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Eigenfunction Expansions for Second-Order Boundary Value Problems with Separated Boundary Conditions

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Abstract

In this paper, we investigate some properties of eigenvalues and eigenfunctions of boundary value problems with separated boundary conditions. Also, we obtain formal series solutions for some partial differential equations associated with the second order differential equation, and study necessary and sufficient conditions for the negative and positive eigenvalues of the boundary value problem. Finally, by the sequence of orthogonal eigenfunctions, we provide the eigenfunction expansions for twice continuously differentiable functions.

Keywords: Boundary value problem, eigenvalue, eigenfunction, completeness, eigenfunction expansion.

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1. Introduction

In the present paper, we consider the boundary value problem L, defined by the following second-order differential equation of the Sturm-Liouville type

$$(k(x)y')' + (\lambda w(x) - g(x))y = 0, \qquad x \in [a, b], \tag{1}$$

with the separated boundary conditions

$$\cos\alpha \ y'(a, \lambda) - \sin\alpha \ y(a, \lambda) = 0, \tag{2}$$

$$\cos\beta \ y'(b, \lambda) - \sin\beta \ y(b, \lambda) = 0, \tag{3}$$

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which have many interesting applications in basic sciences, some branches of natural sciences and engineering. For example, in mathematical physics, the *one dimentional diffusion equation* is the form

$$\frac{\partial v}{\partial t}=\frac{\partial}{\partial x}(k(x)\frac{\partial v}{\partial x})-g(x)v, \tag{4}$$

where $a\leq x\leq b$, $0\leq t<\infty$, and $k(x),g(x)$ are real-valued functions. By the separation of variables technique, we can write the solution $v(x,t)$ in the form

$$v(x,t)=e^{-\lambda t}y(x). \tag{5}$$

In this case, the function y is seen to be a solution of the differential equation (1) with $w(x)=1$. For other examples, we refer to [2, 4, 17].

Studying the properties of spectrum of the boundary value problem L is an interesting subject for many authors. For example, in [6, 11], the authors considered regular Sturm-Liouville problems with $k=w\equiv 1$ and an integrable potential g, or with locally integrable functions k^{-1},g,w on finite intervals, and obtained asymptotic approximations of eigenvalues and eigenfunctions. In [8, 18, 19], inverse spectral problems with non-separated boundary conditions with an integrable function $g(x)$ were investigated. Also, inverse problems for singular differential operators on finite intervals with $k=w\equiv 1$ were studied in [7, 13, 15]. Boundary value problems consisting of (1) with $g\equiv 0$, together with Dirichlet boundary condition or irregular decomposing boundary conditions were investigated in [9, 10], and eigenfunction expansions and their uniformly convergent were studied. In [1], the authors considered (1) in the case when the equation has a singularity and discontinuity inside the interval $[0,T]$, and investigated the properties of the spectrum and its associated inverse problem. Also, in [3, 16], the authors considered the problems consisting of (1)-(2) with $k=w\equiv 1$ or with discontinuous weight function $w(x)$ on symmetric intervals under discontinuity conditions in y and y', together with the boundary condition (3) or an spectral parameter dependent boundary condition at the right endpoint, respectively. They investigated the properties of eigenvalues and eigenfunctions, and obtained asymptotic approximation formulas for fundamental solutions. Finally, in [14], a new type of boundary value problem consisting of a second order differential equation with an abstract operator in a Hilbert space on two disjoint intervals together with eigenparameter dependent boundary conditions and with transmission conditions was investigated.

In the present article, in section 2, we study some properties of eigenvalues and eigenfunctions of the boundary value problem L consisting of the equation (1) together with the boundary conditions (2)-(3), in special cases. Here, λ is the eigenvalue parameter, $k(x)$, $w(x)$ and $g(x)$ are real continuous functions on $[a,b]$, and $k(x),w(x)$ are positive. We will provide the formal series solutions for the function $v(x,t)$ of the diffusion equation (4). Then, we study necessary and sufficient conditions for negative and positive eigenvalues of L, and prove that each twice continuously differentiable function can be expanded in terms of the eigenfunctions of the problem, under a sufficient condition (see section 3).

2. The Eigenfunctions and Formal Series Solutions

In this section, first, we consider some special forms of (1)-(3), and find the eigenfunctions and the formal series solutions in several examples.

