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Special Subgroups of Gyrogroups: Commutators, Nuclei and Radical

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Abstract

A gyrogroup is a nonassociative group-like structure modelled on the space of relativistically admissible velocities with a binary operation given by Einstein's velocity addition law. In this article, we present a few of groups sitting inside a gyrogroup G, including the commutator subgyrogroup, the left nucleus, and the radical of G. The normal closure of the commutator subgyrogroup, the left nucleus, and the radical of G are in particular normal subgroups of G. We then give a criterion to determine when a subgyrogroup H of a finite gyrogroup G, where the index $[G\colon H]$ is the smallest prime dividing $|G|$, is normal in G.

Keywords: Gyrogroup, commutator subgyrogroup, nucleus of gyrogroup, subgyrogroup of prime index, radical of gyrogroup.

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1. Introduction

A gyrogroup, discovered by Abraham A. Ungar [16], is a nonassociative group-like structure modelled on the space of relativistically admissible velocities, together with Einstein's velocity addition [18]. It is remarkable that the gyrogroup structure appears in various fields such as mathematical physics [10, 17], non-Euclidean geometry [19, 20], group theory [6, 7], loop theory [8, 14], harmonic analysis [3, 4], abstract algebra [13, 15], and analysis [1, 2].

This article explores an algebraic aspect of gyrogroups. Recall that in abstract algebra the following theme recurs: given an object X and a subobject Y,

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determine whether the quotient object X/Y has the same algebraic structure as X. It is known, for instance, that a subgroup Ξ of a group Γ gives rise to the quotient group Γ/Ξ if and only if Ξ is normal in Γ. Sometimes, it is possible to use information on a normal subgroup Ξ and on the quotient Γ/Ξ to obtain information about Γ. Therefore, determining the normal subgroups of Γ is useful for studying properties of Γ itself. The situation in gyrogroup theory is analogous. For example, the Lagrange theorem for finite gyrogroups follows from the fact that every gyrogroup G has a normal subgroup Ξ such that G/Ξ is a gyrocommutative gyrogroup [6, Theorem 4.11]. For more details, see Section 5 of [13]. From this point of view, we examine some normal subgyrogroups of a gyrogroup that form groups under the gyrogroup operation.

For basic knowledge of gyrogroup theory, the reader is referred to [13, 15, 19]. Here is the formal definition of a gyrogroup.

Definition 1.1 (Gyrogroup). A groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.

(G3) For all a, $b \in G$, there is an automorphism $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]c \qquad \text{(left gyroassociative law)}$$

for all $c \in G$.

(G4) For all a, $b \in G$, $\mathrm{gyr}\,[a \oplus b, b] = \mathrm{gyr}\,[a, b]$. (left loop property)

2. The Commutator Subgyrogroup

Throughout this section, G is an arbitrary gyrogroup unless otherwise stated.

2.1 The Direct Product and Normal Closure

Recall that the intersection of normal subgroups of a group Γ is again a normal subgroup of Γ. This result continues to hold for gyrogroups, as we will see shortly. Because of the missing of associativity in gyrogroups, it is not straightforward to determine whether a given subgyrogroup H of a gyrogroup G is normal in G. However, according to Theorem 2.3, the smallest (by inclusion) normal subgyrogroup of G that contains H, called the normal closure of H, always exists. The normal closure of H and H have some common features, and sometimes it is possible to obtain information about H from the normal closure of H. See for instance Corollary 2.12.

Given an indexed family of gyrogroups $\{G_i : i \in I\}$, the *direct product of G_i, $i \in I$*, denoted by $\prod_{i\in I} G_i$, consists of all functions $f : I \to \bigcup_{i\in I} G_i$ with the property that $f(i) \in G_i$ for all $i \in I$. For $f, g \in \prod_{i\in I} G_i$, define a function $f \oplus g$ by the equation

$$(f \oplus g)(i) = f(i) \oplus g(i), \qquad i \in I. \tag{1}$$

Theorem 2.1. *Let $\{G_i : i \in I\}$ be an indexed family of gyrogroups. The direct product $\prod_{i\in I} G_i$ with operation defined by $(f, g) \mapsto f \oplus g$ is a gyrogroup.*

Proof. Set $G = \prod_{i\in I} G_i$. The zero function, $i \mapsto 0$, $i \in I$, is a left identity of G. For each $f \in G$, the function $i \mapsto \ominus f(i)$, $i \in I$, is a left inverse of f. The gyroautomorphisms of G are given by

$$(\mathrm{gyr}\,[f, g]h)(i) = \mathrm{gyr}\,[f(i), g(i)]h(i), \qquad i \in I,$$

for all $f, g, h \in G$. It is straightforward to check that the axioms of a gyrogroup are satisfied. □

Theorem 2.2. *Let $\{N_i : i \in I\}$ be an indexed family of normal subgyrogroups of G. Then the intersection $\bigcap_{i\in I} N_i$ is a normal subgyrogroup of G.*

Proof. For each $i \in I$, there exists a gyrogroup homomorphism φ_i of G to a gyrogroup G_i such that $\ker\varphi_i = N_i$. Set $H = \prod_{i\in I} G_i$. For each $a \in G$, define a function $\varphi(a)$ by $\varphi(a)(i) = \varphi_i(a)$ for all $i \in I$. Then $a \mapsto \varphi(a)$, $a \in G$, defines a gyrogroup homomorphism from G to H. Direct computation shows that $\ker\varphi = \bigcap_{i\in I} \ker\varphi_i$. Hence, $\bigcap_{i\in I} N_i = \bigcap_{i\in I} \ker\varphi_i = \ker\varphi \trianglelefteq G$. □

