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An Extension of Poincaré Model of Hyperbolic Geometry with Gyrovector Space Approach

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Abstract

The aim of this paper is to show the importance of analytic hyperbolic geometry introduced in [9]. In [1], Ungar and Chen showed that the algebra of the group $SL(2,\mathbb{C})$ naturally leads to the notion of gyrogroups and gyrovector spaces for dealing with the Lorentz group and its underlying hyperbolic geometry. They defined the Chen addition and then Chen model of hyperbolic geometry. In this paper, we directly use the isomorphism properties of gyrovector spaces to recover the Chen's addition and then Chen model of hyperbolic geometry. We show that this model is an extension of the Poincaré model of hyperbolic geometry. For our purpose we consider the Poincaré plane model of hyperbolic geometry inside the complex open unit disc $\mathbb{D}$. Also we prove that this model is isomorphic to the Poincaré model and then to other models of hyperbolic geometry. Finally, by gyrovector space approach we verify some properties of this model in details in full analogue with Euclidean geometry.

Keywords: Hyperbolic geometry, gyrogroup, gyrovector space, Poincaré model, analytic hyperbolic geometry.

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1. Introduction

Gyrogroups are noncommutative and nonassociative algebraic structures and this noncommutativity-nonassociativity turns out to be generated by the Thomas precession, well-known in the special theory of relativity. Gyrogroups also revealed themselves to be specially fitting in order to deal with formerly unsolved problems in special relativity (e.g. the problem of determining the Lorentz transformation

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that links given initial and final time-like 4-vectors). Gyrogroups are split up into gyrocommutative gyrogroups and nongyrocommutative. It turns out that introducing (gyrocommutative) gyrogroups, Ungar gave a concrete physical realization to formerly well-known algebraic systems called K-loops discovered by Helmut Karzel(e. g. see [2, 3]) in his study of neardomains. Since his 1988 pioneering paper [7] Ungar has studied gyrogroups and gyrovector spaces in several books [8–14] and many papers.

Some gyrocommutative gyrogroups admit a multiplication which turn them to a gyrovector space. Gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified. Armed with a gyrovector space structure, hyperbolic geometry is perfect for use in relativity physics. Abraham A. Ungar introduced the analytic hyperbolic geometry in [9]. The nonassociative algebra of gyrovector spaces is the framework for analytic hyperbolic geometry just as the associative algebra of vector spaces is the framework for analytic Euclidean geometry. Moreover, gyrovector spaces include vector spaces as a special, degenerate case corresponding to trivial gyroautomorphisms. Hence, Ungar gyrovector space approach forms the theoretical framework for uniting Euclidean and hyperbolic geometry.

In this paper, our aim is to use the gyrovector space approach of Ungar to investigate the analytical hyperbolic geometry. For our purpose we consider the Poincaré model of hyperbolic geometry defined inside the complex open unit disc $\mathbb{D} = \{a \in \mathbb{C} \mid |a| = \sqrt{a\bar{a}} < 1\}$ where $\bar{a}$ is the conjugate of a. Using the gyrovector spaces isomorphism, we extend the Poincaré model of hyperbolic geometry to the whole plane $\mathbb{C}$ which is called in [1] Chen model of hyperbolic geometry. We recover Chen gyrogroup and Chen gyrovector space of [1]. But our approach is different from [1]. We directly use the isomorphism properties of gyrovector spaces. As an application of gyrovector spaces as the algebraic settings of analytical hyperbolic geometry, we obtain some concepts of the new model by using gyrovector space properties.

2. Preliminaries and Well-known Results

Definition 2.1. (Gyrogroups). A groupoid $(G, +)$ is a gyrogroup if its binary operation satisfies the following axioms.

G_1. In G there is at least one element, 0, called a left identity, satisfying $0+a = a$ for all $a \in G$.

G_2. There is an element $0 \in G$ satisfying axiom G_1 such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying $-a + a = 0$.

