Mathematics Interdisciplinary Research **2** (2017) 193 – 207

Eigenvalues and Energy of the Cayley Graph of some Groups with respect to a Normal Subset

Maryam Jalali–Rad ⋆

Abstract

Set $X = \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Z_n, T_{4n}, SD_{8n}, Sz(q), G_2(q), V_{8n}\}$, where M_{11}, M_{12}, M_{22}, M_{23}, M_{24} are Mathieu groups and Z_n, T_{4n}, SD_{8n}, $Sz(q)$, $G_2(q)$ and V_{8n} denote the cyclic, dicyclic, semi-dihedral, Suzuki, Ree and a group of order $8n$ presented by

$$V_{8n} = \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b\rangle,$$

respectively. In this paper, we compute all eigenvalues of $Cay(G,T)$, where $G \in X$ and T is minimal, second minimal, maximal or second maximal normal subset of $G \setminus \{e\}$ with respect to its size. In the case that S is a minimal normal subset of $G \setminus \{e\}$, the summation of the absolute value of eigenvalues, energy of the Cayley graph, is evaluated.

Keywords: Simple group, Cayley graph, eigenvalue, energy.

2010 Mathematics Subject Classification: $20D08$, $05C50$, $20C15$.

1. Introduction

Suppose $\Gamma = (V, E)$ is a simple graph, where $V = V(\Gamma)$ is the set of vertices and $E = E(\Gamma)$ is the set of edges of Γ. If G is a finite group, S is a subset of G such that $S = S^{-1}$ and $S \subseteq G \setminus \{e\}$ then the Cayley graph $\Gamma = Cay(G,S)$ is defined by $V(\Gamma) = G$ and $E(\Gamma) = \{\{g, sg\} \mid g \in G, s \in S\}$ [4]. It is clear that a Cayley graph $\Gamma = Cay(G,S)$ is connected if and only if $G = \langle S\rangle$.

A subset S of a finite group G is called normal, if $g^{-1}Sg = S$, for all $g \in G$. It is called symmetric, if $S^{-1} = S$. A normal, symmetric and generating subset of G with this property that $e \notin S$ is said to be an NS of G. A minimal or second minimal NS is denoted by MNS or $SMNS$, respectively.

⋆Corresponding author (E-mail: jalali6834@gmail.com)
Academic Editor: Ivan Gutman
Received 21 August 2017, Accepted 6 November 2017
DOI: 10.22052/mir.2017.101675.1082

The semi-dihedral group SD_{8n}, dicyclic group T_{4n} and the group V_{8n} have the following presentations, respectively:

$$\begin{aligned} SD_{8n} &= \langle a, b \mid a^{4n} = b^2 = e,\ bab = a^{2n-1} \rangle, \\ T_{4n} &= \langle a, b \mid a^{2n} = 1, a^n = b^2, b^{-1}ab = a^{-1} \rangle, \\ V_{8n} &= \langle a, b \mid a^{2n} = b^4 = e,\ aba = b^{-1},\ ab^{-1}a = b \rangle. \end{aligned}$$

It is easy to see the dicyclic group T_{4n} has order $4n$ and the cyclic subgroup $\langle a \rangle$ of T_{4n} has index 2 [13]. The groups SD_{8n} and V_{8n} have order $8n$ and their character tables computed in [11, 6], respectively.

Set $X = \{M_{11}, M_{12}, M_{22}, M_{23}, M_{24}, Z_n, T_{4n}, SD_{8n}, Sz(q), G_2(q), V_{8n}\}$, where M_{11}, M_{12}, M_{22}, M_{23}, M_{24} are denoted Mathieu groups, $Sz(q)$ is Suzuki group of order $(q-1)(q^2+1)$ and $G_2(q)$ is Ree group. The energy $E(G)$ of a graph G is defined as the sum of the absolute values of the eigenvalues of the adjacency matrix [10]. The aim of this paper is to compute the eigenvalues of $Cay(G, S)$, where S is an MNS or $SMNS$ and G is isomorphic to a group in X. In the case that S is a minimal normal subset of $G \setminus \{e\}$, the summation of the absolute value of eigenvalues, energy of the Cayley graph, is evaluated.

