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Normed Gyrolinear Spaces: A Generalization of Normed Spaces Based on Gyrocommutative Gyrogroups

Toshikazu Abe[⋆]

Abstract

In this paper, we consider a generalization of the real normed spaces and give some examples.

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1. Introduction

A magma $(S, \circ)$ is a set S with a binary operation $\circ : S \times S \to S$, $(a, b) \mapsto a \circ b$ for any $a, b \in S$. An automorphism ϕ of a magma $(S, \circ)$ is a bijection $\phi : S \to S$ which preserves the magma operation, that is $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for any $a, b \in S$. The set of all automorphisms of $(S, \circ)$ is denoted by $\mathrm{Aut}(S, \circ)$. If there exists an element $e \in (S, \circ)$ such that $e \circ a = a \circ e = a$ for any $a \in S$, then e is called the identity of $(S, \circ)$. For $a \in (S, \circ)$, if there exists an element $a' \in (S, \circ)$ such that $a \circ a' = a' \circ a = e$, then a' is called the inverse of a.

A magma $(G, \oplus)$ is called a gyrogroup if it satisfies the following (G1) to (G5).

(G1) $(G, \oplus)$ has the identity e.

(G2) For any $a \in (G, \oplus)$, a has the inverse $\ominus a$.

(G3) For any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a, b]c$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$$

[⋆]Corresponding author (E-mail: abebin08@gmail.com)
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(G4) For any $a, b \in G$, the map $\mathrm{gyr}[a, b] : G \to G$ defined by $c \mapsto \mathrm{gyr}[a, b]c$ for any c is an automorphism of the magma $(G, \oplus)$, that is $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$. The map $\mathrm{gyr}[a, b]$ is called a gyroautomorphism of $(G, \oplus)$ generated by a and b.

(G5) For any $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.

A gyrogroup $(G, \oplus)$ is gyrocommutative if the following (G6) is satisfied.

(G6) For any $a, b \in G$, $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$.

A concrete example of a gyrocommutative gyrogroup is provided by the addition of relativistically admissible velocities in Einstein's special relativity, and another concrete example is provided by the Poincaré disk model of hyperbolic geometry. Certain gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces. The gyrovector spaces are a generalization of the real inner product spaces, where addition is not necessarily a commutative group but a gyrocommutative gyrogroup. Ungar studied gyrogroups and gyrovector spaces in several books [4, 5, 6, 7, 8, 9, 10]).

The author and O. Hatori define in [1] the generalized gyrovector spaces and give a Mazur-Ulam type theorem for generalized gyrovector spaces. The generalized gyrovector spaces is a common generalization of the gyrovector spaces and of the real normed spaces. A typical example of a generalized gyrovector space is the positive cone of a unital C^*-algebra. The definition of the generalized gyrovector spaces is as follows.

Definition 1.1. [1] Let $(G, \oplus)$ be a gyrocommutative gyrogroup with the map $\otimes : \mathbb{R} \times G \to G$. Let ϕ be an injection from G into a real normed space $(\mathbb{V}, \|\cdot\|)$. We say that $(G, \oplus, \otimes, \phi)$ (or $(G, \oplus, \otimes)$ just for a simple notation) is a generalized gyrovector space or a GGV in short if the following conditions (GGV0) to (GGV8) are fulfilled:

(GGV0) $\|\phi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a})\| = \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$;

(GGV1) $1 \otimes \boldsymbol{a} = \boldsymbol{a}$ for every $\boldsymbol{a} \in G$;

(GGV2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV4) $(\phi(|r| \otimes \boldsymbol{a}))/\|\phi(r \otimes \boldsymbol{a})\| = \phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G \setminus \{\boldsymbol{e}\}, r \in \mathbb{R} \setminus \{0\}$, where $\boldsymbol{e}$ denotes the identity element of the gyrogroup $(G, \oplus)$;

(GGV5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R}$;

(GGV6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G$ for any $\boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGVV) $\|\phi(G)\| = \{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\|\phi(r \otimes \boldsymbol{a})\| = |r| \otimes' \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(GGV8) $\|\phi(\boldsymbol{a} \oplus \boldsymbol{b})\| \leq \|\phi(\boldsymbol{a})\| \oplus' \|\phi(\boldsymbol{b})\|$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

One may feel that this definition is complicated. In this paper, we give a definition of a generalization of the real normed spaces, which is simpler and more general than the generalized gyrovector spaces. Also, we give some examples of such a space.

2. Definitions and Examples

In the following definition 2.1 we extract the algebraic structures from a gyrovector space (or a generalized gyrovector space). For consistency, we use the term "gyrolinear space" in this paper.

Definition 2.1. Let $(X, \oplus)$ be a gyrocommutative gyrogroup. Let $\otimes$ be a map $\otimes : \mathbb{R} \times X$, $(r, \boldsymbol{x}) \mapsto r \otimes \boldsymbol{x}$. We say that $(X, \oplus, \otimes)$ is a gyrolinear space if it satisfies the following conditions:

(GL1) $1 \otimes \boldsymbol{x} = \boldsymbol{x}$;

(GL2) $(r_1 + r_2) \otimes \boldsymbol{x} = (r_1 \otimes \boldsymbol{x}) \oplus (r_2 \otimes \boldsymbol{x})$;

(GL3) $(r_1 r_2) \otimes \boldsymbol{x} = r_1 \otimes (r_2 \otimes \boldsymbol{x})$;

(GL4) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{x}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{x}$;

(GL5) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = \mathrm{id}_X$;

for any $r, r_1, r_2 \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in X$.

We consider a generalization of normed spaces in Definition 2.2. For convenience, we use the term "normed gyrolinear space" in this paper.

Definition 2.2. Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let $\|\cdot\|$ be a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$, $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|$. Let f be a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\geq 0}$, where $\|X\| = \{\|\boldsymbol{x}\| \in \mathbb{R}_{\geq 0}; \boldsymbol{x} \in X\}$. We say that $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space if it satisfies the following conditions:

(NG1) $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$;

(NG2) $f(\|\boldsymbol{x} \oplus \boldsymbol{y}\|) \leq f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|)$;

(NG3) $f(\|r \otimes \boldsymbol{x}\|) = |r| f(\|\boldsymbol{x}\|)$;

(NG4) $\|\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$;

for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

Lemma 2.3. *Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let $\|\cdot\|$ be a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$, $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|$. Put $\|X\| = \{\|\boldsymbol{x}\| \in \mathbb{R}_{\geq 0}; \boldsymbol{x} \in X\}$ and $\pm\|X\| = \{\pm\|\boldsymbol{x}\| \in \mathbb{R}; \boldsymbol{x} \in X\}$. Then the following three properties are equivalent.*

(a1) *There is a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\geq 0}$ which satisfies the conditions* (NG1) *to* (NG4) *for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.*

(a2) *There is a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $f(0) = 0$, which satisfies the conditions* (NG1) *to* (NG4) *for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.*

(a3) *There is a one dimensional real linear space $(\pm\|X\|, \oplus', \otimes')$ with addition $\oplus'$ and scalar multiplication $\otimes'$, which satisfies the following conditions:*

(R1) $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$;

(R2) $\|\boldsymbol{x} \oplus \boldsymbol{y}\| \leq \|\boldsymbol{x}\| \oplus' \|\boldsymbol{y}\|$;

(R3) $\|r \otimes \boldsymbol{x}\| = |r| \otimes' \|\boldsymbol{x}\|$;

(R4) $\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$;

for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

Proof. (a1) $\Rightarrow$ (a2) : Let f be a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\geq 0}$ which satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$. Note that $f(0) = 0$. Define the map $\tilde{f} : \pm\|X\| \to \mathbb{R}$ by

$$\tilde{f}(a) = \begin{cases} f(a) & (a \in \|X\|) \\ -f(-a) & (-a \in \|X\|) \end{cases}$$

then $\tilde{f}$ is a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $\tilde{f}(0) = 0$. It is trivial that $\tilde{f}$ satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

(a2) $\Rightarrow$ (a3) : Let $\tilde{f}$ be a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $\tilde{f}(0) = 0$, which satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$. Define the two operations $\oplus'_f : \pm\|X\| \times \pm\|X\| \to \pm\|X\|$ and $\otimes'_f : \mathbb{R} \times \pm\|X\| \to \pm\|X\|$ by

$$\begin{aligned} a \oplus'_f b &= f^{-1}(f(a) + f(b)), \\ r \otimes'_f a &= f^{-1}(rf(a)) \end{aligned}$$

for any $a, b \in \pm\|X\|$ and $r \in \mathbb{R}$. Then $(\pm\|X\|, \oplus', \otimes')$ is a one dimensional real linear space. It is easy to check $(\pm\|X\|, \oplus', \otimes')$ satisfies the conditions (R1) to (R4).

(a3) $\Rightarrow$ (a1) : Let $(\pm\|X\|, \oplus', \otimes')$ be a one dimensional real linear space which satisfies the conditions (R1) to (R4). Note that 0 is the origin of the linear space $\pm\|X\|$, since $0 \otimes' \|\boldsymbol{x}\| = \|0 \otimes \boldsymbol{x}\| = \|\boldsymbol{e}\| = 0$. Since $(\pm\|X\|, \oplus', \otimes')$ is isomorphic to

$\mathbb{R}$, usual real line, there is an isomorphism $g : \pm\|X\| \to \mathbb{R}$. Since 0 is the origin of $\pm\|X\|$, we have $g(0) = 0$. Note that $-g$ is also an isomorphism from $\pm\|X\|$ to $\mathbb{R}$. Let $\boldsymbol{x}_0 \in X \setminus \{\boldsymbol{e}\}$. We can assume that $a_0 = g(\|\boldsymbol{x}_0\|) > 0$.

