On Eccentricity Version of Laplacian Energy of a Graph

Nilanjan De*

Abstract

The energy of a graph G is equal to the sum of absolute values of the eigenvalues of the adjacency matrix of G, whereas the Laplacian energy of a graph G is equal to the sum of the absolute value of the difference between the eigenvalues of the Laplacian matrix of G and average degree of the vertices of G. Motivated by the work from Sharafdini and Panahbar [R. Sharafdini, H. Panahbar, Vertex weighted Laplacian graph energy and other topological indices, *J. Math. Nanosci.* **6** (2016) $57 - 65$], in this paper we investigate the eccentricity version of Laplacian energy of a graph G.

Keywords: Eccentricity, eigenvalue, energy (of graph), Laplacian energy, topological index.

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1. Introduction

Let G be a simple graph with n vertices and m edges. Let the vertex and edge sets of G are denoted by $V(G)$ and $E(G)$ respectively. The degree of a vertex v, denoted by $d_G(v)$, is the number of vertices adjacent to v. For any two vertices $u, v \in V(G)$, the distance between u and v is denoted by $d_G(u,v)$ and is given by the number of edges in the shortest path connecting u and v. Also we denote the sum of distances between $v \in V(G)$ and every other vertices in G by $d(x|G)$, i.e., $d(x|G) = \sum_{v \in V(G)} d_G(x,v)$. The eccentricity of a vertex v, denoted by $\varepsilon_G(v)$, is the largest distance from v to any other vertex u of G. The total eccentricity of a graph is denoted by $\zeta(G)$ and is equal to sum of eccentricities of all the vertices of the graph. Let $A = [a_{ij}]$ be the adjacency matrix of G and let $\lambda_1, \lambda_2, ..., \lambda_n$ are eigenvalues of A which are the eigenvalues of the graph G. The energy of a graph

is introduced by Ivan Gutman in 1978 [7] and defined as the sum of the absolute values of its eigenvalues and is denoted by E(G). Thus

$$E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

A large number of results on the graph energy have been reported, see for instance [2, 8, 10, 13]. Motivated by the success of the theory of graph energy, other different energy like quantities have been proposed and studied by different researchers. Let $D(G) = [d_{ij}]$ be the diagonal matrix associated with the graph G, where $d_{ii} = d_G(v_i)$ and $d_{ij} = 0$ if $i \neq j$. Define $L(G) = D(G) - A(G)$, where $L(G)$ is called the Laplacian matrix of G. Let $\mu_1, \mu_2, ..., \mu_n$ be the Laplacian eigenvalues of G. Then the Laplacian energy of G is defined as [9]

$$LE(G) = \sum_{i=1}^{n} |\mu_i - \frac{2m}{n}|.$$

Various studies on Laplacian energy of graphs were reported in the literature [1, 3, 6, 12, 18]. Analogues to Laplacian energy of a graph, a different new type of graph energy was introduced and in this present study, inspired by the work in [16], we investigate the eccentricity version of Laplacian energy of a graph denoted by $LE_\varepsilon(G)$. In this case, we define the Laplacian eccentricity matrix as $L_\varepsilon(G) = \varepsilon(G) - A(G)$, where $\varepsilon(G) = [e_{ij}]$ is the $(n \times n)$ diagonal matrix of G with $e_{ii} = \varepsilon_G(v_i)$ and $e_{ij} = 0$ if $i \neq j$. Here, $\varepsilon_G(v_i)$ is the eccentricity of the vertex $v_i, i = 1, 2, ..., n$. Let $\mu_1', \mu_2', ..., \mu_n'$ be the eigenvalues of the matrix $L_\varepsilon(G)$. Then the eccentricity version of Laplacian energy of G is defined as

$$LE_\varepsilon(G) = \sum_{i=1}^{n} |\mu_i' - \frac{\zeta(G)}{n}|.$$

Recall that $\zeta(G)/n$ is the average vertex eccentricity. In this paper, we fist calculate some basic properties and then establish some upper and lower bounds for $EL_\varepsilon(G)$.

2. Main Results

We know that, the ordinary and Laplacian graph eigenvalues obey the following relations:

$\sum_{i=1}^{n} \lambda_i = 0$; $\sum_{i=1}^{n} \lambda_i^2 = 2m$

$\sum_{i=1}^{n} \mu_i = 2m$; $\sum_{i=1}^{n} \mu_i^2 = 2m + \sum_{i=1}^{n} d(v_i)^2.$

As the Laplacian spectrum is denoted by $\mu_1', \mu_2', ..., \mu_n'$, let $\nu_i' = |\mu_i' - \frac{\zeta(G)}{n}|$. Recall that the first Zagreb eccentricity index of a graph is denoted by $E_1(G)$ and

is equal to the sum of square of all the vertices of the graph G. Thus, we have $E_1(G) = \sum_{i=1}^{n} \varepsilon(v_i)^2$ (for details see [4, 11, 17]). In the following, now we investigate some basic properties of μ_i' and ν_i'.