G_3. For any $a, b, c \in G$ there exists a unique element $\mathrm{gyr}[a,b]c \in G$ such that the binary operation obeys the left gyroassociative law $a + (b + c) = (a + b) + \mathrm{gyr}[a,b]c$.

G_4. The map $\mathrm{gyr}[a,b] : G \to G$ given by $c \mapsto \mathrm{gyr}[a,b]c$ is an automorphism of the groupoid $(G,+)$, i.e. $\mathrm{gyr}[a,b] \in Aut(G,+)$ and the automorphism $\mathrm{gyr}[a,b]$ of G is called the gyroautomorphism of G generated by $a, b \in G$. The operator $\mathrm{gyr} : G \times G \to Aut(G,+)$ is called the gyrator of G.

G_5. Finally, the gyroautomorphism $\mathrm{gyr}[a,b]$ generated by any $a, b \in G$ possesses the left loop property $\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b+a]$.

Definition 2.2. A gyrogroup $(G,+)$ is a gyrocommutative gyrogroup if its binary operation obeys the gyrocommutative law $a + b = \mathrm{gyr}[a,b](b + a)$

Remark 1. Another equivalent definition of gyrocommutative gyrogroup, which also are called K-loops, comes from H. Karzel(cf., [2,3]) as follows:
A loop $(P,+)$ is said to be a K-loop if the following properties hold:
For all $a, b \in P$,

$\mathbf{K_1}$: $\mathrm{gyr}[a,b] \in Aut(P,+)$

$\mathbf{K_2}$: $\mathrm{gyr}[a,b] = \mathrm{gyr}[a, b+a]$

$\mathbf{K_3}$: $-(a+b) = (-a) + (-b)$

Example 2.3. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the complex open unit disc and $\oplus_E$ be the Einstein's velocity addition in Beltrami-Klein model of hyperbolic geometry, hence for $a, b \in \mathbb{D}$,

$$a \oplus_E b = \frac{a+b}{1+<a,b>} + \frac{\gamma_a}{1+\gamma_a}\left(\frac{<a,b>\ a - |a|^2\ b}{1+<a,b>}\right)$$

where $\gamma_a = \frac{1}{\sqrt{1-|a|^2}}$. It is proved that $(\mathbb{D}, \oplus_E)$ is a gyrocommutative gyrogroup(e.g see [4] and [1,9]).

Example 2.4. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the complex open unit disc of Poincaré hyperbolic plane. By the Möbius transformation $z \mapsto e^{i\theta}\frac{a+z}{1+a\bar{z}}$ we define $\oplus_M$ on $\mathbb{D}$ as $a \oplus_M b = \frac{a+b}{1+\bar{a}b}$. Then $(\mathbb{D}, \oplus_M)$ is a gyrocommutative gyrogroup, which is called Möbius gyrogroup(e.g. see [1,9]).

2.1 Gyrovector Space

Gyrovector spaces provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry. The elements of a gyrovector space are called points. Any two points of a gyrovector space give rise to a gyrovector.

Definition 2.5. (Real Inner Product Gyrovector Spaces). A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:
(1) G is a subset of a real inner product vector space V called the carrier of $G, G \subset V$, from which it inherits its inner product, $< ., . >$, and norm, $|\cdot|$, which are invariant under gyroautomorphisms, that is, $< \mathrm{gyr}[u, v]a, \mathrm{gyr}[u, v]b > = < a, b >$ for all points $a, b, u, v \in G$.
(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $a \in G$:
V_1 $1 \otimes a = a$.
V_2 Scalar Distributive Law: $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$.
V_3 Scalar Associative Law: $r_1 \otimes (r_2 \otimes a) = (r_1 r_2) \otimes a$.
V_4 Scaling Property: $\frac{|r| \otimes a}{|r \otimes a|} = \frac{a}{|a|}$.
V_5 Gyroautomorphism Property: $\mathrm{gyr}[\mathrm{u}, \mathrm{v}](r \otimes a) = r \otimes \mathrm{gyr}[u, v]a$.
V_6 Identity Automorphism: $\mathrm{gyr}[r_1 \otimes v, r_2 \otimes v] = I$.
(3) Real vector space structure $(|G|, \oplus, \otimes)$ for the set $|G|$ of one dimensional "vectors" $|G| = \{\pm|a| \mid a \in G\} \subset \mathbb{R}$ with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $a, b \in G$,
V_7 Homogeneity Property: $|r \otimes a| = |r| \otimes |a|$.
V_8 Gyrotriangle Inequality: $|a \oplus b| \leq |a| \oplus |b|$.