First, we prove that $g(\|y\|) > 0$ for any $\boldsymbol{y} \in X \setminus \{\boldsymbol{e}\}$. Assume that there is $\boldsymbol{y} \in X$ such that $g(\|\boldsymbol{y}\|) < 0$. Put $A = \{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}$ and $B = \{\|r \otimes \boldsymbol{y}\|; r \in \mathbb{R}\}$. Clearly, $A \cup B \subset \|X\|$. Since $g(\|r \otimes \boldsymbol{x}_0\|) = g(|r| \otimes' \|\boldsymbol{x}_0\|) = |r|g(\|\boldsymbol{x}_0\|)$, we have $g(A) = \mathbb{R}_{\geq 0}$. Similarly, since $g(\|r \otimes \boldsymbol{y}\|) = g(|r| \otimes' \|\boldsymbol{y}\|) = |r|g(\|\boldsymbol{y}\|)$, we have $g(B) = \mathbb{R}_{\leq 0}$. Thus we have $g(\|X\|) \supset g(A \cup B) = \mathbb{R}$. However, g is a bijection from $\pm\|X\|$ to $\mathbb{R}$, and $\|X\|$ is a proper subset of $\pm\|X\|$. It is a contradiction. So, we have $g(\|\boldsymbol{y}\|) > 0$ for any $\boldsymbol{y} \in X \setminus \{\boldsymbol{e}\}$.

Since g is a bijecton, $g(\{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}) = \mathbb{R}_{\geq 0}$ and $g(\boldsymbol{y}) \in \mathbb{R}_{\geq 0}$ for any $\boldsymbol{y} \in X$, we have $\|X\| = \{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}$. Put $f = g|_{\|X\|}$ then f is a bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$.

Next, we prove that f is a strictly monotone increasing function. For $\boldsymbol{x} \in X$ and $0 \leq \alpha \leq \beta$, we have

$$\begin{aligned}
\alpha \otimes \|\boldsymbol{x}\| &= \|\alpha \otimes \boldsymbol{x}\| \\
&= \left\| \left(\frac{\beta + \alpha}{2} - \frac{\beta - \alpha}{2} \right) \otimes \boldsymbol{x} \right\| \\
&= \left\| \left(\frac{\beta + \alpha}{2} \right) \otimes \boldsymbol{x} \oplus \left(-\frac{\beta - \alpha}{2} \right) \otimes \boldsymbol{x} \right\| \\
&\leq \left(\frac{\beta + \alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \oplus' \left(\frac{\beta - \alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \\
&= \left(\frac{\beta + \alpha}{2} + \frac{\beta - \alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \\
&= \beta \otimes' \|\boldsymbol{x}\|.
\end{aligned}$$

Therefore, we have

$$0 < \alpha < \beta \iff 0 < \alpha \otimes' \|\boldsymbol{x}\| < \beta \otimes' \|\boldsymbol{x}\| \tag{1}$$

for any $\boldsymbol{x} \in X \setminus \{\boldsymbol{e}\}$. Let $a, b \in \|X\|$ and let $\alpha = f(a)/f(\|\boldsymbol{x}_0\|)$, $\beta = f(b)/f(\|\boldsymbol{x}_0\|)$. Then we have

$$\alpha \otimes' \|\boldsymbol{x}_0\| = f^{-1}(\alpha f(\|\boldsymbol{x}_0\|)) = a.$$

Similarly, we have $\beta \otimes' \|\boldsymbol{x}_0\| = b$. Clearly, $\alpha, \beta \geq 0$ and hence

$$0 < \alpha < \beta \iff 0 < a < b$$

as (1). By the definition of α and β, it is trivial that $0 < f(a) < f(b) \iff 0 < \alpha < \beta$. Thus we have,

$$0 < a < b \iff 0 < f(a) < f(b),$$

f is a strictly monotone increasing function.

Recall that

$$f(\|\boldsymbol{x}\| \oplus' \|\boldsymbol{y}\|) = f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|)$$

and

$$f(r \otimes' \|\boldsymbol{x}\|) = rf(\|\boldsymbol{x}\|)$$

for any $\boldsymbol{x}, \boldsymbol{y} \in X$ and $r \in \mathbb{R}$ as f is a restriction of g. Since f is a strictly monotone increasing function, $(\pm\|X\|, \oplus', \otimes')$ satisfies the conditions (R1) to (R4), it is clear that f satisfies the conditions (NG1) to (NG4). $\square$

In the sequel, for a normed gyrolinear space $(X, \oplus, \otimes, \|\cdot\|, f)$, $\tilde{f}$ denotes the function $\tilde{f} : \pm\|X\| \to \mathbb{R}$ which is defined by

$$\tilde{f}(a) = \begin{cases} f(a) & (a \in \|X\|) \\ -f(-a) & (-a \in \|X\|) \end{cases}.$$

Moreover, $(\pm\|X\|, \oplus'_f, \otimes'_f)$ denotes the one dimensional real vector space which is defined by

$$\begin{aligned} a \oplus'_f b &= \tilde{f}^{-1}(\tilde{f}(a) + \tilde{f}(b)), \\ r \otimes'_f a &= \tilde{f}^{-1}(r\tilde{f}(a)) \end{aligned}$$

for any $a, b \in \pm\|X\|$ and $r \in \mathbb{R}$. The following proposition 2.4 is an immediate consequence of Lemma 2.3. The proposition is followed by examples 2.5, 2.6, 2.7 and 2.8.

Proposition 2.4. *Let $(G, \oplus, \otimes, \phi)$ be a GGV with $\phi : G \to (\mathbb{V}, \|\cdot\|)$. Then there is a bijection $f : \|\phi(G)\| \to \mathbb{R}$ which satisfies $\oplus'_f = \oplus'$ and $\otimes'_f = \otimes'$ as Proposition 2.3. We have $(G, \oplus, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = \|\phi(\cdot)\|$. Note that, if $(G, \oplus, \otimes)$ is a gyrovector space, then G is a subset of $\mathbb{V}$ and ϕ is the identity map. Hence $\|\cdot\|' = \|\cdot\|$.*

Example 2.5. A normed vector space $(\mathbb{V}, \|\cdot\|)$ is a normed gyrolinear space $(\mathbb{V}, +, \times, \|\cdot\|, \mathrm{id})$, where $+$ is the vector addition of $\mathbb{V}$, $\times$ is the scalar multiplication of $\mathbb{V}$ and id is the identity map $\mathrm{id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

The admissible velocities in special relativity is a gyrovector space (cf. [6]). Following Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.6. The Einstein gyrovector space is a normed gyrolinear space $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|, \tanh^{-1}\frac{\cdot}{c})$. Note that c is a speed of light in vacuum, $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^3$ and $\mathbb{R}^3_c = \{\boldsymbol{u} \in \mathbb{R}^3; \|\boldsymbol{u}\| < c\}$. The Einstein gyrogroup addition $\oplus_E$ is given by

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle}{c^2}} \left\{ \boldsymbol{u} + \frac{1}{\gamma_u}\boldsymbol{v} + \frac{1}{c^2}\frac{\gamma_u}{1+\gamma_u}\langle \boldsymbol{u}, \boldsymbol{v}\rangle \boldsymbol{u} \right\}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3_c,$$

where $\langle \cdot, \cdot, \rangle$ is the Euclidean inner product of $\mathbb{R}^3$ and γ_u is a Lorenz factor of $\boldsymbol{u}$,

$$\gamma_u = (1 - \|\boldsymbol{u}\|^2/c^2)^{-\frac{1}{2}}.$$

The Einstein scalar multiplication $\otimes_E$ is given by

$$r \otimes_E \boldsymbol{u} = \begin{cases} c \tanh(r \tanh^{-1} \frac{\|\boldsymbol{u}\|}{c}) \frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} & (\boldsymbol{u} \in \mathbb{R}^3_c \setminus \{\boldsymbol{0}\}) \\ 0 & (\boldsymbol{u} = \boldsymbol{0}) \end{cases}$$

for any $r \in \mathbb{R}$.

The Poincaré disk model is an example of a gyrovector space, and it is called the Möbius gyrovector space (cf. [6]). Following Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.7. The Möbius gyrovector space is a normed gyrolinear space $(\mathbb{D}, \oplus_M, \otimes_M, |\cdot|, \tanh^{-1})$. Note that $\mathbb{D}$ is the open unit disc of complex plane $\mathbb{C}$. The Möbius gyrogroup addition is given by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b}, \qquad \forall a, b \in \mathbb{D}.$$

The Möbius scalar multiplication $\otimes_M$ is given by

$$r \otimes_E \boldsymbol{u} = \begin{cases} \tanh(r \tanh^{-1} |a|) \frac{a}{|a|} & (a \in \mathbb{D} \setminus \{0\}) \\ 0 & (a = 0) \end{cases}$$

for any $r \in \mathbb{R}$.

The positive cone of a unital C^*-algebra is a exapmle of a generalized gyrovector space (cf [1]). As Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.8. Let $\mathscr{A}$ be a unital C^*-algebra with the norm $\|\cdot\|$ and $\mathscr{A}_+^{-1}$ be the positive cone of $\mathscr{A}$. Define the binary operation $\oplus_A$ on $\mathscr{A}_+^{-1}$ by

$$a \oplus_A b = a^{\frac{1}{2}} b a^{\frac{1}{2}}, \quad a, b \in \mathscr{A}_+^{-1}.$$

Define the scalar multiplication $\otimes_A : \mathbb{R} \times \mathscr{A}_+^{-1} \to \mathscr{A}_+^{-1}$ by

$$r \otimes_A a = a^r, \quad r \in \mathbb{R}, a \in \mathscr{A}_+^{-1}$$

and the norm $\|\cdot\|' = \|\log \cdot\|$. Then $(\mathscr{A}_+^{-1}, \oplus_A, \otimes_A, \|\cdot\|', \mathrm{id})$ is a normed gyrolinear space, where id is the identity map $\mathrm{id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

The density matrices is an example of a gyrolinear space.

