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The Intrinsic Beauty, Harmony and Interdisciplinarity in Einstein Velocity Addition Law: Gyrogroups and Gyrovector Spaces

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Abstract

The only justification for the Einstein velocity addition law appeared to be its empirical adequacy, so that the intrinsic beauty and harmony in Einstein addition remained for a long time a mystery to be conquered. Accordingly, the aim of this expository article is to present (i) the Einstein relativistic vector addition, (ii) the resulting Einstein scalar multiplication, (iii) the Einstein relativistic mass, and (iv) the Einstein relativistic kinetic energy, along with remarkable analogies with classical results in groups and vector spaces that these Einstein concepts capture in gyrogroups and gyrovector spaces. Making the unfamiliar familiar, these analogies uncover the intrinsic beauty and harmony in the underlying Einstein velocity addition law of relativistically admissible velocities, as well as its interdisciplinarity.

Keywords: Einstein addition, gyrogroup, gyrovector space, hyperbolic geometry, special relativity.

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1. Introduction

A major obstacle to the widespread adoption of hyperbolic geometry is its complexity, which contrasts the simplicity of Euclidean geometry. Hence, the mere mention of hyperbolic geometry is enough to strike fear in the heart of the undergraduate mathematics and physics student. Some regard themselves as excluded from the profound insights of hyperbolic geometry so that this enormous portion of

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human achievement is a closed door to them. However, Einstein velocity addition law of relativistically admissible velocities opens that door, making the hyperbolic geometry of Lobachevsky and Bolyai accessible to a wider audience in terms of novel analogies that the modern and unknown share with the classical and familiar.

Einstein velocity addition law gives rise to a binary operation in the ball of relativistically admissible velocities, called *Einstein addition*. The intrinsic beauty and harmony in Einstein addition has several features, one of which is the gyrogroup isomorphism relation it shares with Möbius addition that results from the Möbius transformation of the complex disk. Einstein introduced the relativistic velocity addition law in his 1905 paper [17] that founded the special relativity theory. The only justification for the Einstein velocity addition law appeared to be its empirical adequacy, so that the intrinsic beauty and harmony in Einstein addition remained for a long time a mystery to be conquered. The discovery of the intrinsic beauty and harmony in Einstein addition is an ongoing process initiated in 1988 by the discovery of the parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces of signature $(1, n)$, $n \in \mathbb{N}$ in [54], resulting in many articles as well as seven related books [60,64,67,69,71,72,79], [39,40,83]. Recently, the parametric realization of Lorentz groups has been extended to pseudo-Euclidean spaces of any signature (m, n), $m, n \in \mathbb{N}$ in [80].

Most texts on special relativity, with a few outstanding exceptions including [3], [24], and [41,42], present the Einstein velocity addition only for parallel velocities. In this simplified special case Einstein velocity addition is both commutative and associative. In general, however, Einstein addition of velocities that need not be parallel is neither commutative nor associative.

Einstein velocity addition law gives rise to a binary operation $\oplus$, called Einstein addition, in the ball of all relativistically admissible velocities. Einstein addition, in turn, gives rise to the Einstein scalar multiplication $\otimes$. Einstein addition and scalar multiplication give rise to hyperbolic vector spaces called *gyrovector spaces*. Applications of gyrogroups and gyrovector spaces are presented in many publications as, for instance, in [60, 64, 67, 69, 71, 72, 79] and in [4–7, 38, 44], [15, 16], [19], [20–23], [48–52], [27], [37], [46], [81], [29] and [1, 25]. Evidently, gyrovector spaces form the algebraic setting for analytic hyperbolic geometry, just as vector spaces form the algebraic setting for analytic Euclidean geometry.

One of the remarkable analogies that Einstein scalar multiplication captures in Einstein's special theory of relativity is the novel analogy that classical and relativistic kinetic energy share, presented in Sect. 11. This analogy, in turn, augments the standard analogies that the classical, Newtonian mass shares with the Einstein relativistic, velocity dependent mass [73]. Being noncommutative and nonassociative, initially Einstein addition was viewed as a structureless binary operation. The subsequent discovery of the rich gyrostructure and the interdisciplinarity that Einstein addition possesses results in the emergence of intrinsic beauty and harmony that underlies Einstein addition, as evidenced from this article.

2. Einstein Addition

Let $c > 0$ be an arbitrarily fixed positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n \in \mathbb{N}$, equipped with the common vector addition, $+$, and inner product, $\cdot$. The home of all n-dimensional Einsteinian velocities is the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{1}$$

of its ambient space $\mathbb{R}^n$. The c-ball $\mathbb{R}^n_c$ is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ given by the equation [60], [42, Eq. 2.9.2], [34, p. 55], [24],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_\mathbf{u}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_\mathbf{u}}{1+\gamma_\mathbf{u}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\}, \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Here $\gamma_\mathbf{v}$ is the Lorentz gamma factor given by the equation

$$\gamma_\mathbf{v} = \frac{1}{\sqrt{1-\frac{\|\mathbf{v}\|^2}{c^2}}}, \tag{3}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its ambient space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its ambient space $\mathbb{R}^n$, as we see from (1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (2) and (3).

When the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{R}^n_c$ of $\mathbb{R}^n$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, Einstein addition (2) specializes to the Einstein addition of parallel velocities,

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1+\frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}}, \qquad \mathbf{u}\|\mathbf{v}, \tag{4}$$

which was partially confirmed experimentally by the Fizeau's 1851 experiment [33].

The *restricted Einstein addition* in (4) is both commutative and associative. Accordingly, the restricted Einstein addition is a commutative group operation, as Einstein noted in [17]; see [18, p. 142]. In contrast, Einstein made no remark about group properties of his addition (2) of velocities that need not be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure discovered more than 80 years later, in 1988 [54, 55, 58], formally defined in Sect. 4.

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. The constant c represents in physical applications the vacuum speed of light. Since we are interested in both physics and geometry, we allow n to be any positive integer.

Einstein addition (2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [17] [18, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [17] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2) of Einstein addition. Einstein was aware of the nonassociativity of his velocity addition law of relativistically admissible velocities that need not be collinear. He therefore emphasized in his 1905 paper that his velocity addition law of relativistically admissible *collinear velocities* forms a group operation [17, p. 907].

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$ and

$$\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}\,. \tag{5}$$

Einstein addition and subtraction satisfy the equations

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{6}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{7}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (6) is called the *gyroautomorphic inverse property* of Einstein addition, and Identity (7) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (7) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u}\,. \tag{8}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is naturally remedied in (29) − (30), p. 14.

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

A frequently used identity that follows immediately from (3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2}\,. \tag{10}$$

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|\,, \tag{11}$$

in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}\,, \tag{12}$$

$\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$. Moreover, Einstein addition is also nonassociative since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\,, \tag{13}$$

$\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$.

As an application of the gamma identity (9), we prove the Einstein gyrotriangle inequality (14).

Theorem 2.1. (The Gyrotriangle Inequality).

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{14}$$

for all $\mathbf{u},\mathbf{v}$ *in an Einstein gyrogroup* $(\mathbb{R}^n_c,\oplus)$.

Proof. By the gamma identity (9) with $\mathbf{u}$ and $\mathbf{v}$ replaced by $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$, and by the Cauchy-Schwarz inequality [32], we have

$$\begin{aligned}
\gamma_{\|\mathbf{u}\|\oplus\|\mathbf{v}\|} &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\|\mathbf{u}\|\|\mathbf{v}\|}{c^2}\right) \\
&\geq \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}} \\
&= \gamma_{\|\mathbf{u}\oplus\mathbf{v}\|}
\end{aligned} \tag{15}$$

for all $\mathbf{u},\mathbf{v}$ in an Einstein gyrogroup $(\mathbb{R}^n_c,\oplus)$. But $\gamma_{\mathbf{x}} = \gamma_{\|\mathbf{x}\|}$ is a monotonically increasing function of $\|\mathbf{x}\|$, $0\leq\|\mathbf{x}\|<c$. Hence (15) implies (14). $\square$

3. Einstein Addition Vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && \text{Commutative Law} \\
\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{Associative Law}
\end{aligned} \tag{16}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ is neither commutative nor associative. Rather, Einstein addition is both gyrocommutative and gyroassociative, as stated in (19) below.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are self maps of $\mathbb{R}^n$ that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$ the gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] \in \mathrm{Aut}(\mathbb{R}^n_c, \oplus)$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{17}$$

for all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^n_c, \oplus)$, $\mathrm{gyr}[\mathbf{u}, \mathbf{v}] : \mathbb{R}^n_c \to \mathbb{R}^n_c$.

We recall that an automorphism of a groupoid $(S, \oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b) = f(a)\oplus f(b)$ for all $a, b \in S$. The set of all automorphisms of a groupoid $(S, \oplus)$ forms a group, under automorphism composition, denoted $\mathrm{Aut}(S, \oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]$, $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, is *trivial* if $\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} = \mathbf{w}$ for all $\mathbf{w} \in \mathbb{R}^n_c$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0}, \mathbf{v}]$, $\mathrm{gyr}[\mathbf{v}, \mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}]$ are trivial, that is,

$$\begin{aligned} \mathrm{gyr}[\mathbf{0}, \mathbf{v}] &= \mathrm{gyr}[\mathbf{v}, \mathbf{0}] = I \\ \mathrm{gyr}[\mathbf{v}, \ominus\mathbf{v}] &= \mathrm{gyr}[\ominus\mathbf{v}, \mathbf{v}] = I \\ \mathrm{gyr}[\mathbf{v}, \mathbf{v}] &= I \end{aligned} \tag{18}$$

for all $\mathbf{v} \in \mathbb{R}^n_c$, I being the identity map, as we see from (17) and (7).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and from associativity. We see this from the gyrocommutative and the gyroassociative laws of Einstein addition in the following list of elegant identities that involve Einstein addition, $\oplus$, and gyrations [60, 64, 67]. For all $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n_c$,

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v} &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law} \\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w}) &= (\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u}, \mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law} \\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} &= \mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v}, \mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law} \\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v}, \mathbf{v}] &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Left Reduction Property} \\
\mathrm{gyr}[\mathbf{u}, \mathbf{v}\oplus\mathbf{u}] &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Right Reduction Property} \\
\mathrm{gyr}[\ominus\mathbf{u}, \ominus\mathbf{v}] &= \mathrm{gyr}[\mathbf{u}, \mathbf{v}] && \text{Gyration Even Property} \\
(\mathrm{gyr}[\mathbf{u}, \mathbf{v}])^{-1} &= \mathrm{gyr}[\mathbf{v}, \mathbf{u}] && \text{Gyration Inversion Law}
\end{aligned} \tag{19}$$

Einstein addition is thus regulated by the gyrations to which it gives rise owing to its nonassociativity. As such, Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [54]. Interestingly, gyrations are the mathematical abstraction of the relativistic phenomenon known as *Thomas precession* [67, Sect. 10.3] [74] [79, Chap. 13]. Thomas precession, in turn, is related to the *mixed state geometric phase*, as Lévay discovered in his work [30] which, according to [30], was motivated by the author work in [61, 62].

The left and right reduction properties in (19) present important gyration identities since they trigger a remarkable reduction in complexity, as Chatelin noted in [11]. These two gyration identities are, however, just the tip of a giant iceberg. The identities in (19) and many other useful gyration identities are studied, for instance, in [60, 64, 67, 69, 71, 72, 79].

