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From the Lorentz Transformation Group in Pseudo-Euclidean Spaces to Bi-Gyrogroups

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Abstract

The Lorentz transformation of order $(m = 1, n)$, $n \in \mathbb{N}$, is the well-known Lorentz transformation of special relativity theory. It is a transformation of time-space coordinates of the pseudo-Euclidean space $\mathbb{R}^{m=1,n}$ of one time dimension and n space dimensions ($n = 3$ in physical applications). A Lorentz transformation without rotations is called a *boost*. Commonly, the special relativistic boost is parametrized by a relativistically admissible velocity parameter $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}_c^n$, whose domain is the c-ball $\mathbb{R}_c^n$ of all relativistically admissible velocities, $\mathbb{R}_c^n = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}$, where the ambient space $\mathbb{R}^n$ is the Euclidean n-space, and $c > 0$ is an arbitrarily fixed positive constant that represents the vacuum speed of light. The study of the Lorentz transformation composition law in terms of parameter composition reveals that the group structure of the Lorentz transformation of order $(m = 1, n)$ induces a gyrogroup and a gyrovector space structure that regulate the parameter space $\mathbb{R}_c^n$. The gyrogroup and gyrovector space structure of the ball $\mathbb{R}_c^n$, in turn, form the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry, which underlies the ball $\mathbb{R}_c^n$. The aim of this article is to extend the study of the Lorentz transformation of order (m, n) from $m = 1$ and $n \geq 1$ to all $m, n \in \mathbb{N}$, obtaining algebraic structures called a *bi-gyrogroup* and a *bi-gyrovector space*. A bi-gyrogroup is a gyrogroup each gyration of which is a pair of a left gyration and a right gyration. A bi-gyrovector space is constructed from a bi-gyrocommutative bi-gyrogroup that admits a scalar multiplication.

Keywords: Bi-gyrogroup, bi-gyrovector space, eigenball, gyrogroup, inner product of signature (m,n), Lorentz transformation of order (m,n), Pseudo-Euclidean space, special relativity.

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1. Introduction

Following the parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces [46], the aim of this article is to study Lorentz transformations in pseudo-Euclidean spaces, where each of the resulting generalized Lorentz transformation group is parametrized by a generalized relativistically admissible velocity. A generalized relativistically admissible velocity, in turn, is an element of the *eigenball* $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ of all real $n \times m$ matrices, just as a relativistically admissible velocity in special relativity is an element of the ball $\mathbb{R}_c^n = \{V \in \mathbb{R}^n : \|V\| < c\}$ of the ambient Euclidean n-space $\mathbb{R}^n$. Here $c > 0$ is an arbitrarily fixed positive constant, analogous to the vacuum speed of light in special relativity.

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n), $m, n \in \mathbb{N}$, is an $(m + n)$-dimensional space with the pseudo-Euclidean inner product of signature (m, n). A Lorentz transformation of order (m, n) is a special linear transformation $\Lambda \in SO(m, n)$ of $\mathbb{R}^{m,n}$ that leaves the pseudo-Euclidean inner product invariant. It is special in the sense that the determinant of the $(m + n) \times (m + n)$ matrix representation of Λ is 1, and the determinant of its first m rows and columns is positive [21, p. 478]. The group $SO(m, n)$ of all Lorentz transformations of order (m, n) is also known as the special pseudo-orthogonal group [21, p. 478], or the group of pseudo-rotations [7]. A Lorentz transformation without rotations is called a boost when $m = 1$ and a bi-boost when $m > 1$. Bi-boosts are studied in [46].

In [46], the bi-boost $B(P)$ in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ is parametrized by $P \in \mathbb{R}^{n\times m}$, $\mathbb{R}^{n\times m}$ being the space of all real rectangular matrices of order $n \times m$. In the special case when $m = 1$, the parameter P is a column vector in $\mathbb{R}^n$ that represents a proper velocity. In physical applications $n = 3$, and a proper velocity in $\mathbb{R}^3$ is a velocity measured by proper (or, traveler's) time rather than observer's time, as explained in [37, 40].

In Sects. 2-5 we review part of the study in [46] of the bi-boost $B(P)$ in order to reach the position enabling us to change the parameter $P \in \mathbb{R}^{n\times m}$ to a new parameter $V \in \mathbb{R}_c^{n\times m}$ in Sect. 6. Here, the space $\mathbb{R}_c^{n\times m}$ of the new parameter V is the *c-eigenball* of the ambient space $\mathbb{R}^{n\times m}$, given by

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\} \\ &= \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0 \le \lambda < c^2\}\end{aligned}$$

where $c > 0$ is an arbitrarily fixed positive constant, said to be the *eigenradius* of the eigenball.

In the special case when $m = 1$, the space $\mathbb{R}^{n\times m}$ specializes to the Euclidean n-space $\mathbb{R}^{n\times 1} = \mathbb{R}^n$ of n-dimensional column vectors. Accordingly, as shown in Example 8.2 (for c normalized to $c = 1$), the eigenball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ specializes to the c-ball of the ambient space $\mathbb{R}^n$, given by $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n = \{V \in \mathbb{R}^n : \|V\| < c\}$. Thus, when $m = 1$, the concepts of the c-eigenball and the c-ball coincide, and the Lorentz transformation of order (m, n) specializes to the Lorentz transformation of special relativity theory in one time dimension and n space dimensions ($n = 3$ in physical applications).

Eigenballs $\mathbb{R}_c^{n\times m}$ are studied in Sect. 7, and in Sect. 8 any eigenball $\mathbb{R}_c^{n\times m}$ forms the parameter space of a Lorentz transformation Λ of order (m, n). It is then shown in Sects. 9-10 that the resulting bi-boosts $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, and $B_c(P)$, $P \in \mathbb{R}^{n\times m}$, leave invariant the inner product of signature (m, n), as expected.

The crucial step of this article is performed in Sects. 11-12, where the composition law of two successive Lorentz transformations of order (m, n) is expressed in terms of a resulting parameter composition law, $V_1 \oplus V_2$, in the parameter space $\mathbb{R}_c^{n\times m}$. The parameter composition law, in turn, gives rise in Sects. 13-16 to the novel bi-gyrogroup and bi-gyrovector space structure of the eigenball $\mathbb{R}_c^{n\times m}$. These novel algebraic structures, finally, pave in Sect. 17 the road leading to the novel non-Euclidean geometry of the eigenball $\mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$.

The algebraic and geometric structure of the parameter space $\mathbb{R}_c^{n\times m}$ is of interest in nonassociative algebra, non-Euclidean geometry, and relativity physics. In the special case when $m = 1$, it gives rise to

1. the group-like structure called a *gyrogroup*; to
2. the vector space-like structure called a *gyrovector space*; to
3. improved understanding of the hyperbolic geometry of Lobachevsky and Bolyai in terms of novel analogies with Euclidean geometry; and to
4. improved understanding of the way hyperbolic geometry regulates Einstein's special theory of relativity.

These structures and their use in hyperbolic geometry and in special relativity, along with other applications, are studied in many papers as, for instance, [2–5,29], [9–13], [31–34], [8,22–24,26–28,47], [25,38,44,46], and in seven books [36,37,40–43,45]. Hence, the extension of these structures from $m = 1$ and all $n \in \mathbb{N}$ to all $m, n \in \mathbb{N}$ is a most promising step towards revealing the non-Euclidean geometry that underlies the eigenball $\mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$. Accordingly, along with [46], this article initiates the extension of the exploration of the algebraic and geometric structure of the eigenball $\mathbb{R}_c^{n\times m}$ from $m = 1$ to $m \geq 1$, for all $n \geq 1$, and the related extension from gyrogroups and gyrovector spaces to bi-gyrogroups and bi-gyrovector spaces.

2. On the Generalized Lorentz Transformation

The (generalized) Lorentz transformation group $SO(m, n)$, $m, n \in \mathbb{N}$, is a group of special linear transformations in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n) that leave the pseudo-Euclidean inner product invariant. A Lorentz transformation Λ of order (m, n), $\Lambda = SO(m, n)$, is special in the sense that the determinant of the $(m + n) \times (m + n)$ matrix representation of Λ is 1, and the determinant of its first m rows and columns is positive [21, p. 478]. In the first part of this paper we present results from [46], where the set $SO(m,n)$ is described in detail.

Let $\mathbb{R}^{n\times m}$ be the set of all $n\times m$ real matrices, let $SO(n)$ be the special orthogonal group of order n, let I_n be the $n\times n$ identity matrix, and let $0_{m,n}$ be the $m\times n$ zero matrix.

Theorem 2.1 below realizes the Lorentz transformations $\Lambda\in SO(m,n)$ parametrically, with the three matrix parameters $P\in\mathbb{R}^{n\times m}$, $O_m\in SO(m)$ and $O_n\in SO(n)$.

Embedding each matrix parameter in an $(m+n)\times(m+n)$ matrix, we define (i) bi-boosts; (ii) right rotations; and (iii) left rotations as follows:

Bi-Boosts: A bi-boost is an $(m+n)\times(m+n)$ matrix $B(P)$ parametrized by $P\in\mathbb{R}^{n\times m}$,

$$B(P) := \begin{pmatrix} \sqrt{I_m+P^tP} & P^t \\ P & \sqrt{I_n+PP^t} \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,, \tag{1}$$

where P^t is the transpose of P.

Right Rotations: A right rotation is an $(m+n)\times(m+n)$ block orthogonal matrix $\rho(O_m)$ parametrized by $O_m\in SO(m)$,

$$\rho(O_m) := \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,. \tag{2}$$

Left Rotations: A left rotation is an $(m+n)\times(m+n)$ block orthogonal matrix $\lambda(O_n)$ parametrized by $O_n\in SO(n)$,

$$\lambda(O_n) := \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,. \tag{3}$$

Theorem 2.1. (Lorentz Transformation Bi-Gyration Decomposition, P**)** ([46, Theorem 8]). *A matrix $\Lambda\in\mathbb{R}^{(m+n)\times(m+n)}$ is a Lorentz transformation of order (m,n), $\Lambda\in SO(m,n)$, $m,n\in\mathbb{N}$, if and only if it is given uniquely by the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m+P^tP} & P^t \\ P & \sqrt{I_n+PP^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{4}$$

or, prarmetrically in short,

$$\Lambda = \Lambda(O_m,P,O_n) = \rho(O_m)B(P)\lambda(O_n) = \begin{pmatrix} P \\ O_n \\ O_m \end{pmatrix}\,. \tag{5}$$

Results (4) – (5) of Theorem 2.1 indicate the notations we use with the generic Lorentz transformation Λ of order (m,n).

We now take the results in ([46, Theorem 13]) as definitions in Def. 2.2 below, giving rise to a binary operation, $\oplus$, in $\mathbb{R}^{n\times m}$ along with two families of automorphisms of $\mathbb{R}^{n\times m}$, called *bi-gyrations*, which are the left gyrations $\mathrm{lgyr}[\cdot,\cdot]$ and the right gyrations $\mathrm{lgyr}[\cdot,\cdot]$.

Definition 2.2. (Bi-gyroaddition and Bi-gyration). The bi-gyroaddition $\oplus$ and bi-gyration (lgyr, rgyr) in the parameter bi-gyrogroupoid $(\mathbb{R}^{n\times m},\oplus)$ are given by the equations

$$\begin{aligned}
&P_1\oplus P_2 = P_1\sqrt{I_m+P_2^tP_2}+\sqrt{I_n+P_1P_1^t}P_2 \in \mathbb{R}^{n\times m}\\
&\mathrm{lgyr}[P_1,P_2]=\\
&\sqrt{I_n+(P_1\oplus P_2)(P_1\oplus P_2)^t}^{\;-1}\left\{P_1P_2^t+\sqrt{I_n+P_1P_1^t}\sqrt{I_n+P_2P_2^t}\right\}\in SO(n)\\
&\mathrm{rgyr}[P_1,P_2]=\\
&\left\{P_1^tP_2+\sqrt{I_m+P_1^tP_1}\sqrt{I_m+P_2^tP_2}\right\}\sqrt{I_m+(P_1\oplus P_2)^t(P_1\oplus P_2)}^{\;-1}\in SO(m)
\end{aligned} \tag{6}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$.

Def. 2.2 proves useful in Theorem 2.3 below, which presents the Lorentz transformation composition law in terms of parameter composition.

Theorem 2.3. (Lorentz Transformation Product Law) ([46, Theorem 21]) *The product of two generic Lorentz transformations*

$$\begin{aligned}\Lambda_1&=(P_1,O_{n,1},O_{m,1})^t\\ \Lambda_2&=(P_2,O_{n,2},O_{m,2})^t\end{aligned} \tag{7}$$

of order (m,n), $m,n\in\mathbb{N}$, is given by

$$\Lambda_1\Lambda_2=\begin{pmatrix}P_1\\O_{n,1}\\O_{m,1}\end{pmatrix}\begin{pmatrix}P_2\\O_{n,2}\\O_{m,2}\end{pmatrix}=\begin{pmatrix}P_1O_{m,2}\oplus O_{n,1}P_2\\ \mathrm{lgyr}[P_1O_{m,2},O_{n,1}P_2]O_{n,1}O_{n,2}\\ O_{m,1}O_{m,2}\mathrm{rgyr}[P_1O_{m,2},O_{n,1}P_2]\end{pmatrix}. \tag{8}$$

where $\oplus$, lgyr and rgyr are given by (6) in terms of the parameters $P_1,P_2\in\mathbb{R}^{n\times m}$.

Illustrative examples follow.

Example 2.4. *In the special case when $P_1=P_2=0_{n,m}$ and $O_{m,1}=O_{m,2}=I_m$, the parameter composition law (8) yields the equation*

$$\lambda(O_{n,1})\lambda(O_{n,2})=\begin{pmatrix}0_{n,m}\\O_{n,1}\\I_m\end{pmatrix}\begin{pmatrix}0_{n,m}\\O_{n,2}\\I_m\end{pmatrix}=\begin{pmatrix}0_{n,m}\\O_{n,1}O_{n,2}\\I_m\end{pmatrix}=\lambda(O_{n,1}O_{n,2}) \tag{9}$$

demonstrating that under the parameter composition law (8) *the parameter* O_n *forms the special orthogonal group* $SO(n)$.