Example 2.9. A qubit density matrix is a 2×2 positive semidefinite Hermitian matrix with trace 1. Let $\boldsymbol{D}$ be the set of all invertible qubit density matrices. Define a binary operation on $\boldsymbol{D}$ by

$$A \oplus B = \frac{A^{\frac{1}{2}} B A^{\frac{1}{2}}}{\mathrm{Tr}(A^{\frac{1}{2}} B A^{\frac{1}{2}})},$$

then $(\boldsymbol{D}, \oplus)$ is a gyrocommutative gyrogroup ([2]). The identity of $(\boldsymbol{D}, \oplus)$ is $\dfrac{1}{2}E$, where E is the identity matrix. The inverse of $A \in \boldsymbol{D}$ is $\ominus A = \dfrac{A^{-1}}{\mathrm{Tr}A^{-1}}$. The gyroautomorphism $\mathrm{gyr}[A,B]$ is given by $\mathrm{gyr}[A,B]C = \dfrac{XCX^*}{\mathrm{Tr}(XCX^*)}$ for any $C \in \boldsymbol{D}$, where $X = X(A,B)$ is a unitary matrix given by $X = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{-\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}$. Define the map $\otimes \mathbb{R} \times \boldsymbol{D} \to \boldsymbol{D}$ by

$$r \otimes A = \frac{A^r}{\mathrm{Tr}A^r}$$

then $(\boldsymbol{D}, \oplus, \otimes)$ is a gyrolinear space. Actually, $(\boldsymbol{D}, \oplus, \otimes)$ satisfies the conditions (GL1) to (GL5) as follows.

(GL1): $1 \otimes A = \frac{A^1}{\mathrm{Tr}A^1} = A$, since $\mathrm{Tr}A = 1$.

(GL2): We have

$$\begin{aligned}(r \otimes A) \oplus (s \otimes A) &= \frac{(\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}} (\frac{A^s}{\mathrm{Tr}A^s}) (\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}}}{\mathrm{Tr}((\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}} (\frac{A^s}{\mathrm{Tr}A^s}) (\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}})} \\ &= \frac{A^{r+s}}{\mathrm{Tr}A^{r+s}} = (r+s) \otimes A.\end{aligned}$$

(GL3): We have

$$r \otimes (s \otimes A) = \frac{(\frac{A^s}{\mathrm{Tr}A^s})^r}{\mathrm{Tr}(\frac{A^s}{\mathrm{Tr}A^s})^r} = \frac{A^{rs}}{\mathrm{Tr}A^{rs}} = (rs) \otimes A.$$

(GL4): Put $X = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{-\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}$, then

$$\begin{aligned}\mathrm{gyr}[A,B](r \otimes C) &= \frac{X \frac{C^r}{\mathrm{Tr}C^r} X^*}{\mathrm{Tr}(X \frac{C^r}{\mathrm{Tr}C^r} X^*)} \\ &= \frac{XC^rX^*}{\mathrm{Tr}XC^rX^*}\end{aligned}$$

Since X is unitary, we have $XC^rX^* = (XCX^*)^r$ and hence

$$\mathrm{gyr}[A,B](r \otimes C) = \frac{XC^rX^*}{\mathrm{Tr}XC^rX^*} = \frac{(XCX^*)^r}{\mathrm{Tr}(XCX^*)^r} = r \otimes \mathrm{gyr}[A,B]C.$$

(GL5): Put $X = ((r \otimes A)^{\frac{1}{2}} (s \otimes A)(r \otimes A)^{\frac{1}{2}})^{-\frac{1}{2}} (r \otimes A)^{\frac{1}{2}} (s \otimes A)^{\frac{1}{2}}$. Then

$$X = \left\{ \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \left(\frac{A^s}{\mathrm{Tr}A^s} \right) \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \right\}^{-\frac{1}{2}} \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \left(\frac{A^s}{\mathrm{Tr}A^s} \right)^{\frac{1}{2}} = E$$

and hence $\operatorname{gyr}[A,B]C = \dfrac{ECE^*}{\operatorname{Tr}(ECE^*)} = C$ for any $C \in \boldsymbol{D}$.

3. Constructing Normed Gyrolinear Spaces

In this section we construct new normed gyrolinear spaces from given normed gyrolinear spaces.

Proofs of the following Lemma 3.1 and 3.2 are elementary, easy and omitted.

Lemma 3.1. *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be (gyrocommutative) gyrogroups. Define the binary operation $\oplus$ on $G = G_1 \times G_2$ by*

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)$$

for any $(x_1, x_2), (y_1, y_2) \in G_1 \times G_2$. Then $(G, \oplus)$ is a (gyrocommutative) gyrogroup. The identity of $(G, \oplus)$ is (e_1, e_2), where e_i is the identity of $(G_i, \oplus_i)$ $(i = 1, 2)$. The inverse of $x = (x_1, x_2) \in G$ is $\ominus x = (\ominus_1 x_1, \ominus_2 x_2)$. The gyroautomorphisms are

$$\operatorname{gyr}[(x_1, x_2), (y_1, y_2)](a_1, a_2) = (\operatorname{gyr}[x_1, y_1]a_1, \operatorname{gyr}[x_2, y_2]a_2)$$

for any $(x_1, x_2), (y_1, y_2), (a_1, a_2) \in G$.

Lemma 3.2. *Let $(X_1, \oplus_1, \otimes_1)$ and $(X_2, \oplus_2, \otimes_2)$ be gyrolinear spaces. Define the binary operation $\oplus$ on $X = X_1 \times X_2$ by*

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)$$

for any $(x_1, x_2), (y_1, y_2) \in G$. Define the scalar multiplication $\otimes$ on G by

$$r \otimes (x_1, x_2) = (r \otimes_1 x_1, r \otimes_2 x_2)$$

for any $r \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in G$. Then $(G, \oplus, \otimes)$ is a gyrolinear space.

Proposition 3.3. *Let $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ and $(X_2, \oplus_2, \otimes_2, \|\cdot\|_2, f)$ be normed gyrolinear spaces. Then $X = X_1 \times X_2$ is a gyrolinear space with $\oplus$ and $\otimes$ as in Lemma 3.2. Put*

$$\|(x_1, x_2)\| = f^{-1}(f(\|x_1\|_1) + f(\|x_2\|_2))$$

for any $(x_1, x_2) \in X$. Then $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space.

Proof. Since $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ is a normed gyrolinear space, f is a bijection from $\|X_1\|_1$ to $\mathbb{R}$. Similarly, f is also a bijection from $\|X_2\|_2$ to $\mathbb{R}$. It means that $\|X_1\|_1 = \|X_2\|_2$. Since

$$f^{-1}(f(a) + f(b)) \in \|X_1\|_1 = \|X_2\|_2$$

for any $a, b \in \|X_1\|_1 = \|X_2\|_2$, we have $\|X\| = \|X_1\|_1 = \|X_2\|_2$. Thus f is a monotone increasing bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$.

(NG1): Let e_i be the identity of X_i for $i = 1, 2$, then $\boldsymbol{e} = (e_1, e_2)$ is the identity of G. Since $f(0) = 0$,

$$\begin{aligned}
\|(x_1, x_2)\| = 0 &\iff f^{-1}(f(\|x_1\|_1) + f(\|x_2\|_2)) \\
&\iff f(\|x_1\|_1) + f(\|x_2\|_2) \\
&\iff f(\|x_1\|_1) = 0 \text{ and } f(\|x_2\|_2) = 0 \\
&\iff \|x_1\|_1 = 0 \text{ and } \|x_2\|_2 = 0 \\
&\iff x_1 = e_1,\ x_2 = e_2.
\end{aligned}$$

It follows that $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$.

(NG2): Let $\boldsymbol{x} = (x_1, x_2), \boldsymbol{y} = (y_1, y_2) \in X$. We have

$$\begin{aligned}
f(\|\boldsymbol{x} \oplus \boldsymbol{y}\|) &= f(\|(x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)\|) \\
&= f(\|x_1 \oplus_1 y_1\|_1) + f(\|x_2 \oplus_2 y_2\|_2) \\
&\leq f(\|x_1\|_1) + f(\|y_1\|_1) + f(\|x_2\|_2) + f(\|y_2\|_2) \\
&= f(\|(x_1, x_2)\|) + f(\|(y_1, y_2)\|) \\
&= f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|).
\end{aligned}$$

(NG3): We have

$$\begin{aligned}
f(\|\alpha \otimes (x_1, x_2)\|) &= f(\|(\alpha \otimes_1 x_1, \alpha \otimes_2 x_2)\|) \\
&= f(\|\alpha \otimes_1 x_1\|) + f(\|\alpha \otimes_2 x_2\|) \\
&= |\alpha| f(\|x_1\|_1) + |\alpha| f(\|x_2\|_2) \\
&= |\alpha|(f(\|x_1\|_1) + f(\|x_2\|_2)) \\
&= |\alpha| f(\|(x_1, x_2)\|)
\end{aligned}$$

for any $(x_1, x_2) \in X$.

(NG4): Let $\boldsymbol{x} = (x_1, x_2), \boldsymbol{u} = (u_1, u_2), \boldsymbol{v} = (v_1, v_2) \in X$. We have

$$\begin{aligned}
f(\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\|) &= f(\|(\operatorname{gyr}[u_1, v_1](x_1), \operatorname{gyr}[u_2, v_2](x_2))\|) \\
&= f(\|(\operatorname{gyr}[u_1, v_1](x_1)\|_1) + f(\|\operatorname{gyr}[u_2, v_2](x_2))\|_2) \\
&= f(\|x_1\|_1) + f(\|x_2\|_2) \\
&= f(\|\boldsymbol{x}\|).
\end{aligned}$$

Thus $\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$. □

Proposition 3.4. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let h be a strictly monotone increasing injection (not necessarily bijection) $h : \|G\| \to \mathbb{R}_{\geq 0}$ with $h(0) = 0$. Put $\|\cdot\|' = h(\|\cdot\|)$. Then $(X, \oplus, \otimes, \|\cdot\|', fh^{-1})$ is a normed gyrolinear space.*

Proof. Note that h is a bijection from $\|X\|$ to $h(\|X\|) = \|X\|'$. Since f is a bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$, we have fh^{-1} is a bijection from $\|X\|'$ to $\mathbb{R}_{\geq 0}$. Moreover, fh^{-1} is also a strictly monotone increasing function as f and h are strictly monotone increasing.

(NG1): $\|x\|' = 0 \iff h(\|x\|) = 0 \iff \|x\| = 0 \iff x = e$.

