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On Relation between the Kirchhoff Index and Laplacian-Energy-Like Invariant of Graphs

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Abstract

Let G be a simple connected graph with $n \geq 2$ vertices and m edges, and let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ be its Laplacian eigenvalues. The Kirchhoff index and Laplacian–energy–like invariant (LEL) of graph G are defined as $Kf(G) = n\sum_{i=1}^{n-1} \frac{1}{\mu_i}$ and $LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}$, respectively. In this paper we consider relationship between $Kf(G)$ and $LEL(G)$.

Keywords: Kirchhoff index, Laplacian-energy-like invariant, Laplacian eigenvalues of graph.

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1. Introduction

Let $G = (V, E)$, $V = \{1, 2, \ldots, n\}$, be a simple connected graph with $n \geq 2$ vertices and m edges, with vertex degrees $\Delta = d_1 \geq d_2 \geq \cdots \geq d_n = \delta > 0$. Further, let $\mathbf{A}$ be the adjacency matrix of G. Eigenvalues of matrix $\mathbf{A}$, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are ordinary eigenvalues of graph G. Some of their well known properties are (see for example [5])

$$\sum_{i=1}^{n} \lambda_i = 0 \quad \text{and} \quad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i = 2m.$$

Let $\mathbf{D}$ be the diagonal matrix of order n, whose diagonal elements are $d_1, d_2, \ldots, d_n$. Then $\mathbf{L} = \mathbf{D} - \mathbf{A}$ is the Laplacian matrix of graph G. Eigenvalues of the matrix $\mathbf{L}$, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{n-1} > \mu_n = 0$ form the so-called Laplacian spectrum of graph G. The Laplacian matrix of a graph and its eigenvalues can be used in several areas of mathematical research and have a physical interpretation in

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various physical and chemical theories. Laplacian eigenvalues satisfy the following identities:

$$\sum_{i=1}^{n-1} \mu_i = \sum_{i=1}^{n} d_i = 2m \quad \text{and} \quad \sum_{i=1}^{n-1} \mu_i^2 = \sum_{i=1}^{n} d_i^2 + \sum_{i=1}^{n} d_i = M_1 + 2m,$$

where $M_1 = M_1(G)$ is the first Zagreb index, introduced in [20] by Gutman and Trinajstić. More about this topological index, as well as the second Zagreb index, one can found in [3, 4, 14, 34].

A concept related to the spectrum of a graph is that of energy. As its name suggests, it is inspired by energy in chemistry. In 1978, Gutman [11] defined energy mathematically as the sum of absolute values of the eigenvalues of the adjacency matrix of graph:

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i|.$$

In the past decade, interest in graph energy has increased and similar definitions have been formulated for other matrices associated with a graph, such as the Laplacian, normalized Laplacian, distance matrices, and even for a general matrix not associated with a graph [41].

In 2006, Gutman and Zhou [21] defined the Laplacian energy of a graph as the sum of the absolute deviations (i.e. distance from the mean) of the eigenvalues of its Laplacian matrix:

$$LE = LE(G) = \sum_{i=1}^{n} \left| \mu_i - \frac{2m}{n} \right| .$$

Liu and Liu [26] introduced the Laplacian–energy–like invariant, shortly LEL, defined as

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.$$

For more information on these, as well as other graph and matrix energies, the interested reader can refer to [11–13, 16, 17, 19, 21, 23, 24, 26, 31, 32, 39] and the references cited therein.

The Wiener index, $W(G)$, originally termed as a "path number", is a topological graph index defined for a graph on n nodes by

$$W(G) = \sum_{i<j} d_{ij},$$

where d_{ij} is the number of edges in the shortest path between vertices i and j in graph G. The first investigations into the Wiener index were made by Harold Wiener in 1947 [40] who realized that there are correlations between the boiling points of paraffin and the structure of the molecules.

In analogy to the Wiener index, Klein and Randić [22] defined the Kirchhoff index, $Kf(G)$, as

$$Kf(G) = \sum_{i<j} r_{ij},$$

where r_{ij} is the resistance distance between the vertices i and j of a simple connected graph G, i.e. r_{ij} is equal to the resistance between two equivalent points on an associated electrical network, obtained by replacing each edge of G by a unit (1 ohm) resistor.