Example 2.5. *In the special case when* $P_1 = P_2 = 0_{n,m}$ *and* $O_{n,1} = O_{n,2} = I_n$, *the parameter composition law* (8) *yields the equation*

$$\rho(O_{m,1})\rho(O_{m,2}) = \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1} \end{pmatrix} \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,2} \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1}O_{m,2} \end{pmatrix} = \rho(O_{m,1}O_{m,2}) \tag{10}$$

demonstrating that under the parameter composition law (8) *the parameter* O_m *forms the special orthogonal group* $SO(m)$.

Example 2.6. *In the special case when* $O_{n,1} = O_{n,2} = I_n$ *and* $O_{m,1} = O_{m,2} = I_m$ *the parameter composition law* (8) *yields the equation*

$$B(P_1)B(P_2) = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \mathrm{lgyr}[P_1, P_2] \\ \mathrm{rgyr}[P_1, P_2] \end{pmatrix} . \tag{11}$$

Clearly, under the parameter composition law (8) *the parameter* $P \in \mathbb{R}^{n\times m}$ *does not form a group, owing to the presence of bi-gyrations. Indeed,* (11) *demonstrates that, in general, the composition of two bi-boosts is not a bi-boost but, rather, a bi-boost associated with a bi-gyration.*

In the special case when $P_1 = P$ *and* $P_2 = \ominus P$, (11) *gives*

$$B(P)B(\ominus P) = \begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} \ominus P \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P \ominus P \\ \mathrm{lgyr}[P, \ominus P] \\ \mathrm{rgyr}[P, \ominus P] \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ I_m \end{pmatrix} , \tag{12}$$

so that the inverse of $B(P)$ *is* $B(\ominus P) = B(-P)$. *In* (12) *we use the results* $\mathrm{lgyr}[P, \ominus P] = I_n$ *and* $\mathrm{rgyr}[P, \ominus P] = I_m$, *which are verified in* ([46, Eq. (114)]).

The product rule (8) is neither commutative nor associative. However, it possesses a rich algebraic structure. Thus, in particular, it obeys a commutative-like and an associative-like laws, called the bi-gyrocommutative and the bi-gyroassociative law of the bi-gyrogroupoid $(\mathbb{R}^{n\times m}, \oplus)$.

Theorem 2.7. (Bi-gyrocommutative Law in $(\mathbb{R}^{n\times m}, \oplus)$**)** ([46, Theorem 25]). *The binary operation* $\oplus$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyrocommutative law*

$$P_1 \oplus P_2 = \mathrm{lgyr}[P_1, P_2](P_2 \oplus P_1)\mathrm{rgyr}[P_1, P_2] \tag{13}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

In Theorem 2.7 the bi-gyration $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$ takes $P_2 \oplus P_1$ into $P_1 \oplus P_2$. It rotates the $n \times m$ matrix $P_2 \oplus P_1 \in \mathbb{R}^{n\times m}$ from the left by the orthogonal matrix $\mathrm{lgyr}[P_1, P_2] \in SO(n)$, and from the right by the orthogonal matrix $\mathrm{rgyr}[P_1, P_2] \in SO(m)$.

Theorem 2.8. (Bi-gyroassociative Law in $(\mathbb{R}^{n\times m}, \oplus)$**)** ([46, Theorem 27]). *The binary operation $\oplus$ in $\mathbb{R}^{n\times m}$ possesses the bi-gyroassociative law*

$$(P_1 \oplus P_2) \oplus \mathrm{lgyr}[P_1, P_2] P_3 = P_1 \mathrm{rgyr}[P_2, P_3] \oplus (P_2 \oplus P_3) \tag{14}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

Note that P_1 and P_2 are grouped together on the left side of (14), while P_2 and P_3 are grouped together on the right side of (14).

3. Bi-Gyrogroups

It proves useful in [46] to replace the binary operation $\oplus$ in $\mathbb{R}^{n\times m}$ by a new binary operation, $\oplus'$, according to the following definition.

Definition 3.1. (Bi-gyrogroup Operation, Bi-gyrogroups) ([46, Definition 35]). Let $(\mathbb{R}^{n\times m}, \oplus)$ be a bi-gyrogroupoid. A new bi-gyrogroup binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ is given by

$$P_1 \oplus' P_2 = (P_1 \oplus P_2) \mathrm{rgyr}[P_2, P_1] \tag{15}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$. The resulting groupoid $(\mathbb{R}^{n\times m}, \oplus')$ is called a bi-gyrogroup.

The *bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ is defined in Def. 3.1 in terms of the *bi-gyrogroupoid* $(\mathbb{R}^{n\times m}, \oplus)$.

It is shown in [46] that (15) implies the following four identities that exhibit an interesting symmetry between the binary operations $\oplus$ and $\oplus'$ in $\mathbb{R}^{n\times m}$.

$$\begin{aligned} P_1 \oplus' P_2 &= (P_1 \oplus P_2) \mathrm{rgyr}[P_2, P_1] \\ P_1 \oplus P_2 &= (P_1 \oplus' P_2) \mathrm{rgyr}[P_1, P_2] \\ P_1 \oplus' P_2 &= \mathrm{lgyr}[P_1, P_2](P_2 \oplus P_1) \\ P_1 \oplus P_2 &= \mathrm{lgyr}[P_1, P_2](P_2 \oplus' P_1) \end{aligned} \tag{16}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

Theorem 3.2. (Bi-gyrocommutative Law in $(\mathbb{R}^{n\times m}, \oplus')$**)** ([46, Theorem 42]). *The binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ possesses the bi-gyrocommutative law*

$$P_1 \oplus' P_2 = \mathrm{lgyr}[P_1, P_2](P_2 \oplus' P_1) \mathrm{rgyr}[P_2, P_1] \tag{17}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

It follows from (13) and (17) that the binary operations $\oplus$ and $\oplus'$ possess the same bi-gyrocommutative law. This is, however, not the case with the bi-gyroassociative law, as shown in Theorem 3.3.

Theorem 3.3. (Bi-gyrogroup Left and Right Bi-gyroassociative Law) ([46, Theorem 41]). *The binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ possesses the left bi-gyroassociative law*

$$P_1\oplus'(P_2\oplus' X) = (P_1\oplus' P_2)\oplus'\mathrm{lgyr}[P_1,P_2]X\mathrm{rgyr}[P_2,P_1] \tag{18}$$

and the right bi-gyroassociative law

$$(P_1\oplus' P_2)\oplus' X = P_1\oplus'(P_2\oplus'\mathrm{lgyr}[P_2,P_1]X\mathrm{rgyr}[P_1,P_2]) \tag{19}$$

for all $P_1, P_2, X \in \mathbb{R}^{n\times m}$.

4. Gyrogroup Gyrations

The bi-gyroassociative laws (18) – (19) and the bi-gyrocommutative law (17) suggest the following definition of gyrations in terms of left and right gyrations.

Definition 4.1. (Gyrogroup Gyrations) ([46, Definition 43]). The gyrator

$$\mathrm{gyr} : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathrm{Aut}(\mathbb{R}^{n\times m}, \oplus')$$

generates automorphisms called gyrations, $\mathrm{gyr}[P_1, P_2] \in \mathrm{Aut}(\mathbb{R}^{n\times m}, \oplus')$, given by the equation

$$\mathrm{gyr}[P_1,P_2]X = \mathrm{lgyr}[P_1,P_2]X\mathrm{rgyr}[P_2,P_1] \tag{20}$$

for all $P_1, P_2, X \in \mathbb{R}^{n\times m}$, where left gyrations, $\mathrm{lgyr}[P_1, P_2]$, and right gyrations, $\mathrm{rgyr}[P_2, P_1]$, are given in (6), p. 233. The gyration $\mathrm{gyr}[P_1, P_2]$ is said to be the gyration generated by $P_1, P_2 \in \mathbb{R}^{n\times m}$. Being automorphisms of $(\mathbb{R}^{n\times m}, \oplus')$, gyrations are also called gyroautomorphisms.

Def. 4.1 will turn out rewarding, leading to the elegant result that any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ is a gyrocommutative gyrogroup.

Theorem 4.2. (Gyrogroup Gyroassociative and gyrocommutative Laws) ([46, Theorem 44]). *The binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ obeys the left and the right gyroassociative law*

$$P_1\oplus'(P_2\oplus' X) = (P_1\oplus' P_2)\oplus'\mathrm{gyr}[P_1,P_2]X \tag{21}$$

and

$$(P_1\oplus' P_2)\oplus' X = P_1\oplus'(P_2\oplus'\mathrm{gyr}[P_2,P_1]X) \tag{22}$$

and the gyrocommutative law

$$P_1\oplus' P_2 = \mathrm{gyr}[P_1,P_2](P_2\oplus' P_1)\,. \tag{23}$$

Proof. Identities (21) – (22) follow immediately from Def. 4.1 and the left and right bi-gyroassociative law (18) – (19). Similarly, (23) follow immediately from Def. 4.1 and the bi-gyrocommutative law (17). □

Lemma 4.3. ([46, Lemma 45]). *The relation* (20) *between gyrations* $\mathrm{gyr}[P_1, P_2]$ *and corresponding bi-gyrations* $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_2, P_1])$*,* $P_1, P_2 \in (\mathbb{R}^{n\times m}, \oplus')$*, is bijective.*

It is obvious from (20) that a gyration $\mathrm{gyr}[P_1, P_2]$ is determined uniquely by the bi-gyration $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$. It follows from Lemma 4.3 that also the converse is true, that is, a bi-gyration $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$ is determined uniquely by the gyration $\mathrm{gyr}[P_1, P_2]$.

It is anticipated in Def. 4.1 that gyrations are automorphisms. The following theorem asserts that this is indeed the case.

Theorem 4.4. (Gyroautomorphism) ([46, Theorem 46]). *Gyrations* $\mathrm{gyr}[P_1, P_2]$ *of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *are automorphisms of the bi-gyrogroup.*

Theorem 4.5. (Left Gyration Reduction Properties) ([46, Theorem 47]). *Left gyrations of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *possess the left gyration left reduction property*

$$\mathrm{lgyr}[P_1, P_2] = \mathrm{lgyr}[P_1 \oplus' P_2, P_2] \tag{24}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[P_1, P_2] = \mathrm{lgyr}[P_1, P_2 \oplus' P_1]\,. \tag{25}$$

Theorem 4.6. (Right Gyration Reduction Properties) ([46, Theorem 48]). *Right gyrations of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *possess the right gyration left reduction property*

$$\mathrm{rgyr}[P_1, P_2] = \mathrm{rgyr}[P_1 \oplus' P_2, P_2] \tag{26}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[P_1, P_2] = \mathrm{rgyr}[P_1, P_2 \oplus' P_1]\,. \tag{27}$$

Theorem 4.7. (Gyration Reduction Properties) ([46, Theorem 49]). *The gyrations of any bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$*,* $m, n \in \mathbb{N}$*, possess the left and right reduction property*

$$\mathrm{gyr}[P_1, P_2] = \mathrm{gyr}[P_1 \oplus' P_2, P_2] \tag{28}$$

and

$$\mathrm{gyr}[P_1, P_2] = \mathrm{gyr}[P_1, P_2 \oplus' P_1]\,. \tag{29}$$

Proof. Identities (28) and (29) follow from Def. 4.1 of gyr in terms of lgyr and rgyr, and from Theorems 4.5 and 4.6. □

5. Gyrogroups and Bi-Gyrogroups

We are now in a position to present the definition of the abstract (gyrocommutative) gyrogroup, and note the proof in [46, Theorem 52] that any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, is a gyrocommutative gyrogroup.

Forming a natural generalization of groups, gyrogroups emerged in the 1988 study of the parametrization of the Lorentz group of Einstein's special relativity theory [35, 36]. Einstein velocity addition, thus, provides a concrete example of a gyrocommutative gyrogroup operation in the ball of all relativistically admissible velocities.

Definition 5.1. (Gyrogroups) ([46, Definition 50]). *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms (G1) – (G5). In G there is at least one element,* 0*, called a left identity, satisfying*

(G1) $\qquad 0\oplus a = a$

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of* a*, satisfying*

(G2) $\qquad \ominus a\oplus a = 0\,.$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\qquad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c\,.$

The map $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

(G4) $\qquad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a, b]$ *of* G *is called the gyroautomorphism, or the gyration, of* G *generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of* G*. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left reduction property*

(G5) $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]\,,$

called the reduction axiom.

The gyrogroup axioms $(G1) – (G5)$ in Definition 5.1 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.

2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.

3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 5.2. (Gyrocommutative Gyrogroups) ([46, Definition 51]). *A gyrogroup* $(G, \oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*
(G6) $\qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$
for all $a, b \in G$.

Theorem 5.3. (Gyrocommutative Gyrogroup) ([46, Theorem 52]).
Any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$, $n, m \in \mathbb{N}$, *is a gyrocommutative gyrogroup.*

Following the definition of the abstract (gyrocommutative) gyrogroup, we are now in the position to present the definition of the abstract (bi-gyrocommutative) bi-gyrogroup.

Definition 5.4. (Bi-gyrogroups) ([46, Definition 53]). *A (gyrocommutative) gyrogroup whose gyrations are bi-gyrations is said to be a (bi-gyrocommutative) bi-gyrogroup.*

A detailed study of the abstract bi-gyrogroup is presented in [32].

A concrete example of a nontrivial bi-gyrogroup is provided by the Einstein bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ that stems in Sect. 2 from the (generalized) Lorentz transformation of order (m, n), $m, n \in \mathbb{N}$. In the special case when $m = 1$ we have $\mathbb{R}^{n\times m} = \mathbb{R}^n$, and the Einstein bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ specializes to the Einstein gyrogroup $(\mathbb{R}^n, \oplus')$. It turns out that $(\mathbb{R}^n, \oplus')$=$(\mathbb{R}^n, \oplus_{\mathrm{U}})$, $(\mathbb{R}^n, \oplus_{\mathrm{U}})$ being the Einstein *proper velocity* gyrogroup associated with the Einstein addition law of proper (traveler's) velocities rather than the common observer's velocities. Einstein PV (proper velocity) gyrogroups, in turn, stem from the *proper velocity Lorentz group* studied, for instance, in [38, 40] and [1]. We, therefore, call $(\mathbb{R}^{n\times m}, \oplus')$ *a PV-bi-gyrogroup.*

As a goal of this paper, we now face the task of changing the parameter $P \in \mathbb{R}^{n\times m}$, which represents generalized proper (traveler's) relativistic velocities, to a new parameter, V, which represents generalized relativistically admissible (observer's) velocities. Achieving the goal, we will obtain Einstein bi-gyrogroups associated with generalized observer's, rather than traveler's, velocities.