(NG2): For any $x, y \in X$, we have

$$\begin{aligned} fh^{-1}(\|x \oplus y\|') &= f(\|x \oplus y\|) \\ &\leq f(\|x\|) + f(\|y\|) = fh^{-1}(\|x\|') + fh^{-1}(\|y\|'). \end{aligned}$$

(NG3): $fh^{-1}(\|\alpha \otimes x\|') = f(\|\alpha \otimes x\|) = |\alpha| f(\|x\|) = |\alpha| fh^{-1}(\|x\|)$.

(NG4): $\|\operatorname{gyr}[u, v]x\|' = h(\|\operatorname{gyr}[u, v]x\|) = h(\|x\|) = \|x\|'$. □

Example 3.5. The Einstein gyrovector space $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|, \tanh^{-1}\frac{\cdot}{c})$ is a normed gyrolinear space. Put $\|\cdot\|' = \tanh^{-1}\frac{\|\cdot\|}{c}$, then $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|', id)$ is also a normed gyrolinear space as Proposition 3.4.

Example 3.6. The Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M, |\cdot|, \tanh^{-1})$ is a normed gyrolinear space. Put $\|\cdot\| = \tanh^{-1}|\cdot|$, then $(\mathbb{D}, \oplus_M, \otimes_M, \|\cdot\|', id)$ is also a normed gyrolinear space as Proposition 3.4.

Example 3.7. Let $1 \leq p \leq \infty$ and $L_p(X)$ be the L_p space on a measure space X with a measure μ. The L_p-norm $\|\cdot\|_p$ is given by

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

Since $(L_p, \|\cdot\|_p)$ is a normed space, $(L_p, +, \times, \|\cdot\|_p, id)$ is a normed gyrolinear space. By Proposition 3.4, $(L_p, +, \times, \|\cdot\|_p^p, k)$ is also a normed gyrolinear space, where $k(x) = x^{\frac{1}{p}}$.

Proposition 3.8. *Let $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ and $(X_2, \oplus_2, \otimes_2, \|\cdot\|_2, g)$ be normed gyrolinear spaces. Then $X = X_1 \times X_2$ is a gyrolinear space with $\oplus$ and $\otimes$ as in Lemma 3.2. Let k be a strictly monotone increasing injection $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $k(0) = 0$. Put*

$$\|(a, b)\|_k = k(f(\|a\|_1) + g(\|b\|_2))$$

for any $(a, b) \in X$. Then $(X, \oplus, \otimes, \|\cdot\|_k, k^{-1})$ is a normed gyrolinear space. In particular, if $k = \mathrm{id}$, then

$$\|(a, b)\| = f(\|a\|_1) + g(\|b\|_2).$$

Proof. Put $\|\cdot\|_1' = f(\|\cdot\|_1)$ then $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1', id)$ is a gyrolinear space as Proposition 3.4. Similarly, put $\|\cdot\|_2' = g(\|\cdot\|_2)$ then $(X_2, \oplus_2, \otimes_2, \|\cdot\|_2', id)$ is also a gyrolinear space. Put $\|(a, b)\| = \|a\|_1' + \|b\|_2'$ for any $(a, b) \in X$. Then

$(X, \oplus, \otimes, \|\cdot\|, id)$ is a gyrolinear space as Proposition 3.3. Note that $\|X\| = \mathbb{R}_{\geq 0}$. Let k be a strictly monotone increasing injection $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $k(0) = 0$. Since $\|X\| = \mathbb{R}_{\geq 0}$, k is a strictly monotone increasing bijection from $\|X\|$ to $k(\|X\|)$. Put $\|\cdot\|_k = k(\|\cdot\|)$ then $(X, \oplus, \otimes, \|\cdot\|_k, k^{-1})$ is a gyrolinear space as Proposition 3.4. Note that

$$\|(a, b)\| = f(\|a\|_1) + g(\|b\|_2)$$

and

$$\|(a, b)\|_k = k(f(\|a\|_1) + g(\|b\|_2))$$

for any $(a, b) \in X$. □

The following Lemma 3.9 is trivial.

Lemma 3.9. *Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let Y be a set and ϕ be a injection $\phi : X \to Y$. Define the binary operation $\oplus_\phi$ on $\phi(X)$ by*

$$\phi(a) \oplus_\phi \phi(b) = \phi(a \oplus b)$$

and the map $\otimes_\phi : \mathbb{R} \times \phi(X) \to \phi(X)$ by

$$r \otimes_\phi \phi(a) = \phi(r \otimes a)$$

for any $r \in \mathbb{R}$ and $a, b \in X$. Then $(\phi(X), \oplus_\phi, \otimes_\phi)$ is a gyrolinear space. Moreover, if $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space, then $(\phi(X), \oplus_\phi, \otimes_\phi, \|\cdot\|', f)$ is a normed gyrolinear space, where

$$\|\phi(a)\|' = \|a\|$$

for any $a \in X$. Note that the identity of $(\phi(X), \oplus_\phi)$ is $\phi(e)$, where e is the identity of $(X, \oplus)$

Proposition 3.10. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number. Define the binary operation $\oplus_\alpha$ on G by*

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in G$. Then $(X, \oplus_\alpha, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = |\alpha| \otimes'_f \|\cdot\|$.

Proof. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number and ϕ be a map $\phi : X \to X$ which is defined by $\phi(x) = \dfrac{1}{\alpha} \otimes x$ for any $x \in X$. Note that ϕ is a bijection. Actually, $\phi^{-1}(x) = \alpha \otimes x$ as $\alpha \otimes (\dfrac{1}{\alpha} \otimes x) = 1 \otimes x = x$

since the conditions (GL3) and (GL1). By Lemma 3.9, $(X, \oplus_\alpha, \otimes_\alpha, \|\cdot\|', f)$ is a normed gyrolinear space, where

$$(\frac{1}{\alpha} \otimes x) \oplus_\alpha (\frac{1}{\alpha} \otimes y) = \frac{1}{\alpha} \otimes (x \oplus y), \tag{2}$$

$$r \otimes_\alpha (\frac{1}{\alpha} \otimes x) = \frac{1}{\alpha} \otimes (r \otimes x) \tag{3}$$

$$\|\frac{1}{\alpha} \otimes x\|' = \|x\| \tag{4}$$

for any $x, y \in X$ and $r \in \mathbb{R}$. Put $a = \frac{1}{\alpha} \otimes x$ and $b = \frac{1}{\alpha} \otimes y$ then $x = \alpha \otimes a$ and $y = \alpha \otimes b$. Hence we have

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in X$ as (2). Note that $\frac{1}{\alpha} \otimes (r \otimes x) = r \otimes (\frac{1}{\alpha} \otimes x)$ for any $r \in \mathbb{R}$ and $x \in X$ as the condition (GL3). It follows that $r \otimes_\alpha a = r \otimes_\alpha (\frac{1}{\alpha} \otimes x) = r \otimes (\frac{1}{\alpha} \otimes x) = r \otimes a$ since (3). So, we have $\oplus_\alpha = \oplus$. The equation (4) follows that $\|a\|' = \|\alpha \otimes a\| = |\alpha| \otimes'_f \|a\|$. □

Proposition 3.11. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number. Define the binary operation $\oplus_\alpha$ on G by*

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in G$. Then $(X, \oplus_\alpha, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space.

Proof. By Proposition 3.10, $(X, \oplus_\alpha, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = |\alpha| \otimes'_f \|\cdot\|$. Note that $f(\|\cdot\|) = f(\frac{1}{\alpha} \otimes'_f \|\cdot\|') = |\frac{1}{\alpha}| f(\|\cdot\|')$. Put $h(a) = f^{-1}(|\frac{1}{\alpha}| f(a))$, then h is a strictly monotone increasing bijection from $\|X\|$ to $\|X\|$. Since $\|\cdot\| = h(\|\cdot\|')$, $(X, \oplus_\alpha, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space as Proposition 3.4. □

4. Structures on a Normed Gyrolinear Space and a Mazur-Ulam Theorem

Definition 4.1. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. The gyrometric ϱ on X is defined by

$$\varrho(a, b) = \|a \ominus b\|$$

for any $a, b \in X$.

Note that the gyrometric ϱ on $(X, \oplus, \otimes, \|\cdot\|, f)$ is not necessarily a metric on X, but $d = f\varrho$ is a metric on X.

Definition 4.2. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a gyrolinear space. Put

$$L[a,b](s) = a \oplus s \otimes (\ominus a \oplus b)$$

for any $a, b \in X$ and $s \in \mathbb{R}$. We call $L[a,b](\mathbb{R})$ the unique gyroline that passes through a and b. We call $L[a,b]([0,1])$ the gyrosegment ab. We call $\boldsymbol{p}(a,b) = L[a,b](\dfrac{1}{2})$ the gyromidpoint of a and b.

The gyromidpoint $\boldsymbol{p}(a,b)$ can be rewritten by $\frac{1}{2}\otimes(a \boxplus b)$, where $\boxplus$ is a coaddition of $(X, \oplus)$.

Example 4.3. Let $(\mathbb{V}, +, \times, \|\cdot\|, id)$ be a normed space. The gyrometric $\varrho(a,b) = \|a-b\|$ is the usual metric induced by its norm. $L[a,b](s) = a + s(-a+b)$ and hence the gyroline is the line, the gyrosegment is the segment, the gyromidpoint is the arithmetic mean $\dfrac{a+b}{2}$.

Example 4.4. Let $(\mathscr{A}_+^{-1}, \oplus_A, \otimes_a, \|\cdot\|', id)$ be a normed gyrolinear space of the positive cone. The gyrometric

$$\varrho(a,b) = \|a \ominus b\|' = \|\log a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\|$$

is the Thompson metric.

$$L[a,b](s) = a^{\frac{1}{2}}(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-s} a^{\frac{1}{2}}$$

and hence the gyrosegment is the geodesic. The gyromidpoint

$$\boldsymbol{p}(a,b) = a^{\frac{1}{2}}(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-\frac{1}{2}} a^{\frac{1}{2}}$$

is the geometric mean.