Definition 2.6. (Gyrovector Space Isomorphisms). Let $(G, \oplus_G, \otimes_G)$ and $(H, \oplus_H, \otimes_H)$ be two gyrovector spaces. A bijective map $\phi : G \to H$ is an isomorphism from G to H if for all $u, v \in G$ and $r \in \mathbb{R}$,
(1) $\phi(u \oplus_G v) = \phi(u) \oplus_H \phi(v)$,
(2) $\phi(r \otimes_G u) = r \otimes_H \phi(u)$ and
(3) $< \frac{u}{|u|}, \frac{v}{|v|} > = < \frac{\phi(u)}{|\phi(u)|}, \frac{\phi(v)}{|\phi(v)|} >$.

Example 2.7. We can form by

$$\otimes : \mathbb{R} \times \mathbb{D} \to \mathbb{D}; (r, a) \mapsto r \otimes a := \tanh(r \cdot \tanh^{-1}(|a|)) \cdot \frac{a}{|a|}, \text{ if } r \neq 0 \text{ and } r \otimes 0 := 0$$

a multiplication of scalars with elements of $\mathbb{D}$. Then $\otimes$ turns gyrogroups $(\mathbb{D}, \oplus_E)$ and $(\mathbb{D}, \oplus_M)$ into gyrovector spaces $(\mathbb{D}, \oplus_E, \otimes_E)$ and $(\mathbb{D}, \oplus_M, \otimes_M)$. The gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ provide algebraic settings for the Beltrami-Klein model of hyperbolic geometry and $(\mathbb{D}, \oplus_M, \otimes_M)$ provide algebraic settings for Poincaré model of hyperbolic geometry. Since $a \oplus_M b = \frac{1}{2} \otimes (2 \otimes a \oplus_E 2 \otimes b)$, Einstein gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ and Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$ are isomorphic. It means Beltrami-Klein model and Poincaré model are isomorphic. The coincidence $\otimes_E = \otimes_M = \otimes$ stems from the fact that for parallel vectors in $\mathbb{D}$, Möbius addition and Einstein addition coincide (cf., [11]).

3. Results

3.2 An Extension of Möbius Gyrovector Space to the Whole Space $\mathbb{C}$

In this section we give a gyrovector space isomorphic to the Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$. Actually we extend the gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$ to the whole gyrovector space $(\mathbb{C}, \oplus, \otimes)$ as follows. Let for $a \in \mathbb{D}$,

$$|a|^2 = a\bar{a}, \ \gamma_a := \frac{1}{\sqrt{1-|a|^2}}, \ \beta_a = \frac{1}{\sqrt{1+|a|^2}}$$

Define $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ by $a \mapsto a\gamma_a$. Therefore $\phi^{-1} : \mathbb{C} \to \mathbb{D}$ is given by $a \mapsto a\beta_a$. Now we extend the Möbius addition $\oplus_M$ to $\oplus$ on $\mathbb{C}$ by the bijection map ϕ as follows:

$$\forall a, b \in \mathbb{C}, \quad a \oplus b := \phi(\phi^{-1}(a) \oplus_M \phi^{-1}(b))$$

Therefore we have

$$a \oplus b = \lambda_{a,b} \frac{a\beta_a + b\beta_b}{1 + \bar{a}b\beta_a\beta_b} = \lambda_{a,b}(a\beta_a \oplus_M b\beta_b)$$

where $\lambda_{a,b} = \sqrt{\frac{1}{\beta_a^2\beta_b^2} + \frac{2<a,b>}{\beta_a\beta_b} + |a|^2|b|^2}$, or equivalently

$$a \oplus b = \frac{1 + a\bar{b}\beta_a\beta_b}{|1 + a\bar{b}\beta_a\beta_b|}(\frac{a}{\beta_b} + \frac{b}{\beta_a})$$