6. Bi-Boost Parameter Change, $P \to V$

It is now useful to introduce a positive parameter $c > 0$ into the bi-boost $B(P)$ in (1), obtaining the bi-boost $B_c(P)$,

$$B_c(P) = \begin{pmatrix} \sqrt{I_m + c^{-2}P^tP} & \frac{1}{c^2}P^t \\ P & \sqrt{I_n + c^{-2}PP^t} \end{pmatrix}, \tag{30}$$

so that $B(P) = B_{c=1}(P)$ is a normalized form of $B_c(P)$.

Definition 6.1. For any $m, n \in \mathbb{N}$ let $\phi : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ be the map given by

$$\phi : \; P \;\mapsto\; V = \sqrt{I_n + c^{-2}PP^t}^{\;-1}P \tag{31}$$

where $c > 0$ is an arbitrarily fixed positive constant.

The image

$$\mathbb{B}^{n\times m} := \phi(\mathbb{R}^{n\times m}) \subset \mathbb{R}^{n\times m} \tag{32}$$

of ϕ in $\mathbb{R}^{n\times m}$ is called the (open) c-eigenball of $\mathbb{R}^{n\times m}$.

The term "eigenball" will be justified in Theorem 7.2 following the observation that $\mathbb{B}^{n\times m} \subset \mathbb{R}^{n\times m}$ is the set $\mathbb{R}_c^{n\times m}$ of all $V \in \mathbb{R}^{n\times m}$ such that each eigenvalue λ of the Gramian matrices VV^t and V^tV is nonnegative and smaller than c, $0 \le \lambda < c$. We will see in Example 8.2 that in the special case when $m = 1$, the c-eigenball $\mathbb{R}_c^{n\times m}$ of $\mathbb{R}^{n\times 1}$ specializes to the c-ball $\mathbb{R}_c^n$ of $\mathbb{R}^n$, that is, $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$.

Lemma 6.2. *The map ϕ in Def. 6.1 can be written equivalently as*

$$\phi : \; P \;\mapsto\; V = P\sqrt{I_m + c^{-2}P^tP}^{\;-1} . \tag{33}$$

Proof. The proof follows immediately from (31) and from the commuting relation

$$P\sqrt{I_m + c^{-2}P^tP} = \sqrt{I_n + c^{-2}PP^t}P \tag{34}$$

for all $P \in \mathbb{R}^{n\times m}$, proved in [46, Eq. (53)] for $c = 1$. The passage from $c = 1$ to $c > 0$ is immediate in this case. □

Theorem 6.3. *For any $P \in \mathbb{R}^{n\times m}$,*

$$V = \sqrt{I_n + c^{-2}PP^t}^{\;-1}P = P\sqrt{I_m + c^{-2}P^tP}^{\;-1} \tag{35}$$

if and only if

$$P = \sqrt{I_n - c^{-2}VV^t}^{\;-1}V = V\sqrt{I_m - c^{-2}V^tV}^{\;-1} . \tag{36}$$

Proof. The proof is divided into four parts. In Parts I_A and I_B we prove that (35) implies (36), and in Parts II_A and II_B we prove that (36) implies (35).

Part $\boldsymbol{I_A}$: Assuming (35), we have

$$P = \sqrt{I_n + c^{-2}PP^t}V \tag{37}$$

and the commuting relation

$$P^t\sqrt{I_n + c^{-2}PP^t}^{\;-1} = \sqrt{I_m + c^{-2}P^tP}^{\;-1}P^t \tag{38}$$

so that by (38) and (35)

$$PP^t\sqrt{I_n + c^{-2}PP^t}^{\;-1} = P\sqrt{I_m + c^{-2}P^tP}^{\;-1}P^t = \sqrt{I_n + c^{-2}PP^t}^{\;-1}PP^t . \tag{39}$$

Then, by (35) and (39),

$$\begin{aligned} VV^t &= \sqrt{I_n + c^{-2}PP^t}^{\,-1} PP^t \sqrt{I_n + c^{-2}PP^t}^{\,-1} \\ &= PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)^{-1}PP^t\,. \end{aligned} \tag{40}$$

Hence,

$$PP^t = (I_n + c^{-2}PP^t)VV^t \tag{41}$$

and

$$PP^t = VV^t(I_n + c^{-2}PP^t)\,. \tag{42}$$

A rearrangement of (41) yields

$$VV^t = PP^t(I_n - c^{-2}VV^t) \tag{43}$$

implying

$$PP^t = VV^t(I_n - c^{-2}VV^t)^{-1}\,. \tag{44}$$

Similarly, a rearrangement of (42) yields

$$VV^t = (I_n - c^{-2}VV^t)PP^t \tag{45}$$

implying

$$PP^t = (I_n - c^{-2}VV^t)^{-1}VV^t\,. \tag{46}$$

Following (44) we have

$$\begin{aligned} I_n + c^{-2}PP^t &= I_n + c^{-2}VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)(I_n - c^{-2}VV^t)^{-1} + c^{-2}VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t + c^{-2}VV^t)(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)^{-1} \end{aligned} \tag{47}$$

so that

$$\sqrt{I_n + c^{-2}PP^t} = \sqrt{I_n - c^{-2}VV^t}^{\,-1}\,. \tag{48}$$

Hence, by (37) and (48),

$$P = \sqrt{I_n + c^{-2}PP^t}V = \sqrt{I_n - c^{-2}VV^t}^{\,-1}V \tag{49}$$

thus validating the first equation in (36), In Part I_B of the proof we validate the second equation in (36),

Part I_B: Assuming (35), we have

$$P = V\sqrt{I_m + c^{-2}P^tP} \tag{50}$$

and the commuting relation, as in (38),

$$P^t\sqrt{I_n+c^{-2}PP^t}^{\,-1}=\sqrt{I_m+c^{-2}P^tP}^{\,-1}P^t \tag{51}$$

so that by (35) and (51),

$$P^tP\sqrt{I_m+c^{-2}P^tP}^{\,-1}=P^t\sqrt{I_n+c^{-2}PP^t}^{\,-1}P=\sqrt{I_m+c^{-2}P^tP}^{\,-1}P^tP\,. \tag{52}$$

Then, by (35) and (52),

$$\begin{aligned}V^tV&=\sqrt{I_m+c^{-2}P^tP}^{\,-1}P^tP\sqrt{I_m+c^{-2}P^tP}^{\,-1}\\&=P^tP(I_m+c^{-2}P^tP)^{-1}\\&=(I_m+c^{-2}P^tP)^{-1}P^tP\,.\end{aligned} \tag{53}$$

Hence,

$$P^tP=(I_m+c^{-2}P^tP)V^tV \tag{54}$$

and

$$P^tP=V^tV(I_m+c^{-2}P^tP) \tag{55}$$

A rearrangement of (54) yields

$$V^tV=P^tP(I_m-c^{-2}V^tV) \tag{56}$$

implying

$$P^tP=V^tV(I_m-c^{-2}V^tV)^{-1}\,. \tag{57}$$

Similarly, a rearrangement of (55) yields

$$V^tV=(I_m-c^{-2}V^tV)P^tP \tag{58}$$

implying

$$P^tP=(I_m-c^{-2}V^tV)^{-1}V^tV\,. \tag{59}$$

Following (57) we have

$$\begin{aligned}I_m+c^{-2}P^tP&=I_m+c^{-2}V^tV(I_m-c^{-2}V^tV)^{-1}\\&=(I_m-c^{-2}V^tV)(I_m-c^{-2}V^tV)^{-1}+c^{-2}V^tV(I_m-c^{-2}V^tV)^{-1}\\&=(I_m-c^{-2}V^tV+c^{-2}V^tV)(I_m-c^{-2}V^tV)^{-1}\\&=(I_m-c^{-2}V^tV)^{-1}\end{aligned} \tag{60}$$

so that

$$\sqrt{I_m+c^{-2}P^tP}=\sqrt{I_m-c^{-2}V^tV}^{\,-1}\,. \tag{61}$$

Hence, by (50) and (61),

$$P = V\sqrt{I_m + c^{-2}P^tP} = V\sqrt{I_m - c^{-2}V^tV}^{\;-1}. \tag{62}$$

Equations (49) and (62) validate the two equations in (36).

Conversely, in Parts II_A and II_B we show that (36) implies (35).

Part $\boldsymbol{II_A}$: Assuming (36), we have

$$V = \sqrt{I_n - c^{-2}VV^t}P \tag{63}$$

and the commuting relation

$$V^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} = \sqrt{I_m - c^{-2}V^tV}^{\;-1}V^t \tag{64}$$

so that, by (64) and (36)

$$\begin{aligned} VV^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} &= V\sqrt{I_m - c^{-2}V^tV}^{\;-1}V^t \\ &= \sqrt{I_n - c^{-2}VV^t}^{\;-1}VV^t\,. \end{aligned} \tag{65}$$

Then, by (36) and (65),

$$\begin{aligned} PP^t &= \sqrt{I_n - c^{-2}VV^t}^{\;-1}VV^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} \\ &= VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)^{-1}VV^t\,. \end{aligned} \tag{66}$$

Hence,

$$VV^t = (I_n - c^{-2}VV^t)PP^t \tag{67}$$

and

$$VV^t = PP^t(I_n - c^{-2}VV^t)\,. \tag{68}$$

A rearrangement of (67) yields

$$PP^t = VV^t(I_n + c^{-2}PP^t) \tag{69}$$

implying

$$VV^t = PP^t(I_n + c^{-2}PP^t)^{-1} \tag{70}$$

Similarly, a rearrangement of (68) yields

$$PP^t = (I_n + c^{-2}PP^t)VV^t \tag{71}$$

implying

$$VV^t = (I_n + c^{-2}PP^t)^{-1}PP^t\,. \tag{72}$$

Following (70) we have

$$\begin{aligned} I_n - c^{-2}VV^t &= I_n - c^{-2}PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)(I_n + c^{-2}PP^t)^{-1} - c^{-2}PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t - c^{-2}PP^t)(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)^{-1} \end{aligned} \tag{73}$$

so that

$$\sqrt{I_n - c^{-2}VV^t} = \sqrt{I_n + c^{-2}PP^t}^{\,-1} . \tag{74}$$

Hence, by (63) and (74),

$$V = \sqrt{I_n - c^{-2}VV^t}P = \sqrt{I_n + c^{-2}PP^t}^{\,-1}P \tag{75}$$

thus validating the first equation in (35). In Part II_B of the proof we validate the second equation in (35).

Part $\boldsymbol{II_B}$: Assuming (36), we have

$$V = P\sqrt{I_m - c^{-2}V^tV} \tag{76}$$

and the commuting relation, as in (64),

$$V^t\sqrt{I_n - c^{-2}VV^t}^{\,-1} = \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^t \tag{77}$$

so that by (36) and (77),

$$V^tV\sqrt{I_m - c^{-2}V^tV}^{\,-1} = V^t\sqrt{I_n - c^{-2}VV^t}^{\,-1}V = \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^tV . \tag{78}$$

Then, by (36) and (78),

$$\begin{aligned} P^tP &= \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^tV\sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ &= V^tV(I_m - c^{-2}V^tV)^{-1} \\ &= (I_m - c^{-2}V^tV)^{-1}V^tV . \end{aligned} \tag{79}$$

Hence,

$$V^tV = (I_m - c^{-2}V^tV)P^tP \tag{80}$$

and

$$V^tV = P^tP(I_m - c^{-2}V^tV) . \tag{81}$$

A rearrangement of (80) yields

$$P^tP = V^tV(I_m + c^{-2}P^tP) \tag{82}$$

implying

$$V^tV = P^tP(I_m + c^{-2}P^tP)^{-1} \,. \tag{83}$$

Similarly, a rearrangement of (81) yields

$$P^tP = (I_m + c^{-2}P^tP)V^tV \tag{84}$$

implying

$$V^tV = (I_m + c^{-2}P^tP)^{-1}P^tP \,. \tag{85}$$

Following (83) we have

$$\begin{aligned} I_m - c^{-2}V^tV &= I_m - c^{-2}P^tP(I_m + c^{-2}P^tP)^{-1} \\ &= (I_m + c^{-2}P^tP)(I_m + c^{-2}P^tP)^{-1} - c^{-2}P^tP(I_m + c^{-2}P^tP)^{-1} \\ &= (I_m + c^{-2}P^tP - c^{-2}P^tP)(I_m + c^{-2}P^tP)^{-1} \\ &= (I_m + c^{-2}P^tP)^{-1} \end{aligned} \tag{86}$$

so that

$$\sqrt{I_m - c^{-2}V^tV} = \sqrt{I_m + c^{-2}P^tP}^{\,-1} \,. \tag{87}$$

Hence, by (76) and (87),

$$V = P\sqrt{I_m - c^{-2}V^tV} = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} \,. \tag{88}$$

Equations (75) and (88) validate the two equations in (35), and the proof is complete. □

Theorem 6.4. *Let $\phi:\ \mathbb{R}^{n\times m} \to \mathbb{B}^{n\times m}$, $m, n \in \mathbb{N}$, be the map given by each of the two mutually equivalent equations*

$$\begin{aligned} \phi:\ P \ &\mapsto\ V = \sqrt{I_n + c^{-2}PP^t}^{\,-1}P \\ \phi:\ P \ &\mapsto\ V = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} \end{aligned} \tag{89}$$

where $\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$ is the image of $\mathbb{R}^{n\times m}$ under ϕ.