Lemma 2.1. *The eigenvalues $\mu_1', \mu_2', ..., \mu_n'$ satisfy the following relations*
(i) $\sum_{i=1}^{n} \mu_i' = \zeta(G)$ and (ii) $\sum_{i=1}^{n} \mu_i'^2 = E_1(G) + 2m$.

Proof. (i) Since, the trace of a square matrix is equal to the sum of its eigenvalues, we have

$$\sum_{i=1}^{n} \mu_i' = tr(EL_\varepsilon(G)) = \sum_{i=1}^{n} [\varepsilon(v_i) - a_{ii}] = \zeta(G).$$

(ii) Again we have,

$$
\begin{aligned}
\sum_{i=1}^{n} \mu_i'^2 &= tr[(\varepsilon(G) - A(G))(\varepsilon(G) - A(G))^T] \\
&= tr[(\varepsilon(G) - A(G))(\varepsilon(G)^T - A(G)^T)] \\
&= tr[\varepsilon(G)\varepsilon(G)^T - A(G)\varepsilon(G)^T - A(G)^T\varepsilon(G) + A(G)A(G)^T] \\
&= \sum_{i=1}^{n} \varepsilon(v_i)^2 + \sum_{i=1}^{n} \lambda_i^2 \\
&= E_1(G) + 2m.
\end{aligned}
$$

$\square$

Lemma 2.2. *The eigenvalues $\nu_1', \nu_2', ..., \nu_n'$ satisfy the following relations*
(i) $\sum_{i=1}^{n} \nu_i' = 0$ and (ii) $\sum_{i=1}^{n} \nu_i'^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m$.

Proof. (i) We have from definition,

$$\sum_{i=1}^{n} \nu_i' = \sum_{i=1}^{n} \left(\mu_i' - \frac{\zeta(G)}{n} \right) = \sum_{i=1}^{n} \mu_i' - \zeta(G) = 0.$$

(ii) Again, similarly we have

$$
\begin{aligned}
\sum_{i=1}^{n} \nu_i'^2 &= \sum_{i=1}^{n} \left(\mu_i' - \frac{\zeta(G)}{n} \right)^2 \\
&= \sum_{i=1}^{n} \left[\mu_i'^2 + \frac{\zeta(G)^2}{n^2} - 2\mu_i'\frac{\zeta(G)}{n} \right] \\
&= E_1(G) + 2m + \frac{\zeta(G)^2}{n} - 2\frac{\zeta(G)^2}{n},
\end{aligned}
$$

which proves the desired result.   □

Lemma 2.3. *The eigenvalues* $\nu_1', \nu_2', ..., \nu_n'$ *satisfy the following relations*

$$\left| \sum_{i<j} \nu_i' \nu_j' \right| = \frac{1}{2} \left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right].$$

Proof. Since $\sum_{i=1}^{n} \nu_i' = 0$, so we can write $\sum_{i=1}^{n} \nu_i'^2 = -2 \sum_{i<j} \nu_i' \nu_j'$. Thus,

$$2 \left| \sum_{i<j} \nu_i' \nu_j' \right| = \sum_{i=1}^{n} \nu_i'^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m.$$

Hence the desired result follows.   □

As an example, in the following, we now calculate the eccentric version of Laplacian energy and corresponding spectrum of two particular type of graphs, namely, complete graph and complete bipartite graph.

Example 2.4. Let K_n be the complete graph with n vertices, then

$$L_\varepsilon(K_n) = \begin{pmatrix} 1 & -1 & -1 & ... & -1 \\ -1 & 1 & -1 & ... & -1 \\ -1 & -1 & 1 & ... & -1 \\ ... & ... & ... & ... & ... \\ -1 & -1 & -1 & ... & 1 \end{pmatrix}.$$

Its characteristic equation is

$$(\mu' - 2)^{(n-1)}(\mu' - (2-n)) = 0.$$

Thus the eccentricity Laplacian spectrum of K_n is given by

$$spec_\varepsilon(G) = \begin{pmatrix} (1-n) & 1 \\ 1 & (n-1) \end{pmatrix}$$

and hence $EL_\varepsilon(G) = 2(n-1)$.