It is not difficult to show that $(\mathbb{C}, \oplus)$ is a gyrocommutative gyrogroup with identity 0 and

$$\text{gyr}[a, b] = \frac{1 + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b}$$

We only prove G_5. Firstly, note that $\beta_{a\oplus b} = \frac{\beta_a\beta_b}{|1+\bar{a}b\beta_a\beta_b|} = \beta_{b\oplus a}$. Therefore

$$\begin{aligned}
\text{gyr}[a, b \oplus a] &= \frac{1 + a(\overline{b \oplus a})\beta_a\beta_{b\oplus a}}{1 + \bar{a}(b \oplus a)\beta_a\beta_{b\oplus a}} \\
&= \frac{1 + a\frac{1+a\bar{b}\beta_a\beta_b}{|1+ab\beta_a\beta_b|}(\frac{\bar{a}}{\beta_b} + \frac{\bar{b}}{\beta_a})\beta_a\beta_{b\oplus a}}{1 + \bar{a}\frac{1+\bar{a}b\beta_a\beta_b}{|1+ab\beta_a\beta_b|}(\frac{a}{\beta_b} + \frac{b}{\beta_a})\beta_a\beta_{b\oplus a}} \\
&= \frac{1 + a\frac{\frac{\bar{a}}{\beta_b} + \frac{\bar{b}}{\beta_a}}{1+\bar{a}b\beta_a\beta_b}\beta_a\beta_a\beta_b}{1 + \bar{a}\frac{\frac{a}{\beta_b} + \frac{b}{\beta_a}}{1+a\bar{b}\beta_a\beta_b}\beta_a\beta_a\beta_b} \\
&= \frac{1 + \bar{a}b\beta_a\beta_b + a\bar{a}\beta^2{}_a + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b + a\bar{a}\beta^2{}_a + a\bar{b}\beta_a\beta_b} \times \frac{1 + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b} \\
&= \text{gyr}[a, b]
\end{aligned}$$

The addition $\oplus$ for parallel velocities reduces to

$$a \oplus b = \frac{a}{\beta_b} + \frac{b}{\beta_a}$$

Now we define the scalar multiplication as follows:

$$r \otimes v := \phi(r \otimes_M \phi^{-1}(v))$$

So we have

$$r \otimes v = \sinh(r \sinh^{-1}(|v|)) \frac{v}{|v|}$$

or equivalently,

$$r \otimes v = \frac{1}{2}\{(\sqrt{1+|v|^2} + |v|)^r - (\sqrt{1+|v|^2} - |v|)^r\} \frac{v}{|v|}$$

if $v \neq 0$ and $r \otimes 0 := 0$. In particular,

$$a' := \frac{1}{2} \otimes a = \frac{\sqrt{\beta_a}}{\sqrt{1+|a|\beta_a} + \sqrt{1-|a|\beta_a}} \cdot a \quad \text{and} \quad 2 \otimes a = \frac{2a}{\beta_a}.$$

$(\mathbb{C}, \oplus, \otimes)$ inherits its inner product from $\mathbb{C}$ such that its gyroautomorphism preserves the inner product $< \cdot, \cdot >$, hence

$$< \mathrm{gyr}[a,b]u, \mathrm{gyr}[a,b]v >= \frac{1}{2}(\mathrm{gyr}[a,b]u \cdot \overline{\mathrm{gyr}[a,b]v} + \overline{\mathrm{gyr}[a,b]u} \cdot \mathrm{gyr}[a,b]v)$$