Then, ϕ is bijective, and the inverse $\phi^{-1}: \mathbb{B}^{n\times m} \to \mathbb{R}^{n\times m}$ of ϕ is given by each of the two mutually equivalent equations

$$\begin{aligned} \phi^{-1}:\ V \ &\mapsto\ P = \sqrt{I_n - c^{-2}VV^t}^{\,-1}V \\ \phi^{-1}:\ V \ &\mapsto\ P = V\sqrt{I_m - c^{-2}V^tV}^{\,-1} \,. \end{aligned} \tag{90}$$

Proof. The proof follows immediately from Theorem 6.3. □

7. Eigenballs

In order to characterize the image $\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$ of $\mathbb{R}^{n\times m}$ under ϕ in terms of eigenvalues, we present the following well-known theorem.

Theorem 7.1. ([6, p. 56]). *If a square matrix A has the eigenvalue λ and the corresponding eigenvector* $\mathbf{x}$*, then any rational function $R(A)$ of A has the eigenvalue $R(\lambda)$ and the eigenvector* $\mathbf{x}$*.*

Theorem 7.1 enables us to prove the following theorem, which characterizes $\mathbb{B}^{n\times m}$ in terms of eigenvalues.

Theorem 7.2. *Let*

$$\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m}) \tag{91}$$

and

$$\mathbb{R}_c^{n\times m} = \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\}. \tag{92}$$

Then,

$$\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m} . \tag{93}$$

Proof. Let $V \in \mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$. Then there exists $P \in \mathbb{R}^{n\times m}$ such that

$$V = \phi(P) = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P \tag{94}$$

and, hence, by (72),

$$VV^t = (I_n + c^{-2}PP^t)^{-1}PP^t . \tag{95}$$

Let λ_i, $i = 1, \dots, n$, be the eigenvalues of PP^t. Then $\lambda_i \ge 0$ and, by (95) and Theorem 7.1, the eigenvalues μ_i of VV^t are

$$\mu_i = \frac{\lambda_i}{1 + \lambda_i/c^2} \tag{96}$$

so that $0 \le \mu_i < c^2$. Hence $V \in \mathbb{R}_c^{n\times m}$, implying the inclusion $\mathbb{B}^{n\times m} \subseteq \mathbb{R}_c^{n\times m}$.

To prove the reverse inclusion, let $V \in \mathbb{R}_c^{n\times m}$ and let μ_i, $i = 1, \dots, n$ be the eigenvalues of VV^t. Then $0 \le \mu_i < c^2$, so that we can define $P \in \mathbb{R}^{n\times m}$ by the equation

$$P = \sqrt{I_n - c^{-2}VV^t}^{\,-1} V . \tag{97}$$

By means of Theorem 6.3, (97) implies

$$V = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P \tag{98}$$

so that $V = \phi(P) \in \mathbb{B}^{n\times m}$, implying the reverse inclusion $\mathbb{R}_c^{n\times m} \subseteq \mathbb{B}^{n\times m}$. Hence, $\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m}$, as desired. $\square$

For any $V \in \mathbb{R}^{n\times m}$ the set of nonzero eigenvalues of VV^t equals the set of nonzero eigenvalues of V^tV. Hence, following (92) we have

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m} :\ \text{Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\} \\ &= \{V \in \mathbb{R}^{n\times m} :\ \text{Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0 \le \lambda < c^2\}\end{aligned} \tag{99}$$

Result (93) of Theorem 7.2 suggests calling $\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m}$ the *eigenball of* $\mathbb{R}^{n\times m}$ *of eigenradius* c, or the c-eigenball in short.

8. Reparametrizing the Bi-Boost

We now wish to change the bi-boost parameter $P \in \mathbb{R}^{n\times m}$, the domain of which is the set $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices, to the new parameter $V \in \mathbb{R}_c^{n\times m}$, the domain of which is the eigenball $\mathbb{R}_c^{n\times m}$ of $\mathbb{R}^{n\times m}$. We, therefore, recall the following equations, which are taken from (36), (61) and (48).

$$\begin{aligned} P = \sqrt{I_n - c^{-2}VV^t}^{\,-1} V &= V\sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ \sqrt{I_m + c^{-2}P^tP} &= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ \sqrt{I_n + c^{-2}PP^t} &= \sqrt{I_n - c^{-2}VV^t}^{\,-1}. \end{aligned} \tag{100}$$

A generic parameter $V \in \mathbb{R}_c^{n\times m}$ in the eigenball $\mathbb{R}_c^{n\times m}$ is constructed by constructing a generic parameter $P \in \mathbb{R}^{n\times m}$ and employing (35).

The equations in (100) along with analogies with the gamma factor of special relativity theory suggest the definition of a *left gamma factor* $\Gamma^L_{n,V}$ and a *right gamma factor* $\Gamma^R_{m,V}$ by the following equations.

$$\begin{aligned} \Gamma^L_{n,V} &:= \sqrt{I_n - c^{-2}VV^t}^{\,-1} \in \mathbb{R}^{n\times n} \\ \Gamma^R_{m,V} &:= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \in \mathbb{R}^{m\times m}. \end{aligned} \tag{101}$$

Naturally, the pair $(\Gamma^L_{n,V},\ \Gamma^R_{m,V})$ of a left and a right gamma factor is called a *bi-gamma factor*. Practically, it is sometimes convenient to use the short notation

$$\gamma_{m,V} := \Gamma^R_{m,V}\,, \qquad \gamma_{n,V} := \Gamma^L_{n,V} \tag{102}$$

in which a left (right) gamma factor is implicitly indicated by the subscript n (m). It proves useful to use *interchangeably* the short notation with γ and the full notation with Γ in (102). We will use the short notation mainly in lengthy intermediate results as, for instance, in (146), p. 255.

Following (36), the left and right gamma factors are related by the first commuting relation in (103) below. The remaining commuting relations in (103) follow immediately from the first one, noting that left and right gamma factors are symmetric matrices.

$$\begin{aligned}
\Gamma^L_{n,V} V &= V \Gamma^R_{m,V} \\
\Gamma^R_{m,V} V^t &= V^t \Gamma^L_{n,V} \\
\Gamma^L_{n,V} V V^t &= V V^t \Gamma^L_{n,V} \\
\Gamma^R_{m,V} V^t V &= V^t V \Gamma^R_{m,V} \,.
\end{aligned} \tag{103}$$

Moreover, by Theorem 6.3 with P replaced by E,

$$\begin{aligned}
E = \Gamma^L_{n,V} V = V \Gamma^R_{m,V} \quad \Longleftrightarrow \quad V &= \sqrt{I_n + c^{-2} E E^t}^{\,-1} E \\
&= E \sqrt{I_m + c^{-2} E^t E}^{\,-1} \,.
\end{aligned} \tag{104}$$

The result in (104) will prove useful in (163), p. 259.

In the *bi-gamma notation* (101), the equations in (100) take the form

$$\begin{aligned}
P = \Gamma^L_{n,V} V = V \Gamma^R_{m,V} &\in \mathbb{R}^{n\times m} \\
\sqrt{I_n + c^{-2} P P^t} = \Gamma^L_{n,V} &\in \mathbb{R}^{n\times n} \\
\sqrt{I_m + c^{-2} P^t P} = \Gamma^R_{m,V} &\in \mathbb{R}^{m\times m} \,.
\end{aligned} \tag{105}$$

Introducing the arbitrarily fixed positive constant $c > 0$ into the bi-boost $B(P)$ in (1) we obtain the bi-boost $B_c(P)$, shown in (106) below, parametrized by $P \in \mathbb{R}^{n\times m}$. The bi-boost $B_c(P)$ leaves invariant the inner product of signature (m, n), $m, n \in \mathbb{N}$, shown in (138), p. 254, as we will prove straightforwardly in Theorem 10.1, p. 256.

The bi-boost $B_c(P)$ can be written as a bi-boost $\bar{B}_c(V)$ parametrized by the new parameter $V \in \mathbb{R}^{n\times m}_c$. Abusing notation, instead of $\bar{B}_c(V)$ we write $B_c(V)$ since no confusion may arise. Thus, following the change of parameter from $B_c(P)$ with parameter $P \in \mathbb{R}^{n\times m}$ to $B_c(V)$ with parameter $V \in \mathbb{R}^{n\times m}_c$ we have by means of (30) and (105),

$$\begin{aligned}
B_c(P) &= \begin{pmatrix} \sqrt{I_m + c^{-2} P^t P} & \frac{1}{c^2} P^t \\ P & \sqrt{I_n + c^{-2} P P^t} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2} \Gamma^R_{m,V} V^t = \frac{1}{c^2} V^t \Gamma^L_{n,V} \\ \Gamma^L_{n,V} V = V \Gamma^R_{m,V} & \Gamma^L_{n,V} \end{pmatrix} =: B_c(V) \,.
\end{aligned} \tag{106}$$

It can be shown that when $m = 1$ the bi-boost $B_c(V)$ specializes to the standard Lorentz boost in one time dimension and n space dimensions, studied in [35]. It, therefore, proves useful to replace the bi-boost $B_c(P)$ parametrized by $P \in \mathbb{R}^{n\times m}$

by the equivalent bi-boost $B_c(V)$ parametrized by $V \in \mathbb{R}_c^{n\times m}$, obtaining

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \tag{107}$$

as we see from (106).

Accordingly, the generic Lorentz transformation $\Lambda(P, O_n, O_m)$ of order (m, n), $m, n \in \mathbb{N}$, in (4) becomes $\Lambda = \Lambda(V, O_n, O_m)$ given by the unique bi-gyration decomposition in theorem 8.1 below.

Owing to the bijective correspondence between the old parameter $P \in \mathbb{R}^{n\times m}$ and the new parameter $V \in \mathbb{R}_c^{n\times m}$, Theorem 2.1 can be translated into the following theorem.

Theorem 8.1. (Lorentz Transformation Bi-gyration Decomposition, V), *A matrix $\Lambda \in \mathbb{R}^{(m+n)\times(m+n)}$ is the matrix representation of a Lorentz transformation of order (m, n), $\Lambda \in SO(m, n)$, if and only if it is given uniquely by the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{108}$$

or, prarmetrically in short,

$$\Lambda = \Lambda(O_m, V, O_n) = \rho(O_m)B(V)\lambda(O_n) = \begin{pmatrix} V \\ O_n \\ O_m \end{pmatrix} \tag{109}$$

for any $V \in \mathbb{R}_c^{n\times m}$, $O_m \in SO(m)$ and $O_n \in SO(n)$.

Example 8.2. In this example we show that in the special case when $m = 1$ the eigenball $\mathbb{R}_c^{n\times 1}$ specializes to the open c-ball $\mathbb{R}_c^n$ of $\mathbb{R}^{n\times 1} = \mathbb{R}^n$.

For $m = 1$, $V \in \mathbb{R}^{n\times m} = \mathbb{R}^n$ is a column vector in the Euclidean n-space $\mathbb{R}^n$, and $V^tV = \|V\|^2$ is a 1×1 matrix the eigenvalue of which is $\lambda = \|V\|^2$. Hence, following (99) and Theorem 7.2 we have $V \in \mathbb{R}_c^{n\times m}$ and

$$\begin{aligned} \mathbb{R}_c^{n\times 1} &= \{V \in \mathbb{R}^n : \text{ The eigenvalue } \|\mathrm{V}\|^2 \text{ of } \mathrm{V^tV} \text{ satisfies } 0 \le \|V\|^2 < c^2\} \\ &= \{V \in \mathbb{R}^n : \ 0 \le \|V\| < c\} \\ &=: \mathbb{R}_c^n \,. \end{aligned} \tag{110}$$

Indeed, in special relativity, the relativistically admissible velocities are elements of the c-ball $\mathbb{R}_c^3$, where c represents the vacuum speed of light.

Example 8.3. In this example we show that when $m = 1$ the right gamma factor equals the gamma factor of special relativity theory.

When $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector so that $P^tP = \|P\|^2$. Then, by (35),

$$V = \phi(P) = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} = \frac{P}{\sqrt{1 + c^{-2}\|P\|^2}} \tag{111}$$

so that $V \in \mathbb{R}^n$ is a column vector and

$$\|V\|^2 = V^tV = \frac{\|P\|^2}{1 + c^{-2}\|P\|^2} \,. \tag{112}$$

Hence, $0 \le \|V\| < c$ and, by (101),

$$\Gamma^R_{m=1,V} = \frac{1}{\sqrt{1 - c^{-2}\|V\|^2}} =: \gamma_V \tag{113}$$

for all $V \in \mathbb{B}^{n\times 1} = \phi(\mathbb{R}^{n\times 1})$. Here γ_V is the gamma factor that plays an important role in special relativity and in its underlying hyperbolic geometry [36, 37, 40, 42, 43, 45].

Example 8.4. Extending (113) to $m \ge 1$, it can be shown that the left and right gamma factors,

$$\Gamma^L_{n,V} := \sqrt{I_n - c^{-2}VV^t}^{\,-1} = \sqrt{I_n + c^{-2}PP^t} \tag{114}$$

and

$$\Gamma^R_{m,V} := \sqrt{I_m - c^{-2}V^tV}^{\,-1} = \sqrt{I_m + c^{-2}P^tP}\,, \tag{115}$$

are related by the equation

$$-I_n + \Gamma^L_{n,V} = \frac{1}{c^2}P(I_m + \Gamma^R_{m,V})^{-1}P^t \tag{116}$$

where P and V are related by Theorem 6.3. Note that by means of (114)–(115), (116) is equivalent to the elegant matrix identity (117), which we prove in the following lemma.

Lemma 8.5. *The matrix identities*

$$-I_n + \sqrt{I_n + c^{-2}PP^t} = \frac{1}{c^2}P\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^{-1}P^t \tag{117}$$

and

$$-I_m + \sqrt{I_m + c^{-2}P^tP} = \frac{1}{c^2}P^t\left(I_n + \sqrt{I_n + c^{-2}PP^t}\,\right)^{-1}P \tag{118}$$

hold for all $P \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$.