As Gutman and Mohar [18] (see also [44]) proved, the Kirchhoff index can also be represented as

$$Kf(G) = n \sum_{i=1}^{n-1} \frac{1}{\mu_i}.$$

More on the Kirchhoff index, as well as its applications in various areas, such as in spectral graph theory, molecular chemistry, computer science, etc. can be found, for example, in [9, 15, 25, 28–30, 35, 42].

In this paper we prove some inequalities that establish relationship between graph invariants $Kf(G)$ and $LEL(G)$. This problem was considered in many papers (see for example [1, 7, 8, 36, 37]). This work was motivated by the results obtained in [8].

2. Preliminaries

In this section we recall some analytic inequalities for real number sequences that will be needed in the subsequent considerations.

Let $p = (p_i)$ and $a = (a_i)$, $i = 1, 2, \ldots, n-1$, be two sequences of positive real numbers with the properties $p_1 + p_2 + \cdots + p_{n-1} = 1$ and $0 < r \le a_i \le R < +\infty$. Rennie [38] proved the following inequality

$$\sum_{i=1}^{n-1} p_i a_i + rR \sum_{i=1}^{n-1} \frac{p_i}{a_i} \le r + R. \tag{1}$$

Let $a = (a_i)$, $i = 1, 2, \ldots, n-1$, be positive real numbers sequence with the property $0 < r \le a_i \le R < +\infty$. In [27] Lupas proved the inequality

$$\sum_{i=1}^{n-1} a_i \sum_{i=1}^{n-1} \frac{1}{a_i} \le (n-1)^2 \left(1 + \frac{\alpha(n-1)(R-r)^2}{rR}\right), \tag{2}$$

where

$$\alpha(n-1) = \frac{1}{4}\left(1 - \frac{(-1)^n + 1}{2(n-1)^2}\right).$$

Zhou, Gutman, and Aleksić [43] proved the following inequality for positive real numbers sequence $a = (a_i)$, $i = 1, 2, \dots, n-1$,

$$\begin{aligned}(n-1)\left(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i - \left(\prod_{i=1}^{n-1} a_i\right)^{\frac{1}{n-1}}\right) &\le (n-1)\sum_{i=1}^{n-1} a_i - \left(\sum_{i=1}^{n-1}\sqrt{a_i}\right)^2 \\ \le (n-1)(n-2)\left(\frac{1}{n-1}\sum_{i=1}^{n-1} a_i - \left(\prod_{i=1}^{n-1} a_i\right)^{\frac{1}{n-1}}\right). \end{aligned} \tag{3}$$

Let $p = (p_i)$, $i = 1, 2, \dots, n-1$, be positive real numbers sequence, and $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n-1$, sequences of non-negative real numbers of similar monotonicity. Then [33]

$$\sum_{i=1}^{n-1} p_i \sum_{i=1}^{n-1} p_i a_i b_i \ge \sum_{i=1}^{n-1} p_i a_i \sum_{i=1}^{n-1} p_i b_i. \tag{4}$$

If sequences $a = (a_i)$ and $b = (b_i)$ are of opposite monotonicity, then the sense of (4) reverses.

Let $a = (a_i)$ and $b = (b_i)$, $i = 1, 2, \dots, n-1$, be two positive real numbers sequences with the properties

$$0 < r_1 \le a_i \le R < +\infty \quad \text{and} \quad 0 < r_2 \le b_i \le R_2 < +\infty\,.$$

In [2] (see also [33]) the following inequality was proven:

$$\left|(n-1)\sum_{i=1}^{n-1} a_i b_i - \sum_{i=1}^{n-1} a_i \sum_{i=1}^{n-1} b_i\right| \le (n-1)^2\alpha(n-1)(R_1 - r_1)(R_2 - r_2)\,, \tag{5}$$

where

$$\alpha(n-1) = \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\left(1 - \frac{1}{n-1}\left\lfloor\frac{n-1}{2}\right\rfloor\right) = \frac{1}{4}\left(1 - \frac{(-1)^n + 1}{2(n-1)^2}\right).$$