4. From Einstein Addition to Gyrogroups

Taking the key features of Einstein groupoids $(\mathbb{R}^n_c, \oplus)$, $n \in \mathbb{N}$, as axioms, and guided by analogies with groups, we are led to the following formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups.

Definition 4.1. (Gyrogroups [67, p. 17]**).** *A groupoid $(G, \oplus)$ is a gyrogroup if its binary operation satisfies the following axioms. In G there is at least one element, 0, called a left identity, satisfying*

(G1) $\qquad 0\oplus a = a$

for all $a \in G$. There is an element $0 \in G$ satisfying axiom $(G1)$ such that for each $a \in G$ there is an element $\ominus a \in G$, called a left inverse of a, satisfying

(G2) $\qquad \ominus a\oplus a = 0\,.$

Moreover, for any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law

(G3) $\qquad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c\,.$

The map $\mathrm{gyr}[a, b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a, b]c$ is an automorphism of the groupoid $(G, \oplus)$, that is,

(G4) $\qquad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a, b]$ of G is called the gyroautomorphism, or the gyration, of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ is called the gyrator of G. Finally, the gyroautomorphism $\mathrm{gyr}[a, b]$ generated by any $a, b \in G$ possesses the left reduction property

(G5) $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]\,.$

The gyrogroup axioms $(G1)-(G5)$ in Definition 4.1 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a \ominus b = a \oplus (\ominus b)$ in gyrogroup theory as well. In full analogy with groups, gyrogroups split up into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 4.2. (Gyrocommutative Gyrogroups). *A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law*
(G6) $\quad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$
for all $a, b \in G$.

First gyrogroup properties are studied in [72, Chap. 1], and more gyrogroup theorems are studied in [60, 64, 67]. Thus, for instance, as in group theory, any gyrogroup possesses a unique identity element, which is both left and right, and any element of a gyrogroup possesses a unique inverse, which is both left and right.

In order to illustrate the power and elegance of the gyrogroup structure, we solve below the two basic gyrogroup equations (20) and (27).

Let us consider the gyrogroup equation

$$\mathbf{a} \oplus \mathbf{x} = \mathbf{b} \tag{20}$$

in a gyrogroup $(G, \oplus)$ for the unknown $\mathbf{x}$. If $\mathbf{x}$ exists, then by the right gyroassociative law (19) we have

$$\begin{aligned} \mathbf{x} &= \mathbf{0} \oplus \mathbf{x} \\ &= (\ominus \mathbf{a} \oplus \mathbf{a}) \oplus \mathbf{x} \\ &= \ominus \mathbf{a} \oplus (\mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{a}]\mathbf{x}) \\ &= \ominus \mathbf{a} \oplus (\mathbf{a} \oplus \mathbf{x}) \\ &= \ominus \mathbf{a} \oplus \mathbf{b}\,, \end{aligned} \tag{21}$$

noting that $\mathrm{gyr}[\mathbf{a}, \ominus \mathbf{a}]$ is trivial by (18).

Thus, if a solution to (20) exists, it must be given uniquely by

$$\mathbf{x} = \ominus \mathbf{a} \oplus \mathbf{b}\,. \tag{22}$$

Conversely, if $\mathbf{x} = \ominus \mathbf{a} \oplus \mathbf{b}$, then $\mathbf{x}$ is indeed a solution to (20) since by the left gyroassociative law and (18) we have

$$\begin{aligned} \mathbf{a} \oplus \mathbf{x} &= \mathbf{a} \oplus (\ominus \mathbf{a} \oplus \mathbf{b}) \\ &= (\mathbf{a} \oplus (\ominus \mathbf{a})) \oplus \mathrm{gyr}[\mathbf{a}, \ominus \mathbf{a}]\mathbf{b} \\ &= \mathbf{0} \oplus \mathbf{b} \\ &= \mathbf{b}\,. \end{aligned} \tag{23}$$

Substituting the solution (22) into its equation (20) and replacing $\mathbf{a}$ by $\ominus \mathbf{a}$ we recover the left cancellation law (7) for Einstein addition,

$$\ominus \mathbf{a} \oplus (\mathbf{a} \oplus \mathbf{b}) = \mathbf{b}\,. \tag{24}$$

The gyrogroup operation (or, addition) of any gyrogroup has an associated dual operation, called the *gyrogroup cooperation* (or, *coaddition*), which is defined below.

Definition 4.3. (The Gyrogroup Cooperation (Coaddition)). *Let $(G, \oplus)$ be a gyrogroup with gyrogroup operation (or, addition) $\oplus$. The gyrogroup cooperation (or, coaddition) $\boxplus$ is a second binary operation in G given by the equation*

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathbf{b} \tag{25}$$

for all $\mathbf{a}, \mathbf{b} \in G$.

Replacing b by $\ominus b$ in (25) we have the *cosubtraction* identity

$$\mathbf{a} \boxminus \mathbf{b} := \mathbf{a} \boxplus (\ominus\mathbf{b}) = \mathbf{a} \ominus \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{b} \tag{26}$$

for all $\mathbf{a}, \mathbf{b} \in G$, noting that $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ is an automorphism of $(G, \oplus)$ so that $\mathrm{gyr}[\mathbf{a}, \mathbf{b}](\ominus\mathbf{b}) = \ominus\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{b}$.

To motivate the introduction of the gyrogroup cooperation and to illustrate the use of the left reduction property in (19), we solve the equation

$$\mathbf{x} \oplus \mathbf{a} = \mathbf{b} \tag{27}$$

for the unknown $\mathbf{x}$ in a gyrogroup $(G, \oplus)$.

Equation (27) results from (20) by interchanging $\mathbf{a}$ and $\mathbf{x}$. Surprisingly, however, the solution of (27) is quite different from the solution of (20), suggesting the introduction of the second binary operation, the cooperation $\boxplus$ in G. We will find that Einstein coaddition, $\boxplus$, proves crucially important (i) in the understanding of Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ in terms of analogies with vector addition in $\mathbb{R}^n$, and (ii) in our mission to capture analogies with classical results.

Assuming that a solution $\mathbf{x}$ to (27) exists, we have the following obvious chain of equations

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} \oplus \mathbf{0} \\
&= \mathbf{x} \oplus (\mathbf{a} \ominus \mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus \mathrm{gyr}[\mathbf{x}, \mathbf{a}](\ominus\mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x} \oplus \mathbf{a}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \boxminus \mathbf{a}\,.
\end{aligned} \tag{28}$$

The gyrogroup cosubtraction, (26), comes into play in (28) in order to capture an analogy with the classical result $\mathbf{x} + \mathbf{a} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbf{b} - \mathbf{a}$. Thus, if a solution $\mathbf{x}$ to the gyrogroup equation (27) exists, it must be given uniquely by (28). One can show that the latter is indeed a solution to (27) [67, Sect. 2.4]. The use of the gyration left reduction property in (28) indicates the remarkable reduction of complexity that this property offers.

The gyrogroup cooperation is introduced into gyrogroups in order to capture useful analogies between gyrogroups and groups, and it results in the emergence of duality symmetries that the two gyrogroup operations, $\oplus$ and $\boxplus$, share. Thus, for instance, the gyrogroup cooperation uncovers the seemingly missing right counterpart of the left cancellation law (7), giving rise to the right cancellation law,

$$(\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{a} = \mathbf{b} \tag{29}$$

for all $\mathbf{a}, \mathbf{b} \in G$, which is obtained by substituting the result of (28) into (27).

Remarkably, the right cancellation law (29) can be dualized, giving rise to the dual right cancellation law

$$(\mathbf{b} \ominus \mathbf{a}) \boxplus \mathbf{a} = \mathbf{b}\,. \tag{30}$$

As an example, and for later reference, we note that it follows from the right cancellation law (29) that

$$\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a} \qquad \Longleftrightarrow \qquad \mathbf{b} \boxplus \mathbf{c} = \mathbf{d} \boxplus \mathbf{a} \tag{31}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in any gyrocommutative gyrogroup $(G, \oplus)$.

An elegant gyrocommutative gyrogroup identity that involves the gyrogroup cooperation, verified in [67, Theorem 3.12], is

$$\mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{a}) = \mathbf{a} \boxplus (\mathbf{a} \oplus \mathbf{b})\,. \tag{32}$$

5. Émile Borel's Dream Comes True

It is not well-known that the famous mathematician Émile Borel was interested in Einstein's special theory of relativity, particularly in the relativistic phenomenon known as Thomas precession [79, Chap. 13] and in Einstein addition. Being noncommutative, Émile Borel considered Einstein addition as "defective". He, therefore, proposed an alternative, commutative addition of relativistically admissible velocities.

The gyrocommutative law of Einstein velocity addition was already known to Silberstein in 1914 [43] in the following sense: According to his 1914 book, Silberstein knew that the Thomas precession generated by $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$ is the unique rotation that takes $\mathbf{v} \oplus \mathbf{u}$ into $\mathbf{u} \oplus \mathbf{v}$ about an axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ in $\mathbb{R}^3$, thus giving rise to the gyrocommutative law. However, obviously, Silberstein did not use the terms "Thomas precession" and "gyrocommutative law". These terms have been coined later, respectively, (i) following Thomas' 1926 paper [53], and (ii) in 1991 [58,59], following the discovery of the accompanying gyroassociative law of Einstein addition in 1988 [54, 55].

A description of the 3-space rotation, which since 1926 is named after Thomas, is found in Silberstein's 1914 book [43]. In 1914 Thomas precession did not have a name, and Silberstein called it in his 1914 book a "certain space-rotation" [43, p. 169]. An early study of Thomas precession, made by Émile Borel in 1913, is described in his 1914 book [10] and, more recently, in [47].

The almost forgotten attempt of Émile Borel to "repair" the seemingly "defective" Einstein's velocity addition law in the years following 1912 is described by Walter in [82, p. 117]:

> "Borel could construct a tetrahedron in kinematic space, and determined thereby both the direction and magnitude of relative [composite] velocity in a symmetric [commutative] manner."

Borel has, thus, "repaired" the breakdown of commutativity in Einstein addition by proposing an alternative, commutative addition. But he did not pay attention to the accompanying breakdown of associativity in Einstein addition. Accordingly, it seemed appropriate to consider the Lorentz transformation group, rather than the groupoid of Einstein addition, as a primitive notion of special relativity [63].

It turns out that Einstein coaddition is commutative. Hence. Émile Borel's dream to construct a viable commutative relativistic velocity addition comes true with the discovery of Einstein coaddition, $\boxplus$. Unlike Borel's commutative addition, the commutative Einstein coaddition does not replace Einstein addition. Rather, it captures analogies with classical results *jointly* with Einstein addition, as the study of the Einstein gyroparallelogram addition law in Sect. 6 reveals.

A gyrogroup cooperation is commutative if and only if its associated gyrogroup operation is gyrocommutative [64, Theorem 3.4] [67, Theorem 3.4]. Hence, in particular, Einstein coaddition is commutative. Indeed, Einstein coaddition, $\boxplus$, in an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$, abstractly defined in (25), can be manipulated in Einstein gyrogroups, obtaining the following chain of equations [67, Eq. (3.195)],

$$
\begin{aligned}
\mathbf{u} \boxplus \mathbf{v} &= \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) - 1}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \\
&= \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{(\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}})^2 - (\gamma_{\mathbf{u}\ominus\mathbf{v}} + 1)}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \\
&= 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \\
&= 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{2 + (\gamma_{\mathbf{u}} - 1) + (\gamma_{\mathbf{v}} - 1)}
\end{aligned}
\tag{33}
$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, demonstrating that the cooperation $\boxplus$ in Einstein gyrogroups $(\mathbb{R}^n_c, \oplus)$ is commutative.