Example 2.1. Let $a = 0$, $b = \pi$, $k(x) = w(x) = 1$, $g(x) = 0$, $\alpha = \frac{\pi}{2}$ and $\beta = -\frac{\pi}{4}$. Then, we can rewrite the problem L as follows:

$$\begin{aligned} y'' + \lambda y &= 0, \qquad 0 \le x \le \pi, && (6)\\ y(0) &= 0, \qquad y'(\pi) + y(\pi) = 0. && (7) \end{aligned}$$

The differential equation (6) has two linearly independent solutions

$$y_1(x) = \cos(\sqrt{\lambda}x), \qquad y_2(x) = \sin(\sqrt{\lambda}x).$$

Thus, it follows from the first condition of (7) that the solution $y(x,\lambda)$ of (6)-(7) is the form

$$y(x,\lambda) = c\sin(\sqrt{\lambda}x),$$

where c is constant. Therefore, according to the secondary condition of (7), we get

$$\sqrt{\lambda} + \tan(\sqrt{\lambda}\pi) = 0. \tag{8}$$

Example 2.2. Let $a = 0$, $k(x) = w(x) = 1$, $g(x) = 0$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. In this case, the problem L be transformed to

$$\begin{aligned} y'' + \lambda y &= 0, \qquad 0 \le x \le b,\\ y'(0) &= 0, \qquad y(b) = 0. \end{aligned}$$

Therefore,

$$y(x,\lambda) = c\cos(\sqrt{\lambda}x).$$

Moreover, L has a countable set of the eigenvalues

$$\lambda_n = (\frac{(2n+1)\pi}{2b})^2, \qquad n = 1, 2, 3, ...,$$

and so, their corresponding eigenfunctions are

$$y_n(x) = \cos(\frac{(2n+1)\pi x}{2b}), \qquad n = 1, 2, 3, ...$$

Definition 2.3. We define the function $< .,. >_w$: $C[a,b] \to C[a,b]$ by

$$< y, z >_w = \int_a^b y(t)\overline{z(t)}dt$$

for all complex functions $y, z \in C[a,b]$. Here, $\overline{z(t)}$ is the conjugate of $z(t)$.

The function $< .,. >_w$ defines an *inner product* on $C[a,b]$.

Theorem 2.4. Suppose that λ_1 and λ_2 are distinct eigenvalues of the boundary value problem L with corresponding eigenfunctions y_1 and y_2, respectively. Then, y_1 and y_2 are orthogonal with respect to the weight function $w(x)$.

Proof. Since y_i is the corresponding eigenfunction of λ_i, we have

$$(k(x)y_i')' - g(x)y_i = -\lambda_i w(x)y_i, \qquad i = 1,2. \tag{9}$$

For $i = 1$, multiply (9) by y_2 and integrate from a to b with respect to x, then

$$\int_a^b y_2(k(x)y_1')'dx - \int_a^b y_1y_2g(x)dx = -\lambda_1 \int_a^b y_1y_2w(x)dx.$$

Integrate by parts yields

$$y_2y_1'k(x)|_a^b - \int_a^b y_1'y_2'k(x)dx - \int_a^b y_1y_2g(x)dx = -\lambda_1 \int_a^b y_1y_2w(x)dx. \tag{10}$$

Similarly, for $i = 2$, multiply (9) by y_1 and integrate from a to b, we get

$$y_1y_2'k(x)|_a^b - \int_a^b y_1'y_2'k(x)dx - \int_a^b y_1y_2g(x)dx = -\lambda_2 \int_a^b y_1y_2w(x)dx. \tag{11}$$

Subtracting the two equations (10) and (11), we obtain

$$y_1y_2'k(x)|_a^b - y_2y_1'k(x)|_a^b = (\lambda_1 - \lambda_2)\int_a^b y_1y_2w(x)dx. \tag{12}$$

Now, let $\alpha, \beta = k\pi + \pi/2$, $k \in Z$, then it follows from (2)-(3) that

$$y_i(a) = 0 = y_i(b), \qquad i = 1,2. \tag{13}$$

Applying (12)-(13) we conclude that

$$(\lambda_1 - \lambda_2)\int_a^b y_1y_2w(x)dx = 0. \tag{14}$$

Similarly, let $\alpha, \beta \neq k\pi + \pi/2$, $k \in Z$, then we have

$$y_i'(a) = \tan\alpha\ y_i(a), \qquad y_i'(b) = \tan\beta\ y_i(b), \qquad i = 1,2. \tag{15}$$

Substituting (15) into (12), we arrive at (14). Also, in the case when $\alpha = k\pi + \pi/2$ and $\beta \neq k\pi + \pi/2$, (14) is valid. Hence, in general, (14) together with $\lambda_1 \neq \lambda_2$ yields

$$\int_a^b y_1y_2w(x)dx = 0,$$

and our proof is complete. □

Example 2.5. Consider the equation (1) with $k(x) = w(x) = 1$, $g(x) = 0$, $a = 0$, $\alpha, \beta = k\pi + \pi/2$, $k \in Z$. In this case, applying the method which used for finding the eigenvalues in Example 2.1, we obtain the eigenvalues and the corresponding eigenfunctions as follows

$$\lambda_n = \frac{n^2\pi^2}{b^2}, \qquad y_n(x) = \sin(\frac{n\pi x}{b}), \qquad n = 1, 2, 3, \ldots.$$