Let $a_1 \ge a_2 \ge \cdots \ge a_{n-1} > 0$ are positive real numbers. The following was proven in [6]:

$$\sum_{i=1}^{n-1} a_i - (n-1)\left(\prod_{i=1}^{n-1} a_i\right)^{\frac{1}{n-1}} \ge \left(\sqrt{a_1} - \sqrt{a_{n-1}}\right)^2. \tag{6}$$

3. Main Results

In the following lemma we establish upper bound for $Kf(G)$ in terms of $LEL(G)$, $M_1(G)$, graph parameters n, m, and Laplacian eigenvalues μ_1 and μ_{n-1}.

Lemma 3.1. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$Kf(G) \leq \frac{n\left(\left(\mu_1^{3/2} + \mu_{n-1}^{3/2}\right) LEL(G) - M_1(G) - 2m\right)}{(\mu_1\mu_{n-1})^{3/2}}. \tag{7}$$

Equality holds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, or for any s, $1 \leq s \leq n-2$, holds $\mu_1 = \mu_2 = \cdots = \mu_s \geq \mu_{s+1} = \cdots = \mu_{n-1}$.

Proof. Setting $p_i = \frac{\sqrt{\mu_i}}{\sum_{i=1}^{n-1}\sqrt{\mu_i}}$, $a_i = \mu_i^{3/2}$, $i = 1, 2, \ldots, n-1$, $R = \mu_1^{3/2}$, $r = \mu_{n-1}^{3/2}$, in (1), we get

$$\frac{\sum_{i=1}^{n-1}\mu_i^2}{\sum_{i=1}^{n-1}\sqrt{\mu_i}} + (\mu_1\mu_{n-1})^{3/2}\frac{\sum_{i=1}^{n-1}\frac{1}{\mu_i}}{\sum_{i=1}^{n-1}\sqrt{\mu_i}} \leq \mu_1^{3/2} + \mu_{n-1}^{3/2},$$

i.e.

$$\frac{M_1(G) + 2m}{LEL(G)} + (\mu_1\mu_{n-1})^{3/2}\frac{\frac{1}{n}Kf(G)}{LEL(G)} \leq \mu_1^{3/2} + \mu_{n-1}^{3/2},$$

where from inequality (7) follows.

Equality in (1) holds if and only if $a_1 = a_2 = \cdots = a_n$, or for any s, $1 \leq s \leq n-2$, holds $a_1 = \cdots = a_s \geq a_{s+1} = \cdots = a_{n-1}$. Therefore equality in (7) holds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, or for any s, $1 \leq s \leq n-2$, holds $\mu_1 = \cdots = \mu_s \geq \mu_{s+1} = \cdots = \mu_{n-1}$. □

In the following theorem we determine an upper bound for $Kf(G)$ in terms of $LEL(G)$, $M_1(G)$, n, m, and lower bound, k, of algebraic connectivity of G, μ_{n-1}.

Theorem 3.2. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then, for any real k with the property $\mu_{n-1} \geq k > 0$, holds*

$$Kf(G) \leq \frac{\left(n^{3/2} + k^{3/2}\right) LEL(G) - M_1(G) - 2m}{n^{1/2}k^{3/2}}. \tag{8}$$

Equality holds if and only if $k = n$, and $G \cong K_n$, or for any s, $1 \leq s \leq n-2$, holds $n = \mu_1 = \cdots = \mu_s \geq \mu_{s+1} = \cdots = \mu_{n-1} = k$.

Proof. Consider the function

$$f(x) = \frac{\mu_{n-1}^{3/2} LEL(G) - M_1(G) - 2m}{x^{3/2}}, \quad x > 0.$$

Since

$$M_1(G) + 2m = \sum_{i=1}^{n-1} \mu_i^2 \geq \mu_{n-1}^{3/2} \sum_{i=1}^{n-1} \sqrt{\mu_i} = \mu_{n-1}^{3/2} LEL(G),$$

it follows that $f(x)$ is an increasing function for $x > 0$. Thus, for $x = \mu_1 \leq n$ holds $f(\mu_1) \leq f(n)$. From (7) we get

$$Kf(G) \leq \frac{n\left(\left(n^{3/2} + \mu_{n-1}^{3/2}\right) LEL(G) - M_1(G) - 2m\right)}{n^{3/2}\mu_{n-1}^{3/2}}. \tag{9}$$