Example 2.5. Let $K_{m,n}$ be the complete graph with $(m+n)$ vertices and mn

edges, then

$$L_\varepsilon(K_{n,n}) = \begin{pmatrix}
2 & 0 & 0 & ... & 0 & -1 & -1 & -1 & ... & -1 \\
0 & 2 & 0 & ... & 0 & -1 & -1 & -1 & ... & -1 \\
0 & 0 & 2 & ... & 0 & -1 & -1 & -1 & ... & -1 \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
0 & 0 & 0 & ... & 2 & -1 & -1 & -1 & ... & -1 \\
-1 & -1 & -1 & ... & -1 & 2 & 0 & 0 & ... & 0 \\
-1 & -1 & -1 & ... & -1 & 0 & 2 & 0 & ... & 0 \\
-1 & -1 & -1 & ... & -1 & 0 & 0 & 2 & ... & 0 \\
.. & .. & .. & .. & .. & .. & .. & .. & .. & .. \\
-1 & -1 & -1 & ... & -1 & 0 & 0 & 0 & ... & 2
\end{pmatrix}.$$

Its characteristic equation is

$$(\mu' - 2)^{2(n-1)}(\mu' - (2 + n))(\mu' - (2 - n)) = 0.$$

So, the eccentricity Laplacian spectrum of $K_{n,n}$ is given by

$$spec_\varepsilon(G) = \begin{pmatrix} -n & n & 0 \\ 1 & 1 & 2(n-1) \end{pmatrix}$$

and hence $EL_\varepsilon(K_{n,n}) = 2n$.

Note that, from the above two examples, the properties of the eigenvalues $\nu'_1, \nu'_2, ..., \nu'_n$ can be verified easily. In the following, next we investigate some upper and lower bounds of eccentricity version of Laplacian energy of a graph G.

Theorem 2.6. *Let G be a connected graph of order n and size m, then*

$$EL_\varepsilon(G) \geq 2\sqrt{m + \frac{1}{2}(E_1(G) - \frac{1}{n}\zeta(G)^2)}.$$

Proof. We have from definition, $EL_\varepsilon(G) = \sum\limits_{i=1}^{n} |\nu'_i|$. So we can write,

$$EL_\varepsilon(G)^2 = \sum_{i=1}^{n} \nu'^2_i + 2\sum_{i<j} |\nu'_i \nu'_j| \geq \left(E_1(G) - \frac{\zeta(G)^2}{n} + 2m \right) + 2\sum_{i<j} |\nu'_i \nu'_j|.$$

Hence using Lemma 2.3, the desired result follows.   □

Theorem 2.7. *Let G be a connected graph of order n and size m; and ν'_1 and ν'_n are maximum and minimum absolute values of $\nu'_i s$, then*

$$EL_\varepsilon(G) \geq \sqrt{nE_1(G) - \zeta(G)^2 + 2mn - \frac{n^2}{4}(\nu'_1 - \nu'_n)^2}.$$

Proof. Let a_i and b_i, $1 \leq i \leq n$ are non-negative real numbers, then using the Ozeki's inequality [14], we have

$$\sum_{i=1}^{n} a_i{}^2 \sum_{i=1}^{n} b_i{}^2 - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \leq \frac{n^2}{4}(M_1 M_2 - m_1 m_2)^2$$

where $M_1 = \max(a_i)$, $m_1 = \min(a_i)$, $M_2 = \max(b_i)$ and $m_2 = \min(b_i)$. Let $a_i = |\nu_i'|$ and $b_i = 1$, then from above we have

$$\sum_{i=1}^{n} |\nu_i'|^2 \sum_{i=1}^{n} 1^2 - \left(\sum_{i=1}^{n} |\nu_i'|\right)^2 \leq \frac{n^2}{4}(\nu_1' - \nu_n')^2.$$

Thus, we can write

$$
\begin{aligned}
EL_\varepsilon(G)^2 \quad & \geq \quad n\sum_{i=1}^{n} |\nu_i'|^2 - \frac{n^2}{4}(\nu_1' - \nu_n')^2 \\
& = \quad n\left(E_1(G) - \frac{\zeta(G)^2}{n} + 2m\right) - \frac{n^2}{4}(\nu_1' - \nu_n')^2,
\end{aligned}
$$

from where the desired result follows. $\qquad\square$

Theorem 2.8. *Let G be a connected graph of order n and size m; and ν_1' and ν_n' are maximum and minimum absolute values of $\nu_i's$, then*

$$EL_\varepsilon(G) \geq \frac{1}{(\nu_1' + \nu_n')}\left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m - n\nu_1'\nu_n'\right].$$