Theorem 2.3. *Let A be a nonempty subset of G. Then there exists a unique normal subgyrogroup of G, denoted by $\langle\overline{A}\rangle$, such that*

1. *$A \subseteq \langle\overline{A}\rangle$, and*
2. *if $N \trianglelefteq G$ and $A \subseteq N$, then $\langle\overline{A}\rangle \subseteq N$.*

Proof. Set $\mathcal{A} = \{K \subseteq G : K \trianglelefteq G \text{ and } A \subseteq K\}$. By Theorem 2.2, $\langle\overline{A}\rangle := \bigcap_{K\in\mathcal{A}} K$ forms a normal subgyrogroup of G satisfying the two conditions. The uniqueness of $\langle\overline{A}\rangle$ follows from condition (2). □

Definition 2.4 (Normal closure). Let A be a nonempty subset of a gyrogroup G. The normal subgyrogroup $\langle\overline{A}\rangle$ in Theorem 2.3 is called the *normal closure of* A or *normal subgyrogroup of* G *generated by* A.

According to Theorem 2.3, the normal closure of A is the smallest (by inclusion) normal subgyrogroup of G that contains A. Note that if A itself is a normal subgyrogroup of G, then $\langle\overline{A}\rangle = A$. In other words, any normal subgyrogroup of G equals its normal closure. The concept of normal closures is needed in studying the commutator subgyrogroup of a gyrogroup in the next section.

2.2 Commutators

In this section, we extend the notion of commutators, which is defined for groups, to gyrogroups. Recall that if Γ is a group, then the commutator subgroup of Γ, denoted by Γ', is the smallest normal subgroup of Γ such that the quotient Γ/Γ' is an abelian group. Unlike the situation in group theory, it is still an open problem whether the commutator subgyrogroup of a gyrogroup G, denoted by G', is normal in G. However, it is true that if G' is normal in G, then the quotient G/G' forms a gyrocommutative gyrogroup. Therefore, we focus attention on the normal closure of G' instead of G'. It turns out that the normal closure of G' is the smallest normal subgyrogroup of G such that the quotient $G/\langle\overline{G'}\rangle$ is gyrocommutative. Further, the normal closure of G' (and hence G') forms a subgroup of G, as we will see shortly.

Let G be a gyrogroup. Given $a, b \in G$, define the *commutator of* a *and* b, denoted by $[a, b]$, by the equation

$$[a, b] = \ominus(a \oplus b) \oplus \operatorname{gyr}[a, b](b \oplus a). \tag{2}$$

Define

$$G' = \langle [a, b] \colon a, b \in G \rangle, \tag{3}$$

the subgyrogroup of G generated by commutators of elements from G, called the *commutator subgyrogroup* of G. Note that if G is a gyrogroup with trivial gyroautomorphisms, then G becomes a group, $[a, b]$ becomes the group-theoretic commutator of a and b, and G' becomes the familiar commutator subgroup of G.

Theorem 2.5. *Let G be a gyrogroup. Then the following hold.*

1. *For all $a, b \in G$, $[a, b] = 0$ if and only if $a \oplus b = \operatorname{gyr}[a, b](b \oplus a)$.*
2. *For all $a, b \in G$, $\ominus(a \oplus b) = (\ominus a \ominus b) \oplus [\ominus a, \ominus b]$.*
3. *If φ is a gyrogroup homomorphism of G, then $\varphi([a, b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in G$.*
4. *If $\tau \in \operatorname{Aut}(G)$, then $\tau(G') = G'$.*
5. *$G' = \{0\}$ if and only if G is gyrocommutative.*

6. *If $G' \trianglelefteq G$, then G/G' is gyrocommutative.*

Proof. Item (1) follows from the left cancellation law.

To verify item (2), we compute

$$\begin{aligned}(\ominus a \ominus b) \oplus [\ominus a, \ominus b] &= \operatorname{gyr}[\ominus a, \ominus b](\ominus b \ominus a)\\ &= \operatorname{gyr}[a, b](\ominus b \ominus a)\\ &= \ominus(a \oplus b).\end{aligned}$$

We have the first equation from the definition of a commutator; the second equation from Theorem 2.34 of [19]; and the last equation from Theorem 2.11 of [19].

(3) By Proposition 23 of [15],

$$\begin{aligned}\varphi([a, b]) &= \varphi(\ominus(a \oplus b) \oplus \operatorname{gyr}[a, b](b \oplus a))\\ &= \ominus(\varphi(a) \oplus \varphi(b)) \oplus \operatorname{gyr}[\varphi(a), \varphi(b)](\varphi(b) \oplus \varphi(a))\\ &= [\varphi(a), \varphi(b)].\end{aligned}$$

(4) Let $\tau \in \operatorname{Aut}(G)$. First, we prove that $G' \subseteq \tau(G')$. For all $a, b \in G$, we have $[a, b] = \tau([\tau^{-1}(a), \tau^{-1}(b)])$ belongs to $\tau(G')$. Hence, $\tau(G')$ contains all the commutators of G. Since G' is the smallest subgyrogroup of G containing the commutators of G and $\tau(G') \leqslant G$, it follows that $G' \subseteq \tau(G')$. Since τ^{-1} is also in $\operatorname{Aut}(G)$, $G' \subseteq \tau^{-1}(G')$. This implies $\tau(G') \subseteq \tau(\tau^{-1}(G')) = G'$ since τ is a bijection. Hence, $\tau(G') = G'$.