The symbol $\otimes$ in (33) represents the Einstein *scalar multiplication* so that, for instance, $2\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v}$, for all $\mathbf{v}$ in a gyrogroup $(G, \oplus)$, as explained in Sect. 9. It turns out that Einstein coaddition $\boxplus$ is more than just a commutative binary operation in the ball. Remarkably, jointly with Einstein addition, $\oplus$, Einstein coaddition, $\boxplus$, gives rise to the hyperbolic parallelogram addition law in the ball. The latter is explained in Sect. 6 and illustrated in Fig. 2.

6. From Parallelograms to Gyroparallelograms

Elements of a real inner product space $\mathbb{V} = (\mathbb{V}, +, \cdot)$, called points and denoted by capital italic letters, A, B, P, Q, etc, give rise to vectors in $\mathbb{V}$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathbb{V}$ give rise to a unique rooted vector $\mathbf{v} \in \mathbb{V}$, rooted at the point P. It has a tail at the point P and a head at the point Q, and it has the value $-P + Q$,

$$\mathbf{v} = -P + Q\,. \tag{34}$$

The length of the rooted vector $\mathbf{v} = -P + Q$ is the distance between the points P and Q, given by the equation

$$\|\mathbf{v}\| = \| - P + Q\|\,. \tag{35}$$

Two rooted vectors $-P + Q$ and $-R + S$ are equivalent if they have the same value, that is,

$$-P + Q \quad \sim \quad - R + S \qquad \text{if and only if} \qquad - P + Q = -R + S \tag{36}$$

The relation $\sim$ in (36) between rooted vectors is reflexive, symmetric and transitive, so that it is an equivalence relation that gives rise to equivalence classes of rooted vectors. To liberate rooted vectors from their roots we define a *vector* to be an equivalence class of rooted vectors. The vector $-P + Q$ is thus a representative of all rooted vectors with value $-P + Q$.

A point $P \in \mathbb{V}$ is identified with the vector $-O + P$, O being the arbitrarily selected origin of the space $\mathbb{V}$. Hence, the algebra of vectors can be applied to points as well. Naturally, geometric and physical properties regulated by a vector space are *origin independent*, that is, independent of the choice of the origin.

Let $A, B, C \in \mathbb{V}$ be three non-collinear points, and let

$$\begin{aligned} \mathbf{u} &= -A + B \\ \mathbf{v} &= -A + C \end{aligned} \tag{37}$$

be two vectors in $\mathbb{V}$ that possess the same tail, A. Furthermore, let D be a point of $\mathbb{V}$ given by the *parallelogram condition*

$$D = B + C - A\,. \tag{38}$$

The quadrangle (also known as a quadrilateral; see [13, p. 52]) $ABDC$ turns out to be a parallelogram in Euclidean geometry, shown in Fig. 1, since its two diagonals, AD and BC, intersect at their midpoints, that is,

$$\tfrac{1}{2}(A + D) = \tfrac{1}{2}(B + C)\,. \tag{39}$$

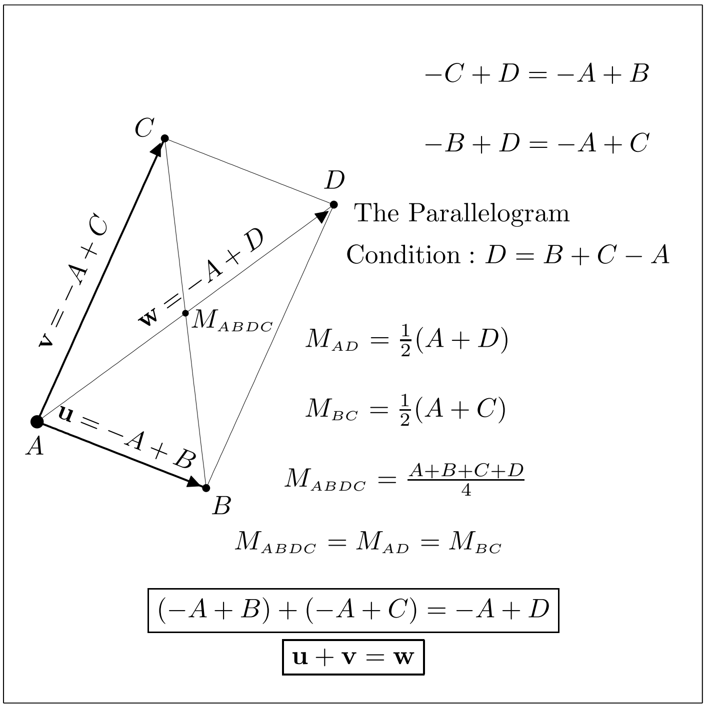

Figure 1: The Euclidean parallelogram and its addition law in a Euclidean vector plane $(\mathbb{R}^2, +, \cdot)$. The diagonals AD and BC of parallelogram $ABDC$ intersect each other at their midpoints. The midpoints of the diagonals AD and BC are, respectively, M_{AD} and M_{BC}, each of which coincides with the parallelogram center M_{ABDC}. This figure sets the stage for its hyperbolic counterpart in Fig. 2.

Clearly, the midpoint equality (39) is equivalent to the parallelogram condition (38).

The vector addition of the vectors $\mathbf{u}$ and $\mathbf{v}$ that generate the parallelogram $ABDC$ according to (37), gives the vector $\mathbf{w}$ by the parallelogram addition law, shown in Fig. 1,

$$\mathbf{w} := -A + D = (-A + B) + (-A + C) = \mathbf{u} + \mathbf{v}\,. \tag{40}$$

Here, by definition, $\mathbf{w}$ is the vector formed by the diagonal AD of the parallelogram $ABDC$, as shown in Fig. 1.

Vectors in the space $\mathbb{V}$ are, thus, equivalence classes of ordered pairs of points, which add according to the parallelogram law, shown in Fig. 1.

Gyrovectors emerge in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ in a way fully analogous to the way vectors emerge in the space $\mathbb{V}$, where $\mathbb{V}_c$ is the c-ball of the space $\mathbb{V}$, $\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$.

Elements of $\mathbb{V}_c$, called points and denoted by capital italic letters, A, B, P, Q, etc, give rise to gyrovectors in $\mathbb{V}_c$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathbb{V}_c$ give rise to a unique rooted gyrovector $\mathbf{v} \in \mathbb{V}_c$, rooted at the point P. It has a tail at the point P and a head at the point Q, and it has the value $\ominus P \oplus Q$,

$$\mathbf{v} = \ominus P \oplus Q\,. \tag{41}$$

The gyrolength of the rooted gyrovector $\mathbf{v} = \ominus P \oplus Q$ is the gyrodistance between the points P and Q, given by the equation

$$\|\mathbf{v}\| = \|\ominus P \oplus Q\|\,. \tag{42}$$

Two rooted gyrovectors $\ominus P \oplus Q$ and $\ominus R \oplus S$ are equivalent if they have the same value, that is,

$$\ominus P \oplus Q \quad \sim \quad \ominus R \oplus S \qquad \text{if and only if} \qquad \ominus P \oplus Q = \ominus R \oplus S \tag{43}$$

The relation $\sim$ in (43) between rooted gyrovectors is reflexive, symmetric and transitive, so that it is an equivalence relation that gives rise to equivalence classes of rooted gyrovectors. To liberate rooted gyrovectors from their roots we define a *gyrovector* to be an equivalence class of rooted gyrovectors. The gyrovector $\ominus P \oplus Q$ is thus a representative of all rooted gyrovectors with value $\ominus P \oplus Q$.

A point P of a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is identified with the gyrovector $\ominus O \oplus P$, O being the arbitrarily selected origin of the space $\mathbb{V}_c$. Hence, the algebra of gyrovectors can be applied to points as well. Naturally, geometric and physical properties regulated by a gyrovector space are expected to be independent of the choice of the origin of the gyrovector space.

Let $A, B, C \in \mathbb{V}_c$ be three non-gyrocollinear points of an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$, and let

$$\begin{aligned} \mathbf{u} &= \ominus A \oplus B \\ \mathbf{v} &= \ominus A \oplus C \end{aligned} \tag{44}$$

be two gyrovectors in $\mathbb{V}$ that possess the same tail, A. Furthermore, let D be a point of $\mathbb{V}_c$ given by the gyroparallelogram condition

$$D = (B \boxplus C) \ominus A\,. \tag{45}$$

Then, the gyroquadrangle $ABDC$ is a gyroparallelogram in the Beltrami-Klein ball model of hyperbolic geometry in the sense that its two gyrodiagonals, AD and BC, intersect at their gyromidpoints, that is,

$$\tfrac{1}{2} \otimes (A \boxplus D) = \tfrac{1}{2} \otimes (B \boxplus C) \tag{46}$$

as illustrated in Fig. 2. Clearly by (31), the gyromidpoint equality (46) is equivalent to the gyroparallelogram condition (45).

The gyrovector addition of the gyrovectors $\mathbf{u}$ and $\mathbf{v}$ that generate the gyroparallelogram $ABDC$ gives the gyrovector $\mathbf{w}$ by the gyroparallelogram addition law, shown in Fig. 2,

$$\mathbf{w} := \ominus A \oplus D = (\ominus A \oplus B) \boxplus (\ominus A \oplus C) =: \mathbf{u} \boxplus \mathbf{v}\,. \tag{47}$$

Here, by definition, $\mathbf{w}$ is the gyrovector formed by the gyrodiagonal AD of the gyroparallelogram $ABDC$. The gyrovector identity in (47), where D is given by (45), is explained in (50) below.

Gyrovectors in the ball $\mathbb{V}_c$ are, thus, equivalence classes of ordered pairs of points, which add according to the gyroparallelogram law shown in Fig. 2.

The equivalence relation in vectors is origin independent. Hence, expressions appropriately derived from vectors are origin independent as well. Thus, in particular, (i) the length of a vector, (ii) the angle between to vectors with a common tail, and (iii) the parallelogram addition of to vectors with a common tail, are origin independent.

In contrast, the equivalence relation in gyrovectors is origin dependent. Fortunately, however, some important expressions derived from gyrovectors are origin independent. Thus, for instance, (i) the gyrolength of a gyrovector, (ii) the gyroangle between to gyrovectors with a common tail, and (iii) the gyroparallelogram addition of two gyrovectors with a common tail, are origin independent. A deep study of origin independence involves the study of *gyroisometries*, found in [79, Sects. 3.11-3.12].

7. The Gyroparallelogram Addition Law

In Euclidean geometry a parallelogram is a quadrangle the two diagonals of which intersect at their midpoints. In full analogy, in hyperbolic geometry a gyroparallelogram is a gyroquadrangle the two gyrodiagonals of which intersect at their gyromidpoints, as shown in Fig. 2. Accordingly, if A, B and C are any three non-gyrocollinear points (that is, they do not lie on a gyroline) in an Einstein gyrovector space, and if a fourth point D is given by the *gyroparallelogram condition*

$$D = (B \boxplus C) \ominus A\,, \tag{48}$$

then the gyroquadrangle $ABDC$ is a gyroparallelogram, shown in Fig. 2.