Since $\mathrm{gyr}[a,b]\overline{\mathrm{gyr[a,b]}} = \frac{1+a\bar{b}\beta_a\beta_b}{1+\bar{a}b\beta_a\beta_b} \frac{1+\bar{a}b\beta_a\beta_b}{1+a\bar{b}\beta_a\beta_b} = 1$, so we have

$$< \mathrm{gyr}[a,b]u,\ \mathrm{gyr}[a,b]v >= \frac{1}{2}(u\bar{v} + \bar{u}v) =< u, v > .$$

V_1 is trivial.
Let $r_1, r_2 \in \mathbb{R}$ and $a \in \mathbb{C}$. Since $\beta_{r_i \otimes a} = \frac{1}{\cosh(r_i \sinh^{-1}(|a|))}$ so

$$\lambda_{r_1 \otimes a, r_2 \otimes a} = \cosh((r_1 + r_2)\sinh^{-1}(|a|)).$$

Therefore we have

$$\begin{aligned}
r_1 &\otimes a \oplus r_2 \otimes a \\
= &\ \lambda_{r_1 \otimes a, r_2 \otimes a} \frac{\tanh(r_1 \sinh^{-1}(|a|)) + \tanh(r_2 \sinh^{-1}(|a|))}{1 + \tanh(r_1 \sinh^{-1}(|a|)) \tanh(r_2 \sinh^{-1}(|a|))} \frac{a}{|a|} \\
= &\ \cosh((r_1+r_2)\sinh^{-1}(|a|)) \tanh((r_1+r_2)\sinh^{-1}(|a|)) \frac{a}{|a|} \\
= &\ \sinh((r_1+r_2)\sinh^{-1}(|a|)) \frac{a}{|a|} \\
= &\ (r_1 + r_2) \otimes a
\end{aligned}$$

Thus V_2 is valid.

For $r_1, r_2 \in \mathbb{R}$ and $a \in \mathbb{C}$ we have:

$$\begin{aligned} r_1 & \otimes (r_2 \otimes a) = r_1 \otimes (\sinh(r_2 \sinh^{-1}(|a|))\frac{a}{|a|}) \\ = & \sinh(r_1 \sinh^{-1}(|\sinh(r_2 \sinh^{-1}(|a|))\frac{a}{|a|})|)\frac{r_2 \otimes a}{|r_2 \otimes a|} \\ = & \sinh(r_1 \sinh^{-1}(|\sinh(r_2 \sinh^{-1}(|a|)))|)\frac{a}{|a|} \\ = & \sinh((r_1 r_2)\sinh^{-1}(|a|))\frac{a}{|a|} \\ = & (r_1 r_2) \otimes a \end{aligned}$$

Therefore V_3 is also valid.
V_4 comes from definition.
Now since $|\operatorname{gyr}[a,b]| = 1$, then for $r \in \mathbb{R}$ and $a, b, u \in \mathbb{C}$ we have:

$$\begin{aligned} r \otimes \operatorname{gyr}[a,b]u &= \sinh(r \sinh^{-1}(|\operatorname{gyr}[a,b]u|))\frac{\operatorname{gyr}[a,b]u}{|\operatorname{gyr}[a,b]u|} \\ &= \operatorname{gyr}[a,b]\sinh(r \sinh^{-1}(|u|))\frac{u}{|u|} \\ &= \operatorname{gyr}[a,b](r \otimes u) \end{aligned}$$

Hence V_5 holds. Straightforward computations shows that V_6 and V_7 are valid.
Since $\oplus_M$ satisfies in triangle inequality, we can write

$$\begin{aligned} |a \oplus b| &= |\lambda_{a,b}(a\beta_a \oplus_M \beta_b b)| \\ &= \lambda_{a,b}|(a\beta_a \oplus_M b\beta_b)| \leq \lambda_{a,b}(|a\beta_a| \oplus_M |b\beta_b|) = |a| \oplus |b| \end{aligned}$$

So $\oplus$ satisfies in V_8. Thus we have proved that $(\mathbb{C}, \oplus, \otimes)$ is a gyrovector space.