Item (5) follows immediately from item (1).

(6) Suppose that $G' \trianglelefteq G$. Then G/G' has the quotient gyrogroup structure. Let $a, b \in G$. According to Theorem 27 of [15], we have

$$\begin{aligned}\ominus((a \oplus G') \oplus (b \oplus G')) &= \ominus((a \oplus b) \oplus G')\\ &= (\ominus(a \oplus b)) \oplus G'\\ &= ((\ominus a \ominus b) \oplus [\ominus a, \ominus b]) \oplus G'\\ &= ((\ominus a \ominus b) \oplus G') \oplus ([\ominus a, \ominus b] \oplus G')\\ &= (\ominus a \ominus b) \oplus G'\\ &= (\ominus a \oplus G') \oplus (\ominus b \oplus G')\\ &= \ominus(a \oplus G') \ominus (b \oplus G').\end{aligned}$$

This proves that G/G' satisfies the automorphic inverse property and so G/G' is gyrocommutative by Theorem 3.2 of [19]. $\square$

A subgyrogroup H of G is called an *L-subgyrogroup* of G, denoted by $H \leqslant_L G$, if $\operatorname{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$. For more information about L-subgyrogroups, see Section 4 of [15].

Theorem 2.6. *The commutator subgyrogroup of G is an L-subgyrogroup of G.*

Proof. By Theorem 2.5 (4), G' is invariant under the gyroautomorphisms of G. Hence, $G' \leqslant_L G$. □

Proposition 2.7. *Let N be a normal subgyrogroup of G. The following are equivalent:*

1. *G/N is gyrocommutative.*
2. *$G' \subseteq N$.*
3. *$[a, b] \in N$ for all $a, b \in G$.*

Proof. (1) ⇒ (2) Let $a, b \in G$. Set $X = a \oplus N$ and $Y = b \oplus N$. Since G/N is gyrocommutative, $X \oplus Y = \mathrm{gyr}\,[X, Y](Y \oplus X)$. From Theorem 27 of [15], we have

$$(a \oplus b) \oplus N = (\mathrm{gyr}\,[a, b](b \oplus a)) \oplus N.$$

It follows that

$$\begin{aligned}
[a, b] \oplus N &= (\ominus(a \oplus b) \oplus \mathrm{gyr}\,[a, b](b \oplus a)) \oplus N \\
&= \ominus((a \oplus b) \oplus N) \oplus (\mathrm{gyr}\,[a, b](b \oplus a) \oplus N) \\
&= \ominus((a \oplus b) \oplus N) \oplus ((a \oplus b) \oplus N) \\
&= 0 \oplus N.
\end{aligned}$$

Hence, $[a, b] \in N$ for all $a, b \in G$ and so $G' \subseteq N$ by the minimality of G'.

The implication (2) ⇒ (3) is trivial.

(3) ⇒ (1) Since $N \trianglelefteq G$, G/N admits the quotient gyrogroup structure. The proof that G/N is gyrocommutative follows the same steps as in the proof of Theorem 2.5 (6). □

Theorem 2.8. *The normal closure of G' is the unique normal subgyrogroup of G such that*

1. *$G/\langle\overline{G'}\rangle$ is gyrocommutative, and*
2. *if $\varphi\colon G \to A$ is a gyrogroup homomorphism into a gyrocommutative gyrogroup A, then φ factors through $\langle\overline{G'}\rangle$ in the sense that $\langle\overline{G'}\rangle \subseteq \ker\varphi$.*

Proof. By Theorem 2.3, $\langle\overline{G'}\rangle \trianglelefteq G$ and $G' \subseteq \langle\overline{G'}\rangle$. Hence, by Proposition 2.7, $G/\langle\overline{G'}\rangle$ is gyrocommutative. Suppose that $\varphi\colon G \to A$ is a gyrogroup homomorphism of G, where A is a gyrocommutative gyrogroup. For $a, b \in G$, we have

$$\varphi([a, b]) = [\varphi(a), \varphi(b)] = 0$$

since $\varphi(a), \varphi(b) \in A$ and A is gyrocommutative. Thus, $[a, b] \in \ker\varphi$ for all $a, b \in G$, which implies $G' \subseteq \ker\varphi$. Since $\ker\varphi \trianglelefteq G$, it follows from the minimality of $\langle\overline{G'}\rangle$ that $\langle\overline{G'}\rangle \subseteq \ker\varphi$.

(Uniqueness) Assume that K_1 and K_2 are normal subgyrogroups of G that satisfy the two conditions. Let $\Pi_1\colon G \to G/K_1$ and $\Pi_2\colon G \to G/K_2$ be the canonical projections. As K_1 satisfies the second condition and Π_2 is a gyrogroup homomorphism, we have $K_1 \subseteq \ker \Pi_2 = K_2$. Interchanging the roles of K_1 and K_2, one obtains that $K_2 \subseteq \ker \Pi_1 = K_1$. Hence, $K_1 = K_2$. $\square$

Theorem 2.8 implies the *universal property* of the normal closure of G': given any gyrogroup homomorphism φ from G to a gyrocommutative gyrogroup A, there is a unique gyrogroup homomorphism $\Phi\colon G/\langle\overline{G'}\rangle \to A$ such that $\Phi \circ \Pi = \varphi$, that is, the following diagram commutes.

