \left\| z^{4N} \right\| \left| x' - x'' \right| \left| \ln\left| x' - x'' \right| \right|, \forall\, x',\, x'' \in \Omega.$$

Therefore, $\left\{ K_{4N}\, z^{4N} \right\} \subset C^4(\Omega)$, and taking into account condition $\left\| z^{4N} \right\| \le M$, we obtain the uniform boundedness and equicontinuity of the sequence $\left\{ K_{4N}\, z^{4N} \right\}$. Then, by the Arzelà–Ascoli theorem, the sequence $\left\{ K_{4N}\, z^{4N} \right\}$ is relatively compact. In addition, by proceeding as in [15], one can show that

$$\left\| K^{4N}\, z^{4N} - p^{4N} \left(\; K_{4N}\, z^{4N} \right) \right\| \to 0 \text{ as } N \to \infty.$$

As a result, applying Theorem 4.2 from [16], we find that Equations (3) and (5) have unique solutions $u_* = \left(\mu_1^*, \mu_2^*, \mu_3^*, \lambda^* \right)^{\mathrm{T}} \in C^4(\Omega)$ and $w^{4N} \in C^{4N}$, respectively, and

$$c_1\, \delta_N \le \left\| w^{4N} - p^{4N} u_* \right\| \le \, c_2\, \delta_N,$$

where

$$c_1 \;= 1/ \sup_{N \ge N_0} \left\| I^{4N} + K^{4N} \right\| > 0,\; c_2 \;= \sup_{N \ge N_0} \left\| \left(I^{4N} + K^{4N} \right)^{-1} \right\| < +\infty,$$

$$\delta_N = \left\| \left(I^{4N} + K^{4N} \right) \left(p^{4N} u_* \right) - 2v^{4N} \right\|.$$

Since

$$2v^{4N} = 2p^{4N} v = p^{4N} \left(u_* + K u_* \right) = p^{4N} u_* + p^{4N} \left(K u_* \right),$$

then taking into account the error estimates of the cubature formulas for the integrals $(K_{mp}\varphi)(x)$, $m, p = \overline{1, 4}$, we obtain

$$\delta_N = \left\| K^{4N} \left(p^{4N} u_* \right) - p^{4N} \left(K u_* \right) \right\| \le M \left(\left\| u_* \right\|_4 N^{-\frac{1}{2}} \ln N + \omega \left(u_*, N^{-\frac{1}{2}} \right) \right).$$

Moreover, following the same approach as in [17], one can show that

$$\omega \, \left(K u_*, \, \delta \right) \le M \left\| \, u_* \right\|_4 \, \delta \, \left| \ln \delta \right|.$$

Then, taking into account the inequalities

$$\omega \left(u_*, \, N^{-\frac{1}{2}} \right) = \omega \left(2v - K u_*, \, N^{-\frac{1}{2}} \right) \le \omega \left(2v, \, N^{-\frac{1}{2}} \right) + \omega \left(K u_*, \, N^{-\frac{1}{2}} \right) \le$$

$$\le M \left(\omega \left(v, \, N^{-\frac{1}{2}} \right) + \left\| \, u_* \right\|_4 N^{-\frac{1}{2}} \ln N \right), \tag{7}$$

and

$$\left\| \, u_* \, \right\|_4 = \left\| 2 \left(I + K \right)^{-1} v \, \right\|_4 \le 2 \left\| \, \left(I + K \right)^{-1} \right\| \, \left\| \, v \, \right\|_4, \tag{8}$$

we obtain

$$\delta_N \le M \left(\omega \left(v, N^{-\frac{1}{2}} \right) + \left\| v \right\|_4 N^{-\frac{1}{2}} \ln N \right).$$

The theorem is proven. □

In a similar way, the following theorem can be proven.

Theorem 2.3. *Let* $Imk > 0$ *and* $v \in C^4(\Omega)$. *Then, the systems of Equations (4) and (6) have unique solutions* $\tilde{u}_* = \left(\tilde{\mu}_1^*, \tilde{\mu}_2^*, \tilde{\mu}_3^*, \tilde{\lambda}^* \right)^{\mathrm{T}} \in C^4(\Omega)$ *and* $\tilde{w}^{4N} \in C^{4N}$, *respectively, and the following estimate holds:*

$$\left\| \, \tilde{w}^{4N} - p^{4N} \tilde{u}_* \right\| \le M \left(\omega \left(v, N^{-\frac{1}{2}} \right) + \left\| v \right\|_4 N^{-\frac{1}{2}} \ln N \right).$$

3. On the approximate solutions of magnetic boundary value problems

In this section, sequences are constructed that converge to the exact solutions of the interior and exterior magnetic boundary value problems for the vector Helmholtz equation.