Indeed, the two gyrodiagonals of gyroquadrangle $ABDC$ are the gyrosegments AD and BC, shown in Fig. 2, the gyromidpoints of which coincide, that is,

$$\tfrac{1}{2} \otimes (A \boxplus D) = \tfrac{1}{2} \otimes (B \boxplus C) \tag{49}$$

where, by (31), the result in (49) is equivalent to the gyroparallelogram condition (48).

The analogies that equations (48) – (49) in gyrovector spaces share with their counterpart equations (38) – (39) in vector spaces indicate that both the gyrogroup

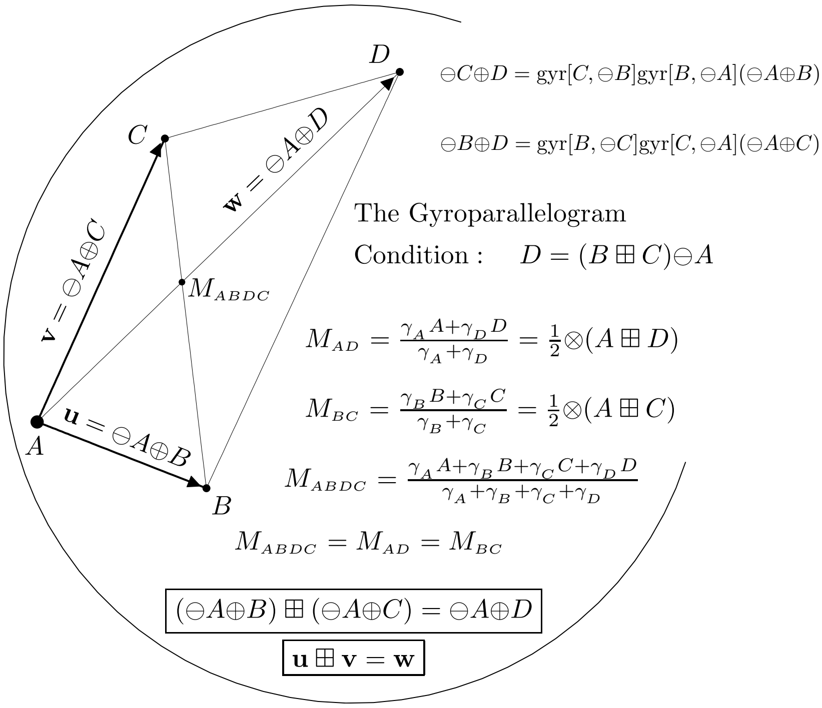

Figure 2: The Einstein gyroparallelogram and its addition law in an Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. The gyrodiagonals AD and BC of gyroparallelogram $ABDC$ intersect each other at their gyromidpoints. The gyromidpoints of the gyrodiagonals AD and BC are, respectively, M_{AD} and M_{BC}, each of which coincides with the gyroparallelogram gyrocenter M_{ABDC}. The analogies that this figure shares with Fig. 1 are obvious.

operation and cooperation, $\oplus$ and $\boxplus$, are necessary for our mission to capture analogies between vector and gyrovector spaces.

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ and let D be the point that augments gyrotriangle ABC into the gyroparallelogram $ABDC$, as shown in Fig. 2. Then, D is determined uniquely by the gyroparallelogram condition (48), obeying the gyroparallelogram addition law [72, Theorem 5.5]

$$(\ominus A\oplus B)\boxplus(\ominus A\oplus C) = (\ominus A\oplus D) \tag{50}$$

shown in Fig. 2. In full analogy with the parallelogram addition law of vectors in Euclidean geometry, (40), the gyroparallelogram addition law (50) of gyrovectors in hyperbolic geometry can be written as

$$\mathbf{u}\boxplus\mathbf{v} = \mathbf{w} \tag{51}$$

where $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are the *gyrovectors*

$$\begin{aligned}\mathbf{u} &= \ominus A \oplus B \\ \mathbf{v} &= \ominus A \oplus C \\ \mathbf{w} &= \ominus A \oplus D\end{aligned} \tag{52}$$

which emanate from the point A [67, Chap. 5].

In his 1905 paper that founded the special theory of relativity [17], Einstein noted that his velocity addition does not satisfy the Euclidean parallelogram law:

> "Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt also nach unserer Theorie nur in erster Annäherung."
>
> A. Einstein [17]

[English translation: Thus the law of velocity parallelogram is valid according to our theory only to a first approximation.]

Indeed, Einstein velocity addition, $\oplus$, is noncommutative and does not give rise to an exact "velocity parallelogram" in Euclidean geometry. However, as illustrated in Fig. 2, Einstein velocity *coaddition*, $\boxplus$, which is commutative, does give rise to an exact "velocity gyroparallelogram" in hyperbolic geometry.

The breakdown of commutativity in Einstein velocity addition law seemed undesirable to the famous mathematician Émile Borel. Borel's resulting attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described by Walter in [82, p. 117]. Here, however, we see that there is no need to repair Einstein velocity addition law for being noncommutative since, despite of being noncommutative, it gives rise to the gyroparallelogram law of gyrovector addition, which turns out to be commutative. The compatibility of the gyroparallelogram addition law of Einsteinian velocities with cosmological observations of stellar aberration is studied in [67, Chap. 13] and [72, Sect. 10.2]. The extension of the gyroparallelogram addition law of $k = 2$ summands into a higher dimensional gyroparallelotope addition law of $k > 2$ summands is presented in (54) − (55) below and studied in [67, Sect. 10.12] and [79, Sect. 6.4].

8. Gyroparallelotopes

The extreme sides of (33) give the equation

$$\mathbf{u} \boxplus_2 \mathbf{v} = 2 \otimes \frac{\gamma_{\mathbf{u}} \mathbf{u} + \gamma_{\mathbf{v}} \mathbf{v}}{2 + (\gamma_{\mathbf{u}} - 1) + (\gamma_{\mathbf{v}} - 1)} \tag{53}$$

where we replace $\boxplus$ by $\boxplus_2$ to emphasize that the binary operation $\boxplus = \boxplus_2$ is valid only for two summands.

Equation (53) is written in a form that suggests that the extension of the gyroparallelogram addition law (53), which involves two summands, to the gyroparallelepiped addition law, which involves three summands, is given by the following *gyroparallelepiped law*

$$\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 \mathbf{w} := 2\otimes \frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v} + \gamma_{\mathbf{w}}\mathbf{w}}{2 + (\gamma_{\mathbf{u}} - 1) + (\gamma_{\mathbf{v}} - 1) + (\gamma_{\mathbf{w}} - 1)} \tag{54}$$

$\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}_c^n$.

Einstein coaddition (54) of three summands is commutative and associative in the generalized sense that it is a symmetric function of the summands. The gyroparallelepiped that results from the gyroparallelepiped law (54) is studied in detail in [67, Secs. 10.9–10.12].

We may note that by (53)−(54) we have $\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 \mathbf{0} = \mathbf{u} \boxplus_2 \mathbf{v}$, as expected. However, unexpectedly we have $\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 (\ominus\mathbf{v}) \neq \mathbf{u}$.

The extension of (54) to the Einstein coaddition of k summands, $k > 3$, is now straightforward, giving rise to the higher dimensional *gyroparallelotope law* in $\mathbb{R}_c^n$,

$$\mathbf{v}_1 \boxplus_k \mathbf{v}_2 \boxplus_k \ \ldots \ \boxplus_k \mathbf{v}_k := 2\otimes \frac{\sum_{i=1}^k \gamma_{\mathbf{v}_i}\mathbf{v}_i}{2 + \sum_{i=1}^k (\gamma_{\mathbf{v}_i} - 1)} \tag{55}$$

$\mathbf{v}_k \in G$, $k \in \mathbb{N}$. As expected, the gyroparallelotope law (55) is origin independent. An interesting study of parallelotopes in Euclidean geometry is found in [12].

In the Euclidean limit $c \to \infty$, (i) gamma factors tend to 1, and (ii) the hyperbolic scalar multiplication, $\otimes$, of a gyrovector (see Sect. 9) by 2 tends to the common scalar multiplication of a vector by 2. Hence, in the Euclidean limit, the right-hand side of (55) tends to the vector sum $\sum_{i=1}^k \mathbf{v}_i$ in $\mathbb{R}^n$, as expected.

9. Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication (*gyromultiplication*), giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as shown in [60, 64, 67, 69, 71, 72, 79].

Let $k\otimes\mathbf{v}$ be the Einstein addition of k copies of $\mathbf{v} \in \mathbb{R}_c^n$, that is $k\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v} \ldots \oplus\mathbf{v}$ (k terms). Then,

$$k\otimes\mathbf{v} = \frac{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k - \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k}{\left(1 + \frac{\|\mathbf{v}\|}{c}\right)^k + \left(1 - \frac{\|\mathbf{v}\|}{c}\right)^k} \frac{c\mathbf{v}}{\|\mathbf{v}\|} \tag{56}$$

for $\mathbf{v} \neq \mathbf{0}$, and $k\otimes\mathbf{0} = \mathbf{0}$.

The definition of scalar gyromultiplication in an Einstein gyrovector space requires analytically continuing k off the positive integers, thus obtaining the following definition.

Definition 9.1. (Einstein Scalar Multiplication). *An Einstein gyrovector space* $(\mathbb{R}^n_c, \oplus, \otimes)$ *is an Einstein gyrogroup* $(\mathbb{R}^n_c, \oplus)$ *with scalar gyromultiplication* $\otimes$ *given by*

$$r\otimes\mathbf{v} = \frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r} \frac{c\mathbf{v}}{\|\mathbf{v}\|} = \tanh(r \tanh^{-1}\frac{\|\mathbf{v}\|}{c})\frac{c\mathbf{v}}{\|\mathbf{v}\|} \tag{57}$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_c$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

As an example, it follows from Def. 9.1 that *Einstein half* is given by the equation

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}\mathbf{v}}{1+\gamma_{\mathbf{v}}}\,, \tag{58}$$

so that, as expected, $\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} = \mathbf{v}$.

Einstein gyrovector spaces are studied in [60, 64, 67, 69, 71, 72, 79]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer k, and for all real numbers $r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n_c$, we have

$$\begin{aligned} k\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && k \text{ terms} \\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \end{aligned} \tag{59}$$

in any Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{60}$$

$\mathbf{a} \in \mathbb{R}^n_c$, $\mathbf{a} \neq \mathbf{0}$, $r \in \mathbb{R}$, $r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a} \tag{61}$$

$\mathbf{a}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \tag{62}$$

$r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_c$.

Any Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ inherits an inner product and a norm from its ambient vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} \\ \|\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\| \end{aligned} \tag{63}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

10. Gyrovector Spaces

Taking the key features of Einstein scalar multiplication as axioms, and guided by analogies with vector spaces, we are led to the following formal gyrovector space definition in which gyrovector spaces turn out to form a most natural generalization of vector spaces.

Definition 10.1. (Real Inner Product Gyrovector Spaces [67, p. 154]**).** A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) G is a subset of a real inner product vector space $\mathbb{V}$ called the ambient space of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V1) $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$ Inner Product Gyroinvariance

for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V2)	$1\otimes\mathbf{a} = \mathbf{a}$	Identity Scalar Multiplication
(V3)	$(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$	Scalar Distributive Law
(V4)	$(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$	Scalar Associative Law
(V5)	$\dfrac{\lvert r\rvert\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \dfrac{\mathbf{a}}{\|\mathbf{a}\|}, \quad \mathbf{a} \neq \mathbf{0},\ r \neq 0$	Scaling Property
(V6)	$\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}$	Gyroautomorphism Property
(V7)	$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$	Identity Gyroautomorphism.