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\Pi"'] & \arrow[d, phantom, "\circlearrowleft"] & A \\
& G/\langle\overline{G'}\rangle \arrow[ur, dotted, "\Phi"'] &
\end{tikzcd}$$

Here, Π denotes the canonical projection given by $\Pi(a) = a \oplus \langle\overline{G'}\rangle$ for all $a \in G$, and Φ is given by

$$\Phi(a \oplus \langle\overline{G'}\rangle) = \varphi(a) \tag{4}$$

for all $a \in G$.

Theorem 2.9. *Let N be a normal subgyrogroup of G. Then G/N is gyrocommutative if and only if $\langle\overline{G'}\rangle \subseteq N$.*

Proof. Suppose that G/N is gyrocommutative. Then the canonical projection $\Pi\colon G \to G/N$ fits item (2) of Theorem 2.8. Hence, $\langle\overline{G'}\rangle \subseteq \ker \Pi = N$. Conversely, if $\langle\overline{G'}\rangle \subseteq N$, then $G' \subseteq N$ and so G/N is gyrocommutative by Proposition 2.7. $\square$

Proposition 2.10. $\langle\overline{G'}\rangle = \{0\}$ *if and only if G is gyrocommutative.*

Proof. If $\langle\overline{G'}\rangle = \{0\}$, then $G \cong G/\langle\overline{G'}\rangle$ via the canonical projection. Hence, G is gyrocommutative. Conversely, if G is gyrocommutative, then so is $G/\{0\}$. Hence, $\langle\overline{G'}\rangle \subseteq \{0\}$ by Theorem 2.9. This implies $\langle\overline{G'}\rangle = \{0\}$. $\square$

By a *subgroup* of a gyrogroup G we mean a subgyrogroup of G that forms a group under the operation of G [13, Proposition 3.3]. One of the remarkable consequences of Theorem 2.8 is that the normal closure of G' (and hence G') is a subgroup of G.

Theorem 2.11. *The normal closure of G' is a subgroup of G.*

Proof. By Theorem 4.11 of [6], G has a normal subgroup Ξ such that G/Ξ is a gyrocommutative gyrogroup. By Theorem 2.9, $\langle\overline{G'}\rangle \subseteq \Xi$. Since Ξ is a subgroup of G, so is $\langle\overline{G'}\rangle$. $\square$

Corollary 2.12. *The commutator subgyrogroup of G is a subgroup of G.*

Proof. The corollary follows from the fact that $G' \subseteq \langle\overline{G'}\rangle$. $\square$

3. Nuclei and the Radical of a Gyrogroup

Throughout this section, G is an arbitrary gyrogroup. We follow [5] in presenting a few normal subgroups sitting inside a gyrogroup. The main goal of this section is to prove that the left nucleus and radical of G are normal subgyrogroups of G that form groups under the gyrogroup operation. The key idea is as follows. Every gyrogroup can be embedded into its left multiplication group, and normality of the subgyrogroup under consideration follows from normality of the corresponding subgroup of the left multiplication group. This in particular shows a remarkable connection between groups and gyrogroups.

As in loop theory, the *left nucleus*, *middle nucleus*, and *right nucleus of* G are defined, respectively, by

$$\begin{aligned} N_l(G) &= \{a \in G\colon \forall b, c \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\},\\ N_m(G) &= \{b \in G\colon \forall a, c \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\},\\ N_r(G) &= \{c \in G\colon \forall a, b \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\}. \end{aligned}$$

Since G satisfies the left gyroassociative law and the general left cancellation law, the left nucleus, middle nucleus, and right nucleus of G can be restated in terms of gyroautomorphisms as follows:

$$\begin{aligned} N_l(G) &= \{a \in G\colon \forall b \in G,\ \mathrm{gyr}\,[a, b] = \mathrm{id}_G\},\\ N_m(G) &= \{b \in G\colon \forall a \in G,\ \mathrm{gyr}\,[a, b] = \mathrm{id}_G\},\\ N_r(G) &= \{c \in G\colon \forall a, b \in G,\ \mathrm{gyr}\,[a, b]c = c\}. \end{aligned}$$

By Theorem 2.34 of [19], $\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}\,[b, a]$ for all $a, b \in G$. It follows that the left nucleus and middle nucleus of G are identical.

Theorem 3.1. *The left nucleus, middle nucleus, and right nucleus of G are L-subgyrogroups of G. Furthermore, they are subgroups of G.*

Proof. Because $\mathrm{gyr}\,[0, a] = \mathrm{id}_G$ for all $a \in G$, $0 \in N_l(G)$. Let $a \in N_l(G)$ and let $b \in G$. By Theorem 2.34 of [19], $\mathrm{gyr}\,[\ominus a, b] = \mathrm{gyr}\,[\ominus a, \ominus(\ominus b)] = \mathrm{gyr}\,[a, \ominus b] = \mathrm{id}_G$. Hence, $\ominus a$ is in $N_l(G)$. Let $a, b \in N_l(G)$ and let $c, x \in G$. According to the gyrator identity [19, Theorem 2.10], we compute

$$\begin{aligned} \mathrm{gyr}\,[a \oplus b, c]x &= \ominus((a \oplus b) \oplus c) \oplus ((a \oplus b) \oplus (c \oplus x))\\ &= \ominus((a \oplus b) \oplus c) \oplus (a \oplus (b \oplus \mathrm{gyr}\,[b, a](c \oplus x)))\\ &= \ominus((a \oplus b) \oplus c) \oplus (a \oplus (b \oplus (c \oplus x)))\\ &= \ominus((a \oplus b) \oplus c) \oplus (a \oplus ((b \oplus c) \oplus x))\\ &= \ominus((a \oplus b) \oplus c) \oplus ((a \oplus (b \oplus c)) \oplus x)\\ &= \ominus((a \oplus b) \oplus c) \oplus (((a \oplus b) \oplus c) \oplus x)\\ &= x. \end{aligned}$$