A subset $A \subseteq G$ is called rational if, for every character $\chi \in Irr(G)$, $\chi(A) = \sum_{x \in A} \chi(x)$ is an integer. Alpering in some recent papers [1, 2], proved some very nice results on integrality of Cayley graphs. Among these results, two are very important as follows:

- if the Cayley graph of G on a set S is an integral Cayley graph then S is a rational set.
- for abelian groups a set S is rational if and only if the Cayley graph on S is an integral Cayley graph.

In this paper we study the integral Cayley graphs on simple group. Our calculations are made by computer algebra system GAP [17]. Our notation is standard and can be taken from [5, 12, 13].

2. Main Results

If Γ is a graph then $Spec(\Gamma)$ denotes the multi$-$set of all $\Gamma-$eigenvalues. Two graphs Γ_1 and Γ_2 are said to be co$-$spectral, if they have the same spectrum. Diaconis and Shahshahani [7], was the first mathematician considered the problem of computing eigenvalues of Cayley graphs into account. They used the character table of the group under consideration to calculate the eigenvalues of $Cay(G, S)$, where G is a finite group and S is an NS of G. We refer to [14, 19] for more information on this topic. The aim of this section is to compute all eigenvalues of the Cayley graph of groups in the set X with respect to a minimal, second minimal, maximal or second maximal normal subset of G.

To compute the eigenvalues of a Cayley graph $Cay(G, S)$ with respect to a normal subset S of group $G \setminus \{e\}$, we apply a result of Zieschang [19, Theorem 1]. For the sake of completeness, we mention here this result.

Proposition 2.1. *Let $Cay(G, T)$ denote the Cayley graph of a finite group G with respect to a normal subset T of $G \setminus \{e\}$. Let further $\{\chi_1, \ldots, \chi_s\}$ be the set of all irreducible complex characters of G and define $\lambda_j = \frac{1}{\chi_j(e)} \sum_{t \in T} \chi_j(t)$, where $1 \leq j \leq s$. Then $\{\lambda_1, \ldots, \lambda_s\}$ is the set of all values of the spectrum of $Cay(G, T)$. Moreover, if m_j is the multiplicity of λ_j, then*

$$m_j = \sum_{\substack{k=1 \\ \lambda_k = \lambda_j}}^{s} (\chi_k(e))^2.$$

Suppose $Irr(G)$ denotes the set of all irreducible characters of G and $Cl(G)$ is the set of conjugacy classes of G. We also assume that $\mathcal{A}(G)$ is the set of all normal subsets S of $G \setminus \{e\}$. Define $S' = G \setminus (S \cup \{e\})$. One can see that, S is a minimal element of $\mathcal{A}(G)$ if and only if S' is a second maximal element of $\mathcal{A}(G)$. Notice that the first maximal is $G \setminus \{e\}$ and the Cayley graph $Cay(G, G \setminus \{e\})$ is $(|G| - 1)-$regular. This implies that $Cay(G, G \setminus \{e\})$ is complete and $Spec(Cay(G, G \setminus \{e\})) = \{-1(|G| - 1 \ times),\ |G| - 1\}$.

Proposition 2.2. *Suppose G is a finite group with exactly n conjugacy classes and S, S' are normal subsets of $G \setminus \{e\}$ such that $S' = G \setminus S \cup \{e\}$. Moreover, we assume that $\Gamma = Cay(G, S)$, $\Gamma' = Cay(G, S')$. Set $Spec(\Gamma) = \{\lambda_1, \ldots, \lambda_n\}$ and $Spec(\Gamma') = \{\beta_1, \ldots, \beta_n\}$. Then $\beta_i = -\lambda_i - 1$, $1 \leq i \leq n$.*