Theorem 3.1. *Let* $x_* \in D, Imk > 0, g \in H_\alpha(\Omega), f \in H^3_{\perp,\alpha}(\Omega)$*, and* $w^{4N} = \left(w_1^{4N}, w_2^{4N}, \ldots, w_{4N}^{4N}\right)$ *be the solution of the system of algebraic Equations (5). Then the sequence*

$$E^N(x_*) = e_1 E_1^N(x_*) + e_2 E_2^N(x_*) + e_3 E_3^N(x_*) =$$

$$= e_1 \sum_{j=1}^{N} \left(\left(n_2(x(j))\, w_{2N+j}^{4N} - n_3(x(j))\, w_{N+j}^{4N} \right) \Phi_k(x_*, x(j)) + \right.$$

$$\left. + \frac{\partial \Phi_k(x_*, x(j))}{\partial x_1} w_{3N+j}^{4N} \right) mes\Omega_j +$$

$$+ e_2 \sum_{j=1}^{N} \left(\left(n_3(x(j))\, w_j^{4N} - n_1(x(j))\, w_{2N+j}^{4N} \right) \Phi_k(x_*, x(j)) + \right.$$

$$\left. + \frac{\partial \Phi_k(x_*, x(j))}{\partial x_2} w_{3N+j}^{4N} \right) mes\Omega_j +$$

$$+ e_3 \sum_{j=1}^{N} \left(\left(n_1(x(j))\, w_{N+j}^{4N} - n_2(x(j))\, w_j^{4N} \right) \Phi_k(x_*, x(j)) + \frac{\partial \Phi_k(x_*, x(j))}{\partial x_3} w_{3N+j}^{4N} \right) mes\Omega_j,$$

converges to the value at point x_* *of the solution* $E(x)$ *of the interior magnetic boundary value problem, and*

$$\left| E(x_*) - E^N(x_*) \right| \le M \left(N^{-\frac{\alpha}{2}} + N^{-\frac{1}{2}} \ln N \right).$$

Proof. It is known that if the vector function $u_* = (\mu_1^*, \mu_2^*, \mu_3^*, \lambda^*)^{\mathrm{T}}$ is a solution to the system of integral Equations (3), then the vector field

$$E(x) = \int_\Omega \Phi_k(x, y)\, [n(y), \mu^*(y)]\, d\Omega_y + grad \int_\Omega \Phi_k(x, y)\, \lambda^*(y)\, d\Omega_y, \quad x \in D,$$

is the solution of the interior magnetic boundary value problem, where $\mu^* = (\mu_1^*, \mu_2^*, \mu_3^*)$.

Since $x_* \notin \Omega$, it is easy to compute that

$$E(x_*) = e_1 E_1(x_*) + e_2 E_2(x_*) + e_3 E_3(x_*) =$$

$$= e_1 \int_\Omega \left(\left(n_2(y)\, \mu_3^*(y) - n_3(y)\, \mu_2^*(y) \right) \Phi_k(x_*, y) + \frac{\partial \Phi_k(x_*, y)}{\partial x_1} \lambda^*(y) \right) d\Omega_y +$$

$$+ e_2 \int_\Omega \left(\left(n_3(y)\, \mu_1^*(y) - n_1(y)\, \mu_3^*(y) \right) \Phi_k(x_*, y) + \frac{\partial \Phi_k(x_*, y)}{\partial x_2} \lambda^*(y) \right) d\Omega_y +$$

$$+e_3 \int\limits_{\Omega} \left(\left(n_1(y)\, \mu_2^*(y) - n_2(y)\, \mu_1^*(y) \right) \Phi_k(x_*, y) + \frac{\partial \Phi_k(x_*, y)}{\partial x_3} \lambda^*(y) \right) d\Omega_y, \; x \in D.$$