The celebrated Mazur-Ulam theorem asserts that surjective isometry between two normed vector spaces is a real linear isomorphism followed by a translation. In [1], author and Hatori give a generalization of the Mazur-Ulam theorem for generalized gyrovector spaces. This theorem holds for normed gyrolinear space as the following Theorem 4.5 and Corollary 4.6. There are no gaps between proofs for normed gyrolinear spaces and for generalized gyrovector spaces in [1]. Refer to [1] for the proofs.

Theorem 4.5. *Let $(X_i, \oplus_i, \otimes_i, \|\cdot\|_i, f_i)$ be a normed gyrolinear space and ϱ_i be the gyrometric for $i = 1, 2$. Let $T : X_1 \to X_2$ be a surjection. If T preserves the gyrometric,*

$$\varrho_2(Ta, Tb) = \varrho_1(a, b)$$

for any $a, b \in X_1$, then T preserves the gyromidpoint,

$$T\boldsymbol{p}(a,b) = \boldsymbol{p}(Ta, Tb)$$

for any $a, b \in X_1$.

Corollary 4.6. *Let $(X_i, \oplus_i, \otimes_i, \|\cdot\|_i, f_i)$ be a normed gyrolinear space and ϱ_i be the gyrometric for $i = 1, 2$. Let $T : X_1 \to X_2$ be a surjection. Suppose that T preserves the gyrometric,*

$$\varrho_2(Ta, Tb) = \varrho_1(a, b)$$

for any $a, b \in X_1$. Then T is of the form $T(\cdot) = T(e_1) \oplus T(\cdot)$, where e_1 is the identity of X_1 and T_0 is an isometrical isomorphism in the sense that the equalities

$$T_0(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = T_0(\boldsymbol{a}) \oplus_2 T_0(\boldsymbol{b}); \tag{5}$$

$$T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a}); \tag{6}$$

$$\varrho_2(T_0\boldsymbol{a}, T_0\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}). \tag{7}$$

for every $\boldsymbol{a}, \boldsymbol{b} \in G_1$ and $\alpha \in \mathbb{R}$ hold.

5. A Normed Gyrolinear Space Induced by a Metric Space

5.1 A Gyrocommutative Gyrogroup Induced by a Metric Space

A dyadic symset is a magma $(X, \circ)$ satisfying for all $a, b, c \in X$ the following axioms (d1) to (d4):

(d1) $a \circ a = a$;

(d2) $a \circ (a \circ b) = b$;

(d3) $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$;

(d4) the equation $x \circ a = b$ has a unique solution $x \in X$, called the midpoint of a and b, and denoted $a\sharp b$.

In the paper [3], Lawson and Lim show a strong equivalence between pointed dyadic symsets and uniquely 2-divisible gyrocommutative gyrogroups in the following sense.

Let $(X, \circ)$ be a dyadic symset and $e \in X$. Define a new binary operation $\oplus$ on X by $x \oplus y = (e\sharp x) \circ (e \circ y)$ then $(X, \oplus)$ is a uniquely 2-divisible gyrocommutative gyrogroup with the identity e. Conversely, let $(X, \oplus)$ be a uniquely 2-divisible gyrocommutative gyrogroup. Define a new binary operation $\circ$ on X by $x \circ y = 2 \otimes x \ominus y$, then $(X, \circ)$ is a dyadic symset.

As the consequence of the fact, we have a uniquely 2-divisible gyrocommutative gyrogroup which is induced by a metric space as following Lemma 5.3.

Definition 5.1. Let (X, d) be a metric space. We say that (X, d) satisfies the condition K if the following conditions (K1) to (K3) are hold.

(K1) For any pair $x, y \in X$, there exists a unique element $c \in X$ such that

$$d(x, c) = d(c, y) = \frac{1}{2}d(x, y).$$

We call c the metric midpoint of x and y and write

$$c = \mathrm{mid}(x, y).$$

(K2) For any elements $x, y \in X$, there exists a unique element $z \in X$ such that $x = \mathrm{mid}(y, z)$. We write

$$z = \varphi_x(y)$$

and we call the map $\varphi_x : X \to X$ the metric reflection in the point x.

(K3) The metric reflection $\varphi_x : X \to X$ is an isometry for any $x \in X$.

Note that $\mathrm{mid}(x, y) = \mathrm{mid}(y, x)$. Moreover, $z = \varphi_x(y) \iff z = \mathrm{mid}(y, z) \iff y = \varphi_x(z)$ and hence $\varphi_x^{-1} = \varphi_x$.

Definition 5.2. Let (X, d) be a metric space that satisfies the condition K. For fixed $e \in X$, we define the binary operation $\oplus_e$ on X by

$$x \oplus_e y = \varphi_{\tilde{x}}\varphi_e(y),$$

where $\tilde{x} = \mathrm{mid}(e, x)$ for any $x, y \in X$. We call $\oplus_e$ the binary operation induced by the metric d on X at $e \in X$.

Theorem 5.3. *Let (X, d) be a metric space that satisfies the condition K and let $e \in X$. Let $\oplus_e$ is the binary operation on X induced by the metric d at e. Then $(X, \oplus_e)$ is a uniquely 2-divisible gyrocommutative gyrogroup.*

Proof. Let (X, d) be a metric space that satisfies the condition K and let $e \in X$. Define a binary operation $\circ$ by $x \circ y = \varphi_x(y)$.

First, we prove that $(X, \circ)$ is a dyadic symset.

(d1): $a \circ a = \varphi_a(a) = a$.

(d2): $a \circ (a \circ b) = \varphi_a\varphi_a(b) = b$.

(d3): Since φ_a is an isometry,

$$d(x, b) = d(b, y) = \frac{1}{2}d(x, y)$$

implies

$$d(\varphi_a(x), \varphi_a(y)) = d(\varphi_a(b), \varphi_a(y)) = \frac{1}{2}d((\varphi_a(x), \varphi_a(y))).$$

Therefore, $b = \mathrm{mid}(x, y)$ implies that $\varphi_a(b) = \mathrm{mid}(\varphi_a(x), \varphi_a(y))$. It follows that $\varphi_{\varphi_a(b)}(\varphi_a(x)) = \varphi_a(y)$. Since $y = \varphi_b(x)$, we have

$$(a \circ b) \circ (a \circ x) = \varphi_{\varphi_a(b)}(\varphi_a(x)) = \varphi_a(\varphi_b(x)) = a \circ (b \circ x).$$

(d4): $x \circ a = b \iff \varphi_x(a) = b \iff x = \mathrm{mid}(a, b)$. The midpoint of a and b is $a \sharp b = \mathrm{mid}(a, b)$.

Since $(X, \circ)$ is a dyadic symset, we have $(X, \oplus)$ is a uniquely 2-divisible gyrocommutative gyrogroup, where the binary operation $\oplus$ is defined by $x \oplus y = (e \sharp x) \circ (e \circ y)$. Note that $x \oplus y = (e \sharp x) \circ (e \circ y) = \varphi_{\varphi_e(x)}(\varphi_e(y))$. □

Let $(X, \oplus)$ be a gyrogroup. For $a \in X$, the left translation $\lambda_a : X \to X$ is defined by $\lambda_a(x) = a \oplus x$ for any $x \in X$. It is well known that

$$\mathrm{gyr}[a, b] = \lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b \tag{8}$$

for any $a, b \in X$.

Proposition 5.4. *Let (X, d) be a metric space which satisfies the condition K and let $e \in X$. Let $\oplus_e$ is the binary operation on X induced by the metric d at e. Then*

$$d(a \oplus_e x, a \oplus_e y) = d(x, y) \tag{9}$$

and

$$d(\mathrm{gyr}[a, b]x, \mathrm{gyr}[a, b]y) = d(x, y) \tag{10}$$

for any $a, b, x, y \in X$.

Proof. Let $a \in X$ and put $\tilde{a} = \mathrm{mid}(e, a)$. Since the condition (K3), we have $\varphi_{\tilde{a}}$ and φ_e are isometries. Hence

$$d(a \oplus_e x, a \oplus_e y) = d(\varphi_{\tilde{a}} \varphi_e x, \varphi_{\tilde{a}} \varphi_e y) = d(x, y)$$

for $x, y \in X$. It follows that

$$d(\mathrm{gyr}[a, b]x, \mathrm{gyr}[a, b]y) = d(\lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b(x), \lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b(y)) = d(x, y)$$

for any $a, b, x, y \in X$. □

5.2 Preparations

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map δ from $[0, l]$ to X such that $\delta(0) = x$, $\delta(l) = y$ and $d(\delta(t_1), \delta(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, $l = d(x, y)$. (X, d) is called a uniquely geodesic space if any pairs $x, y \in X$ has exactly one geodesic path joining x and y.

In this subsection, (X,d) is a uniquely geodesic space with condition K and $\gamma_{x,y}$ is a map from $[0,1]$ to X defined by

$$\gamma_{x,y}(t)=\delta_{x,y}(td(x,y))$$

for any $t\in[0,1]$, where $\delta_{x,y}$ denotes the geodesic path joining x and y. It is easy to show that $\gamma_{x,y}(0)=x$, $\gamma_{x,y}(1)=y$ and

$$d(\gamma_{x,y}(t_1),\gamma_{x,y}(t_2))=|t_1-t_2|d(x,y)$$

for any $t_1,t_2\in[0,1]$.

Note that $\gamma_{x,y}(s)$ is a unique point c in X which satisfies $d(x,c)=sd(x,y)$ and $d(c,y)=(1-s)d(x,y)$ for any $0\le s\le 1$.

For $x\in X$, define the map ϕ_x on X by the equation

$$\phi_x(y)=\varphi_y(x)$$

for any $y\in X$. Then we have

$$\phi_x(y)=c \iff \varphi_y(x)=c \iff \mathrm{mid}(x,c)=y.$$

It implies that ϕ_x is a bijection on X for any $x\in X$.