Proof. It is clear that S' is a symmetric and generating subset of G. So, by Proposition 2.1,

$$\begin{aligned}
\beta_i &= \frac{1}{\chi_i(e)} \sum_{s' \in S'} \chi_i(s') \\
&= \frac{1}{\chi_i(e)} \Big(\sum_{s' \in G} \chi_i(s') - \sum_{s \in S \cup \{e\}} \chi_i(s) \Big) \\
&= \frac{1}{\chi_i(e)} \sum_{s' \in G} \chi_i(s') - \frac{1}{\chi_i(e)} \sum_{s \in S} \chi_i(s) - \frac{1}{\chi_i(e)} \chi_i(e) \\
&= -\lambda_i - 1,
\end{aligned}$$

proving the result. □

Proposition 2.3. *Suppose $G = \{g_1, g_2, \ldots, g_n\}$ and F is a subset of $\{1, 2, \ldots, n\}$ such that for all $j \in F$, the order of g_j is a power of a prime p_j. Define $S = \bigcup_{j \in F} g_j^G$ and $\Gamma = Cay(G, S)$. We also assume that for each $\chi \in Irr(G)$, $\chi(g_j)$ is an integer. Then $Spec(\Gamma) = \{\lambda_1, \ldots, \lambda_n\}$, where $\lambda_i = \sum_{j \in F} |g_j^G| \Big(1 + \frac{k_j p_j}{\chi_i(e)}\Big)$, for some integer k_j.*

Proof. Suppose χ_l, $1 \leq l \leq |Irr(G)|$, is an arbitrary irreducible character of G. Then by [13, Corollary 22.27], $\chi_l(g_j) \equiv \chi_l(e)\ (mod\, p_j)$. Thus, $\chi_l(g_j) - \chi_l(e) = p_j k_j$, for some integer k_j. This implies that $\frac{\chi_l(g_j)}{\chi_l(e)} = 1 + \frac{p_j k_j}{\chi_l(e)}$ and so,

$$\lambda_l \quad = \quad \sum_{s\in S} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j\in F} \sum_{s\in g_j^G} \frac{\chi_l(s)}{\chi_l(e)} = \sum_{j\in F} \sum_{s\in g_j^G} \Big(1 + \frac{p_j k_j}{\chi_l(e)}\Big) = \sum_{j\in F} |g_j^G| \Big(1 + \frac{p_j k_j}{\chi_l(e)}\Big).$$

□

Corollary 2.4. *Suppose G is a finite group, S is a normal subset of G such that all elements of S are involutions and $\Gamma = Cay(G,S)$. If F is a representative set for $G-$conjugacy classes of S then the following statements hold:*

1. *if G is abelian then all $\Gamma-$eigenvlues are odd or all of them are even,*
2. *if G is a simple group then $\lambda_\chi = \sum_{g\in F} |g^G|(1 + \frac{4k_g}{\chi(e)})$.*

Proof. To prove (1), it is enough to apply Proposition 2.3 and the fact that the irreducible characters of G are linear. We now assume that G is simple, g is an involution in G and $\chi \in Irr(G)$. Then $\chi(g) \equiv \chi(e)\ (mod\ 4)$ or G has a normal subgroup of index 2. Since G is simple, there exists k_g such that $\frac{\chi(g)}{\chi(1)} = 1 + \frac{4k_g}{\chi(1)}$. By Proposition 2.1, $\lambda_\chi = \sum_{g\in S} \frac{\chi(g)}{\chi(e)} = \sum_{x\in F} \sum_{g\in x^G} (1 + \frac{4k_g}{\chi(e)}) = \sum_{g\in F} |g^G|(1 + \frac{4k_g}{\chi(e)})$. □

Suppose G is a group, A is the set of all character values of G, $Q(A)$ denotes the extension of Q by A and Λ is the Galois group of this extension. It is well-known that there exists ε such that $Q(A) \subseteq Q(\varepsilon)$, where ϵ is a primitive $n-th$ root of unity. Thus, if $\alpha \in \Lambda$ then there exists a unique positive integer r such that $(r,n) = 1$ and $\alpha(\varepsilon) = \varepsilon^r$. So, it is well define to use the notation $\alpha = \sigma_r$. The group Λ acts on the set of irreducible characters and conjugacy classes of G by $\chi^\alpha(g) = \alpha(\chi(g))$ and $(x^G)^{\sigma_r} = (x^r)^G$, respectively.