As can be seen, it suffices to show that the sequence $E_1^N(x_*)$ converges to $E_1(x_*)$. Clearly,

$$E_1(x_*) - E_1^N(x_*) = \sum_{j=1}^{N} \int\limits_{\Omega_j} \left(\frac{\partial \Phi_k(x_*, y)}{\partial x_1} - \frac{\partial \Phi_k(x_*, x(j))}{\partial x_1} \right) \lambda^*(y)\, d\Omega_y +$$

$$+ \sum_{j=1}^{N} \int\limits_{\Omega_j} \left(\Phi_k(x_*, y) - \Phi_k(x_*, x(j)) \right) \left(n_2(y)\, \mu_3^*(y) - n_3(y)\, \mu_2^*(y) \right) d\Omega_y +$$

$$+ \sum_{j=1}^{N} \int\limits_{\Omega_j} \left(\lambda^*(y) - w_{3N+j}^{4N} \right) \frac{\partial \Phi_k(x_*, x(j))}{\partial x_1} d\Omega_y +$$

$$+ \sum_{j=1}^{N} \int\limits_{\Omega_j} \left(n_2(y)\, \mu_3^*(y) - n_3(y)\, \mu_2^*(y) - n_2(x(j))\, w_{2N+j}^{4N} + n_3(x(j))\, w_{N+j}^{4N} \right) \times$$

$$\times \Phi_k(x_*, x(j))\, d\Omega_y.$$

Let the terms on the right-hand side of this last equality be denoted by h_1^N, h_2^N, h_3^N and h_4^N, respectively.

Taking into account that $x_* \notin \Omega$, we have:

$$\left| \frac{\partial \Phi_k(x_*, y)}{\partial x_1} - \frac{\partial \Phi_k(x_*, x(j))}{\partial x_1} \right| \leq M R(N), \quad \forall y \in \Omega_j.$$

Then, considering Lemma 2.1 and inequality (8), we obtain

$$\left| h_1^N \right| \leq \sum_{j=1}^{N} \int\limits_{\Omega_j} \left| \frac{\partial \Phi_k(x_*, y)}{\partial x_1} - \frac{\partial \Phi_k(x_*, x(j))}{\partial x_1} \right| \left| \lambda^*(y) \right| d\Omega_y \leq$$

$$\leq M R(N) \int\limits_{\Omega} \left| \lambda^*(y) \right| d\Omega_y \leq M \left\| \lambda^* \right\|_{\infty} R(N) \leq \frac{M}{\sqrt{N}}.$$

Similarly, it can be shown that

$$\left| h_2^N \right| \leq \frac{M}{\sqrt{N}}.$$

Since $g \in H_{\alpha}(\Omega)$ and $f \in H_{\alpha}^3(\Omega)$, then, taking into account Theorem 2.2, inequalities (7) and (8), and Lemma 2.1, we find

$$\left| \lambda(y) - w_{3N+j}^{4N} \right| \leq \left| \lambda(y) - \lambda(x(j)) \right| + \left| \lambda(x(j)) - w_{3N+j}^{4N} \right| \leq \omega\left(\mu_3^*, R(N) \right) +$$

$$+M\left(\omega\left(v,N^{-\frac{1}{2}}\right)+\|v\|_4\,N^{-\frac{1}{2}}\ln N\right)\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right),\quad \forall y\in\Omega_j.$$

It follows that

$$\left|h_3^N\right|\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right)\int\limits_{\Omega}\left|\frac{\partial\Phi_k\left(x_*,\,x\left(j\right)\right)}{\partial x_1}\right|d\Omega_y\le$$

$$\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right).$$

Moreover, also taking into account conditions $g\in H_\alpha\left(\Omega\right)$ and $f\in H_\alpha^3\left(\Omega\right)$, as well as Theorem 2.2, inequalities (7) and (8), and Lemma 2.1, we obtain

$$\left|n_2\left(y\right)\mu_3^*\left(y\right)-n_3\left(y\right)\mu_2^*\left(y\right)-n_2\left(\,x\left(j\right)\right)w_{2N+j}^{4N}+n_3\left(\,x\left(j\right)\right)w_{N+j}^{4N}\right|\le$$