We have the second equation from the right gyroassociative law; the third and forth equations since $b \in N_l(G)$; the fifth and sixth equations since $a \in N_l(G)$; the last equation from the left cancellation law. Since x is arbitrary, $\mathrm{gyr}\,[a \oplus b, c] = \mathrm{id}_G$ and so $a \oplus b \in N_l(G)$. By the subgyrogroup criterion [15, Proposition 14], $N_l(G) \leqslant G$. By definition of $N_l(G)$, $N_l(G) \leqslant_L G$. Since $\mathrm{gyr}\,[a, b]|_{N_l(G)} = \mathrm{id}_{N_l(G)}$ for all $a, b \in N_l(G)$, $N_l(G)$ is a subgroup of G. Since $N_m(G) = N_l(G)$, we have $N_m(G) \leqslant_L G$ and $N_m(G)$ is a subgroup of G as well. The proof that $N_r(G)$ is an L-subgyrogroup and a subgroup of G is straightforward. □

Let a be an arbitrary element of G. Recall that the *left gyrotranslation by* a, L_a, is a permutation of G defined by

$$L_a(x) = a \oplus x, \qquad x \in G.$$

For a given subgyrogroup H of G, define $L(H) = \{L_a : a \in H\}$. In the case $H = G$, we have $L(G) = \{L_a : a \in G\}$. The *left multiplication group of* G, $\mathrm{LMlt}\,(G)$, is the subgroup of the symmetric group on G generated by $L(G)$. In other words,

$$\mathrm{LMlt}\,(G) = \langle L_a : a \in G \rangle.$$

A subset X of a group Γ is a *twisted subgroup* [5, p. 187] of Γ if $1 \in X$, 1 being the identity element of Γ; $x \in X$ implies $x^{-1} \in X$; and $x, y \in X$ implies $xyx \in X$.

Theorem 3.2. *$L(G)$ is a twisted subgroup of* $\mathrm{LMlt}\,(G)$.

Proof. The theorem follows from the fact that $L_a^{-1} = L_{\ominus a}$ and

$$L_a \circ L_b \circ L_a = L_{(a \oplus b) \boxplus a}$$

for all $a, b \in G$. Here, the coaddition $\boxplus$ of G is defined by $a \boxplus b = a \oplus \mathrm{gyr}\,[a, \ominus b]b$ for all $a, b \in G$. □

In light of Theorem 3.2, $L(G)$ is a *generating* twisted subgroup of $\mathrm{LMlt}\,(G)$. This leads to the following theorem.

Theorem 3.3. *Define*

$$L(G)^{\#} = \bigcap_{a \in G} L_a L(G).$$

Then $L(G)^{\#}$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ *contained in $L(G)$.*

Proof. The theorem is an application of Theorem 3.8 of [5]. □

Theorem 3.4. *$L(N_l(G))$ is a normal subgroup of* $\mathrm{LMlt}\,(G)$.

Proof. From Theorem 5.7 of [5], we have $L(N_l(G)) = L(G)^{\#}$. Hence, $L(N_l(G))$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ by Theorem 3.3. □

Following [5], we define

$$L(G)' = \{L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_n} : a_i \in G \text{ and } L_{a_n} \circ L_{a_{n-1}} \circ \cdots \circ L_{a_1} = \mathrm{id}_G\}. \quad (5)$$

Since $L(G)$ is a generating twisted subgroup of $\mathrm{LMlt}\,(G)$, it follows from a result of Foguel, Kinyon, and Phillips [5, p. 189] that $L(G)'$ is a normal subgroup of $\mathrm{LMlt}\,(G)$. In fact, we have the following theorem.

Theorem 3.5. *$L(G)'$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ such that $L(G)' \subseteq L(G)^{\#}$.*

Proof. The theorem follows directly from Proposition 3.10 of [5]. □

Set $\mathrm{Sym}_0(G) = \{\sigma \in \mathrm{Sym}\,(G) \colon \sigma(0) = 0\}$. Note that $L(G) \cap \mathrm{Sym}_0(G) = \{\mathrm{id}_G\}$. This implies that if G is a gyrogroup with a nonidentity gyroautomorphism, say $\mathrm{gyr}\,[a, b]$, then $L(G)$ is a proper twisted subgroup of $\mathrm{LMlt}\,(G)$. In fact, $\mathrm{gyr}\,[a, b] = L^{-1}_{a\oplus b} \circ L_a \circ L_b$ belongs to $\mathrm{LMlt}\,(G)$, but does not belong to $L(G)$ since otherwise $\mathrm{gyr}\,[a, b]$ would belong to $L(G) \cap \mathrm{Sym}_0(G) = \{\mathrm{id}_G\}$. In this case, $L(G)'$ and $L(G)^{\#}$ form *proper* normal subgroups of $\mathrm{LMlt}\,(G)$ for they are contained in $L(G)$.