In the following, we show that $(\mathbb{C}, \oplus, \otimes)$ and $(\mathbb{D}, \oplus_M, \otimes_M)$ are gyroisomorphic.
Consider the map $\phi : (\mathbb{D}, \oplus_M, \otimes_M) \longrightarrow (\mathbb{C}, \oplus, \otimes)$ given by $a \mapsto a\gamma_a$.

(i) For $a,\ b \in \mathbb{D},\ \phi(a \oplus_M b) = \phi(\frac{a+b}{1+\bar{a}b}) = \frac{a+b}{1+\bar{a}b}\gamma_a\gamma_b|1+\bar{a}b|$.
On the other hand, since $\beta_{a\gamma_a} = \frac{1}{\gamma_a}$ and $\lambda_{a\gamma_a, b\gamma_b} = \gamma_a\gamma_b|1+\bar{a}b|$,

$$\begin{aligned} \phi(a) \oplus \phi(b) = a\gamma_a \oplus a\gamma_b &= \lambda_{a\gamma_a, b\gamma_b}\frac{a\gamma_a\beta_{a\gamma_a} + b\gamma_b\beta_{b\gamma_b}}{1 + \overline{a\gamma_a\beta_{a\gamma_a}}b\gamma_b\beta_{b\gamma_b}} \\ &= \gamma_a\gamma_b|1+\bar{a}b|\frac{a+b}{1+\bar{a}b} \end{aligned}$$

Hence $\phi(a \oplus_M b) = \phi(a) \oplus \phi(b)$.

(ii) $\phi(r \otimes_M a) = r \otimes_M a\gamma_{r\otimes_M a} = \sinh(r\tanh^{-1}(|a|))\frac{a}{|a|}$, on the other hand,

$$\begin{aligned} r \otimes \phi(a) = r \otimes a\gamma_a &= \sinh(r\ln(\frac{|a|}{\sqrt{1-|a|^2}} + \frac{1}{\sqrt{1-|a|^2}}))\frac{a}{|a|} \\ &= \sinh(r\tanh^{-1}(|a|))\frac{a}{|a|} \end{aligned}$$

So $\phi(r \otimes_M a) = r \otimes \phi(a)$.

$$\text{(iii)} \quad < \frac{\phi(a)}{|\phi(a)|}, \frac{\phi(b)}{|\phi(b)|} >=< \frac{a\gamma_a}{|a|\gamma_a}, \frac{b\gamma_b}{|b|\gamma_b} >=< \frac{a}{|a|}, \frac{b}{|b|} > .$$

From (i),(ii) and (iii) we conclude that ϕ is a gyrovector space isomorphism. Thus we have proved the following theorem:

Theorem 3.1. $(\mathbb{C}, \oplus, \otimes)$ *is a gyrovector space isomorphic to the Möbius gyrovector space* $(\mathbb{D}, \oplus_M, \otimes_M)$.

Note that by example 2.7, $(\mathbb{C}, \oplus, \otimes)$ is isomorphic to the Einstein's gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ and Ungar gyrovector space $(\mathbb{R}^2_c, \oplus_U, \otimes_U)$ described in [1]. Also note that $(\mathbb{C}, \oplus, \otimes)$ is exactly the Chen gyrovector space introduced in [1] by specifying the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(r) = \sinh(\frac{r}{2})$ in definition of general addition of the group $SL(2, \mathbb{C})$.

3.3 Extension of Poincaré Model of Hyperbolic Geometry

Since $(\mathbb{D}, \oplus_M, \otimes_M)$ provides the algebraic setting for the Poincaré model of hyperbolic geometry, and the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ is an extension of it to $\mathbb{C}$, so $(\mathbb{C}, \oplus, \otimes)$ provides the algebraic settings for a new model of hyperbolic geometry just as vector spaces provide the algebraic setting for Euclidean geometry. Also our model is an extension of the hyperbolic geometry of the Poincaré model to the whole plane $\mathbb{C}$ in which the unique geodesic through two given points a and b in the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ is given by $a \oplus (\ominus a \oplus b) \otimes t$ with $0 \le t \le 1$. This geodesic (or, gyroline), its segment from a to b, and the midpoint $m_{ab} = a \oplus (\ominus a \oplus b) \otimes \frac{1}{2}$, of the segment are shown in Fig. 1. These are Euclidean semi-hyperbolas with asymptotes which intersect at the origin.