(3) Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors".

(V8) $\|G\| = \{\pm\|\mathbf{a}\| : \mathbf{a} \in G\} \subset \mathbb{R}$ Vector Space with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V9) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$ Homogeneity Property

(V10) $\|\mathbf{a}\oplus\mathbf{b}\| \le \|\mathbf{a}\|\oplus\|\mathbf{b}\|$ Gyrotriangle Inequality.

Einstein (gyro)addition and scalar (gyro)multiplication in $\mathbb{R}^n_c$ thus give rise to the Einstein gyrovector spaces $(\mathbb{R}^n_c, \oplus, \otimes)$, $n \ge 2$.

11. Relativistic and Classical Kinetic Energy

Kinetic energy depends on mass and relative velocity. The relativistic mass of an object with Newtonian mass m (also called *relativistically invariant mass*) moving with velocity $\mathbf{v} \in \mathbb{R}^3_c$ relative to an inertial frame Σ_0 is $m\gamma_{\mathbf{v}}$. Having Einstein half (58) in hand, we can recast the relativistic kinetic energy of moving objects into a form that shares analogies with its classical counterpart. The relativistic kinetic energy K_{rel} of an object with rest (Newtonian) mass m that moves uniformly with velocity $\mathbf{v}$ relative to an inertial frame Σ_0 is given by the equation [66]

$$K_{rel} = c^2 m(\gamma_{\mathbf{v}} - 1)\,, \tag{64}$$

where c is the vacuum speed of light. We manipulate (64) in the following chain of equations, some of which are numbered for subsequent explanation.

$$\begin{aligned} K_{rel} &\overset{(1)}{\widehat{=\!=}} c^2 m(\gamma_{\mathbf{v}} - 1) = \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} c^2 m(\gamma_{\mathbf{v}} - 1)\frac{\gamma_{\mathbf{v}}+1}{\gamma_{\mathbf{v}}^2} \\ &=\!= \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} c^2 m \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \\ &\overset{(2)}{\widehat{=\!=}} \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} m\mathbf{v}^2 = \frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}}+1}\cdot m\gamma_{\mathbf{v}}\mathbf{v} \\ &\overset{(3)}{\widehat{=\!=}} (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v})\,, \end{aligned} \tag{65}$$

$m\gamma_{\mathbf{v}}$ being the velocity dependent relativistic mass [73] of the moving object relative to Σ_0.

Derivation of the numbered equalities in (65) follows:

1. Follows from (64).
2. Follows from (10), p. 8.

3. Follows from Einstein half (58).

The relativistic kinetic energy K_{rel} in (65),

$$K_{rel} = (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v})\,, \tag{66}$$

is given by the inner product of a "relativistic half-velocity" and a corresponding relativistic momentum, in full analogy with the classical kinetic energy K_{cls},

$$K_{cls} = \tfrac{1}{2}m\mathbf{v}^2 = (\tfrac{1}{2}\mathbf{v})\cdot(m\mathbf{v})\,, \tag{67}$$

which is given by the inner product of a "classical half-velocity" and a corresponding classical momentum. The ability of Einstein scalar multiplication to capture analogies between modern and classical results thus emerges.

The analogies that (66) and (67) share demonstrate that the relativistic counterpart of the Newtonian mass m is the relativistic, velocity dependent mass $m\gamma_{\mathbf{v}}$. The controversy around the relativistic mass is described in [73]. It is owing to analogies that Newtonian mass and Einsteinian relativistic mass share that the notion of barycentric coordinates in Euclidean geometry can be translated into the notion of gyrobarycentric coordinates in hyperbolic geometry, as shown in [71, 72] and [70, 75, 76], and in Sects. 16-18.

12. Einstein Gyrolines – The Hyperbolic Lines

In applications to geometry, where the letters a, b, c are frequently used, it is convenient to replace the notation $\mathbb{R}^n_c$ for the c-ball of an Einstein gyrovector space by the s-ball, $\mathbb{R}^n_s$. We thus switch from c to s to avoid notational confusion. Moreover, it is understood that $n \geq 2$, unless specified otherwise.

Let $A, B \in \mathbb{R}^n_s$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, the graph L_{AB} of the set of all points

$$L_{AB} = A\oplus(\ominus A\oplus B)\otimes t\,, \tag{68}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}^n_s$. As such, it is a geodesic line of the Beltrami-Klein ball model of hyperbolic geometry, shown in Fig. 3 for $n = 2$. The geodesic line (68) is the unique gyroline that passes through the points A and B. It passes through the point A when $t = 0$ and, owing to the left cancellation law, (7), it passes through the point B when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* AB that joins the points A and B in Fig. 3 is obtained from gyroline (68) with $0 \leq t \leq 1$.

Gyrolines (68) are the geodesics of the Beltrami-Klein ball model of hyperbolic geometry. Similarly, gyrolines (68) with Einstein addition $\oplus$ replaced by Möbius addition $\oplus_{\mathrm{M}}$ are the geodesics of the Poincaré ball model of hyperbolic geometry.

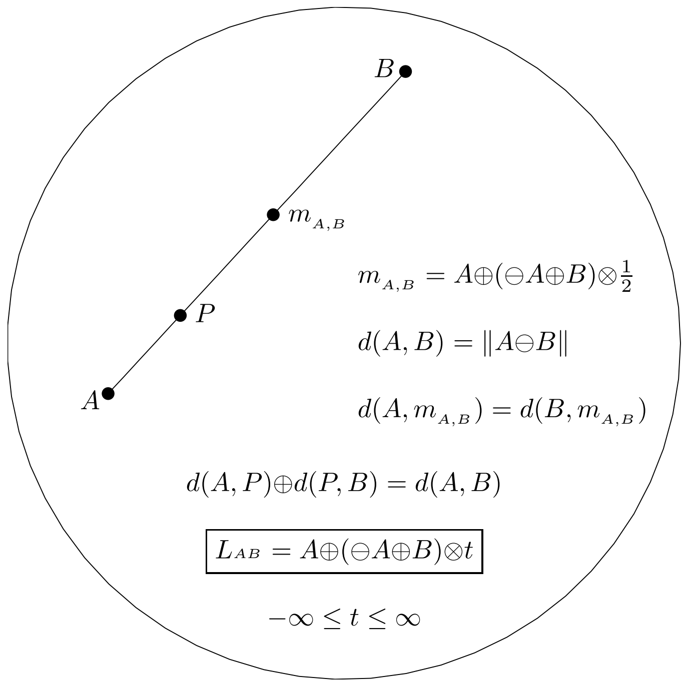

Figure 3: Gyrolines, the hyperbolic lines L_{AB} in Einstein gyrovector spaces, are fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in the Cartesian model of the Euclidean geometry of $\mathbb{R}^n$. Here $\oplus = \oplus_{\mathrm{E}}$ is Einstein addition, as opposed to Fig. 4 where $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition. The figure shows that Einstein gyrolines in the hyperbolic plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are Euclidean segments in the disc $\mathbb{R}^2_s$.

These interesting results are established by methods of differential geometry in [65], and are illustrated in Figs. 3 and 4.

Each point of (68) with $0 < t < 1$ is said to lie *between* A and B. Thus, for instance, the point P in Fig. 3 lies between the points A and B. As such, the points A, P and B obey the *gyrotriangle equality* according to which

$$d(A, P) \oplus d(P, B) = d(A, B)\,, \tag{69}$$

in full analogy with Euclidean geometry. Here

$$d(A, B) = \|\ominus A \oplus B\|\,, \tag{70}$$

$A, B \in \mathbb{R}^n_s$, is the Einstein *gyrodistance function*, also called the Einstein *gyrometric*. This gyrodistance function in Einstein gyrovector spaces corresponds bijectively to a standard hyperbolic distance function, as demonstrated in [67, Sect. 6.19].

A contact between Einstein gyrodistance function and differential geometry is provided by the Riemannian gyroline element of Einstein gyrovector spaces, studied in [64, Sect. 7.5] and [65]. It turns out that the Riemannian gyroline element of Einstein gyrovector spaces, given by

$$ds^2 = \|(\mathbf{v} + d\mathbf{v})\ominus\mathbf{v}\|^2 \tag{71}$$

is identical with the well-known Riemannian line element of the Beltrami-Klein disc model of hyperbolic geometry.

13. Möbius Addition

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : \; |z| < 1\} \tag{72}$$

in the complex plane $\mathbb{C}$ is given by the polar decomposition [2, 28],

$$z \mapsto e^{i\theta} \frac{a + z}{1 + \overline{a}z} = e^{i\theta}(a \oplus_{\mathrm{M}} z) \, . \tag{73}$$

It induces the Möbius addition $\oplus_{\mathrm{M}}$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus_{\mathrm{M}} z = \frac{a + z}{1 + \overline{a}z} \tag{74}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a.

In order to extend Möbius addition from the disk to the ball, let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way,

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2 \, . \tag{75}$$

Then

$$\begin{aligned} \bar{u}v + u\bar{v} &= 2\mathbf{u}{\cdot}\mathbf{v} \\ |u| &= \|\mathbf{u}\| \end{aligned} \tag{76}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc

$$\mathbb{R}^2_{s=1} = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < s = 1\} \tag{77}$$

of $\mathbb{R}^2$. Indeed,

$$\begin{aligned}
\mathbb{D} \ni u \oplus v &= \frac{u+v}{1+\bar{u}v} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^2_{s=1}
\end{aligned} \tag{78}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{s=1}$. The last equation in (78) is a vector equation, so that its restriction to the ball of the Euclidean two-dimensional space is a mere artifact. Suggestively, we thus arrive at the following definition of Möbius addition in the ball $\mathbb{R}^n_s$,

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \frac{(1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\,. \tag{79}$$

Like Einstein groupoids $(\mathbb{R}^n_s, \oplus_{\mathrm{E}})$, Möbius groupoids $(\mathbb{R}^n_s, \oplus_{\mathrm{M}})$ are gyrocommutative gyrogroups. The gyrogroup isomorphism between Einstein addition $\oplus = \oplus_{\mathrm{E}}$ and Möbius addition $\oplus_{\mathrm{M}}$ is given by the equations [67, p. 227]

$$\begin{aligned}
\tfrac{1}{2}\otimes_{\mathrm{E}}(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}) &= \tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{v} \\
2\otimes_{\mathrm{M}}(\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}) &= 2\otimes_{\mathrm{E}}\mathbf{u}\oplus_{\mathrm{E}}2\otimes_{\mathrm{E}}\mathbf{v}
\end{aligned} \tag{80}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$.

The operations $\otimes_{\mathrm{E}}$ and $\otimes_{\mathrm{M}}$ are identical to each other, $\otimes_{\mathrm{E}} = \otimes_{\mathrm{M}} =: \otimes$. Hence, Identities (80) can be written equivalently as

$$\begin{aligned}
\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v}) \\
\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v})
\end{aligned} \tag{81}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$.

The related connection between Möbius transformation and Lorentz transformation of Einstein's special theory of relativity was recognized by H. Liebmann in 1905 [36, pp. 122–123].