Now, consider the function

$$g(x) = \frac{n^{3/2}LEL(G) - M_1(G) - 2m}{x^{3/2}}.$$

Since

$$M_1(G) + 2m = \sum_{i=1}^{n-1} \mu_i^2 \leq \mu_1^{3/2} \sum_{i=1}^{n-1} \sqrt{\mu_i} \leq n^{3/2} LEL(G),$$

the function $g(x)$ is decreasing for $x > 0$. Then, for $x = \mu_{n-1} \geq k > 0$ holds $g(\mu_{n-1}) \leq g(k)$. From (9) follows

$$Kf(G) \leq \frac{n\left(\left(n^{3/2} + k^{3/2}\right) LEL(G) - M_1(G) - 2m\right)}{n^{3/2}k^{3/2}},$$

where from we arrive at (8). □

Remark 1. Equality in (8), depending on the parameter k, is attained for a various classes of graphs. Thus, for example, equality holds for $k = 1$ and $G \cong K_{1,n-1}$, or $k = \frac{n}{2}$ and $G \cong K_{\frac{n}{2},\frac{n}{2}}$, or $k = n-2$ and $G \cong K_n - e$.

Corollary 3.3. *Let G be a simple connected graph with $n \geq 3$ vertices. Then for any real k, $\mu_{n-1} \geq k > 0$, we have*

$$Kf(G) \leq \frac{nLEL(G)}{k^{3/2}}.$$

Equality holds if and only if $G \cong K_n$.

Corollary 3.4. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then for any real k, $\mu_{n-1} \geq k > 0$, holds*

$$4(M_1(G) + 2m)Kf(G) \leq n(LEL(G))^2 \left(\left(\frac{n}{k}\right)^{3/4} + \left(\frac{k}{n}\right)^{3/4}\right)^2, \tag{10}$$

with equality if and only if $G \cong K_n$.

Proof. Inequality (10) is obtained from

$$n^{3/2}k^{3/2}Kf(G)+n(M_1(G)+2m)\le n\left(n^{3/2}+k^{3/2}\right)LEL(G),$$

and the AG (arithmetic-geometric mean) inequality applied on the right side of the above inequality (see, for example, [33]). □

Corollary 3.5. *Let G be a simple connected graph with $n\ge 3$ vertices and m edges. Then for any real k, $\mu_{n-1}\ge k>0$,*

$$Kf(G)\le \frac{n^2(LEL(G))^2}{8m(2m+n)}\left(\left(\frac{n}{k}\right)^{3/4}+\left(\frac{k}{n}\right)^{3/4}\right)^2,$$

with equality if and only if $G\cong K_n$.

Proof. This inequality follows from the inequality (10) and inequality $M_1\ge \frac{4m^2}{n}$ proved in [10].

□

Theorem 3.6. *Let G be a simple connected graph with $n\ge 3$ vertices. Then, for any real k with the property $\mu_{n-1}\ge k>0$, holds*

$$\begin{aligned}&\left(Kf(G)+n(n-1)(n-2)(nt)^{-\frac{1}{n-1}}\right)(LEL(G))^2\\ &\le n(n-1)^4\left(1+\alpha(n-1)\left(\left(\frac{n}{k}\right)^{1/2}+\left(\frac{k}{n}\right)^{1/2}-2\right)\right)^2,\end{aligned}\tag{11}$$

where $t=t(G)=\frac{1}{n}\prod_{i=1}^{n-1}\mu_i$ is the number of spanning trees in G. Equality holds if $k=n$ and $G\cong K_n$.