Proof. Clearly,

$$\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^2 = 2\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right) + c^{-2}P^tP\,. \tag{119}$$

Let

$$R := \left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^{-1} \tag{120}$$

so that (119) can be written as

$$2R^{-1} + c^{-2}P^tP - (R^{-1})^2 = 0_{m,m}\,. \tag{121}$$

Left multiplying and right multiplying (121) by R yields

$$2R + c^{-2}RP^tPR - I_m = 0_{m,m}\,. \tag{122}$$

Left multiplying by P and right multiplying by P^t, (122) yields

$$P(2R + c^{-2}RP^tPR - I_m)P^t = 0_{m,m} \tag{123}$$

so that

$$2PRP^t + c^{-2}PRP^tPRP^t = PP^t \tag{124}$$

and hence,

$$I_n + c^{-2}(2PRP^t + c^{-2}PRP^tPRP^t) = I_n + c^{-2}PP^t\,. \tag{125}$$

Identity (125) can be written as

$$(I_n + c^{-2}PRP^t)^2 = I_n + c^{-2}PP^t \tag{126}$$

implying

$$I_n + c^{-2}PRP^t = \sqrt{I_n + c^{-2}PP^t}\,. \tag{127}$$

Finally, by means of (120), (127) yields (117), as desired.

The proof of (118) is similar to that of (117). □

Example 8.6. In the special case when $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector, $P^tP = \|P\|^2$, and PP^t is an $n \times n$ matrix, so that (117) specializes to

$$\sqrt{I_n + c^{-2}PP^t} = I_n + \frac{1}{c^2}\frac{1}{1 + \sqrt{1 + c^{-2}\|P\|^2}}PP^t\,. \tag{128}$$

We now manipulate (128) in the following chain of equations, which are numbered for subsequent explanation. For all $V \in \mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$,

$$\begin{aligned}
\sqrt{I_n - c^{-2}VV^t}^{\;-1} &\overset{(1)}{=\!=} I_n + \frac{1}{c^2}\frac{1}{1 + \frac{1}{\sqrt{1-c^{-2}\|V\|^2}}}\left(\Gamma^L_{n,V}\right)^2 VV^t \\
&\overset{(2)}{=\!=} I_n + \frac{1}{c^2}\frac{1}{1+\gamma_V}V\left(\Gamma^R_{m,V}\right)^2 V^t \\
&\overset{(3)}{=\!=} I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V}VV^t\,.
\end{aligned} \tag{129}$$

Derivation of the numbered equalities in (129) follows:

1. This equation is equivalent to (128) since (i) the left sides of the two equations are equal by (48); and (ii) their right sides are equal by (61) with $m = 1$, and by (66) along with (101).

2. Follows from Item (1) by (113) and by the first commuting relation in (103).

3. Follows from Item (2) by (113), noting that $m = 1$.

Noting (101), the chain of equation (129) yields the important equation

$$\Gamma^L_{n,V} = I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V}VV^t\,, \qquad (\mathrm{m}=1)\,, \tag{130}$$

which holds for $m = 1$ and all $n \in \mathbb{N}$.

The importance of (130) is revealed in Example 8.7 below, enabling us to show straightforwardly that the bi-boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$, specializes to the Lorentz boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$, of special relativity in the special case when $m = 1$.

Example 8.7. When $m = 1$ the bi-boost $B_c(V)$ in (107) can be manipulated by means of (103) and by means of (113) and (130), obtaining the following chain of equations.

$$\begin{aligned} B_c(V) &= \begin{pmatrix} \Gamma^R_{m=1,V} & \frac{1}{c^2}\Gamma^R_{m=1,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \\ &= \begin{pmatrix} \Gamma^R_{m=1,V} & \frac{1}{c^2}\Gamma^R_{m=1,V}V^t \\ V\Gamma^R_{m=1,V} & \Gamma^L_{n,V} \end{pmatrix} \\ &= \begin{pmatrix} \gamma_V & \frac{1}{c^2}\gamma_V V^t \\ \gamma_V V & I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V}VV^t \end{pmatrix}\,, \qquad (m=1), \end{aligned} \tag{131}$$

where $V \in \mathbb{R}_c^{n\times 1} \subset \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector in the ball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ of $\mathbb{R}^n$,

$$\mathbb{R}_c^n = \{V \in \mathbb{R}^n :\ \|V\| < c\}\,. \tag{132}$$

The extreme right side of (131) turns out to be the standard special relativistic $(n+1)\times(n+1)$ matrix representation of the Lorentz group in one time dimension and n space dimensions [35] [36, p. 254] [40, p. 447]. Accordingly, it follows from (131) that in the special case when $m = 1$ the Lorentz group of order (m, n) specializes to the Lorentz group of special relativity theory.

Example 8.8. In the special case when $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector so that $P^tP = \|P\|^2$. Accordingly, when $m = 1$ Identity (117) specializes to Identity (128),

$$\sqrt{I_n + c^{-2}PP^t} = I_n + \frac{1}{c^2}\frac{PP^t}{1+\sqrt{1+c^{-2}\|P\|^2}}\,. \tag{133}$$

Hence, when $m = 1$, the boost $B_c(P)$ in (106) specializes to the *proper velocity (PV) bi-boost*

$$B_c(P) = \begin{pmatrix} \sqrt{1+c^{-2}\|P\|^2} & \frac{1}{c^2}P^t \\ P & I_n + \frac{1}{c^2}\frac{PP^t}{1+\sqrt{1+c^{-2}\|P\|^2}} \end{pmatrix} \tag{134}$$

in one proper-time dimension and n space dimensions, where $P \in \mathbb{R}^n$ is the *proper velocity* of special relativity (in physical applications $n = 3$).

The PV-bi-boost (134) leaves invariant the relativistic inner product in (138) below.

The PV-bi-boost $B_c(P)$ involves the proper-velocity parameter $P \in \mathbb{R}^n$, which is measured by means of proper-time. The need for a search for a proper-time boost, like the one in (134), arises in several papers as, for instance, [14–20] and [37–39, 46].

The application $B_c(P)(t, \mathbf{x})^t$ of the PV-bi-boost $B_c(P)$ to time space coordinates $(t, \mathbf{x})^t$ is linear, and it keeps the relativistic norm

$$\tau = \sqrt{t^2 - \mathbf{x}^2/c^2} \tag{135}$$

invariant.

Similarly, the application $B_c(P)(\sqrt{\tau^2 + \mathbf{x}^2/c^2}, \mathbf{x})^t$ of the PV-bi-boost $B_c(P)$ to proper-time space coordinates $(\tau, \mathbf{x})^t$ is nonlinear, and it keeps the proper-time τ invariant.

9. The Bi-Boost $B_c(V)$

We know by construction that the bi-boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, of order (m, n), $m, n \in \mathbb{N}$, preserves the inner product of signature (m, n) in the pseudo-Euclidean space $\mathbb{R}^{m,n}$. However, solely owing to the commuting relations in (103), a direct proof is straightforward, simple and, hence, instructive. Accordingly, the aim of this section is to prove directly that the bi-boost $B_c(V)$ in (139) below preserves the pseudo-Euclidean inner product of signature (m, n), $m, n \in \mathbb{N}$, in (138) below.

Let

$$\mathbf{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \in \mathbb{R}^m, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \tag{136}$$

so that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{x} \end{pmatrix} = (t_1, \ldots, t_m, x_1, \ldots, x_n)^t \in \mathbb{R}^{m,n} \tag{137}$$

is a generic point of the pseudo-Euclidean space $\mathbb{R}^{m,n}$. The inner product of signature (m, n) in $\mathbb{R}^{m,n}$ involves the constant $c > 0$ according to the equation

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} := \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix}^t \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -c^{-2} I_n \end{pmatrix} \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{t}_1 \cdot \mathbf{t}_2 - c^{-2} \mathbf{x}_1 \cdot \mathbf{x}_2 \tag{138}$$

for all $(\mathbf{t}_1, \mathbf{x}_1)^t, (\mathbf{t}_2, \mathbf{x}_2)^t \in \mathbb{R}^{m,n}$, where $\mathbf{t}_1 \cdot \mathbf{t}_2 = \mathbf{t}_1^t \mathbf{t}_2$ and $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^t \mathbf{x}_2$ are the standard inner product in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

The bi-boost $B_c(V)$ is given by its $(m+n) \times)(m+n)$ matrix representation (107),

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m,V} & c^{-2} \Gamma^R_{m,V} V^t \\ \Gamma^L_{n,V} V & \Gamma^L_{n,V} \end{pmatrix} \tag{139}$$

$m, n \in \mathbb{N}$, where the left and right gamma factors are given by (101),

$$\begin{aligned} \Gamma^L_{n,V} &= \sqrt{I_n - c^{-2} V V^t}^{\,-1} \in \mathbb{R}^{n \times n} \\ \Gamma^R_{m,V} &= \sqrt{I_m - c^{-2} V^t V}^{\,-1} \in \mathbb{R}^{m \times m}. \end{aligned} \tag{140}$$

The space of the parameter V in (139) – (140) is the c-eigenball $\mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$ which is given by

$$\begin{aligned} \mathbb{R}_c^{n \times m} &= \{V \in \mathbb{R}^{n \times m} : \text{ Each eigenvalue } \lambda \text{ of } V V^t \text{ satisfies } 0 \le \lambda < c^2\} \\ &= \{V \in \mathbb{R}^{n \times m} : \text{ Each eigenvalue } \lambda \text{ of } V^t V \text{ satisfies } 0 \le \lambda < c^2\}. \end{aligned} \tag{141}$$

The generic parameter $V \in \mathbb{R}_c^{n \times m}$ in the c-eigenball $\mathbb{R}_c^{n \times m}$ of $\mathbb{R}^{n \times m}$ is constructed by constructing a generic parameter $P \in \mathbb{R}^{n \times m}$ and employing (35),

$$V = \sqrt{I_n + c^{-2} P P^t}^{\,-1} P = P \sqrt{I_m + c^{-2} P^t P}^{\,-1}. \tag{142}$$

Theorem 9.1. *The bi-boost*

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m,V} & c^{-2} \Gamma^R_{m,V} V^t \\ \Gamma^L_{n,V} V & \Gamma^L_{n,V} \end{pmatrix} \tag{143}$$

$V \in \mathbb{R}_c^{n \times m}$, $m, n \in \mathbb{N}$, *leaves the pseudo-Euclidean inner product* (138) *invariant, that is*

$$B_c(V) \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot B_c(V) \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} \tag{144}$$

for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^m$ *and* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

Proof. For convenient, we use in the proof the short notation in (102).

$$B_c(V)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}\gamma_{m,V} & c^{-2}\gamma_{m,V}V^t\\ \gamma_{n,V}V & \gamma_{n,V}\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}\gamma_{m,V}\mathbf{t} + c^{-2}\gamma_{m,V}V^t\mathbf{x}\\ \gamma_{n,V}V\mathbf{t} + \gamma_{n,V}\mathbf{x}\end{pmatrix}. \tag{145}$$

Hence, by (136) – (138), (103) and (105), we have the following chain of equations.

$$\begin{aligned}
&B_c(V)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(V)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}\\
&= \begin{pmatrix}\gamma_{m,V}\mathbf{t}_1 + c^{-2}\gamma_{m,V}V^t\mathbf{x}_1\\ \gamma_{n,V}V\mathbf{t}_1 + c^{-2}\gamma_{n,V}\mathbf{x}_1\end{pmatrix}^t \begin{pmatrix}I_m & 0_{m,n}\\ 0_{n,m} & -c^{-2}I_n\end{pmatrix}\begin{pmatrix}\gamma_{m,V}\mathbf{t}_2 + c^{-2}\gamma_{m,V}V^t\mathbf{x}_2\\ \gamma_{n,V}V\mathbf{t}_2 + \gamma_{n,V}\mathbf{x}_2\end{pmatrix}\\
&= (\mathbf{t}_1^t\gamma_{m,V} + c^{-2}\mathbf{x}_1^tV\gamma_{m,V},\ -c^{-2}\mathbf{t}_1^tV^t\gamma_{n,V} - c^{-2}\mathbf{x}_1^t\gamma_{n,V})\\
&\qquad\qquad \times\begin{pmatrix}\gamma_{m,V}\mathbf{t}_2 + c^{-2}\gamma_{m,V}V^t\mathbf{x}_2\\ \gamma_{n,V}V\mathbf{t}_2 + \gamma_{n,V}\mathbf{x}_2\end{pmatrix}\\
&= \mathbf{t}_1^t\gamma_{m,V}^2\mathbf{t}_2 + c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2 + c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2 + c^{-4}\mathbf{x}_1^tV\gamma_{m,V}^2V^t\mathbf{x}_2\\
&\quad - c^{-2}(\mathbf{t}_1^tV^t\gamma_{n,V}^2V\mathbf{t}_2 + \mathbf{t}_1^tV^t\gamma_{n,V}^2\mathbf{x}_2 + \mathbf{x}_1^t\gamma_{n,V}^2V\mathbf{t}_2 + \mathbf{x}_1^t\gamma_{n,V}^2\mathbf{x}_2)\\
&= \mathbf{t}_1^t\gamma_{m,V}^2\mathbf{t}_2 + c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2 + c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2 + c^{-4}\mathbf{x}_1^t\gamma_{n,V}^2VV^t\mathbf{x}_2\\
&\quad - (c^{-2}\mathbf{t}_1^tV^tV\gamma_{m,V}^2\mathbf{t}_2 + c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2 + c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2 + c^{-2}\mathbf{x}_1^t\gamma_{n,V}^2\mathbf{x}_2)\\
&= \mathbf{t}_1^t(I_m - c^{-2}V^tV)\gamma_{m,V}^2\mathbf{t}_2 - c^{-2}\mathbf{x}_1^t\gamma_{n,V}^2(I_n - c^{-2}VV^t)\mathbf{x}_2\\
&= \mathbf{t}_1^t\mathbf{t}_2 - c^{-2}\mathbf{x}_1^t\mathbf{x}_2\\
&= \mathbf{t}_1\cdot\mathbf{t}_2 - c^{-2}\mathbf{x}_1\cdot\mathbf{x}_2\\
&= \begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix},
\end{aligned} \tag{146}$$

as desired. □

Example 9.2. Following (140) we have the obvious limits of large c,

$$\begin{aligned}\lim_{c\to\infty}\Gamma_{m,V}^R &= I_m\\ \lim_{c\to\infty}\Gamma_{n,V}^L &= I_n\,.\end{aligned} \tag{147}$$

Hence, in that limit we have

$$B_\infty(V) := \lim_{c\to\infty} B_c(V) = \begin{pmatrix}I_m & 0_{m,n}\\ V & I_n\end{pmatrix} \tag{148}$$

so that the limit of (145) as c approaches infinity yields an obvious generalization of the familiar Galilei transformation in a pseudo-Euclidean space of signature (m,n),

$$B_\infty(V)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}I_m & 0_{m,n}\\ V & I_n\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}\mathbf{t}\\ \mathbf{x}+V\mathbf{t}\end{pmatrix}. \tag{149}$$

10. The Bi-Boost $B_c(P)$

We know by its construction in [46] that the bi-boost $B_{c=1}(P)$, $P\in\mathbb{R}^{n\times m}$, of order (m,n), $m,n\in\mathbb{N}$, preserves the inner product (138) of signature (m,n) in the pseudo-Euclidean space $\mathbb{R}^{m,n}$. However, solely owing to the commuting relations in (154) below, a direct proof is straightforward, simple and, hence, instructive. Accordingly, the aim of this section is to prove directly that the bi-boost $B_c(P)$ preserves the inner product (138) for an arbitrarily fixed positive constant c.