When $\mathbf{u}$ and $\mathbf{v}$ are parallel in $\mathbb{R}^n_s \subset \mathbb{R}^n$, scalar gyromultiplication is distributive over gyroaddition [27]. Hence, in the special case when $\mathbf{u}\|\mathbf{v}$ in $\mathbb{R}^n$ the two equations in (81) degenerate to the single equation

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}\,, \qquad \mathbf{u}\|\mathbf{v} \tag{82}$$

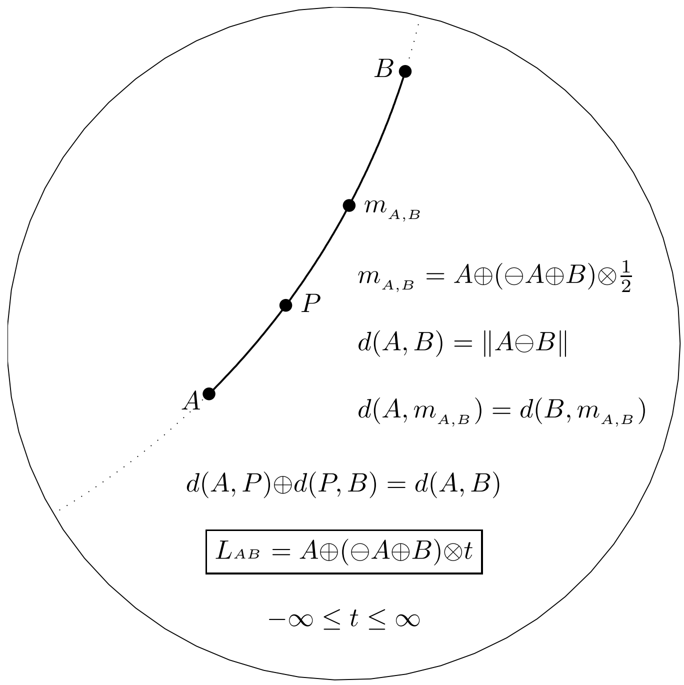

Figure 4: Gyrolines, the hyperbolic lines L_{AB} in Möbius gyrovector spaces, are fully analogous to lines in Euclidean spaces. The gyroline $L_{AB} = A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ is a geodesic line in the Cartesian-Poincaré ball model of hyperbolic geometry. Here $\oplus = \oplus_{\text{M}}$ is Möbius addition, as opposed to Fig. 3 where $\oplus = \oplus_{\text{E}}$ is Einstein addition. The figure indicates that Möbius gyrolines in the hyperbolic plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are Euclidean circular arcs in the disc $\mathbb{R}^2_s$ that approach the boundary of the disc orthogonally.

Accordingly, Einstein scalar multiplication, $\otimes_{\text{E}}$, and Möbius scalar multiplication, $\otimes_{\text{M}}$, share the same formula, $\otimes_{\text{E}} = \otimes_{\text{M}} =: \otimes$, where $\otimes$ is given by (57),

Einstein and Möbius addition are originated from totally two different disciplines. Accordingly, the elegant relationship (81) between Einstein and Möbius addition indicates, once again, the intrinsic beauty, harmony and interdisciplinarity in Einstein addition.

14. Möbius Gyrolines

Replacing Einstein addition $\oplus = \oplus_{\text{E}}$ in Sect. 12 by Möbius addition $\oplus = \oplus_{\text{M}}$ in this section, we obtain the Möbius gyrolines

$$L_{AB} = A\oplus(\ominus A\oplus B)\otimes t\,, \tag{83}$$

$t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Fig. 4 for $n = 2$. As we see from Fig. 4, Möbius gyrolines in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are circular arcs that approach the boundary of the disc $\mathbb{R}^2_s$ orthogonally. These are the well-known geodesics of the Poincaré disc model of hyperbolic geometry.

Along with Möbius gyrolines we have the Möbius gyrodistance function

$$d(A, B) = \|\ominus A \oplus B\| \tag{84}$$

and the gyrotriangle equality

$$d(A, P) \oplus d(P, B) = d(A, B) \tag{85}$$

for any $A, B, P \in (\mathbb{R}^n_s, \oplus)$, where P lies between A and B, as shown in Fig. 4.

A contact between Möbius gyrodistance function and differential geometry is provided by the Riemannian gyroline element of Möbius gyrovector spaces, studied in [64, Sect. 7.3] and [65]. It turns out that the Riemannian gyroline element of Möbius gyrovector spaces, given by

$$ds^2 = \|(\mathbf{v} + d\mathbf{v}) \ominus \mathbf{v}\|^2 \,, \tag{86}$$

is identical with the well-known Riemannian line element of the Poincaré disc model of hyperbolic geometry.

It should be emphasized that Equations (83) – (86) of this section are identical in form with Equations (68) – (71) of Sect. 12. However, $\oplus = \oplus_{\text{E}}$ in Sect. 12, while $\oplus = \oplus_{\text{M}}$ in this section, where Einstein addition $\oplus = \oplus_{\text{E}}$ is given by (2), p. 7, and Möbius addition $\oplus_{\text{M}}$ is given by (79), p. 29.

15. Gyrotrigonometry

Hyperbolic trigonometry is called *gyrotrigonometry* and, similarly, hyperbolic angles are called *gyroangles*. Graphically, gyrotrigonometry is best illustrated in the Poincaré disc model of hyperbolic geometry since the Poincaré ball model is *conformal* in the following sense. A gyroangle between two intersecting Möbius gyrolines equals the angle between corresponding intersecting tangent lines, as shown in Fig. 5. The equations in this section are valid in any gyrovector space. In particular, they are valid in Einstein gyrovector spaces, when $\oplus = \oplus_{\text{E}}$, and in Möbius gyrovector spaces, when $\oplus = \oplus_{\text{M}}$. Graphical illustrations are presented for Möbius gyrovector planes in Figs. 5 and 6.

The gyroangle included by the gyrosegments AB and AC that emanate from the point A, denoted $\angle BAC$, has the measure α given by the equation [64, 67, 69, 71, 72, 79]

$$\cos\alpha = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \,, \tag{87}$$

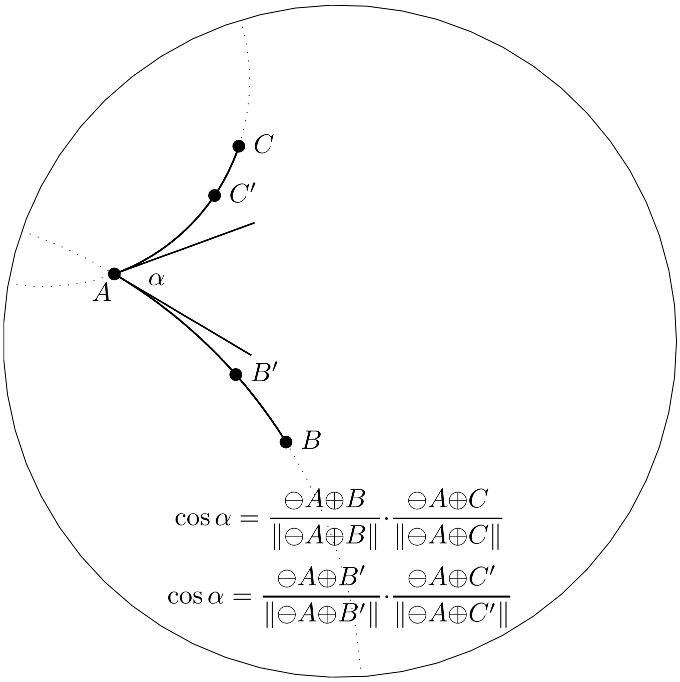

Figure 5: A Möbius gyroangle α generated by two intersecting Möbius geodesic rays (gyrorays). Its measure equals the measure of the Euclidean angle generated by corresponding intersecting tangent lines.

$A, B, C \in \mathbb{R}^n_s$, where "cos" is the common cosine function of trigonometry. Accordingly,

$$\alpha = \cos^{-1} \frac{\ominus A\oplus B}{\|\ominus A\oplus B\|}\cdot\frac{\ominus A\oplus C}{\|\ominus A\oplus C\|}\,, \tag{88}$$

$0 \le \alpha < \pi$. The point A is the vertex of the gyroangle $\angle BAC$. A gyroangle with vertex at the origin, $O = \mathbf{0}$, of the ball coincides with its Euclidean counterpart,

$$\cos\alpha = \frac{\ominus O\oplus B}{\|\ominus O\oplus B\|}\cdot\frac{\ominus O\oplus C}{\|\ominus O\oplus C\|} = \frac{B}{\|B\|}\cdot\frac{C}{\|C\|}\,. \tag{89}$$

The measure of a gyroangle is invariant under the motions of hyperbolic geometry, which are left gyrotranslations and rotations. In particular, any gyroangle with vertex A can be moved by a hyperbolic motion (gyromotion) to a gyroangle with vertex O while keeping the gyroangle measure invariant. Having vertex O, the resulting gyroangle behaves like an angle. Hence, trigonometric identities for angles as, for instance, $\cos^2\alpha + \sin^2\alpha = 1$, remain valid for gyroangles as well. Gyrotrigonometry and its application in analytic hyperbolic geometry are studied in [64, 67, 69, 71, 72, 79]. An elegant application of gyrotriangle gyrotrigonometry, which has no Euclidean counterpart, is presented in Fig. 6.

A Möbius gyrotriangle along with its standard notation and some basic identities is presented in Fig. 6. Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{R}^n_s$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^n_s$ and side gyrolengths $a, b, c \in (-s, s)$,

$$\begin{aligned} \mathbf{a} &= \ominus B \oplus C, & a &= \|\mathbf{a}\|, \\ \mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\|, \\ \mathbf{c} &= \ominus A \oplus B, & c &= \|\mathbf{c}\|. \end{aligned} \tag{90}$$

The gyroangle measures α, β and γ of the gyroangles at the vertices A, B and C are given by the gyrotrigonometric identities

$$\begin{aligned} \cos\alpha &= \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \\ \cos\beta &= \frac{\ominus B \oplus A}{\|\ominus B \oplus C\|} \cdot \frac{\ominus B \oplus A}{\|\ominus B \oplus C\|} \\ \cos\gamma &= \frac{\ominus C \oplus A}{\|\ominus C \oplus A\|} \cdot \frac{\ominus C \oplus B}{\|\ominus C \oplus B\|} \end{aligned} \tag{91}$$

in full analogy with corresponding trigonometric identities.

In Euclidean geometry the triangle angles do not determine its side lengths. In contrast, in hyperbolic geometry the gyrotriangle gyroangles determine uniquely its side gyrolengths according to the gyrotriangle gyrotrigonometric identities (92) of the following theorem 15.1 [64, Theorem 8.48].