Proof. Let a_i and b_i, $1 \leq i \leq n$ are non-negative real numbers, then from Diaz-Metcalf inequality [5], we have

$$\sum_{i=1}^{n} b_i{}^2 + mM\sum_{i=1}^{n} a_i{}^2 \leq (m + M)\left(\sum_{i=1}^{n} a_i b_i\right),$$

where, $ma_i \leq b_i \leq Ma_i$. Let $a_i = 1$ and $b_i = |\nu_i'|$, then from above we have

$$\sum_{i=1}^{n} |\nu_i'|^2 + \nu_1'\nu_n'\sum_{i=1}^{n} 1^2 \leq (\nu_1' + \nu_n')\left(\sum_{i=1}^{n} |\nu_i'|\right).$$

Now, since

$$EL_\varepsilon(G) = \sum_{i=1}^{n} |\nu_i'|$$

and $\sum\limits_{i=1}^{n} {\nu_i'}^2 = E_1(G) - \frac{\zeta(G)^2}{n} + 2m$, we get

$$E_1(G) - \frac{\zeta(G)^2}{n} + 2m + n\nu_1'\nu_n' \leq (\nu_1' + \nu_n')EL_\varepsilon(G),$$

from where the desired result follows. □

Theorem 2.9. *Let G be a connected graph of order n and size m, then*

$$EL_\varepsilon(G) \leq \sqrt{\left[nE_1(G) - \zeta(G)^2 + 2mn\right]}.$$

Proof. Using the Cauchy-Schwarz inequality to the vectors $(|\nu_1'|, |\nu_2'|, ..., |\nu_n'|)$ and $(1, 1, ..., 1)$, we have

$$\sum_{i=1}^{n} |\nu_i'| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\nu_i'|^2}$$

Thus from definition, we have

$$\begin{aligned}
EL_\varepsilon(G) = \sum_{i=1}^{n} |\nu_i'| &\leq \sqrt{n}\sqrt{\sum_{i=1}^{n} |\nu_i'|^2} \\
&= \sqrt{n}\sqrt{\sum_{i=1}^{n} |\mu_i' - \frac{\zeta(G)}{n}|^2} = \sqrt{n}\sqrt{\left[E_1(G) - \frac{\zeta(G)^2}{n} + 2m\right]}.
\end{aligned}$$

Hence the desired result follows. □

Theorem 2.10. *Let G be a connected graph of order n and size m, then*

$$EL_\varepsilon(G) \geq \frac{2\sqrt{\nu_1'\nu_n'}}{\nu_1' + \nu_n'}\sqrt{\left[nE_1(G) - \zeta(G)^2 + 2mn\right]}.$$

Proof. We have, from Polya-Szego inequality [15] for non-negatve real numbers a_i and b_i, $1 \leq i \leq n$

$$\sum_{i=1}^{n} a_i^2 \sum_{i=1}^{n} b_i^2 \leq \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

where $M_1 = \max(a_i)$, $m_1 = \min(a_i)$, $M_2 = \max(b_i)$ and $m_2 = \min(b_i)$. Let $a_i = |\nu_i'|$ and $b_i = 1$, then from above we have

$$\sum_{i=1}^{n} |\nu_i'|^2 \sum_{i=1}^{n} 1^2 \leq \frac{1}{4}\left(\sqrt{\frac{\nu_n'}{\nu_1'}} + \sqrt{\frac{\nu_1'}{\nu_n'}}\right)^2 \left(\sum_{i=1}^{n} |\nu_i'|\right)^2.$$

That is,

$$n \sum_{i=1}^{n} |\nu_i^{'}|^2 \leq \frac{1}{4} \left(\sqrt{\frac{\nu_n^{'}}{\nu_1^{'}}} + \sqrt{\frac{\nu_1^{'}}{\nu_n^{'}}} \right)^2 (EL_\varepsilon(G))^2$$

$$= \frac{1}{4} \frac{(\nu_n^{'} + \nu_1^{'})^2}{\nu_n^{'} \nu_1^{'}} (EL_\varepsilon(G))^2.$$

Since, $\sum\limits_{i=1}^{n} \nu_i^{'2} = E_1(G) - \frac{\zeta(G)^2}{n} + 2m$, we get the desired result from above. $\square$

3. Conclusion

In this paper, we study different properties and bounds of eccentricity version of Laplacian energy of a graph G. It is found that, there is great analogy between the original Laplacian energy and eccentricity version of Laplacian energy, where as also have some distinct difference.

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Nilanjan De
Department of Basic Sciences and Humanities (Mathematics),
Calcutta Institute of Engineering and Management,
Kolkata, West Bengal,
India
E-mail: de.nilanjan@rediffmail.com