Proof. For $a_i=\sqrt{\mu_i}$, $i=1,2,\ldots,n-1$, $r=\sqrt{\mu_{n-1}}$, $R=\sqrt{\mu_1}$, the inequality (2) becomes

$$\left(\sum_{i=1}^{n-1}\sqrt{\mu_i}\right)\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)\le (n-1)^2\left(1+\alpha(n-1)\left(\sqrt[4]{\frac{\mu_1}{\mu_{n-1}}}-\sqrt[4]{\frac{\mu_{n-1}}{\mu_1}}\right)^2\right),$$

i.e.

$$\left(\sum_{i=1}^{n-1}\sqrt{\mu_i}\right)^2\left(\sum_{i=1}^{n-1}\frac{1}{\sqrt{\mu_i}}\right)^2\le (n-1)^4\left(1+\alpha(n-1)\left(\sqrt{\frac{\mu_1}{\mu_{n-1}}}+\sqrt{\frac{\mu_{n-1}}{\mu_1}}-2\right)\right)^2.\tag{12}$$

The function $f(x) = x + \frac{1}{x}$ is increasing for $x \geq 1$. Since $\mu_{n-1} \geq k$ and $\mu_1 \leq n$, it holds $x = \sqrt{\frac{\mu_1}{\mu_{n-1}}} \leq \sqrt{\frac{n}{k}}$. Therefore from (12) we get

$$\left(\sum_{i=1}^{n-1} \sqrt{\mu_i}\right)^2 \left(\sum_{i=1}^{n-1} \frac{1}{\sqrt{\mu_i}}\right)^2 \leq (n-1)^4 \left(1 + \alpha(n-1)\left(\sqrt{\frac{n}{k}} + \sqrt{\frac{k}{n}} - 2\right)\right)^2. \quad (13)$$

For $a_i = \frac{1}{\mu_i}$, $i = 1, 2, \dots, n-1$, inequality on the right side of (3) becomes

$$\left(\sum_{i=1}^{n-1} \frac{1}{\sqrt{\mu_i}}\right)^2 \geq \sum_{i=1}^{n-1} \frac{1}{\mu_i} + (n-1)(n-2)\left(\prod_{i=1}^{n-1} \frac{1}{\mu_i}\right)^{\frac{1}{n-1}},$$

i.e.

$$\left(\sum_{i=1}^{n-1} \frac{1}{\sqrt{\mu_i}}\right)^2 \geq \frac{1}{n} Kf(G) + (n-1)(n-2)(nt)^{-\frac{1}{n-1}}. \quad (14)$$

Now inequality (11) is a direct consequence of inequalities (13) and (14).

□

Corollary 3.7. *Let G be a simple connected graph with $n \geq 3$ vertices. Then, for any real k, $\mu_{n-1} \geq k > 0$,*

$$\left(Kf(G) + n(n-1)(n-2)(nt)^{-\frac{1}{n-1}}\right)(LEL(G))^2$$
$$\leq \frac{n(n-1)^4}{16}\left(\left(\frac{n}{k}\right)^{1/2} + \left(\frac{k}{n}\right)^{1/2} + 2\right)^2.$$

Equality holds if $k = n$ and $G \cong K_n$.

Proof. This inequality can be obtained according to (11) and inequality

$$\alpha(n-1) \leq \frac{1}{4}.$$

□

In the following theorem we prove inequality reverse to (10).

Theorem 3.8. *Let G be a simple connected graph with $n \geq 2$ vertices and m edges. Then*

$$(M_1(G) + 2m)Kf(G) \geq n(LEL(G))^2, \quad (15)$$

with equality if and only if $G \cong K_n$.

Proof. Setting $p_i = \frac{1}{\mu_i}$, $a_i = b_i = \mu_i^{3/2}$, $i = 1, 2, \ldots, n-1$, in (4) we get

$$\left(\sum_{i=1}^{n-1} \frac{1}{\mu_i} \right) \left(\sum_{i=1}^{n-1} \mu_i^2 \right) \geq \left(\sum_{i=1}^{n-1} \sqrt{\mu_i} \right)^2 ,$$

where from directly follows (15).

Equality in (4) holds if and only if $a_1 = a_2 = \cdots = a_{n-1}$ and/or $b_1 = b_2 = \cdots = b_{n-1}$, therefore equality in (15) holds if and only if $\mu_1 = \mu_2 = \cdots = \mu_{n-1}$, i.e. $G \cong K_n$.