The following proposition provides a sufficient condition for normality of a subgyrogroup. As an application of this proposition, we prove that the left nucleus and radical of G are normal subgyrogroups of G.

Proposition 3.6 ([11]). *If H is a subgyrogroup of G such that*

1. $\mathrm{gyr}\,[h, a] = \mathrm{id}_G$ *for all* $h \in H, a \in G$,
2. $\mathrm{gyr}\,[a, b](H) \subseteq H$ *for all* $a, b \in G$, *and*
3. $a \oplus H = H \oplus a$ *for all* $a \in G$,

then H is a normal subgyrogroup of G.

Lemma 3.7. *Let G be a gyrogroup. Then*

1. $\mathrm{gyr}\,[a, b](N_l(G)) \subseteq N_l(G)$ *for all* $a, b \in G$, *and*
2. $N_l(G) \oplus a = a \oplus N_l(G)$ *for all* $a \in G$.

Proof. (1) Set $N = N_l(G)$ and let n be an arbitrary element of N. Let $a, b \in G$. According to the commutation relation [15, Equation (14)], we have

$$L_{\mathrm{gyr}\,[a,b]n} = \mathrm{gyr}\,[a, b] \circ L_n \circ \mathrm{gyr}^{-1}[a, b].$$

Since $\mathrm{gyr}\,[a, b] \in \mathrm{LMlt}\,(G)$ and $L(N) \trianglelefteq \mathrm{LMlt}\,(G)$, it follows that $L_{\mathrm{gyr}\,[a,b]n}$ belongs to $L(N)$. Hence, $L_{\mathrm{gyr}\,[a,b]n} = L_{\tilde{n}}$ for some $\tilde{n} \in N$, which implies $\mathrm{gyr}\,[a, b]n = \tilde{n} \in N$. Since n is arbitrary, we obtain $\mathrm{gyr}\,[a, b](N) \subseteq N$.

(2) Let $a \in G$ and let $n \in N$. By the left cancellation law, $x = \ominus a \oplus (n \oplus a)$ is such that $n \oplus a = a \oplus x$. We compute

$$\begin{aligned} L_x &= L_{\ominus a \oplus (n \oplus a)} \\ &= L_{(\ominus a \oplus n) \oplus a} \\ &= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, a] \\ &= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, a \oplus (\ominus a \oplus n)] \\ &= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, n] \\ &= L_{\ominus a \oplus n} \circ L_a \\ &= L_{\ominus a} \circ L_n \circ \mathrm{gyr}^{-1}[\ominus a, n] \circ L_a \\ &= L_a^{-1} \circ L_n \circ L_a. \end{aligned}$$

We obtain the second equation since $n \in N = N_m(G)$; the third and seventh equations from the identity $L_{a \oplus b} = L_a \circ L_b \circ \mathrm{gyr}^{-1}[a, b]$; the forth equation from the right loop property; the sixth and last equations since $n \in N$. Since $L(N) \trianglelefteq \mathrm{LMlt}\,(G)$, we have $L_x \in L(N)$, which implies $x \in N$. Thus, $N \oplus a \subseteq a \oplus N$.

From Lemma 2.19 of [19], we can let $y \in G$ be such that $a \oplus n = y \oplus a$. To conclude that $a \oplus N \subseteq N \oplus a$, we have to show that y belongs to N. In fact, one obtains similarly that $L_n = L_a^{-1} \circ L_y \circ L_a$, which implies $L_y = L_a \circ L_n \circ L_a^{-1} \in L(N)$. Hence, $y \in N$, as desired. □

Theorem 3.8. *The left nucleus of G is a normal subgroup of G.*

Proof. The theorem follows immediately from Theorem 3.1, Proposition 3.6, the defining property of $N_l(G)$, and Lemma 3.7. □

Corollary 3.9. *The middle nucleus of G is a normal subgroup of G.*

Proof. This is because the left nucleus and middle nucleus of G are the same. □

Following [5], the *radical of G*, denoted by $\mathrm{Rad}\,(G)$, is defined by

$$\mathrm{Rad}\,(G) = \{a \in G \colon L_a \in L(G)'\}. \tag{6}$$

Theorem 3.10. *The radical of G is a subgroup of G contained in the left nucleus of G.*

Proof. First, we prove that $\mathrm{Rad}\,(G) \subseteq N_l(G)$. Let $a \in \mathrm{Rad}\,(G)$. Then $L_a \in L(G)'$. By Theorem 3.5, $L(G)' \subseteq L(G)^{\#}$ and by Theorem 5.7 of [5], $L(G)^{\#} = L(N_l(G))$. It follows that $L_a \in L(N_l(G))$, which implies $a \in N_l(G)$.