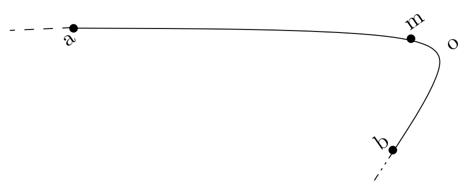

Figure 1. gyroline passing through two points a and b and their midpoint m.

3.4 Trigonometry

One can employ the gyrogroup operation and its gyrovector space to describe the trigonometry of hyperbolic geometry which is called now gyrotrigonometry (e. g. see [6, 11]). In the following by using the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ we verify and obtain some trigonometry relations of our model.
Let $a, b \in \mathbb{C}$ and $a \perp b$. Then

$$\beta_{(a\oplus b)\sqrt{2}} = \beta_{a\sqrt{2}}\beta_{b\sqrt{2}} \qquad (*)$$

(i) By using $(*)$, $\beta_{(a\oplus b)\sqrt{2}}|a \oplus b|^2 = \beta_{a\sqrt{2}}\beta_{b\sqrt{2}}(\frac{|a|^2}{\beta_b^2} + \frac{|b|^2}{\beta_a^2})$.

(ii) $\beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2 = \frac{\beta_{a\sqrt{2}}|a|^2}{\beta_{\beta_{b\sqrt{2}}|b|^2}} + \frac{\beta_{b\sqrt{2}}|b|^2}{\beta_{\beta_{b\sqrt{2}}|a|^2}}$. $\qquad (**)$

But $\beta_{\beta_{a\sqrt{2}}|a|^2} = \frac{1}{\sqrt{1+\frac{|a|^4}{1+2|a|^2}}} = \frac{\beta_a^2}{\beta_{a\sqrt{2}}}$, so $(**)$ is equal to $\beta_{a\sqrt{2}}\beta_{b\sqrt{2}}(\frac{|a|^2}{\beta_b^2} + \frac{|b|^2}{\beta_a^2})$.

From (i) and (ii) we get the hyperbolic Pythagorean theorem

$$\beta_{(a\oplus b)\sqrt{2}}|a \oplus b|^2 = \beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2$$

Thus we have proved the following theorem:

Theorem 3.2. *Let $a, b \in \mathbb{C}$ and $a \perp b$. Then the hyperbolic Pythagorean theorem in $(\mathbb{C}, \oplus, \otimes)$ is of the form*

$$\beta_{c\sqrt{2}}|c|^2 = \beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2.$$

Note that in general, for any $a, b \in \mathbb{C}$ we have the following relations:

$$\beta_a^2\beta_b^2|a \oplus b|^2 = \beta_a^2|a|^2 + \beta_b^2|b|^2 + 2\beta_a\beta_b < a, b >,$$

$$\beta_{a\oplus b} = \frac{\beta_a\beta_b}{|1 + \bar{a}b\beta_a\beta_b|}.$$

Hyperbolic Distance. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^{\geq 0}; (a, b) \mapsto |a \ominus b|$. Equivalently we can write

$$d(a, b) = |\frac{a}{\beta_b} - \frac{b}{\beta_a}|.$$

It is easy to show that d is a metric on $\mathbb{C}$ which is the hyperbolic distance of any two points a and b in our model.

3.4.1 Hyperbolic Angle

For three points a, b and c in gyrovector space $(\mathbb{C}, \oplus, \otimes)$ the cosine of the hyperbolic angle α between two geodesic rays $a \oplus (\ominus a \oplus b) \otimes t$ and $a \oplus (\ominus a \oplus c) \otimes t$ with common point a and respectively containing b and c is given by the equation

$$\cos\alpha = \frac{\ominus a \oplus b}{|\ominus a \oplus b|} \cdot \frac{\ominus a \oplus c}{|\ominus a \oplus c|}$$

This hyperbolic angle α is independent of the choice of the points b and c on the geodesic rays, and it remains invariant under left gyrotranslations and rotations.