Lemma 5.5. *Let $x,y,z,c\in X$. Then the following holds.*

(y1) For any $s\in\mathbb{R}\setminus\{0\}$,

$$\begin{cases}d(x,z)=|s|d(x,y)\\ d(z,y)=|1-s|d(x,y)\end{cases}\iff\begin{cases}d(x,y)=|\frac{1}{s}|d(x,z)\\ d(y,z)=|1-\frac{1}{s}|d(x,z).\end{cases}$$

(y2) For any $0\le s\le 1$,

$$c=\gamma_{x,y}(s)\iff\begin{cases}d(x,c)=sd(x,y)\\ d(c,y)=(1-s)d(x,y).\end{cases}$$

(y3)

$$c=\phi_x(y)\iff\begin{cases}d(x,c)=2d(x,y)\\ d(c,y)=(2-1)d(x,y).\end{cases}$$

(y4) For any natural number n,

$$c=\phi_x^n(y)\iff\begin{cases}d(x,c)=2^nd(x,y)\\ d(c,y)=(2^n-1)d(x,y).\end{cases}$$

(y5) For any $s > 1$,

$$c = \phi_x^n(\gamma_{x,y}(\frac{s}{2^n})) \iff \begin{cases} d(x,c) = sd(x,y) \\ d(c,y) = (s-1)d(x,y), \end{cases}$$

where $n \in \mathbb{N}$ which satisfies $2^{n-1} < s \leq 2^n$.

(y6) For any $s > 0$,

$$\begin{cases} d(x,c) = sd(x,y) \\ d(c,y) = |s-1|d(x,y) \end{cases} \iff \begin{cases} d(x,c') = sd(x,y) \\ d(c',y) = (1+s)d(x,y), \end{cases}$$

where $c' = \varphi_x(c)$.

In particular, for any real number s, there exists a unique point c in X such that $d(e,c) = |s|d(e,x)$ and $d(c,x) = |1-s|d(e,x)$.

Proof. (y1) and (y2) are obvious.

(y3): We have

$$\begin{aligned} c = \phi_x(y) &\iff \operatorname{mid}(x,c) = y \\ &\iff d(x,y) = d(c,y) = \frac{1}{2}d(x,c) \\ &\iff d(x,c) = 2d(x,y) \text{ and } d(c,y) = d(x,y). \end{aligned}$$

(y4): We will prove by induction. When $n = 1$, the argument is true by (y2). Let k be a natural number and suppose that $\phi_x^k(y)$ is a unique point $c_k \in X$ which satisfies

$$d(x,c_k) = 2^k d(x,y) \text{ and } d(c_k,y) = (2^k-1)d(x,y).$$

Note that

$$\begin{aligned} c = \phi_x^{k+1}(y) &\iff c = \phi_x(c_k) \\ &\iff \operatorname{mid}(x,c) = c_k \\ &\iff d(x,c_k) = d(c,c_k) = \frac{1}{2}d(x,c) \\ &\iff d(x,c) = 2d(x,c_k) \text{ and } d(c,c_k) = d(x,c_k) \\ &\iff d(x,c) = 2^{k+1}d(x,y) \text{ and } d(c,c_k) = 2^k d(x,y). \end{aligned}$$

($\Rightarrow$): Let $c = \phi_x^{k+1}(y)$. We have

$$d(x,c) = 2^{k+1}d(x,y)$$

and

$$d(y,c) \leq d(y,c_k) + d(c_k,c) = (2^{k+1}-1)d(x,y),$$

$$d(y,c) \geq d(x,c) - d(x,y) = (2^{k+1}-1)d(x,y).$$

Therefore

$$d(y,c) = (2^{k+1}-1)d(x,y).$$

($\Leftarrow$): Let c be a point of X which satisfies

$$d(x,c) = 2^{k+1}d(x,y) \text{ and } d(c,y) = (2^{k+1}-1)d(x,y).$$

Then

$$d(x,y) = \frac{1}{2^{k+1}}d(x,c) \text{ and } d(y,c) = (1-\frac{1}{2^{k+1}})d(x,c)$$

as (y1). It implies that $y = \gamma_{x,c}(\frac{1}{2^{k+1}})$. Put $c' = \mathrm{mid}(x,c)$, then we have

$$d(x,c') = \frac{1}{2}d(x,c) = 2^k d(x,y)$$

and

$$\begin{aligned} d(y,c') &= d(\gamma_{x,c}(\frac{1}{2k+1}), \gamma_{x,c}(\frac{1}{2})) \\ &= (\frac{1}{2} - \frac{1}{2^{k+1}})d(x,c) = (2^k-1)d(x,y). \end{aligned}$$

By the inductive assumption, we have $c' = \phi_x^k(y)$ and hence $c = \phi_x^{k+1}(y)$.

By the principle of induction, the proof of (y4) is complete.

(y5): Let $s > 1$. Then there exist $n \in \mathbb{N}$ such that $2^{n-1} < s \leq 2^n$. Put $s' = \dfrac{s}{2^n}$, then $\dfrac{1}{2} < s' \leq 1$.

($\Rightarrow$): Let c_0 be a point in X that satisfies

$$d(x,c_0) = s'd(x,y) \text{ and } d(c_0,y) = (1-s')d(x,y).$$

Following (y1) we have $c_0 = \gamma_{x,y}(s')$. Let c be a point in X that satisfies

$$d(x,c) = 2^n d(x,c_0) \text{ and } d(c,c_0) = (2^n-1)d(x,c_0).$$

Since (y3), we have $c = \phi_x^n(c_0)$. Put $b = \mathrm{mid}(x,y)$, then

$$d(c_0,b) = d(\gamma_{x,y}(s'), \gamma_{x,y}(\frac{1}{2})) = (s'-\frac{1}{2})d(x,y) = (1-\frac{1}{2s'})d(x,c_0)$$

and

$$d(x,b) = \frac{1}{2}d(x,y) = \frac{1}{2s'}d(x,c_0) = \frac{1}{2^{n+1}s'}d(x,c) = \frac{1}{2s}d(x,c).$$

Thus,

$$\begin{aligned} d(b,c) \leq d(b,c_0) + d(c_0,c) &= (2^n - \frac{1}{2s'})d(x,c_0) \\ &= (1-\frac{1}{2^{n+1}s'})d(x,c) = (1-\frac{1}{2s})d(x,c), \end{aligned}$$

$$d(b,c) \geq d(x,c) - d(x,b) = (1 - \frac{1}{2^{n+1}s'})d(x,c) = (1 - \frac{1}{2s})d(x,c)$$

and hence

$$d(b,c) = (1 - \frac{1}{2s})d(x,c).$$

Therefore, we have $b = \gamma_{x,c}(\frac{1}{2s})$. Since $b = \mathrm{mid}(x,y)$, we have $y = \gamma_{x,c}(\frac{1}{s})$, that is,

$$d(x,y) = \frac{1}{s}d(x,c) \quad \text{and} \quad d(y,c) = (1 - \frac{1}{s})d(x,c).$$

By (y1) we have

$$d(x,c) = sd(x,y) \quad \text{and} \quad d(c,y) = (s-1)d(x,y).$$

($\Leftarrow$): Let c be a point in X which satisfies

$$d(x,c) = sd(x,y) \quad \text{and} \quad d(c,y) = (s-1)d(x,y).$$

By (y1) and (y2) we have $y = \gamma_{x,c}(\frac{1}{s})$. Let $c_1 = \mathrm{mid}(x,c)$ and $c_{k+1} = \mathrm{mid}(x,c_k)$ for any $k \in \mathbb{N}$. Then $c_k = \gamma_{x,c}(\frac{1}{2^k})$ and $\phi_x^k(c_k) = c$ for any $k \in \mathbb{N}$. Thus we have

$$d(y,c_n) = (\frac{1}{s} - \frac{1}{2^n})d(x,c) = (1 - \frac{s}{2^n})d(x,y)$$

and

$$d(x,c_n) = \frac{1}{2^n}d(x,c) = \frac{s}{2^n}d(x,y).$$

It implies that $c_n = \gamma_{x,y}(\frac{s}{2^n})$ and hence $c = \phi_x^n(\gamma_{x,y}(\frac{s}{2^n}))$.

(y6): Let $s > 0$.

($\Rightarrow$): Let $c \in X$ be a point in X such that

$$d(x,c) = sd(x,y) \quad \text{and} \quad d(c,y) = |1-s|d(x,y).$$

Since φ_x is an isometry and $\varphi_x(x) = x$, we have

$$d(x,\varphi_x(c)) = d(x,c) = sd(x,y)$$

and

$$d(\varphi_x(c),\varphi_x(y)) = d(c,y) = |1-s|d(x,y).$$

By the definition of φ_x, we have $d(a,\varphi_x(a)) = 2d(x,a)$ for any $a \in X$ and hence

$$d(\varphi_x(y),y) = 2d(x,y),$$

$$d(\varphi_x(c),c) = 2d(x,c) = 2sd(x,y).$$

We first assume that $0 < s \leq 1$. Then

$$d(\varphi_x(c),y) \leq d(\varphi_x(c),x) + d(x,y) = (1+s)d(x,y)$$

and

$$d(\varphi_x(c), y) \geq d(\varphi_x(y), y) - d(\varphi_x(y), \varphi_x(c)) = (1+s)d(x, y).$$

It follows that

$$d(\varphi_x(c), y) = (1+s)d(x, y).$$

Next, we assume that $1 < s$. Then

$$d(\varphi_x(c), y) \leq d(\varphi_x(c), x) + d(x, y) = (1+s)d(x, y)$$

and

$$d(\varphi_x(c), y) \geq d(\varphi_x(c), c) - d(c, y) = (1+s)d(x, y).$$

It follows that

$$d(\varphi_x(c), y) = (1+s)d(x, y).$$

($\Leftarrow$): Since (y2) to (y5), there exists a point c in X such that

$$d(x, c) = sd(x, y) \quad \text{and} \quad d(c, y) = |s-1|d(x, y).$$

We have

$$d(x, \varphi_x(c)) = sd(x, y) \quad \text{and} \quad d(\varphi_x(c), y) = (1+s)d(x, y)$$

as opposite direction. Let $c' \in X$ be a point such that

$$d(x, c') = sd(x, y) \quad \text{and} \quad d(c', y) = (1+s)d(x, y).$$

Put $t = 1+s$ then $t > 1$ and

$$d(y, c') = td(y, x) \quad \text{and} \quad d(c', x) = (t-1)d(y, x).$$

Since (y4), such a point c' is unique in X. Thus $c' = \varphi_x(c)$. □

By Lemma 5.5, for any $x, y \in X$ and any $s \in \mathbb{R}$, there exists a unique point $c \in X$ such that $d(x, c) = |s|d(x, y)$ and $d(c, y) = |1-s|d(x, y)$. We will denote the such point c by $\gamma_{x,y}(s)$.