Let $a \in \mathrm{Rad}\,(G)$. Then $L_{\ominus a} = L_a^{-1} \in L(G)'$ for $L(G)' \leqslant \mathrm{LMlt}\,(G)$. Hence, $\ominus a \in \mathrm{Rad}\,(G)$. Let $a, b \in \mathrm{Rad}\,(G)$. Since $\mathrm{Rad}\,(G) \subseteq N_l(G)$, $\mathrm{gyr}\,[a, b] = \mathrm{id}_G$. Thus, $L_{a \oplus b} = L_a \circ L_b \circ \mathrm{gyr}^{-1}[a, b] = L_a \circ L_b \in L(G)'$. This proves $a \oplus b \in \mathrm{Rad}\,(G)$ and by the subgyrogroup criterion, $\mathrm{Rad}\,(G) \leqslant G$. Since $N_l(G)$ is a subgroup of G, so is $\mathrm{Rad}\,(G)$. □

Lemma 3.11. *Let G be a gyrogroup. Then*

1. $\mathrm{gyr}\,[a, b](\mathrm{Rad}\,(G)) \subseteq \mathrm{Rad}\,(G)$ *for all* $a, b \in G$, *and*
2. $\mathrm{Rad}\,(G) \oplus a = a \oplus \mathrm{Rad}\,(G)$ *for all* $a \in G$.

Proof. The proof of this lemma follows the same steps as in the proof of Lemma 3.7 with appropriate modifications. □

Theorem 3.12. *The radical of G is a normal subgroup of G.*

Proof. The theorem follows directly from Proposition 3.6, Theorem 3.10, and Lemma 3.11. □

Note that a gyrogroup G is a group if and only if G equals its left nucleus. Hence, if G is a gyrogroup that is not a group, then $N_l(G)$ and $\mathrm{Rad}\,(G)$ are *proper* normal subgroups of G. Note also that normality of $N_l(G)$ and $\mathrm{Rad}\,(G)$ in G follows from normality of $L(N_l(G))$ and $L(L(G)')$ in the left multiplication group of G, see the proof of Lemma 3.7.

4. Subgyrogroups of Prime Index

Motivated by the study of subgroups of prime index in [9], we study subgyrogroups of prime index. Specifically, we are going to prove a gyrogroup version of the following well-known result in abstract algebra: if Ξ is a subgroup of a finite group Γ such that the index $[\Gamma\colon \Xi]$ is the smallest prime dividing the order of Γ, then Ξ is normal in Γ [9, Theorem 1]. It is notable that normality of a subgyrogroup H of a finite gyrogroup G, where $[G\colon H]$ is the smallest prime dividing the order of G, depends on the invariance of the left cosets of H in G under the gyroautomorphisms of G, see Theorem 4.4.

Unless stated otherwise, G is an arbitrary finite gyrogroup.

Let G be a gyrogroup, let $a \in G$, and let $m \in \mathbb{Z}$. Define recursively the following notation:

$$0a = 0, \quad ma = a \oplus ((m-1)a),\ m \geq 1, \quad ma = (-m)(\ominus a),\ m < 0. \tag{7}$$

By induction, one can verify the following usual rules of integral multiples:

1. $(-m)a = \ominus(ma) = m(\ominus a)$,
2. $(m+k)a = (ma) \oplus (ka)$, and
3. $(mk)a = m(ka)$

for all $a \in G$ and $m, k \in \mathbb{Z}$.

Theorem 4.1. *Suppose that H is a subgyrogroup of a gyrogroup G such that $[G\colon H] = p$, p being a prime. The following are equivalent:*

1. *For any* $a \in G - H$, $pa \in H$.

2. *For any* $a \in G - H$, $na \in H$ *for some positive integer* n, *depending on* a, *with no prime divisor less than* p.

3. *For any* $a \in G - H$, $a, 2a, \ldots, (p-1)a \notin H$.

Proof. (1) $\Rightarrow$ (2) Choosing $n = p$ gives item (2).

(2) $\Rightarrow$ (3) Let $a \in G - H$ and let n be as in item (2). By the well-ordering principle, we can let s be the smallest positive integer such that $sa \in H$. Note that $s > 1$. Write $n = st + r$ with $0 \leq r < s$. Then $ra = (n - st)a = (na) \oplus (-st)a = (na) \ominus (t(sa))$. Thus, $ra \in H$ for $na, sa \in H$. The minimality of s forces $r = 0$, so $n = st$. If $s < p$, then s (and hence n) would have a prime divisor less than p. Hence, $s \geq p$, which implies $a, 2a, \ldots, (p-1)a \notin H$.

(3) $\Rightarrow$ (1) First, we prove that $0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H$ are all distinct. Assume to the contrary that $ra \oplus H = sa \oplus H$ for some integers r and s such that $0 \leq r < s \leq p - 1$. Then $sa = (ra) \oplus h$ for some $h \in H$. It follows that $(-r + s)a = \ominus(ra) \oplus (sa) = h \in H$. This contradicts the assumption because $0 < s - r < p$.

Since $|G/H| = [G\colon H] = p$, we have $G/H = \{0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H\}$. Hence, $pa \oplus H = ta \oplus H$ for some t with $0 \leq t \leq p - 1$. As before, the equality gives $(p - t)a \in H$. Since $0 \leq t \leq p - 1$, we have $1 \leq p - t \leq p$. By assumption, $p - t = p$, which implies $t = 0$. Hence, $pa \oplus H = 0 \oplus H = H$ and so $pa \in H$. □

Proposition 4.2. *Let* H *be a subgyrogroup of a gyrogroup* G *such that* $[G\colon H] = p$, p *being a prime. If* H *satisfies one of the conditions in Theorem 4.1, then*

$$G/H = \{0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H\}$$

for any $a \in G - H$.