$$\le\left|\left(n_2\left(y\right)-n_2\left(\,x\left(j\right)\right)\right)\mu_3^*\left(y\right)\right|+\left|\left(\mu_3^*\left(y\right)-\mu_3^*\left(x\left(j\right)\right)\right)n_2\left(\,x\left(j\right)\right)\right|+$$

$$+\left|\left(\mu_3^*\left(x\left(j\right)\right)-w_{2N+j}^{4N}\right)n_2\left(\,x\left(j\right)\right)\right|+\left|\left(n_3\left(\,x\left(j\right)\right)-n_3\left(y\right)\right)\mu_2^*\left(y\right)\right|+$$

$$+\left|\left(w_{N+j}^{4N}-\mu_2^*\left(x\left(j\right)\right)\right)n_3\left(\,x\left(j\right)\right)\right|+\left|\left(\mu_2^*\left(x\left(j\right)\right)-\mu_2^*\left(y\right)\right)n_3\left(\,x\left(j\right)\right)\right|\le$$

$$\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right).$$

Hence,

$$\left|h_4^N\right|\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right)\int\limits_{\Omega}\left|\Phi_k\left(x_*,\,x\left(j\right)\right)\right|d\Omega_y\le$$

$$\le M\left(N^{-\frac{\alpha}{2}}+N^{-\frac{1}{2}}\ln N\right).$$

As a result, summing the estimates obtained for the expressions h_1^N, h_2^N, h_3^N and h_4^N, we obtain

$$\left|E_1\left(x_*\right)-E_1^N\left(x_*\right)\right|\le M\left(N^{-\frac{\beta}{2}}+N^{-\frac{1}{2}}\ln N\right).$$

The theorem is proven. □

Similarly, one can prove the validity of the following

Theorem 3.2. *Let $x^*\in R^3\backslash\bar{D}, Imk>0, g\in H_\alpha\left(\Omega\right), f\in H_{\perp,\alpha}^3\left(\Omega\right)$ and $\tilde{w}^{4N}=\left(\tilde{w}_1^{4N},\tilde{w}_2^{4N},\ldots,\tilde{w}_{4N}^{4N}\right)^{\mathrm{T}}$ be the solution of the system of algebraic Equations (6). Then the sequence*

$$\tilde{E}^N\left(x^*\right)=e_1\tilde{E}_1^N\left(x^*\right)+e_2\tilde{E}_2^N\left(x^*\right)+e_3\tilde{E}_3^N\left(x^*\right)=$$

$$=e_1\sum_{j=1}^{N}\left(\left(n_2\left(\,x\left(j\right)\right)w_{2N+j}^{4N}-n_3\left(\,x\left(j\right)\right)w_{N+j}^{4N}\right)\Phi_k\left(x^*,\,x\left(j\right)\right)+\right.$$

$$+\frac{\partial \Phi_k\left(x^*,\, x\left(j\right)\right)}{\partial x_1} w_{3N+j}^{4N}\Big) \, mes\Omega_j +$$

$$+e_2 \sum_{j=1}^{N} \Big(\big(n_3\left(\, x\left(j\right)\right) w_j^{4N} - n_1\left(\, x\left(j\right)\right) w_{2N+j}^{4N} \big) \, \Phi_k\left(x^*,\, x\left(j\right)\right) +$$

$$+\frac{\partial \Phi_k\left(x^*,\, x\left(j\right)\right)}{\partial x_2} w_{3N+j}^{4N}\Big) \, mes\Omega_j +$$

$$+e_3 \sum_{j=1}^{N} \Big(\big(n_1\left(\, x\left(j\right)\right) w_{N+j}^{4N} - n_2\left(\, x\left(j\right)\right) w_j^{4N} \big) \, \Phi_k\left(x^*,\, x\left(j\right)\right) +$$

$$+\frac{\partial \Phi_k\left(x^*,\, x\left(j\right)\right)}{\partial x_3} w_{3N+j}^{4N}\Big) \, mes\Omega_j ,$$

converges to the value at point x^* *of the solution* $\tilde{E}\left(x\right)$ *of the exterior magnetic boundary value problem, and*

$$\left| \tilde{E}\left(x^*\right) - \tilde{E}^N\left(x^*\right) \right| \leq M \, \left(N^{-\frac{\alpha}{2}} + N^{-\frac{1}{2}} \ln N \right).$$

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