Therefore, we deduce that

$$< y_m, y_n >_w = \int_a^b \sin(\frac{m\pi x}{b}) \sin(\frac{n\pi x}{b}) dx = 0, \qquad m \neq n.$$

Hence, according to (5), we may construct a formal series solution for $v(x,t)$ by superposition

$$v(x,t) = \sum_{n=1}^{\infty} k_n e^{-\lambda_n t} \sin(\frac{n\pi x}{b}). \tag{16}$$

For finding the Fourier coefficients k_n, substituting $t = 0$ and multiply both sides of (16) by $\sin(\frac{m\pi x}{b})$, and integrate from 0 to b give us

$$\int_0^b v(x,0) \sin(\frac{m\pi x}{b}) dx = \int_0^b \sum_{n=1}^{\infty} k_n \sin(\frac{m\pi x}{b}) \sin(\frac{n\pi x}{b}) dx.$$

Assuming uniform convergence and by orthogonality we obtain (with $m = n$)

$$\int_0^b v(x,0) \sin(\frac{n\pi x}{b}) dx = \int_0^b \sum_{n=1}^{\infty} k_n \sin^2(\frac{n\pi x}{b}) dx,$$

therefore, we get

$$k_n = \frac{2}{b} \int_0^b v(x,0) \sin(\frac{n\pi x}{b}) dx.$$

Hence, for $M = \frac{2}{b} \int_0^b |v(x,0)| dx$,

$$|k_n| \leq M, \qquad n = 1, 2, 3, \ldots.$$

Moreover,

$$|k_n e^{-\lambda_n t} \sin(\frac{n\pi x}{b})| \leq M e^{-(\frac{n\pi x}{b})^2 t} \to 0$$

as $t \to \infty$. This together with (16) yields

$$v(x,t) \to 0 \quad as\ t \to \infty,$$

by uniform convergence.

Example 2.6. Let $k(x) = w(x) = x$, $g(x) = \frac{m^2}{x}$, $m \geq 0$, $a = 0$, $\alpha, \beta = k\pi + \pi/2$, $k \in Z$. Then,

$$\lambda_n = \frac{j_{m,n}^2}{b^2}, \qquad y_n(x) = J_m(\frac{j_{m,n}}{b}x), \qquad n = 1, 2, 3, ...,$$

where $j_{m,n}$ is the n^{th} positive zero of the Bessel function J_m. Moreover,

$$< y_m, y_n >_w = \int_0^b J_m(\frac{j_{m,n}}{b}x) J_m(\frac{j_{m,s}}{b}x) dx = 0, \qquad n \neq s.$$

By (5) and the method applied in Example 2.5, we can write

$$v(x,t) = \sum_{n=1}^{\infty} r_n e^{-\lambda_n t} J_m(\frac{j_{m,n}}{b}x) dx,$$

where the coefficients r_n are obtain as follows

$$r_n = \frac{\int_0^b v(x,0) J_m(\frac{j_{m,n}}{b}x) dx}{\int_0^b \{J_m(\frac{j_{m,n}}{b}x)\}^2 dx}.$$

Theorem 2.7. The eigenvalues of the boundary value problem L, defined by (1)-(3), are real.

Proof. It follows from the relation (14) that for each eigenvalues λ_1, λ_2 of L,

$$(\lambda_1 - \overline{\lambda_2}) \int_a^b y_1 \overline{y_2} w(x) dx = 0. \tag{17}$$

Now, we choose $y_2 = y_1$. This together with (17) yields

$$(\lambda_1 - \overline{\lambda_1}) \int_a^b |y_1|^2 w(x) dx = 0.$$

Since $|y_1|^2 w(x) > 0$, we obtain $\lambda_1 = \overline{\lambda_1}$. Consequently, the eigenvalue λ_1 is real. Since λ_1 was an arbitrary eigenvalue of L, the proof is complete. □

3. The Eigenfunction Expansion

In this section, necessary and sufficient conditions for the negative and positive eigenvalues of the boundary value problem L are obtained. Also, we discuss completeness of the eigenfunctions of L, and prove that each function can be expanded in terms of the eigenfunctions of L, under a sufficient condition.