Theorem 10.1. *The bi-boost*

$$B_c(P)=\begin{pmatrix}\sqrt{I_m+c^{-2}P^tP} & \frac{1}{c^2}P^t\\ P & \sqrt{I_n+c^{-2}PP^t}\end{pmatrix} \tag{150}$$

$P\in\mathbb{R}^{n\times m}$, $m,n\in\mathbb{N}$, *leaves the pseudo-Euclidean inner product* (138) *invariant, that is*

$$B_c(P)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(P)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}=\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix} \tag{151}$$

for any $\mathbf{t}_1,\mathbf{t}_2\in\mathbb{R}^m$ *and* $\mathbf{x}_1,\mathbf{x}_2\in\mathbb{R}^n$.

Proof. It is convenient to use in the proof the short notation

$$\begin{aligned} b_m &:= \sqrt{I_m+c^{-2}P^tP}\\ b_n &:= \sqrt{I_n+c^{-2}PP^t}\end{aligned} \tag{152}$$

so that, by (150),

$$B_c(P)=\begin{pmatrix}b_m & \frac{1}{c^2}P^t\\ P & b_n\end{pmatrix} \tag{153}$$

and, by (34), we have the commuting relations

$$\begin{aligned} Pb_m &= b_nP\\ P^tb_n &= b_mP^t\,.\end{aligned} \tag{154}$$

The application of the bi-boost $B_c(P)$ to $(\mathbf{t},\mathbf{x})^t\in\mathbb{R}^{m,n}$ is given by

$$B_c(P)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}b_m & \frac{1}{c^2}P^t\\ P & b_n\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}b_m\mathbf{t}+c^{-2}P^t\mathbf{x}\\ P\mathbf{t}+b_n\mathbf{x}\end{pmatrix}. \tag{155}$$

Hence, by (136) – (138), (154) and (155) we have the following chain of equations.

$$
\begin{aligned}
&B_c(P)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(P)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}\\
&\quad=\begin{pmatrix}b_m\mathbf{t}_1+c^{-2}P^t\mathbf{x}_1\\ P\mathbf{t}_1+b_n\mathbf{x}_1\end{pmatrix}^t\begin{pmatrix}I_m & 0_{m,n}\\ 0_{n,m} & -c^{-2}I_n\end{pmatrix}\begin{pmatrix}b_m\mathbf{t}_2+c^{-2}P^t\mathbf{x}_2\\ P\mathbf{t}_2+b_n\mathbf{x}_2\end{pmatrix}\\
&\quad=(\mathbf{t}_1^t b_m+c^{-2}\mathbf{x}_1^t P,\quad -c^{-2}(\mathbf{t}_1^t P^t+\mathbf{x}_1^t b_n))\begin{pmatrix}b_m\mathbf{t}_2+c^{-2}P^t\mathbf{x}_2\\ P\mathbf{t}_2+b_n\mathbf{x}_2\end{pmatrix}\\
&\quad=\mathbf{t}_1^t b_m^2\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2+c^{-4}\mathbf{x}_1^t PP^t\mathbf{x}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t P^t b_n\mathbf{x}_2+\mathbf{x}_1^t b_n P\mathbf{t}_2+\mathbf{x}_1^t b_n^2\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t(I_m+c^{-2}P^tP)\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2+c^{-4}\mathbf{x}_1^t PP^t\mathbf{x}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t P^t b_n\mathbf{x}_2+\mathbf{x}_1^t b_n P\mathbf{t}_2+\mathbf{x}_1^t(I_n+c^{-2}PP^t)\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t(I_m+c^{-2}P^tP)\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t b_m P^t\mathbf{x}_2+\mathbf{x}_1^t P b_m\mathbf{t}_2+\mathbf{x}_1^t\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t\mathbf{t}_2-c^{-2}\mathbf{x}_1^t\mathbf{x}_2\\
&\quad=\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}.
\end{aligned}
\tag{156}
$$

as desired. □

11. Bi-Boost Product with the Parameter V

The Lorentz transformation product law, expressed in terms of the old parameter $P \in \mathbb{R}^{n\times m}$ in Theorem 2.3, was derived in [46, Theorem 21]. Accordingly, an important objective of the present article is to derive the Lorentz transformation product law expressed in terms of the new parameter $V \in \mathbb{R}_c^{n\times m}$.

Let $B_c(V_k)$, $k = 1, 2$, be two bi-boosts parametrized by $V_k \in \mathbb{R}_c^{n\times m}$,

$$B_c(V_k)=\begin{pmatrix}\gamma_{m,V_k} & \frac{1}{c^2}\gamma_{m,V_k}V_k^t\\ \gamma_{n,V_k}V_k & \gamma_{n,V_k}\end{pmatrix} \tag{157}$$

where we use the short notation in (102), p. 247.

By matrix multiplication and the commuting relations (103),

$$
\begin{aligned}
B_c(V_1)B_c(V_2) &= \begin{pmatrix} \gamma_{m,V_1} & \frac{1}{c^2}\gamma_{m,V_1}V_1^t \\ \gamma_{n,V_1}V_1 & \gamma_{n,V_1} \end{pmatrix} \begin{pmatrix} \gamma_{m,V_2} & \frac{1}{c^2}\gamma_{m,V_2}V_2^t \\ \gamma_{n,V_2}V_2 & \gamma_{n,V_2} \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{m,V_1}\gamma_{m,V_2} + \frac{1}{c^2}\gamma_{m,V_1}V_1^t\gamma_{n,V_2}V_2 & \frac{1}{c^2}(\gamma_{m,V_1}\gamma_{m,V_2}V_2^t + \gamma_{m,V_1}V_1^t\gamma_{n,V_2}) \\ \gamma_{n,V_1}V_1\gamma_{m,V_2} + \gamma_{n,V_1}\gamma_{n,V_2}V_2 & \frac{1}{c^2}\gamma_{n,V_1}V_1\gamma_{m,V_2}V_2^t + \gamma_{n,V_1}\gamma_{n,V_2} \end{pmatrix} \\
&= \begin{pmatrix} \gamma_{m,V_1}(I_m + \frac{1}{c^2}V_1^tV_2)\gamma_{m,V_2} & \frac{1}{c^2}\gamma_{m,V_1}(V_1+V_2)^t\gamma_{n,V_2} \\ \gamma_{n,V_1}(V_1+V_2)\gamma_{m,V_2} & \gamma_{n,V_1}(I_n + \frac{1}{c^2}V_1V_2^t)\gamma_{n,V_2} \end{pmatrix} =: \begin{pmatrix} E_{11} & \frac{1}{c^2}E_{12} \\ E_{21} & E_{22} \end{pmatrix} .
\end{aligned}
\tag{158}
$$

As we see from (158), the product of two bi-boosts need not be a bi-boost. However, it is a Lorentz transformation and, as such, it uniquely possesses the bi-gyration decomposition (108). Hence, by (108), we can express the bi-boost product $B_c(V_1)B_c(V_2)$ as follows,

$$
\begin{aligned}
&B_c(V_1)B_c(V_2) = \\
&= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2] & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V_{12}} & \frac{1}{c^2}\Gamma^R_{m,V_{12}}V_{12}^t \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & \mathrm{lgyr}[V_1,V_2] \end{pmatrix} \\
&= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}} & \frac{1}{c^2}\mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}}V_{12}^t\mathrm{lgyr}[V_1,V_2] \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}}\mathrm{lgyr}[V_1,V_2] \end{pmatrix} \\
&= \begin{pmatrix} E_{11} & \frac{1}{c^2}E_{12} \\ E_{21} & E_{22} \end{pmatrix}
\end{aligned}
\tag{159}
$$

where the composite parameter $V_{12} \in \mathbb{R}_c^{n\times m}$,

$$
V_{12} =: V_1 \oplus V_2 \tag{160}
$$

and the bi-gyration $(\mathrm{lgyr}[V_1,V_2], \mathrm{rgyr}[V_1,V_2]) \in SO(n) \times SO(m)$ are to be determined in terms of V_1 and V_2.

The uniqueness of the Lorentz transformation bi-gyration decomposition, insured by the Bi-gyration Decomposition Theorem 8.1, implies that the matrix entries E_{ij}, $i, j = 1, 2$, defined in (158), and the matrix entries E_{ij} defined in (159) are identically equal.

Hence, the expressions

$$
\begin{aligned}
&V_{12} =: V_1 \oplus V_2 \in \mathbb{B}^{n\times m} \\
&\mathrm{lgyr}[V_1,V_2] \in SO(n) \\
&\mathrm{rgyr}[V_1,V_2] \in SO(m)
\end{aligned}
\tag{161}
$$

that appear in (159) are uniquely determined by the Bi-gyration Decomposition Theorem 8.1. Employing (158) – (159), in the following Subsections – we determine each of the expressions in (161) in terms of V_1 and V_2.

11.1. E_{21}

In this subsection we study the consequences of the equality between E_{21} in (158) and E_{21} in (159).

With $V_{12} = V_1 \oplus V_2$, we see from (159) that

$$E_{21} = \Gamma^L_{n,V_1\oplus V_2}(V_1 \oplus V_2)\,. \tag{162}$$

Hence, by (104), the binary operation $\oplus$ in $\mathbb{R}^{n\times m}_c$ is given by

$$V_1 \oplus V_2 = \sqrt{I_n + c^{-2}E_{21}E_{21}^t}^{\,-1} E_{21} = E_{21}\sqrt{I_m + c^{-2}E_{21}^t E_{21}}^{\,-1}\,, \tag{163}$$

where, by (158),

$$E_{21} = \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2}\,, \tag{164}$$

$V_1, V_2 \in \mathbb{R}^{n\times m}_c$.

Thus, the *bi-gyrosum* $V_1 \oplus V_2$ is expressed in (163) – (164) in terms of V_1 and V_2.

It is interesting to note that following (141), (163) – (164) and (147), we have the limits

$$\begin{aligned}\lim_{c\to\infty} \mathbb{R}^{n\times m}_c &= \mathbb{R}^{n\times m}\\ \lim_{c\to\infty}(V_1 \oplus V_2) &= V_1 + V_2\,.\end{aligned} \tag{165}$$

Thus, as expected, in the limit of large c, the binary operation $\oplus$ in the eigenball $\mathbb{R}^{n\times m}_c$ tends to the common matrix addition, $+$, in the ambient space $\mathbb{R}^{n\times m}$.

In the special case when $m = 1$, the binary operation $\oplus$ in the eigenball $\mathbb{R}^{n\times m}_c$ specializes to Einstein velocity addition of special relativity in the ball $\mathbb{R}^n_c$, as indicated in Example 8.7. Einstein velocity addition in the ball $\mathbb{R}^n_c$ is studied, for instance, in [36, 40].

Additionally, the equality between E_{21} in (159) and in (158), along with the first commuting relation in (103), yields the elegant equations

$$\begin{aligned}\Gamma^L_{n,V_1\oplus V_2}(V_1 \oplus V_2) &= \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2}\\ (V_1 \oplus V_2)\Gamma^R_{m,V_1\oplus V_2} &= \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2}\end{aligned} \tag{166}$$

which show how closely the binary operations $\oplus$ and $+$ are related to each other.

Lemma 11.1. *The expression* E_{21} *in* (162) *possesses the commuting relations*

$$\begin{aligned} E_{21}E_{21}^t\sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1} &= \sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1}E_{21}E_{21}^t \\ E_{21}^tE_{21}\sqrt{I_m+c^{-2}E_{21}^tE_{21}}^{\,-1} &= \sqrt{I_m+c^{-2}E_{21}^tE_{21}}^{\,-1}E_{21}^tE_{21} \end{aligned} \tag{167}$$

and the identities

$$\begin{aligned} \Gamma^L_{n,V_1\oplus V_2} &:= \sqrt{I_n-c^{-2}(V_1\oplus V_2)(V_1\oplus V_2)^t}^{\,-1} = \sqrt{I_n+c^{-2}E_{21}E_{21}^t} \\ \Gamma^R_{m,V_1\oplus V_2} &:= \sqrt{I_m-c^{-2}(V_1\oplus V_2)^t(V_1\oplus V_2)}^{\,-1} = \sqrt{I_m+c^{-2}E_{21}^tE_{21}}\,. \end{aligned} \tag{168}$$

Proof. The commuting relations in (167) follow immediately from the commuting relation in (163).

By (163) and (167) we have

$$\begin{aligned} (V_1\oplus V_2)(V_1\oplus V_2)^t &= \sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1}E_{21}E_{21}^t\sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1} \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t\,. \end{aligned} \tag{169}$$

Hence,

$$\begin{aligned} I_n-c^{-2}(V_1\oplus V_2)(V_1\oplus V_2)^t &= I_n-c^{-2}(I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}(I_n+c^{-2}E_{21}E_{21}^t)-c^{-2}(I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}(I_n+c^{-2}E_{21}E_{21}^t-c^{-2}E_{21}E_{21}^t) \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1} \end{aligned} \tag{170}$$

thus proving the first identity in (168). The proof of the second identity in (168) is similar. □

11.2. E_{11} and E_{22}

In this subsection we study the consequences of the equality between E_{11} (E_{22}) in (158) and E_{11} (E_{22}) in (159).