Theorem 15.1. (AAA to SSS Conversion Law). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$ with vertices A, B, C, corresponding gyroangles α, β, γ, $0 < \alpha + \beta + \gamma < \pi$, and side gyrolengths a, b, c, as shown in Fig. 6. The side gyrolengths of the gyrotriangle ABC are determined by its gyroangles according to the AAA to SSS conversion equations*

$$\begin{aligned} \frac{a^2}{s^2} &= \frac{\cos\alpha + \cos(\beta+\gamma)}{\cos\alpha + \cos(\beta-\gamma)} \\ \frac{b^2}{s^2} &= \frac{\cos\beta + \cos(\alpha+\gamma)}{\cos\beta + \cos(\alpha-\gamma)} \\ \frac{c^2}{s^2} &= \frac{\cos\gamma + \cos(\alpha+\beta)}{\cos\gamma + \cos(\alpha-\beta)}\,. \end{aligned} \tag{92}$$

In the Euclidean limit $s \to \infty$, the equations in (92) reduce, respectively, to

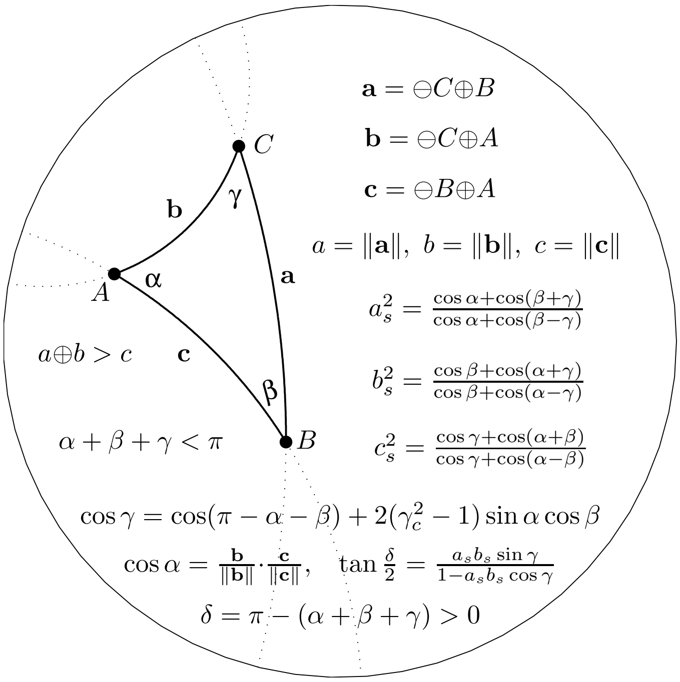

Figure 6: A Möbius gyrotriangle ABC in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}_s^2, \oplus, \otimes)$ is shown. Its sides are formed by gyrovectors that link its vertices, in full analogy with Euclidean triangles. Its hyperbolic side lengths, a, b, c, are uniquely determined in (93) by its gyroangles. The gyrotriangle gyroangle sum is less than π. Here, $a_s = a/s$, etc. Note that in the limit of large s, $s \to \infty$, the $\cos\gamma$ equation reduces to $\cos\gamma = \cos(\pi - \alpha - \beta)$ so that $\alpha + \beta + \gamma = \pi$, implying that both sides of each of the squared side gyrolength equations, shown in the figure and listed in (93), vanish.

the equations

$$\begin{aligned} 0 &= \cos\alpha + \cos(\beta + \gamma) \\ 0 &= \cos\beta + \cos(\alpha + \gamma) \\ 0 &= \cos\gamma + \cos(\alpha + \beta) \end{aligned} \tag{93}$$

each of which is equivalent to the Euclidean identity

$$\alpha + \beta + \gamma = \pi\,. \tag{94}$$

Hence, the AAA (gyroAngle gyroAngle gyroAngle) to SSS (gyroSide gyroSide gyroSide) Conversion Law (92) in Theorem 15.1 is valid in hyperbolic geometry, where $\alpha+\beta+\gamma<\pi$, and it is invalid in Euclidean geometry, where the triangle angle identity (94) holds.

16. Resultant Relativistically Invariant Mass

The relativistic mass $m\gamma_{\mathbf{v}}$, already encountered in Sect. 11, plays an important role in Einstein's special relativity and in analytic hyperbolic geometry [73]. Einstein velocity addition admits the following theorem about relativistic mass.

Theorem 16.1. (Resultant Relativistically Invariant Mass Theorem). *Let $(\mathbb{R}^n_s,\oplus)$ be an Einstein gyrogroup, and let $m_k\in\mathbb{R}$ and $\mathbf{v}_k\in\mathbb{R}^n_s$, $k=1,2,\ldots,N$, be N real numbers and N elements of $\mathbb{R}^n_s$ satisfying*

$$\sum_{k=1}^{N} m_k\gamma_{\mathbf{v}_k} \;\neq\; 0\,. \tag{95}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix}\gamma_{\mathbf{v}_k}\\ \gamma_{\mathbf{v}_k}\mathbf{v}_k\end{pmatrix} = m_0 \begin{pmatrix}\gamma_{\mathbf{v}_0}\\ \gamma_{\mathbf{v}_0}\mathbf{v}_0\end{pmatrix} \tag{96}$$

be an $(n+1)$-vector equation for the two unknowns $m_0\in\mathbb{R}$ and $\mathbf{v}_0\in\mathbb{R}^n$.

Then (96) possesses a unique solution $(m_0,\mathbf{v}_0)$, $m_0\neq 0$, $\mathbf{v}_0\in\mathbb{R}^n_s$, satisfying the following three identities for all $\mathbf{w}\in\mathbb{R}^n_s$ (including, in particular, the interesting special case of $\mathbf{w}=\mathbf{0}$):

$$\mathbf{w}\oplus\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}} \tag{97}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{m_0} \tag{98}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0}(\mathbf{w}\oplus\mathbf{v}_0) = \frac{\sum_{k=1}^{N} m_k\gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{m_0} \tag{99}$$

where

$$m_0 \;=\; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k(\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)}-1)}\;. \tag{100}$$

The proof of Theorem 16.1 is found in [71, Theorem 3.7] and in [72, Theorem 3.2].

It follows from (96) that (i) $m_0\gamma_{\mathbf{v}_0}$ is the resultant relativistic mass of a system of N particles with relativistic masses $m_k\gamma_{\mathbf{v}_k}$, and (ii) $m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0$ is the resultant relativistic momentum of a system of N particles with relativistic momenta $m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k$,

$k = 1, \ldots, N$. In physical applications $n = 3$, and $m_k > 0$, $k = 0, 1, \ldots, N$, are positive real numbers that represent relativistically invariant (Newtonian) masses. In geometry, however, $n \geq 1$ and m_k are any real numbers that need not be positive.

Identities (97) − (99) of Theorem 16.1 are *covariant* in the sense the $\mathbf{v}_0$ and $\mathbf{v}_k$ vary together under left gyrotranslations by any $\mathbf{w} \in \mathbb{R}^n_s$. The constant m_0 in (100) in *invariant* in the sense that it remains invariant under left gyrotranslations of $\mathbf{v}_k$ by any $\mathbf{w} \in \mathbb{R}^n_s$.

It follows from (100) that the relativistically invariant mass m_0 of a particle system of N particles is greater than the sum $\sum_{k=1}^{N} m_k$ of the Newtonian Masses of its constituents. The excessive mass, $m_0 - \sum_{k=1}^{N} m_k$, is *dark* in the sense that (i) it is generated by internal relative velocities between the constituents of the particle system, and that (ii) it reveals its presence only gravitationally, since it emits no radiation and it involves no collisions [68, 73]. Interestingly, the relativistically invariant mass m_0 of a particle system in (100) is precisely what we need in order to adapt the Euclidean notion of barycentric coordinates for use in hyperbolic geometry without losing covariance.

To appreciate the power and elegance of Theorem 16.1 in relativistic mechanics in terms of novel analogies that it shares with familiar results in classical mechanics, we present below the classical counterpart, Theorem 16.2, of Theorem 16.1. Theorem 16.2 is derived from Theorem 16.1 by approaching the Newtonian/Euclidean limit when $s = c$ tends to infinity. The resulting Theorem 16.2 is immediate, and its importance in classical mechanics is well-known. Like Theorem 16.1, Theorem 16.2 involves an expression, (103) below, which is covariant under translations and, as such, fully analogous to (97), which is covariant under left gyrotranslations.

Theorem 16.2. (Resultant Newtonian Invariant Mass Theorem). *Let* $(\mathbb{R}^n, +)$ *be a Euclidean* n*-space, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}^n$, $k = 1, 2, \ldots, N$, *be* N *real numbers and* N *elements of* $\mathbb{R}^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \neq 0 \tag{101}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{v}_0 \end{pmatrix} \tag{102}$$

be an $(n+1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (102) *possesses a unique solution* $(m_0, \mathbf{v}_0)$, $m_0 \neq 0$, *satisfying the following equations for all* $\mathbf{w} \in \mathbb{R}^n$ *(including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$*):*

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{103}$$

and

$$m_0 = \sum_{k=1}^{N} m_k \,. \tag{104}$$

The proof of Theorem 16.2 is immediate.

Unlike Identity (103) of Theorem 16.2, which is immediate, its counterpart in Theorem 16.1, Identity (97), is not immediate and, hence, unexpected. Yet, in full analogy with Theorem 16.2, the validity of Identity (97) in Theorem 16.1 for all $\mathbf{w} \in \mathbb{R}_c^n$ is geometrically important. This geometric importance of Identity (97) stems from its following implication: The velocity $\mathbf{v}_0$ of the center of momentum frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space $\mathbb{R}_s^n$ with its underlying Cartesian-Beltrami-Klein ball model of hyperbolic geometry.

Not unexpectedly, the Newtonian mass m_0 in (104) of a particle system plays an important role in Theorem 17.3, p. 39, on the covariance of barycentric coordinates under the motions of Euclidean geometry, which are translations and rotations. Remarkably, the relativistic invariant mass m_0 in (100) of a particle system plays an analogous important role in Theorem 18.3, p. 42, on the gyrocovariance of gyrobarycentric coordinates under the gyromotions of hyperbolic geometry, which are left gyrotranslations and rotations. Left gyrotranslations, in turn, play an important role in the application of gyrobarycentric coordinates for determining analytically various gyrotriangle gyrocenters in [71, 72, 79].

17. Barycentric Coordinates

The notion of barycentric coordinates dates back to Möbius. The use of barycentric coordinates in Euclidean geometry is described in [84], and the historical contribution of Möbius' barycentric coordinates to vector analysis is described in [14, pp. 48–50].

In this section we set the stage for the introduction in Sect. 18 of barycentric coordinates into hyperbolic geometry by illustrating the way Theorem 16.2, p. 36, suggests the introduction of barycentric coordinates into Euclidean geometry.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers such that

$$\sum_{k=1}^{N} m_k \ \neq\ 0 \tag{105}$$

and let $A_k \in \mathbb{R}^n$ be N given points in the Euclidean n-space $\mathbb{R}^n$, $k = 1, \ldots, N$. Theorem 16.2, p. 36, states the trivial, but geometrically significant, result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ A_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ P \end{pmatrix} \tag{106}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}^n$ possesses the unique solution given by

$$m_0 = \sum_{k=1}^{N} m_k \tag{107}$$

and

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{108}$$

satisfying for all $X \in \mathbb{R}^n$,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}\,. \tag{109}$$

We view (108) as the representation of a point $P \in \mathbb{R}^n$ in terms of its *barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of points $S = \{A_1, \ldots, A_N\}$. Identity (109), then, insures that the barycentric coordinate representation (108) of P with respect to the set S is *covariant* (or, *invariant in form*) in the following sense. The point P and the points of the set S of its barycentric coordinate representation vary together under translations. Indeed, a translation $X + A_k$ of A_k by X, $k = 1, \ldots, N$, in (109) results in the translation $X + P$ of P by X.

In order to insure that barycentric coordinate representations with respect to a set S are unique, we require S to be pointwise independent.