Lemma 5.6. *For any $x, y \in X$ and $s, t \in \mathbb{R}$, the equation*

$$d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t|d(x, y)$$

holds.

Proof. Put $a_r = \gamma_{x,y}(r)$ for any $r \in \mathbb{R}$. We can assume that $s \leq t$.
(the case: $0 \leq s \leq t \leq 1$): trivial.
(the case: $0 \leq s \leq 1 \leq t$): By the definition of a_r, we have

$$\begin{aligned}
d(x, a_s) &= |s|d(x, y) = sd(x, y), \\
d(a_s, y) &= |1-s|d(x, y) = (1-s)d(x, y) \\
d(x, a_t) &= |t|d(x, y) = td(x, y), \\
d(a_t, y) &= |1-t|d(x, y) = (t-1)d(x, y).
\end{aligned}$$

Then

$$\begin{aligned} d(a_s, a_t) &\le d(a_s, y) + d(y, a_t) = (t-s)d(x,y), \\ d(a_s, a_t) &\ge d(x, a_t) - d(x, a_s) = (t-s)d(x,y) \end{aligned}$$

and hence

$$d(a_s, a_t) = (t-s)d(x,y).$$

(the case: $1 \le s \le t$): Since (y1), we have $y = \gamma_{x,a_t}(\frac{1}{t})$. Let $c = \gamma_{x,a_t}(\frac{s}{t})$, then, since $0 \le \frac{1}{t}, \frac{s}{t} \le 1$, we have

$$\begin{aligned} d(x,c) &= \frac{s}{t} d(x, a_t) = s d(x,y), \\ d(y,c) &= |\frac{1}{t} - \frac{s}{t}| d(x, a_t) = |1-s| d(x,y). \end{aligned}$$

It implies that $c = a_s$. Thus

$$d(a_s, a_t) = |\frac{s}{t} - 1| d(x, a_t) = |s-t| d(x,y).$$

As the above part of the proof, we have

$$0 \le s, t \Rightarrow d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t| d(x,y).$$

(the case: $s \le 0 \le 1 \le t$): By the definition of a_t we have

$$\begin{aligned} d(x, a_t) &= t d(x,y), \\ d(a_t, y) &= (t-1) d(x,y). \end{aligned}$$

It follows that

$$\begin{aligned} d(a_t, x) &= \frac{t}{t-1} d(a_t, y), \\ d(x,y) &= (1 - \frac{t}{t-1}) d(a_t, y). \end{aligned}$$

It implies that $x = \gamma_{a_t,y}(\frac{t}{t-1})$. Let $c = \gamma_{a_t,y}(\frac{t-s}{t-1})$. Since $0 \le \frac{t}{t-1}, \frac{t-s}{t-1}$, we have

$$\begin{aligned} d(y,c) &= |1 - \frac{t-s}{t-1}| d(y, a_t) = (1-s) d(x,y), \\ d(c,x) &= |\frac{t-s}{t-1} - \frac{t}{t-1}| d(y, a_t) = |s| d(x,y). \end{aligned}$$

Hence $c = a_s$. Thus

$$d(a_t, a_s) = d(a_t, c) = \frac{t-s}{t-1} d(a_t, y) = (t-s) d(x,y).$$

(the case: $s \leq 0 \leq t \leq 1$): By the definition of a_r and triangle inequality, we have

$$\begin{aligned} d(a_s, a_t) &\leq d(a_s, x) + d(x, a_t) \\ &= -sd(x, y) + td(x, y) = (t-s)d(x, y), \\ d(a_s, a_t) &\geq d(y, a_s) + d(y, a_t) \\ &= (1-s)d(x, y) + (1-t)d(x, y) = (t-s)d(x, y) \end{aligned}$$

and hence

$$d(a_s, a_t) = (t-s)d(x, y).$$

(the case: $s \leq t \leq 0$): For any $r \in \mathbb{R}$, we have

$$\gamma_{x,y}(r) = \gamma_{y,x}(1-r)$$

by the definition. Since $0 \leq 1-s, 1-t$, we have

$$d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = d(\gamma_{y,x}(1-s), \gamma_{y,x}(1-t)) = (t-s)d(y, x).$$

□

Lemma 5.7. *Let $x, y \in X$. The equation*

$$\varphi_{\gamma_{x,y}(s)}(\gamma_{x,y}(t)) = \gamma_{x,y}(2s-t)$$

holds for any $s, t \in \mathbb{R}$.

Proof. Since Lemma 5.6, we have

$$\begin{aligned} d(\gamma_{x,y}(2s-t), \gamma_{x,y}(s)) &= |s-t|d(x, y), \\ d(\gamma_{x,y}(s), \gamma_{x,y}(t)) &= |s-t|d(x, y) \end{aligned}$$

and

$$d(\gamma_{x,y}(2s-t), \gamma_{x,y}(t)) = |2s-2t|d(x, y).$$

Thus

$$d(\gamma_{x,y}(2s-t), \gamma_{x,y}(s)) = d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = \frac{1}{2}d(\gamma_{x,y}(2s-t), \gamma_{x,y}(t))$$

and hence

$$\gamma_{x,y}(s) = \mathrm{mid}(\gamma_{x,y}(t), \gamma_{x,y}(2s-t)).$$

Therefore,

$$\varphi_{\gamma_{x,y}(s)}(\gamma_{x,y}(t)) = \gamma_{x,y}(2s-t).$$

□

5.3 A Normed Gyrolinear Space Induced by a Metric Space

In this subsection, let (X,d) be a uniquely geodesic space which satisifies the condition K. For $x,y \in X$ and $s \in \mathbb{R}$, $\gamma_{e,x}(s)$ denote the unique point $c \in X$ that satisfies $d(x,c) = |s|d(x,y)$ and $d(c,y) = |1-s|d(x,y)$.

Definition 5.8. Let (X,d) be a uniquely geodesic space that satisfies the condition K. For fixed $e \in X$, we define the scalar multiplication $\otimes_e$ on X by

$$s \otimes_e x = \gamma_{e,x}(s)$$

for any $x \in X$ and $s \in \mathbb{R}$. We call $\otimes_e$ the scalar multiplication induced by the metric d on X at e.

In the following part of this subsection e is a point in X. $\oplus_e$ is the binary operation induced by the metric d on X at e, and $\otimes_e$ is the scalar multiplication induced by the metric d on X at e.

Proposition 5.9. *Let (X,d) be a uniquely geodesic space which satisfies the condition K. Let $e \in X$ and put $\|x\|_e = d(e,x)$ for any $x \in X$. Assume that (X,d) satisfies the following condition:*

(K4) $x \to y$ implies $\varphi_x(a) \to \varphi_y(a)$ for any $x,y,a \in X$.

Then $(X, \oplus_e, \otimes_e, \|\cdot\|_e, id)$ is a normed gyrolinear space with gyrometric d.

Proof. Recall that $(X,\oplus_e)$ is a gyrocommutative gyrogroup by Lemma 5.3.

(GL1): It is an immediate consequence of $\gamma_{e,x}(1) = x$.

(GL2): By Lemma 5.7, we have

$$\begin{aligned}
(r \otimes_e x) \oplus_e (s \otimes_e x) &= \varphi_{\mathrm{mid}(e, r\otimes_e x)}\varphi_e(s \otimes_e x) \\
&= \varphi_{\mathrm{mid}(e,\gamma_{e,x}(r))}((-s) \otimes_e x) \\
&= \varphi_{\gamma_{e,x}(\frac{r}{2})}(\gamma_{e,x}(-s)) \\
&= \gamma_{e,x}(\frac{2r}{2} - (-s)) \\
&= \gamma_{e,x}(r+s) \\
&= (r+s) \otimes_e x
\end{aligned}$$

(GL3): Put $z = r \otimes_e (s \otimes_e x)$, then $z = \gamma_{e,s\otimes_e x}(r)$. Since (y1), $x = \gamma_{e,s\otimes_e x}(\frac{1}{s})$. We have $z = (rs) \otimes_e x$ as

$$d(e,z) = d(e, r \otimes_e (s \otimes_e x)) = |r|d(e, s \otimes_e x) = |rs|d(e,x)$$

and

$$\begin{aligned}
d(x,z) &= d(\gamma_{e,s\otimes_e x}(\frac{1}{s}), \gamma_{e,s\otimes_e x}(r)) \\
&= |r - \frac{1}{s}|d(e, s\otimes_e x) = |rs-1|d(e,x).
\end{aligned}$$

(GL4): Since $\mathrm{gyr}[x,y]$ is an automorphism, $\mathrm{gyr}[x,y]e = e$. By equation (10) of Proposition 5.4, $\mathrm{gyr}[x,y]$ is a isometry. Hence we have

$$\begin{aligned}
c = r \otimes_e a \quad &\iff \quad \begin{cases} d(e,c) = |r| d(e,a) \\ d(c,a) = |1-r| d(e,a) \end{cases} \\
&\iff \quad \begin{cases} d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]c) = |r| d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]a) \\ d(\mathrm{gyr}[x,y]c, \mathrm{gyr}[x,y]a) = |1-r| d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]a) \end{cases} \\
&\iff \quad \begin{cases} d(e, \mathrm{gyr}[x,y]c) = |r| d(e, \mathrm{gyr}[x,y]a) \\ d(\mathrm{gyr}[x,y]c, \mathrm{gyr}[x,y]a) = |1-r| d(e, \mathrm{gyr}[x,y]a) \end{cases} \\
&\iff \quad \mathrm{gyr}[x,y]c = \gamma_{e, \mathrm{gyr}[x,y]a}(r) \\
&\iff \quad \mathrm{gyr}[x,y]c = r \otimes_e \gamma[x,y]a
\end{aligned}$$