Proof. There is no loss in assuming that H satisfies condition (3) of Theorem 4.1. As proved in Theorem 4.1, $G/H = \{0 \oplus H, \ldots, (p-1)a \oplus H\}$ for any $a \notin H$. □

Proposition 4.3. *Let* H *be a subgyrogroup of* G. *If* $[G\colon H]$ *is the smallest prime dividing the order of* G, *then* H *satisfies condition* (2) *of Theorem 4.1.*

Proof. By Proposition 6.1 of [13], $|G|a = 0 \in H$. Since $|G|$ has no prime divisors less than $[G\colon H]$, condition (2) of Theorem 4.1 holds. □

Theorem 4.4. *Let* H *be a subgyrogroup of* G *such that* $[G\colon H]$ *is the smallest prime dividing the order of* G. *Then* $H \trianglelefteq G$ *if and only if there is an element* $y \in G - H$ *such that*

$$\mathrm{gyr}\,[a, b](iy \oplus H) \subseteq iy \oplus H$$

for all $a, b \in G$ *and* $i \in \{0, 1, \ldots, p-1\}$.

Proof. Set $[G\colon H] = p$.

($\Rightarrow$) Suppose that $H \trianglelefteq G$. Then G/H admits the gyrogroup structure and becomes a gyrogroup of order p. By Theorem 6.2 of [13], G/H forms a cyclic group. In particular, $\mathrm{gyr}\,[X,Y]Z = Z$ for all $X,Y,Z \in G/H$. Let a,b,c be arbitrary elements of G. Set $X = a \oplus H$ and $Y = b \oplus H$. From Theorem 27 of [15], we have $c \oplus H = \mathrm{gyr}\,[X,Y](c \oplus H) = (\mathrm{gyr}\,[a,b]c) \oplus H$. Since $H \trianglelefteq G$, $\mathrm{gyr}\,[a,b](H) = H$, which implies $(\mathrm{gyr}\,[a,b]c) \oplus H = \mathrm{gyr}\,[a,b](c \oplus H)$. Hence, $\mathrm{gyr}\,[a,b](c \oplus H) = c \oplus H$.

($\Leftarrow$) Let y be as in the assumption. By Propositions 4.2 and 4.3,

$$G/H = \{0 \oplus H, y \oplus H, \dots, (p-1)y \oplus H\}.$$

For each $x \in G$, $x \oplus H = iy \oplus H$ for some $i \in \{0,1,\dots,p-1\}$. By assumption,

$$\mathrm{gyr}\,[a,b](x \oplus H) = \mathrm{gyr}\,[a,b](iy \oplus H) \subseteq iy \oplus H = x \oplus H.$$

By Theorem 4.5 of [12], G acts on G/H by left gyroaddition. By Proposition 3.5 (2) and Theorem 4.6 of [12], $\ker\dot{\varphi} \subseteq H$, where $\dot{\varphi}$ is the associated permutation representation of G. By the first isomorphism theorem [15, Theorem 28],

$$G/\ker\dot{\varphi} \cong \mathrm{Im}\,\dot{\varphi} \leqslant \mathrm{Sym}\,(G/H).$$

Hence, $[G\colon \ker\dot{\varphi}]$ divides $p!$. Since $\ker\dot{\varphi} \leqslant_L G$ and $H \leqslant_L G$, we have

$$[G\colon \ker\dot{\varphi}] = [G\colon H][H\colon \ker\dot{\varphi}] = p[H\colon \ker\dot{\varphi}],$$

which implies $[H\colon \ker\dot{\varphi}]$ divides $(p-1)!$. If $[H\colon \ker\dot{\varphi}] > 1$, one would find a prime q dividing $[H\colon \ker\dot{\varphi}]$ and would have $q|(p-1)!$. Thus, $q < p$ and q divides $|H|$. Since $|H|$ divides $|G|$, we have q divides $|G|$, a contradiction. Hence, $[H\colon \ker\dot{\varphi}] = 1$ and so $H = \ker\dot{\varphi} \trianglelefteq G$. □

Recall from abstract algebra that a subgroup of a group Γ of index two is normal in Γ. This result can be generalized to the case of gyrogroups as follows.

Theorem 4.5. *If H is a subgyrogroup of G such that $\mathrm{gyr}\,[a,b](H) \subseteq H$ for all $a,b \in G$ and $[G\colon H] = 2$, then $H \trianglelefteq G$.*

Proof. Let $y \in G - H$ be fixed. By Propositions 4.2 and 4.3, $G/H = \{H, y \oplus H\}$. To complete the proof, we show that $\mathrm{gyr}\,[a,b](y \oplus H) \subseteq y \oplus H$ for all $a,b \in G$. If $z \in \mathrm{gyr}\,[a,b](y \oplus H)$, then $z = \mathrm{gyr}\,[a,b](y \oplus h) = (\mathrm{gyr}\,[a,b]y) \oplus (\mathrm{gyr}\,[a,b]h)$ for some $h \in H$. By assumption, $z \in (\mathrm{gyr}\,[a,b]y) \oplus H$. Note that $\mathrm{gyr}\,[a,b]y \notin H$ since otherwise $\mathrm{gyr}\,[a,b]y = \tilde{h} \in H$ would imply $y = \mathrm{gyr}^{-1}[a,b]\tilde{h} = \mathrm{gyr}\,[b,a]\tilde{h} \in H$, a contradiction. Hence, $(\mathrm{gyr}\,[a,b]y) \oplus H = y \oplus H$ and so $z \in y \oplus H$. This proves $\mathrm{gyr}\,[a,b](x \oplus H) \subseteq x \oplus H$ for all $a,b,x \in G$. By Theorem 4.4, $H \trianglelefteq G$. □

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