Theorem 3.3. *Let $\triangle(a,b,c)$ be any triangle in hyperbolic plane $\mathbb{C}$ with angles α, β and γ in a, b and c respectively and denote the opposite sides of a, b and c respectively with a, b and c. Then*

(i) If $\gamma = \frac{\pi}{2}$ then

$$\cos(\alpha) = \frac{|b|\beta_b(2-\beta_c^2)}{|c|\beta_c(2-\beta_b^2)} = \frac{|b|}{|c|}\frac{\beta^2_{b\sqrt{2}}\beta_c}{\beta^2_{c\sqrt{2}}\beta_b}$$

and

$$\sin(\alpha) = \frac{|a|\beta_c}{|c|\beta_a}$$

(ii) $\cos(\gamma) = \frac{\beta^2_{a\sqrt{2}}\beta^2_{b\sqrt{2}} - \beta^2_{c\sqrt{2}}}{4|a||b|}\beta_a\beta_b$

(iii)

$$\frac{\sin(\alpha)\beta_a}{|a|} = \frac{\sin(\beta)\beta_b}{|b|} = \frac{\sin(\gamma)\beta_c}{|c|}$$

(iv) $\beta^2_{c\sqrt{2}} = \frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)}$ *or equivalently,*

$$|c|^2 = \frac{\cos(\alpha+\beta)+\cos(\gamma)}{2\sin(\alpha)\sin(\beta)}$$

3.5 Defect and Area

Let $\triangle(a,b,c)$ be a triangle in $(\mathbb{C},\oplus)$, without loss of generality we can assume that $c = o$. It is shown in Proposition 3.3 of [3] that the defect of $\triangle(o,a,b)$, hence δ, is the measure of $\mathrm{gyr}[a,-b]$. Since $\mathrm{gyr}[a,-b] = \frac{1-a\bar{b}\beta_a\beta_b}{1-\bar{a}b\beta_a\beta_b}$ and $\cos(\delta) = \frac{1}{2}(\ \mathrm{gyr}[a,-b] + \overline{\mathrm{gyr}[a,-b]})$, so we obtain

$$\cos(\delta) = \frac{1-2<a,b>\beta_a\beta_b + [(a\bar{b})^2+(\bar{a}b)^2]\beta_a^2\beta_b^2}{1-2<a,b>\beta_a\beta_b + |a|^2|b|^2\beta_a^2\beta_b^2}$$

Thus if we set $b^\perp := ib$ where $i = \sqrt{-1}$, we have

$$\tan(\frac{\delta}{2}) = \frac{<a,b^\perp>\beta_a\beta_b}{1-2<a,b>\beta_a\beta_b}$$

In particular if $a \perp b$, then

$$\tan(\frac{\delta}{2}) = |a||b|\beta_a\beta_b$$

We define area equal to defect, so the area of $\triangle(a, b, c)$ with defect δ is

$$S := 2\tan^{-1}\left(\frac{< a, b^{\perp} > \beta_a\beta_b}{1 - 2 < a, b > \beta_a\beta_b}\right)$$

By similar arguments described in [5] we have the following result about circles in this model:

Theorem 3.4. *Let C_r be any circle of radius r in hyperbolic plane $\mathbb{C}$ with circumference P and area S. Then*

$$P = \frac{4\pi r}{\beta_r}, \qquad S = 4\pi r^2$$

Theorem 3.5. *Let $\triangle(A, B, C)$ be any triangle and C_r be its circumscribed circle with radius r in hyperbolic plane $\mathbb{C}$. If δ be its defect then*

$$\sin(\frac{\delta}{2}) = \frac{|a||b||c|}{2r\beta_r}\beta_a\beta_b\beta_c(2 - \beta_r^2).$$

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