Proposition 2.5. *Suppose G is a finite group and S,T are two $\Lambda-$conjugate conjugacy classes of G. Then $\Gamma_1 = Cay(G,S)$ and $\Gamma_2 = Cay(G,T)$ are co$-$spectral.*

Proof. Let G be a finite group, χ be an irreducible character of G and S,T are two $\Lambda-$conjugate conjugacy classes of G. If $\{\chi_1, \ldots, \chi_r\}$ is the orbit of χ under action of the Galois group, then $\{\chi_1(S), ..., \chi_r(S)\} = \{\chi_1(T), ..., \chi_r(T)\}$. Suppose $Spec(\Gamma_1) = \{\lambda_1, \ldots, \lambda_r\}$ and $Spec(\Gamma_2) = \{\mu_1, \ldots, \mu_r\}$. If $\chi_i(S) = \chi_i(T)$, $\chi_i \in Irr(G)$, then $\lambda_i = \frac{1}{\chi_i(e)}|T|\chi_i(T) = \frac{1}{\chi_i(e)}|S|\chi_i(S) = \mu_i$. In other case, if $\{\chi_{n_1}, \ldots, \chi_{n_s}\}$ is the orbit of χ_i under the action of Galois group then the values of $\chi_{n_1}(S), \ldots, \chi_{n_r}(S)$ can be permuted to find $\chi_{n_1}(T), \ldots, \chi_{n_r}(T)$. This shows that there exists positive integer k such that $\chi_i(e) = \chi_k(e)$. Thus, $\lambda_i = \frac{1}{\chi_i(e)}|T|\chi_i(T)$ $= \frac{1}{\chi_k(e)}|S|\chi_k(S) = \mu_k$. Therefore, $\Gamma_1 = Cay(G,S)$ and $\Gamma_2 = Cay(G,T)$ are co$-$spectral. □

The converse of Proposition 2.5 is not generally correct. To do this, we consider the following example:

Example 2.6. Consider the alternating group A_6 which has exactly two conjugacy classes S and T of elements of order 3. It is easy to see that these classes are not Galois conjugate, but the Cayley graphs $\Gamma_1 = Cay(G,S)$ and $\Gamma_2 = Cay(G,T)$ are co$-$spectral. In fact,

$$Spec(\Gamma_1) = \begin{pmatrix} -8 & -5 & 0 & 4 & 16 & 40 \\ 25 & 128 & 81 & 100 & 25 & 1 \end{pmatrix} = Spec(\Gamma_2).$$

Example 2.7. Consider the cyclic group $G = \langle a \rangle$ of order n and $S = \{a, a^{-1}\}$. Then S is an MNS with the following spectrum:

$$Spec(Cay(G,S)) = \{w^j + w^{-j} \mid 0 \leq j \leq n-1\},$$

where $w = e^{\frac{2\pi i}{n}}$. The energy of this Cayley graph approximately is $\frac{4n}{\pi}$, see [9]. By a similar argument, one can prove $T = S \cup \{a^2, a^{-2}\}$ is an $SMNS$ for G, all eigenvalues of $Cay(G,T)$ are $4Cos(\frac{3\pi j}{n})Cos(\frac{\pi j}{n}))$, $0 \leq j \leq n-1$.

By method of subgroups of index 2 [13, p. 420], the group T_{4n} has exactly $n+3$ conjugacy classes $\{e\}$, $\{a^n\}$, $\{a^r, a^{-r}\}$ $(1 \leq r \leq n-1)$, $\{a^{2j}b \mid 0 \leq j \leq n-1\}$ and $\{a^{2j+1}b \mid 0 \leq j \leq n-1\}$. It is easy to see that T_{4n} has exactly four linear characters and $n-1$ non-linear irreducible characters recorded in Table 1.