□

Corollary 3.9. *Let G be a simple connected graph with $n \geq 2$ vertices. Then*

$$Kf(G) \geq \frac{LEL(G)}{\sqrt{n}}.$$

Equality holds if and only if $G \cong K_n$.

By a similar procedure as in case of Theorem 3.8, the following result can be proved.

Theorem 3.10. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges. Then*

$$(Kf(G) - 1)(M_1 + 2m - (1 + \Delta)^2) \geq n \left(LEL(G) - \sqrt{n} \right)^2 ,$$

with equality if and only if $G \cong K_n$, or $G \cong K_{1,n-1}$.

In the next theorem we establish lower and upper bounds for LEL.

Theorem 3.11. *Let G be a simple connected graph with $n \geq 3$ vertices and m edges, and k be an arbitrary real number so that $\mu_{n-1} \geq k > 0$. Then*

$$\begin{aligned} \left((1+\Delta)^{1/4} - \left(\frac{2m}{n-1} \right)^{1/4} \right)^2 &\leq LEL - (n-1)(nt)^{\frac{1}{2(n-1)}} \\ &\leq (n-1)^2 \alpha (n-1) \left(n^{1/4} - k^{1/4} \right)^2 , \end{aligned} \tag{16}$$

where $t = t(G) = \frac{1}{n} \prod_{i=1}^{n-1} \mu_i$ is the number of spanning trees in G.

Equality on the left side of (16) *holds if $k = n$ and $G \cong K_n$, and on the right side when $G \cong K_n$.*

Proof. For $a_i = b_i = \mu_i^{1/4}$, $i = 1, 2, \dots, n-1$, $R_1 = R_2 = \mu_1^{1/4}$ and $r_1 = r_2 = \mu_{n-1}^{1/4}$, the inequality (16) transforms into

$$(n-1)\sum_{i=1}^{n-1}\sqrt{\mu_i} - \left(\sum_{i=1}^{n-1}\sqrt[4]{\mu_i}\right)^2 \le (n-1)^2\alpha(n-1)\left(\mu_1^{1/4} - \mu_{n-1}^{1/4}\right)^2,$$

i.e.

$$(n-1)LEL - \left(\sum_{i=1}^{n-1}\sqrt[4]{\mu_i}\right)^2 \le (n-1)^2\alpha(n-1)\left(\mu_1^{1/4} - \mu_{n-1}^{1/4}\right)^2. \tag{17}$$

For $a_i = \mu_i^{1/2}$ from the left side of (3) we get

$$\left(\sum_{i=1}^{n-1}\mu_i^{1/4}\right)^2 \le (n-2)\sum_{i=1}^{n-1}\sqrt{\mu_i} + (n-1)\left(\prod_{i=1}^{n-1}\mu_i^{1/2}\right)^{\frac{1}{n-1}},$$

i.e.

$$\left(\sum_{i=1}^{n-1}\mu_i^{1/4}\right)^2 \le (n-2)LEL - (n-1)(nt)^{-\frac{1}{2(n-1)}}. \tag{18}$$

Now, from (17) and (18) we obtain

$$LEL - (n-1)(nt)^{\frac{1}{2(n-1)}} \le (n-1)^2\alpha(n-1)\left(\mu_1^{1/4} - \mu_{n-1}^{1/4}\right)^2.$$

From the above inequality and $\mu_1 \le n$, $\mu_{n-1} \ge k > 0$, the right side of (16) is obtained.

For $a_i = \sqrt{\mu_i}$, $i = 1, 2, \dots, n-1$, the inequality (6) becomes

$$\sum_{i=1}^{n-1}\sqrt{\mu_i} - (n-1)\left(\prod_{i=1}^{n}\sqrt{\mu_i}\right)^{\frac{1}{n-1}} \ge \left(\mu_1^{1/4} - \mu_{n-1}^{1/4}\right)^2,$$

i.e.

$$LEL - (n-1)(nt)^{\frac{1}{2(n-1)}} \ge \left(\mu_1^{1/4} - \mu_{n-1}^{1/4}\right)^2.$$

From the above, and inequalities $\mu_1 \ge 1 + \Delta$ and $\mu_{n-1} \le \frac{2m}{n-1}$, the left side of (16) is obtained.

□

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