Lemma 3.1. Suppose that λ is an eigenvalue of L with corresponding eigenfunction y. Then,

$$\lambda = \frac{\int_a^b \{k(x)(y')^2 + g(x)y^2\}dx - (k(x)yy')|_a^b}{\int_a^b y^2(x)w(x)dx}. \tag{18}$$

Proof. First, multiply (1) by y and integrate the result with respect to x on the interval $[a, b]$, we obtain

$$\int_a^b (k(x)y')'y - \int_a^b g(x)y^2)dx = -\lambda \int_a^b y^2 w(x)dx. \tag{19}$$

Now, after an integration by parts from (19), we arrive at (18). □

According to Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let* $\alpha, \beta = k\pi + \pi/2$*,* $k \in Z$*. If* $\tan\alpha \geq 0$*,* $\tan\beta \leq 0$*, and* $g(x) \geq 0$ *for every* $x \in [a, b]$*, then the eigenvalue* λ *of* L *is always positive.*

Remark 1. We note that all eigenvalues of L are simple, because otherwise, let y_1 and y_2 be the linearly independent eigenfunctions correspond to the eigenvalue λ. Thus, y_1 and y_2 satisfy the boundary conditions (2)-(3). Moreover, we can write every solution $y(x, \lambda)$ of (1) corresponding to λ in the form

$$y(x, \lambda) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1, c_2 are arbitrary constants, and $y(x, \lambda)$ must be satisfied in the boundary conditions (2)-(3). On the other hand, we know that the problem consisting of (1) together with arbitrary initial conditions that be incompatible with (2)-(3), has a unique solution. Hence, these give us a contradiction.

The following assertion can be proved like Theorem 4.8, p. 157 of [12].

Lemma 3.3. If $H(x, t)$ is a continuous, real-valued and symmetric function, $f : [a, b] \to R$ is a continuous function defined by

$$f(x) = \int_a^b H(x, t)r(t)dt, \qquad x \in [a, b]$$

for some continuous $r : [a, b] \to R$, then f may be expanded in the uniformly convergent series $f = \sum_{n=1}^{\infty} \alpha_n y_n$ on $[a, b]$, where $\{y_n\}$ is the sequence of the the eigenfunctions of L, and

$$\alpha_n = \frac{\int_a^b f(x)y_n(x)dx}{\int_a^b y_n^2(x)dx}, \qquad n \geq 1.$$

The Lemma 3.3 plays an important role for proving the following theorem which is the main result of this section.

Theorem 3.4. If $h : [a, b] \to R$ is an arbitrary twice continuously differentiable function satisfy (2)-(3), then h can be expanded in the uniformly convergent series $h = \sum_{n=1}^{\infty} \beta_n y_n$ on $[a, b]$, where

$$\beta_n = \frac{\int_a^b h(x) y_n(x) w(x) dx}{\int_a^b y_n^2(x) w(x) dx}, \qquad n \geq 1. \tag{20}$$

Proof. First, since h is twice continuously differentiable, there exists a continuous function $p : [a, b] \to R$ such that

$$\ell(h(x)) = -p(x)\sqrt{k(x)},$$

where

$$\ell h := \frac{d}{dx}(k(x)\frac{dh}{dx}) - g(x)h.$$

Second, h satisfies (2)-(3), thus by the method of variation of parameters, $h(x)$ can be written as

$$h(x) = -\int_a^b G(x, t) p(t) \sqrt{w(t)} dt, \qquad x \in [a, b],$$

where

$$G(x, t) = \begin{cases} \frac{u(x)v(t)}{k(t)W(u,v)(t)}, & a \leq x \leq t \leq b, \\ \frac{u(t)v(x)}{k(t)W(u,v)(t)}, & a \leq t \leq x \leq b, \end{cases}$$

where u and v are two arbitrary linearly independent solutions of (1), and $W(u, v)(x)$ is the Wronskian of u and v. Therefore,

$$h(x)\sqrt{w(x)} = -\int_a^b G(x, t) p(t) \sqrt{w(x) w(t)} dt, \qquad x \in [a, b]. \tag{21}$$

Since,

$$\begin{aligned} \frac{d}{dx}\{k(x)W(u, v)(x)\} &= u(x)(k(x)v'(x))' - v(x)(k(x)u'(x))' \\ &= 0, \end{aligned}$$

thus, $k(x)W(u, v)(x)$ is constant. Hence, $G(x, t)\sqrt{w(x)w(t)}$ is continuous and symmetric. Therefore, by the relation (21) and Lemma 3.3 we derive the following uniformly convergent expansion

$$h\sqrt{w} = \sum_{n=1}^{\infty} \beta_n y_n \sqrt{w},$$

where β_n is of the form (20). This completes the proof of Theorem 3.4. $\square$

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