With $V_{12}=V_1\oplus V_2$, we see from (159) that

$$\begin{aligned} E_{11} &= \mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_1\oplus V_2} \\ E_{22} &= \Gamma^L_{n,V_1\oplus V_2}\mathrm{lgyr}[V_1,V_2]\,, \end{aligned} \tag{171}$$

so that for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$,

$$\begin{aligned}\mathrm{lgyr}[V_1, V_2] &= (\Gamma^L_{n,V_1\oplus V_2})^{-1}E_{22}\\ \mathrm{rgyr}[V_1, V_2] &= E_{11}(\Gamma^R_{m,V_1\oplus V_2})^{-1}\,,\end{aligned} \tag{172}$$

where, by (158),

$$\begin{aligned}E_{11} &= \Gamma^R_{m,V_1}(I_m + \frac{1}{c^2}V_1^tV_2)\Gamma^R_{m,V_2}\\ E_{22} &= \Gamma^L_{n,V_1}(I_n + \frac{1}{c^2}V_1V_2^t)\Gamma^L_{n,V_2}\end{aligned} \tag{173}$$

and where, by (101) and Lemma 11.1,

$$\begin{aligned}\Gamma^L_{n,V_1\oplus V_2} &= \sqrt{I_n - \frac{1}{c^2}(V_1{\oplus}V_2)(V_1{\oplus}V_2)^t}^{\,-1} = \sqrt{I_n + \frac{1}{c^2}E_{21}E_{21}^t}\\ \Gamma^R_{m,V_1\oplus V_2} &= \sqrt{I_m - \frac{1}{c^2}(V_1{\oplus}V_2)^t(V_1{\oplus}V_2)}^{\,-1} = \sqrt{I_m + \frac{1}{c^2}E_{21}^tE_{21}}\,.\end{aligned} \tag{174}$$

Following (172) - (174) we have

$$\begin{aligned}&\mathrm{lgyr}[V_1, V_2] =\\ &\sqrt{I_n + \frac{1}{c^2}E_{21}E_{21}^t}^{\,-1}\sqrt{I_n - \frac{1}{c^2}V_1V_1^t}^{\,-1}(I_n + \frac{1}{c^2}V_1V_2^t)\sqrt{I_n - \frac{1}{c^2}V_2V_2^t}^{\,-1}\\ &\mathrm{rgyr}[V_1, V_2] =\\ &\sqrt{I_m - \frac{1}{c^2}V_1^tV_1}^{\,-1}(I_m + \frac{1}{c^2}V_1^tV_2)\sqrt{I_m - \frac{1}{c^2}V_2^tV_2}^{\,-1}\sqrt{I_m + \frac{1}{c^2}E_{21}^tE_{21}}^{\,-1}\,.\end{aligned} \tag{175}$$

Equations (171) and (173) yield the *bi-gamma identities*

$$\begin{aligned}\mathrm{rgyr}[V_1, V_2]\Gamma^R_{m,V_1\oplus V_2} &= \Gamma^R_{m,V_1}(I_m + \frac{1}{c^2}V_1^tV_2)\Gamma^R_{m,V_2}\\ \Gamma^L_{n,V_1\oplus V_2}\mathrm{lgyr}[V_1, V_2] &= \Gamma^L_{n,V_1}(I_n + \frac{1}{c^2}V_1V_2^t)\Gamma^L_{n,V_2}\,.\end{aligned} \tag{176}$$

For $V \in \mathbb{R}^{n\times m}$, the left (right) gamma factor $\Gamma^L_{n,V}$ ($\Gamma^R_{m,V}$) is real if and only if $V \in \mathbb{R}_c^{n\times m}$, as we see from (99) and (101), p. 247. Hence, each of the two equations in (176) yields the following implication: $V_1, V_2 \in \mathbb{R}_c^{n\times m} \Rightarrow V_1{\oplus}V_2 \in \mathbb{R}_c^{n\times m}$, so that $\oplus$ is a binary operation in $\mathbb{R}_c^{n\times m}$ as expected.

Example 11.2. In the special case when $m = 1$, $\mathrm{rgyr}[V_1, V_2] \in SO(1) = \{1\}$, so that $\mathrm{rgyr}[V_1, V_2] = 1$. Hence, the first identity in (176) specializes to the gamma identity,

$$\gamma_{V_1\oplus V_2} = \gamma_{V_1}\gamma_{V_2}(1 + \frac{1}{c^2}V_1{\cdot}V_2), \qquad (m = 1)\,, \tag{177}$$

which plays an important role in special relativity and its underlying hyperbolic geometry [36, 40, 43].

In fact, The gamma identity (177) signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [30] and Varičak [48, 49] in terms of *rapidities* [40, p. 90].

11.3. E_{12}

In this subsection we study the consequences of the equality between E_{12} in (158) and E_{12} in (159).

The equality between E_{12} in (159) and in (158) yields the equation

$$\mathrm{rgyr}[V_1, V_2]\Gamma^R_{m,V_1\oplus V_2}(V_1\oplus V_2)^t \mathrm{lgyr}[V_1, V_2] = \Gamma^R_{m,V_1}(V_1 + V_2)^t \Gamma^L_{n,V_2} \tag{178}$$

for all $V_1, V_2 \in \mathbb{R}^{n\times m}_c$, $m, n \in \mathbb{N}$. Transposing (178), noting that

$$\begin{aligned}(\mathrm{lgyr}[V_1, V_2])^t &= (\mathrm{lgyr}[V_1, V_2])^{-1} = \mathrm{lgyr}[V_2, V_1]\\ (\mathrm{rgyr}[V_1, V_2])^t &= (\mathrm{rgyr}[V_1, V_2])^{-1} = \mathrm{rgyr}[V_2, V_1]\end{aligned} \tag{179}$$

we obtain the equation

$$\mathrm{lgyr}[V_2, V_1](V_1\oplus V_2)\Gamma^R_{m,V_1\oplus V_2}\mathrm{rgyr}[V_2, V_1] = \Gamma^L_{n,V_2}(V_1 + V_2)\Gamma^R_{m,V_1} \,. \tag{180}$$

Manipulating the left side of (180) by means of the first commuting relation in (103), and manipulating the right side of (180) by means of (166) we obtain the equation

$$\mathrm{lgyr}[V_2, V_1]\Gamma^L_{n,V_1\oplus V_2}(V_1\oplus V_2)\mathrm{rgyr}[V_2, V_1] = \Gamma^L_{n,V_2\oplus V_1}(V_2\oplus V_1) \tag{181}$$

for all $V_1, V_2 \in \mathbb{R}^{n\times m}_c$. The resulting elegant equation demonstrates that the application of the bi-gyration $(\mathrm{lgyr}[V_2, V_1], \mathrm{rgyr}[V_2, V_1])$ takes $\Gamma^L_{n,V_1\oplus V_2}(V_1\oplus V_2)$ into $\Gamma^L_{n,V_2\oplus V_1}(V_2\oplus V_1)$. Equation (181) thus gives rise to a nice bi-gyrocommutative-like law.

12. Product of Lorentz Transformations, V

Techniques have been developed in [46] enabling the product of Lorentz transformations in the parameter P to be determined by Theorem 2.3, p. 233. By similar techniques one can determine the product of Lorentz transformations in the parameter V as well, obtaining the following theorem.

Theorem 12.1. (Lorentz Transformation Product Law, V) *The product of two generic Lorentz transformations*

$$\begin{aligned}\Lambda_1 &= (V_1, O_{n,1}, O_{m,1})^t\\ \Lambda_2 &= (V_2, O_{n,2}, O_{m,2})^t\end{aligned} \tag{182}$$

of order (m,n), $m,n\in\mathbb{N}$, *in terms of parameter composition is given by*

$$\Lambda_1\Lambda_2=\begin{pmatrix}V_1\\O_{n,1}\\O_{m,1}\end{pmatrix}\begin{pmatrix}V_2\\O_{n,2}\\O_{m,2}\end{pmatrix}=\begin{pmatrix}V_1O_{m,2}\oplus O_{n,1}V_2\\ \mathrm{lgyr}[V_1O_{m,2},O_{n,1}V_2]O_{n,1}O_{n,2}\\ O_{m,1}O_{m,2}\mathrm{rgyr}[V_1O_{m,2},O_{n,1}V_2]\end{pmatrix}, \tag{183}$$

where $\oplus$, lgyr *and* rgyr *are given by* (163) – (164) *and* (175) *in terms of the parameters* $V_1,V_2\in\mathbb{R}_c^{n\times m}$.

Interestingly, the Lorentz transformation product laws in (183) and (8) of Theorem 12.1 and of Theorem 2.3, p. 233, respectively, have the same form when we interchange V_i and P_i, $i=1,2$. Note, however, that the definitions of $\oplus$, lgyr and rgyr in Theorems 12.1 and 2.3 do not share the same form.

Similarly, as one can check, the gyrogroupoid $(\mathbb{R}_c^{n\times m},\oplus)$ possesses the same bi-gyrocommutative law as that of the gyrogroupoid $(\mathbb{R}^{n\times m},\oplus)$, with the parameter $P\in\mathbb{R}^{n\times m}$ replaced by the parameter $V\in\mathbb{R}_c^{n\times m}$. We thus obtain the following Theorem 12.2 from its P-counterpart Theorem 2.7, p. 234, by replacing (P_1,P_2) by (V_1,V_2).

Theorem 12.2. (Bi-gyrocommutative Law in $(\mathbb{R}_c^{n\times m},\oplus)$**).** *The binary operation* $\oplus$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyrocommutative law*

$$V_1\oplus V_2=\mathrm{lgyr}[V_1,V_2](V_2\oplus V_1)\mathrm{rgyr}[V_1,V_2] \tag{184}$$

for all $V_1,V_2\in\mathbb{R}_c^{n\times m}$.

Similarly, as one can check, the gyrogroupoid $(\mathbb{R}_c^{n\times m},\oplus)$ possesses the same bi-gyroassociative law as that of the gyrogroupoid $(\mathbb{R}^{n\times m},\oplus)$, with the parameter $P\in\mathbb{R}^{n\times m}$ replaced by the parameter $V\in\mathbb{R}_c^{n\times m}$. We thus obtain the following Theorem 12.3 from its P-counterpart Theorem 2.8, p. 235, by replacing (P_1,P_2) by (V_1,V_2).

Theorem 12.3. (Bi-gyroassociative Law in $(\mathbb{R}_c^{n\times m},\oplus)$**).** *The binary operation* $\oplus$ *in* $\mathbb{R}_c^{n\times m}$ *possesses the bi-gyroassociative law*

$$(V_1\oplus V_2)\oplus\mathrm{lgyr}[V_1,V_2]V_3=V_1\mathrm{rgyr}[V_2,V_3]\oplus(V_2\oplus V_3) \tag{185}$$

for all $V_1,V_2\in\mathbb{R}_c^{n\times m}$.

13. Bi-Gyrogroups

As in Sect. 3 with the parameter $P\in\mathbb{R}^{n\times m}$, it proves useful with the parameter $V\in\mathbb{R}_c^{n\times m}$, as well, to replace the binary operation $\oplus$ in $\mathbb{R}_c^{n\times m}$ by a new binary operation, $\oplus'$, according to the following definition.

Definition 13.1. (Bi-gyrogroup Operation, Bi-gyrogroups). Let $(\mathbb{R}_c^{n\times m}, \oplus)$ be a bi-gyrogroupoid. A new bi-gyrogroup binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ is given by

$$V_1 \oplus' V_2 = (V_1 \oplus V_2)\mathrm{rgyr}[V_2, V_1] \tag{186}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$. The resulting groupoid $(\mathbb{R}_c^{n\times m}, \oplus')$ is called a bi-gyrogroup.

Having the form of Def. 3.1, Def. 13.1 defines the *bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ in terms of the *bi-gyrogroupoid* $(\mathbb{R}_c^{n\times m}, \oplus)$.

Remark 1. In the special case when $m = 1$, the binary operations $\oplus'$ and $\oplus$ coincide since $\mathrm{rgyr}[V_2, V_2] = 1$, as noted in Example 11.2. Accordingly, when $m = 1$, the two binary operations $\oplus'$ and $\oplus$ in $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ coincide with Einstein velocity addition of special relativity.

It is shown in [46] that (186) implies the following four identities that exhibit an interesting symmetry between the binary operations $\oplus$ and $\oplus'$ in $\mathbb{R}_c^{n\times m}$.

$$\begin{aligned} V_1 \oplus' V_2 &= (V_1 \oplus V_2)\mathrm{rgyr}[V_2, V_1] \\ V_1 \oplus V_2 &= (V_1 \oplus' V_2)\mathrm{rgyr}[V_1, V_2] \\ V_1 \oplus' V_2 &= \mathrm{lgyr}[V_1, V_2](V_2 \oplus V_1) \\ V_1 \oplus V_2 &= \mathrm{lgyr}[V_1, V_2](V_2 \oplus' V_1) \end{aligned} \tag{187}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$.

Bi-gyrogroups $(\mathbb{R}_c^{n\times m}, \oplus')$ possess a commutative-like and an associative-like law. Indeed, by [46, Theorems 42, 41] with P replaced by V we have the following two theorems.

Theorem 13.2. (Bi-gyrocommutative Law in $(\mathbb{R}_c^{n\times m}, \oplus')$). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ possesses the bi-gyrocommutative law*

$$V_1 \oplus' V_2 = \mathrm{lgyr}[V_1, V_2](V_2 \oplus' V_1)\mathrm{rgyr}[V_2, V_1] \tag{188}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$.

Theorem 13.3. (Bi-gyrogroup Left and Right Bi-gyroassociative Law of $\oplus'$). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ possesses the left bi-gyroassociative law*

$$V_1 \oplus' (V_2 \oplus' X) = (V_1 \oplus' V_2) \oplus' \mathrm{lgyr}[V_1, V_2] X \mathrm{rgyr}[V_2, V_1] \tag{189}$$

and the right bi-gyroassociative law

$$(V_1 \oplus' V_2) \oplus' X = V_1 \oplus' (V_2 \oplus' \mathrm{lgyr}[V_2, V_1] X \mathrm{rgyr}[V_1, V_2]) \tag{190}$$

for all $V_1, V_2, X \in \mathbb{R}_c^{n\times m}$.