(GL5): For any $x \in X$, since $(X, \oplus_e)$ satisfies the condition (G5), we have

$$\begin{aligned}
\mathrm{gyr}[n \otimes_e x, x] &= \mathrm{gyr}[((n-1) \otimes_e x) \oplus_e x, x] \\
&= \mathrm{gyr}[(n-1) \otimes_e x, x]
\end{aligned}$$

for any integer n. It follows that

$$\mathrm{gyr}[n \otimes_e x, x] = \mathrm{gyr}[0 \otimes_e x, x] = \mathrm{gyr}[e, x] = id_X$$

for any integer n. Also, we have

$$\mathrm{gyr}[x, m \otimes_e x] = \mathrm{gyr}^{-1}[m \otimes_e x, x] = id_X$$

for any integer m. Since gyrocommutative gyrogroup satisfies the equation

$$\mathrm{gyr}[a,b]\,\mathrm{gyr}[b,c] gyr[c,a] = id$$

for any a, b, c ([6]Theorem 3.31), we have

$$\mathrm{gyr}[n \otimes_e x, m \otimes_e x] = \mathrm{gyr}[n \otimes_e x, x]\,\mathrm{gyr}[x, m \otimes_e x]\,\mathrm{gyr}[m \otimes_e x, n \otimes_e x] = id_X$$

for any integers n, m. It follows that

$$\mathrm{gyr}[\frac{n}{m} \otimes_e y, y] = \mathrm{gyr}[n \otimes_e (\frac{1}{m} \otimes_e y), m \otimes_e (\frac{1}{m} \otimes_e y)] = id_X \tag{11}$$

for any $y \in X$ and rational number $\dfrac{n}{m}$.

Let $\{k_n\}$ be a sequence of rational numbers such that $k_n \to \alpha$. Then we have $k_n \otimes_e x = \gamma_{e,x}(k_n) \to \gamma_{e,x}(\alpha) = \alpha \otimes_e x$ by Lemma 5.6. By the condition (K4) we have

$$\lambda_{k_n \otimes_e x}(a) = \varphi_{k_n \otimes_e x} \varphi_e(a) \to \varphi_{\alpha \otimes_e x} \varphi_e(a) = \lambda_{\alpha \otimes_e x}(a)$$

for any $x, a \in X$. Thus we have

$$\begin{aligned} \mathrm{gyr}[k_n \otimes_e x, x](a) &= \lambda_{\ominus(k_n \otimes_e x \oplus_e x)} \lambda_{k_n \otimes_e x} \lambda_x(a) \\ &= \lambda_{(-k_n - 1)\otimes_e x} \lambda_{k_n \otimes_e x} \lambda_x(a) \\ &\to \lambda_{(-\alpha-1)\otimes_e x} \lambda_{\alpha \otimes_e x} \lambda_x(a) \\ &= \lambda_{\ominus(\alpha \otimes_e x \oplus_e x)} \lambda_{\alpha \otimes_e x} \lambda_x(a) \\ &= \mathrm{gyr}[\alpha \otimes_e x, x](a) \end{aligned}$$

for any real number α and $a, x \in X$, where $\{k_n\}$ is a sequence of rational numbers such that $k_n \to \alpha$. Following (11) we have $\mathrm{gyr}[\alpha \otimes_e x, x] = id_X$ for any $x \in X$ and $\alpha \in \mathbb{R}$. Thus we have

$$\mathrm{gyr}[r \otimes_e x, s \otimes_e x] = \mathrm{gyr}[\frac{r}{s} \otimes_e (s \otimes_e x), s \otimes_e x] = id_X$$

for any $x \in X$ and $r, s \in \mathbb{R}$.

(NG1): $\|x\|_e = 0 \iff d(e, x) = 0 \iff x = e$

(NG2): Following Proposition 5.4, we have

$$\begin{aligned} \|x \oplus_e y\|_e &= d(e, x \oplus_e y) \\ &\leq d(e, x) + d(x, x \oplus_e y) \\ &= d(e, x) + d(e, y) = \|x\|_e + \|y\|_e \end{aligned}$$

for any $x, y \in X$.

(NG3): For any $x \in X$ and $r \in \mathbb{R}$, we have

$$\begin{aligned} \|r \otimes_e x\|_e &= d(e, r \otimes_e x) \\ &= d(\gamma_{e,x}(0), \gamma_{e,x}(r)) \\ &= |0 - r| d(e, x) = |r| \|x\|_e \end{aligned}$$

by Lemma 5.6.

(NG4): Since any gyroautomorphism preserves the identity e and Proposition 5.4, we have

$$\begin{aligned} \|\,\mathrm{gyr}[x, y]a\|_e &= d(e, \mathrm{gyr}[x, y]a) \\ &= d(\mathrm{gyr}[x, y]e, \mathrm{gyr}[x, y]a) \\ &= d(e, a) = \|a\|_e \end{aligned}$$

for any $a, x, y \in X$.

Finally, since Proposition 5.4, we have

$$d(x, y) = d(e, x \ominus_e y) = \|x \ominus_e y\|.$$

□

The following Corollary 5.10 is a immediately consequence of Proposition 5.9 and Proposition 3.4.

Corollary 5.10. *Let X be a set and ϱ be a function $\varrho : X \times X \to X$ that satisfies $\varrho(x, y) = 0$ if and only if $x = y$. Let $e \in X$ and put $\|x\|_e' = \varrho(e, x)$ for any $x \in X$. Let f be a monotone increasing bijection $f : \|X\|_e' \to \mathbb{R}_{\geq 0}$, where $\|X\|_e' = \{\|x\|_e' : x \in X\}$. Put $d = f\varrho$. Suppose that (X, d) is a uniquely geodesic space that satisfies the condition K and the condition (K4). Then $(X, \oplus_e, \otimes_e, \|\cdot\|_e', f)$ is a normed gyrolinear space with gyrometric ϱ.*

Proof. Put $\|x\|_e = d(e, x)$. By Proposition 5.9, $(X, \oplus_e, \otimes_e, \|\cdot\|_e, id)$ is a normed gyrolinear space. Since f is a monotone increasing bijection $f : \|X\|_e' \to \mathbb{R}_{\geq 0}$, we have

$$\|X\|_e = \{\|x\|_e : x \in X\} = \{f\|x\|_e' : x \in X\} = \mathbb{R}_{\geq 0}$$

and hence f^{-1} is a monotone increasing bijection $f^{-1} : \|X\| \to \|X\|_e'$. Note that $0 \in \|X\|'$ as $\|e\|' = 0$. Since f^{-1} is strictly monotone increasing, we have $f^{-1}(0) = 0$. By Proposition 3.4, we have $(X, \oplus_e, \otimes_e, \|\cdot\|', f)$. Finally, we have

$$\varrho(x, y) = f^{-1}d(x, y) = f^{-1}\|x \ominus_e y\| = \|x \ominus_e y\|'.$$

□

5.4 Examples

Example 5.11. Let $\|\cdot\|$ be the Euclidean norm and d be the Euclidean metric on $\mathbb{R}^n$. Then the Euclidean space $(\mathbb{R}^n, d)$ is a uniquely geodesic metric space that satisfies the condition K with

$$\mathrm{mid}(x, y) = \frac{x + y}{2}$$

and

$$\varphi_x(y) = 2x - y$$

for any $x, y \in \mathbb{R}$. In this case, $(\mathbb{R}^n, \oplus_0) = (\mathbb{R}^n, +)$ as

$$\begin{aligned} x \oplus_0 y &= \varphi_{\mathrm{mid}(0,x)}\varphi_0(y) \\ &= \varphi_{\frac{x}{2}}(-y) \\ &= x + y \end{aligned}$$

for any $x, y \in X$. Moreover, $\otimes_0$ coincides with the usual scalar multiplication on $\mathbb{R}^n$ as

$$d(0, rx) = \|rx\| = |r|\|x\| = rd(0, x)$$

and

$$d(x, rx) = \|x - rx\| = |1 - r|\|x\| = |1 - r|d(0, x)$$

for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

Example 5.12. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Möbius addition $\oplus$ in $\mathbb{D}$ is given by the equation

$$a \oplus b = \frac{a+b}{1+\overline{a}b}$$

for any $a, b \in \mathbb{D}$. $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup (see [6]) and is called the Möbius gyrogroup. The identity of $(\mathbb{D}, \oplus)$ is the origin of $\mathbb{C}$ and $\ominus a = -a$ for every $a \in \mathbb{D}$. Moreover, the Möebius multiplication is given by

$$r \otimes a = \tanh(r \tanh^{-1} |a|) \frac{a}{|a|}$$

for any $a \in \mathbb{D}$ and $r \in \mathbb{R}$. The Möbius gyrometric ϱ is given by the equation

$$\varrho(a, b) = |(\ominus a) \oplus b| \left(= \left| \frac{-a+b}{1-\overline{a}b} \right| \right)$$

for every $a, b \in \mathbb{D}$. $(\mathbb{D}, \varrho)$ is a metiric space in itself, and $(\mathbb{D}, \tanh^{-1} \varrho)$ is a metric space again. $(\mathbb{D}, \varrho)$ doesn't satisfy the condition K. However, $(\mathbb{D}, \tanh^{-1} \varrho)$ satisfies the condition K with

$$\mathrm{mid}(a, b) = \frac{1}{2} \otimes (a \boxplus b)$$

and

$$\varphi_a(b) = (2 \otimes a) \oplus (-b)$$

for any $a, b \in \mathbb{D}$. Let $\oplus_0$ be the binary operation on $\mathbb{D}$ induced by $\tanh^{-1} \varrho$ at 0 then $\oplus_0 = \oplus$. Moreover, $(\mathbb{D}, \tanh^{-1} \varrho)$ is a uniquely geodesic metric space. Let $\otimes_0$ be the scalar multiplication on $\mathbb{D}$ induced by $\tanh^{-1} \varrho$ at 0 then $\otimes_0 = \otimes$.

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Toshikazu Abe
Information Engeineering,
Niigata University,
Niigata, Japan
E-mail: abebin08@gmail.com