Definition 17.1. (Pointwise Independence). A set S of N points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \geq 2$, is *pointwise independent* if the $N-1$ vectors $-A_1 + A_k$, $k = 2, \ldots, N$, are linearly independent.

Definition 17.2. (Barycentric Coordinates). Let

$$S = \{A_1, \ldots, A_N\} \tag{110}$$

be a pointwise independent set of N points in $\mathbb{R}^n$. The real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \neq 0 \tag{111}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}\,. \tag{112}$$

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \ldots, m_N)$ of the point P in (112) are equivalent to the barycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \dots, m_N)$ are also written as $(m_1 : \dots : m_N)$.

Barycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{113}$$

are called *special barycentric coordinates.*

Equation (112) is said to be the (unique) barycentric coordinate representation of P with respect to the set S.

Theorem 17.3. (Covariance of Barycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{114}$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean n-space $\mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \dots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (114) *is covariant, that is,*

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \tag{115}$$

for all $X \in \mathbb{R}^n$, and

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \tag{116}$$

for all $R \in SO(n)$.

Proof. The proof is immediate, noting that rotations $R \in SO(n)$ of $\mathbb{R}^n$ about its origin are linear maps of $\mathbb{R}^n$. □

Following the vision of Felix Klein in his *Erlangen Program* [8, 35], it is owing to the covariance with respect to translations and rotations that barycentric coordinate representations possess geometric significance. Indeed, translations and rotations in Euclidean geometry form the *group of motions* of the geometry, studied in [79], and according to Felix Klein's Erlangen Program [8], a geometric property is a property that remains invariant in form under the group of motions of the geometry.

18. Gyrobarycentric Coordinates

Guided by analogies with Sect. 17, in this section we introduce barycentric coordinates into hyperbolic geometry where, naturally, they are called *gyrobarycentric coordinates* [70–72, 75, 77, 78]. Gyrobarycentric coordinates prove useful in the analytic determination of various gyrotriangle gyrocenters, just as barycentric coordinates prove useful in the analytic determination of various triangle centers.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers, and let $A_k \in \mathbb{R}^n_s$ be N given gyropoints in an Einstein gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, $k = 1, \ldots, N$, satisfying,

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \; > \; 0 \tag{117}$$

Theorem 16.1, p. 35 presents the result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{A_k} \\ \gamma_{A_k} A_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_P \\ \gamma_P P \end{pmatrix} \tag{118}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}^n_s$ possesses the unique solution given by

$$m_0 \; = \; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)} \tag{119}$$

$m_0 > 0$, satisfying

$$m_0 \; = \; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(X \oplus A_j) \oplus (X \oplus A_k)} - 1)} \tag{120}$$

for all $X \in \mathbb{R}^n_s$, and

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{121}$$

satisfying

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{122}$$

for all $X \in \mathbb{R}^n_s$.

Furthermore, Theorem 16.1, p. 35, also asserts that P and m_0 satisfy the two identities

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{123}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{124}$$

and, more generally,

$$\gamma_{X\oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}}{m_0} \tag{125}$$

and

$$\gamma_{X\oplus P}(X\oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}(X\oplus A_k)}{m_0} \tag{126}$$

for all $X \in \mathbb{R}^n_s$.

We view (121) as the representation of a gyropoint $P \in \mathbb{R}^n_s$ in terms of its *hyperbolic barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of gyropoints $S = \{A_1, \ldots, A_N\}$. Naturally in gyrolanguage, hyperbolic barycentric coordinates are called *gyrobarycentric coordinates*. Identity (122) insures that the *gyrobarycentric coordinate representation* (121) of P with respect to the set S is *gyrocovariant* as stated in Theorem 18.3 below. The gyropoint P and the gyropoints of the set S of its gyrobarycentric coordinate representation vary together under left gyrotranslations. Indeed, a left gyrotranslation $X\oplus A_k$ of A_k by X, $k = 1, \ldots, N$ in (122) results in the left gyrotranslation $X\oplus P$ of P by X.

In order to insure that gyrobarycentric coordinate representations with respect to a set S are unique, we require S to be hyperbolically pointwise independent or, in gyrolanguage, *gyropointwise Independent*.

Definition 18.1. (Gyropointwise Independence). A set S of N gyropoints $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n_s$, $n \geq 2$, is *gyropointwise independent* if the $N-1$ gyrovectors in $\mathbb{R}^n_s$, $\ominus A_1\oplus A_k$, $k = 2, \ldots, N$, considered as vectors in $\mathbb{R}^n$, are linearly independent.

We are now in the position to present the formal definition of gyrobarycentric coordinates, as motivated by mass and center of momentum velocity of Einsteinian particle systems and by analogies with barycentric coordinates.

Definition 18.2. (Gyrobarycentric Coordinates). Let

$$S = \{A_1, \ldots, A_N\} \tag{127}$$

be a gyropointwise independent set of N gyropoints in $\mathbb{R}^n_s$. The real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \gamma_{A_k} > 0 \tag{128}$$

are gyrobarycentric coordinates of a gyropoint $P \in \mathbb{R}^n_s$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \,. \tag{129}$$

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ of the gyropoint P in (129) are equivalent to the gyrobarycentric coordinates $(\lambda m_1, \dots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates $(m_1, \dots, m_N)$ are also written as $(m_1 : \dots : m_N)$.

Gyrobarycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{130}$$

are called *special gyrobarycentric coordinates*.

Equation (129) is said to be the gyrobarycentric coordinate representation of P with respect to the set S.

Finally, the constant of the gyrobarycentric coordinate representation of P in (129) is $m_0 > 0$, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ . \tag{131}$$

Theorem 18.3. (Gyrocovariance of Gyrobarycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{132a}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}_s^n$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with respect to a gyropointwise independent set $S = \{A_1, \dots, A_N\} \subset \mathbb{R}_s^n$.

Then

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{132b}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{132c}$$

where m_0, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ , \tag{132d}$$

$m_0 > 0$, *is the constant of the gyrobarycentric coordinate representation* (132a).

Furthermore, the gyrobarycentric coordinate representation (132a) *and its associated identities in* (132b) – (132d) *are gyrocovariant, that is,*

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{133a}$$

$$\gamma_{X \oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}}{m_0} \tag{133b}$$

$$\gamma_{X \oplus P} (X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{m_0} \tag{133c}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(X \oplus A_j) \oplus (X \oplus A_k)} - 1)} \tag{133d}$$

for all $X \in \mathbb{R}_s^n$, *and*

$$RP = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} RA_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \tag{134a}$$

$$\gamma_{RP} = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}}{m_0} \tag{134b}$$

$$\gamma_{RP} (RP) = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} (RA_k)}{m_0} \tag{134c}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(RA_j) \oplus (RA_k)} - 1)} \tag{134d}$$

for all $R \in SO(n)$.

The proof of Theorem 18.3 is found in [72, Theorem 4.6].

Following the vision of Felix Klein in his *Erlangen Program* [8, 35], it is owing to the gyrocovariance, that is, covariance with respect to left gyrotranslations and rotations, that gyrobarycentric coordinate representations are geometrically significant. Indeed, left gyrotranslations and rotations in hyperbolic geometry form the group of motions of the geometry, studied in [79, Sect. 3.12] and, according to Felix Klein's Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

The following two corollaries of Theorem 18.3 prove useful.

Corollary 18.4. *Let $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$ be a gyropointwise independent set of N gyropoints in $\mathbb{R}_s^n$, and let*

$$P = \frac{\sum_{k=1}^N m_k \gamma_{A_k} A_k}{\sum_{k=1}^N m_k \gamma_{A_k}} \tag{135}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}^n$ with respect to the set S. Furthermore, let m_0 be the representation constant, given by

$$m_0^2 = \left(\sum_{k=1}^N m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^N m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)\,. \tag{136}$$

Then, the point P lies in the ball $\mathbb{R}_s^n$, $P \in \mathbb{R}_s^n$, if and only if $m_0^2 > 0$ (In other words, the point P is a gyropoint if and only if $m_0^2 > 0$).

The proof of Corollary 18.4 is found in [72, Corollary 4.9].

Corollary 18.5. *Let $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$ be a gyropointwise independent set of N gyropoints in $\mathbb{R}_s^n$, and let*

$$P = \frac{\sum_{k=1}^N m_k \gamma_{A_k} A_k}{\sum_{k=1}^N m_k \gamma_{A_k}} \tag{137}$$

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}^n$ with respect to the set S, with positive gyrobarycentric coordinates $m_k > 0$, $k = 1, \ldots, N$. Then, $P \in \mathbb{R}_s^n$. Moreover, P lies on the the convex span of S if and only if $m_k > 0$, $k = 1, \ldots, N$.

The proof of Corollary 18.5 is found in [72, Corollary 4.10]

19. Gyrolanguage

The checkered history of gyrolanguage begins in 1988 [54] with the discovery of the parametric realization of the Lorentz transformation group of special relativity theory in terms of relativistically admissible velocities. It turned out that the group structure of Lorentz transformations induces the gyrocommutative gyrogroup structure of the space $\mathbb{R}_c^3$ of all relativistically admissible velocities with the binary operation $\oplus$ given by Einstein's velocity addition law.

The gyrocommutative gyrogroup structure $(\mathbb{R}_c^3, \oplus)$ that regulates Einstein addition was initially called a *nonassociative group* [55]. In the initial study of the concrete example $(\mathbb{R}_c^3, \oplus)$, the gyrocommutative and gyroassociative laws of Einstein addition were called weakly commutative and weakly associative laws and, accordingly, gyrocommutative gyrogroups were called *weakly associative-commutative groups* (WACGs, in short) [57]. Furthermore, in this initial study of gyrocommutative gyrogroups the rich algebra of the gyrations that are associated with Einstein addition was discovered. Gyrations were called Thomas rotations for being related to the special relativistic phenomenon known as Thomas precession [55]. The term *K-loop* with "K" after Karzel, which refers to the gyrocommutative gyrogroup, was coined by the author in [56] as evidenced from [26, pp. 169-170]. The term *K-loop* is in use by some authors, and its prehistory is unfolded in [42, p. 142] and in [60, Remark 6.12].

Prior to its introduction by the author, the term "K-loop" has already been in use by Soĭkis, in 1970 [45] and later, but independently, by Basarab, in 1992 [9]. Unlike the term "K-loop" that Ungar coined, the "K" in each of the terms "K-loop" coined by Soĭkis and by Basarab does not refer to "Karzel".

Finally, in 1991 [58] the author has realized that a most appropriate term for the abstract Thomas precession is *Thomas gyration* (or gyration, in short) so that, accordingly, the weakly commutative and weakly associative laws of Einstein addition became the gyrocommutative and the gyroassociative laws. Hence, consistently, the extension by abstraction of the Einstein groupoid $(\mathbb{R}^3_c, \oplus)$ is now called a gyrocommutative gyrogroup.

Merging gyroterminology with terminology [59], the emergence of gyrolanguage is thus natural. It is a language in which we prefix a gyro to terms that describe concepts in algebra and geometry to mean the analogous concepts in gyroalgebra and gyrogeometry. An interesting example is provided by the term *gyrolayout*, which has been coined by D. K. Urribarri, S. M. Castro and S. R. Martig in the title of their paper [81], where the 3-dimensional Einstein gyrovector space is employed for the generation of computer hyperbolic visualization.

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