14. Gyrogroup Gyrations

The bi-gyroassociative laws (189) – (190) and the bi-gyrocommutative law (188) suggest the following definition of gyrations in terms of left and right gyrations.

Definition 14.1. (Gyrogroup Gyrations) ([46, Definition 43]). The gyrator gyr,

$$\mathrm{gyr} : \mathbb{R}_c^{n\times m} \times \mathbb{R}_c^{n\times m} \to \mathrm{Aut}(\mathbb{R}_c^{n\times m}, \oplus') \tag{191}$$

generates automorphisms called gyrations, $\mathrm{gyr}[V_1, V_2] \in \mathrm{Aut}(\mathbb{R}_c^{n\times m}, \oplus')$, given by the equation

$$\mathrm{gyr}[V_1, V_2]X = \mathrm{lgyr}[V_1, V_2]X\mathrm{rgyr}[V_2, V_1] \tag{192}$$

for all $V_1, V_2, X \in \mathbb{R}_c^{n\times m}$, where left gyrations, $\mathrm{lgyr}[V_1, V_2]$, and right gyrations, $\mathrm{rgyr}[V_2, V_1]$, are given in (175). The gyration $\mathrm{gyr}[V_1, V_2]$ is said to be the gyration generated by $V_1, V_2 \in \mathbb{R}_c^{n\times m}$. Being automorphisms of $(\mathbb{R}_c^{n\times m}, \oplus')$, gyrations are also called gyroautomorphisms.

Def. 14.1 will turn out rewarding, leading to the elegant result that any bi-gyrogroup $(\mathbb{R}_c^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, is a gyrocommutative gyrogroup.

Theorem 14.2. (Gyrogroup Gyroassociative and gyrocommutative Laws). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ obeys the left and the right gyroassociative law*

$$V_1\oplus'(V_2\oplus'X) = (V_1\oplus'V_2)\oplus'\mathrm{gyr}[V_1, V_2]X \tag{193}$$

and

$$(V_1\oplus'V_2)\oplus'X = V_1\oplus'(V_2\oplus'\mathrm{gyr}[V_2, V_1]X) \tag{194}$$

and the gyrocommutative law

$$V_1\oplus'V_2 = \mathrm{gyr}[V_1, V_2](V_2\oplus'V_1)\,. \tag{195}$$

Proof. Identities (193) – (194) follow immediately from Def. 14.1 and the left and right bi-gyroassociative law (189) – (190). Similarly, (195) follow immediately from Def. 14.1 and the bi-gyrocommutative law (188). □

Lemma 14.3. ([46, Lemma 45]). *For any $V_1, V_2 \in (\mathbb{R}_c^{n\times m}, \oplus')$, the relation* (192) *between bi-gyrations* $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_2, V_1])$ *and gyrations* $\mathrm{gyr}[V_1, V_2]$ *is bijective.*

It is obvious from (192) that a gyration $\mathrm{gyr}[V_1, V_2]$ is determined uniquely by the bi-gyration $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_1, V_2])$. It follows from Lemma 14.3 that also the converse is true, that is, a bi-gyration $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_1, V_2])$ is determined uniquely by the gyration $\mathrm{gyr}[V_1, V_2]$.

It is anticipated in Def. 14.1 that gyrations are automorphisms. The following theorem asserts that this is indeed the case.

Theorem 14.4. (Gyroautomorphism) ([46, Like Theorem 46]). *For all* $V_1, V_2 \in \mathbb{R}_c^{n\times m}$*, gyrations* $\mathrm{gyr}[V_1, V_2]$ *of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *are automorphisms of the bi-gyrogroup.*

Theorem 14.5. (Left Gyration Reduction Properties) ([46, Like Theorem 47]). *Left gyrations of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *possess the left gyration left reduction property*

$$\mathrm{lgyr}[V_1, V_2] = \mathrm{lgyr}[V_1 \oplus' V_2, V_2] \tag{196}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[V_1, V_2] = \mathrm{lgyr}[V_1, V_2 \oplus' V_1]\,. \tag{197}$$

Theorem 14.6. (Right Gyration Reduction Properties) ([46, Like Theorem 48]). *Right gyrations of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *possess the right gyration left reduction property*

$$\mathrm{rgyr}[V_1, V_2] = \mathrm{rgyr}[V_1 \oplus' V_2, V_2] \tag{198}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[V_1, V_2] = \mathrm{rgyr}[V_1, V_2 \oplus' V_1]\,. \tag{199}$$

Theorem 14.7. (Gyration Reduction Properties) ([46, Like Theorem 49]). *The gyrations of any bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$*,* $m, n \in \mathbb{N}$*, possess the left and right reduction property*

$$\mathrm{gyr}[V_1, V_2] = \mathrm{gyr}[V_1 \oplus' V_2, V_2] \tag{200}$$

and

$$\mathrm{gyr}[V_1, V_2] = \mathrm{gyr}[V_1, V_2 \oplus' V_1]\,. \tag{201}$$

Proof. Identities (200) and (201) follow from Def. 14.1 of gyr in terms of lgyr and rgyr, and from Theorems 14.5 and 14.6. □

Finally, we have the most important theorem, which is the V-counterpart of Theorem 5.3..

Theorem 14.8. (Gyrocommutative Gyrogroup) ([46, Like Theorem 52]). *Any bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$*,* $n, m \in \mathbb{N}$*, is a gyrocommutative gyrogroup.*

15. Scalar Multiplication for the Parameter V

Let M_1 and M_2 be two square matrices such that the inverse, M_2^{-1}, of M_2 exists. If the two matrices satisfy the commuting relation

$$M_1 M_2^{-1} = M_2^{-1} M_1\,, \tag{202}$$

then we may use the convenient notation

$$\frac{M_1}{M_2} := M_1 M_2^{-1} = M_2^{-1} M_1\,. \tag{203}$$

We are motivated by the scalar multiplication in $(\mathbb{R}_c^{n\times m}, \oplus')$, with $m = 1$, which is the scalar multiplication in $(\mathbb{R}_c^n, \oplus_{\mathrm{E}})$ studied, for instance, in [37, Eq. (6.267), p. 195]. We wish to extend it from $m = 1$ to all $m \geq 1$. Accordingly, we define scalar multiplication in $(\mathbb{R}_c^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, by each of the following two equations, which are mutually equivalent.

$$
\begin{aligned}
r\otimes V &:= \frac{I_n - \left(\Gamma_{n,V}^L - \sqrt{(\Gamma_{n,V}^L)^2 - I_n}\,\right)^{2r}}{I_n + \left(\Gamma_{n,V}^L - \sqrt{(\Gamma_{n,V}^L)^2 - I_n}\,\right)^{2r}} \frac{\Gamma_{n,V}^L}{\sqrt{(\Gamma_{n,V}^L)^2 - I_n}} V \\
&= V \frac{I_m - \left(\Gamma_{m,V}^R - \sqrt{(\Gamma_{m,V}^R)^2 - I_m}\,\right)^{2r}}{I_m + \left(\Gamma_{m,V}^R - \sqrt{(\Gamma_{m,V}^R)^2 - I_m}\,\right)^{2r}} \frac{\Gamma_{m,V}^R}{\sqrt{(\Gamma_{m,V}^R)^2 - I_n}}
\end{aligned}
\tag{204}
$$

for all $r \in \mathbb{R}$ and $V \in \mathbb{R}_c^{n\times m}$. In the special case when $m = 1$, the scalar multiplication in (204) specializes to the one in [37, Eq. (6.267), p. 195].

As expected, the scalar multiplication in (204) satisfies the equation

$$
B_c(r\otimes V) = B_c(V)^r \tag{205}
$$

so that $V \in \mathbb{R}_c^{n\times m} \Rightarrow r\otimes V \in \mathbb{R}_c^{n\times m}$. In fact, (204) is derived from (205) by calculating the matrix $B_c(V)^r$ for $r \in \mathbb{N}$ and then analytically continuing r off the positive integers.

Furthermore, (205) implies the *scalar distributive law* and the *scalar associative law*

$$
\begin{aligned}
(r_1 + r_2)\otimes V &= r_1\otimes V \oplus' r_2\otimes V \\
(r_1 r_2)\otimes V &= r_1\otimes(r_2\otimes V)
\end{aligned}
\tag{206}
$$

and, hence, the *monodistributive law*

$$
r\otimes(r_1\otimes V \oplus' r_2\otimes V) = r\otimes(r_1\otimes V) \oplus' r\otimes(r_2\otimes V) \tag{207}
$$

for all $r, r_1, r_2 \in \mathbb{R}$ and all $V \in \mathbb{R}_c^{n\times m}$.

Naturally in gyrolanguage, the triple $(\mathbb{R}_c^{n\times m}, \oplus', \otimes)$ is said to be a *bi-gyrovector space*. Here $\oplus'$ is the binary operation in $\mathbb{R}_c^{n\times m}$ given by (186).

16. Scalar Multiplication for the Parameter P

In this section we continue using the notation in (202) − (203).

We introduce the following β-notation,

$$
\begin{aligned}
\beta_{n,P}^L &:= \sqrt{I_n + c^{-2}PP^t}^{\;-1} \in \mathbb{R}^{n\times n} \\
\beta_{m,P}^R &:= \sqrt{I_m + c^{-2}P^tP}^{\;-1} \in \mathbb{R}^{m\times m}\,,
\end{aligned}
\tag{208}
$$

in analogy with the Γ-notation in (140).

We are motivated by the scalar multiplication in $(\mathbb{R}^{n\times m},\oplus')$ with $m=1$, which is the scalar multiplication in $(\mathbb{R}^n,\oplus_{\text{U}})$ studied, for instance, in [37, Eq. (6.285), p. 200]. We wish to extend it from $m=1$ to all $m\geq 1$. Accordingly, we define scalar multiplication in $(\mathbb{R}^{n\times m},\oplus')$, $m,n\in\mathbb{N}$, by each of the following two equations, which are mutually equivalent.

$$\begin{aligned} r\otimes P &:= \frac{1}{2}\frac{\left(I_n+\sqrt{I_n-(\beta^L_{n,P})^2}\,\right)^r-\left(I_n-\sqrt{I_n-(\beta^L_{n,P})^2}\,\right)^r}{(\beta^L_{n,P})^{r-1}\sqrt{I_n-(\beta^L_{n,P})^2}}P \\ &= \frac{1}{2}P\frac{\left(I_m+\sqrt{I_m-(\beta^R_{m,P})^2}\,\right)^r-\left(I_m-\sqrt{I_m-(\beta^R_{m,P})^2}\,\right)^r}{(\beta^R_{m,P})^{r-1}\sqrt{I_m-(\beta^R_{m,P})^2}} \end{aligned} \tag{209}$$

for all $r\in\mathbb{R}$ and $P\in\mathbb{R}^{n\times m}$. In the special case when $m=1$, the scalar multiplication in (209) specializes to the one in [37, Eq. (6.285), p. 200].

As expected, the scalar multiplication in (209) satisfies the equation

$$B_c(r\otimes P)=B_c(P)^r \tag{210}$$

where $B_c(P)$ is the bi-boost in (134). In fact, (209) is derived from (210) by calculating the matrix $B_c(P)^r$ for $r\in\mathbb{N}$ and then analytically continuing r off the positive integers.

Identity (210) implies the *scalar distributive law* and the *scalar associative law*

$$\begin{aligned} (r_1+r_2)\otimes P &= r_1\otimes P\oplus' r_2\otimes P \\ (r_1r_2)\otimes P &= r_1\otimes(r_2\otimes P) \end{aligned} \tag{211}$$

and, hence, the *monodistributive law*

$$r\otimes(r_1\otimes P\oplus' r_2\otimes P)=r\otimes(r_1\otimes P)\oplus' r\otimes(r_2\otimes P) \tag{212}$$

for all $r,r_1,r_2\in\mathbb{R}$ and all $P\in\mathbb{R}^{n\times m}$.

Hence, the triple $(\mathbb{R}^{n\times m},\oplus',\otimes)$ is a bi-gyrovector space. Here $\oplus'$ is the binary operation in $\mathbb{R}^{n\times m}$ given by (15).

17. Paving the Road to the Eigenball Geometry

We have exposed the structure of the bi-gyrovector space $(\mathbb{R}_c^{n\times m},\oplus',\otimes)$ of the eigenball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ of all rectangular real matrices of order $n\times m$, $m,n\in\mathbb{N}$. The bi-gyrovector space structure forms the algebraic setting for the non-Euclidean geometry that underlies the eigenball, just as the vector space structure forms the algebraic setting for the standard model of Euclidean geometry [41]. Indeed, in the special case when $m=1$ the situation is well-known:

In this special case, when $m=1$,

1. the eigenball $\mathbb{R}_c^{n\times m}$ specializes to the ball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ of the Euclidean n-space $\mathbb{R}^n$, as shown in Example 8.2, p. 249; and

2. the binary operation $\oplus'$ in $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ specializes to the binary operation given by Einstein's velocity addition law of relativistically admissible velocities in special relativity, as indicated in Example 8.7, p. 252.

Thus, when $m = 1$ the bi-gyrovector space $(\mathbb{R}_c^{n\times m}, \oplus', \otimes)$ specializes to the gyrovector space $(\mathbb{R}_c^n, \oplus', \otimes)$. The latter, in turn, forms the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry that underlies the ball $\mathbb{R}_c^n$, where $\oplus'$ in $\mathbb{R}_c^{n\times m}$ specializes to Einstein addition in $\mathbb{R}_c^n$. The resulting analytic hyperbolic geometry has been studied since 2001 in the seven books [36,37,40–43, 45] and in many articles.

It is, therefore, expected that the bi-gyrovector space structure, studied in [46] and in the present article, paves the road to to the discovery of the extended analytic hyperbolic geometry that regulates the eigenball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ for any $m, n \in \mathbb{N}$.

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