Example 2.8. An MNS of T_{4n} can be computed as $S = (ab)^{T_{4n}} \cup b^{T_{4n}}$ and $S = a^{T_{4n}} \cup b^{T_{4n}}$, when n is odd and even, respectively. The simple eigenvalues of Cayley graph $Cay(T_{4n}, S)$ are $\{n+2, -n-2, n-2, -n+2\}$ and $\{2n, -2n\}$, for even and odd n, respectively. Moreover, if n is even then this graph has eigenvalues $2Cos(\frac{\pi j}{n})$ with multiplicity four, where $1 \leq j \leq n-1$. If n is odd, 0 is an eigenvalue of $Cay(T_{4n}, S)$ with multiplicity $4n-2$. To see this, it is easy to see that S is an $MNSG$ of T_{4n}. By Proposition 2.1, if n is even then $\lambda_{\chi_i} = \frac{1}{\chi_i(e)}(2\chi_i(a) + n\chi_i(b))$. If n is odd, then $\lambda_{\chi_i} = \frac{n}{\chi_i(e)}(\chi_i(b) + \chi_i(ab))$. So, by Table 1, one can easily compute the eigenvalues of $Cay(T_{4n}, S)$. If n is odd then $E(Cay(T_{4n}, S)) = 4n$ and if n is even then the energy of this Cayley graph is computed as follows:

$$\begin{aligned} E(Cay(T_{4n}, S)) &= 4n + 8\sum_{j=1}^{n-1} |Cos(\frac{\pi j}{n})| \\ &\approx 4n + \frac{8n}{\pi}\int_0^{\pi} |Cos(x)|dx - 8 = 4n + \frac{16n}{\pi} - 8. \end{aligned}$$

Therefore we have

$$E(Cay(T_{4n}, S))) \approx \begin{cases} 4n & if\ n\ is\ odd \\ 4n + \frac{16n}{\pi} - 8 & if\ n\ is\ even. \end{cases}$$

By a similar argument, one can see that $T = S \cup \{a^n\}$ is an $SMNS$ for T_{4n}. If n is odd then the simple eigenvalues of $Cay(T_{4n}, T)$ are $1 \pm 2n$. Other eigenvalues can be computed as -1 with multiplicity $2n$ and 1 with multiplicity $2n-2$. If n is even then the simple eigenvalues of $Cay(T_{4n}, T)$ are $3 \pm n$ and $-1 \pm n$. Other eigenvalues can be computed as $2Cos(\frac{\pi j}{n}) + (-1)^j$, $1 \leq j \leq n-1$, with multiplicity 4.

Example 2.9. The character table of V_{8n} computed in [13, 421], when n is odd. Darafsheh and Poursalavati [6] generalized this group in the case that n is even and computed its character table. This group has exactly $2n+3$ conjugacy classes, if n is odd, and $2n+6$ conjugacy classes, for even n. If n is odd then the conjugacy classes are: $\{1\}$, $\{b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}(0 \leq r \leq n-1)$, $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ $(1 \leq s \leq \frac{n-1}{2})$, $\{a^j b^k \mid j \textit{ is even} \ \& \ k = 1, 3\}$, $\{a^j b^k \mid j \textit{ is odd} \ \& \ k = 1, 3\}$. The conjugacy classes, for even n, are: $\{1\}$, $\{b^2\}$, $\{a^n\}$, $\{a^n b^2\}$, $\{a^{2r+1}, a^{-2r-1}b^2\}(0 \leq r \leq n-1)$, $\{a^{2s}, a^{-2s}\}$, $\{a^{2s}b^2, a^{-2s}b^2\}$ $(1 \leq s \leq \frac{n}{2} - 1)$, $\{a^{2k}b^{(-1)^k} \mid 0 \leq k \leq n-1\}$, $\{a^{2k}b^{(-1)^{k+1}} \mid 0 \leq k \leq n-1\}$, $\{a^{2k+1}b^{(-1)^k} \mid 0 \leq k \leq n-1\}$, $\{a^{2k+1}b^{(-1)^{k+1}} \mid 0 \leq k \leq n-1\}$. An MNS of V_{8n} can be computed at $S = a^{V_{8n}} \cup (a^{-1})^{V_{8n}} \cup b^{V_{8n}}$ and $S = a^{V_{8n}} \cup b^{V_{8n}} \cup (a^{-1})^{V_{8n}} \cup (b^{-1})^{V_{8n}}$, when n is odd and even, respectively. The simple eigenvalues of Cayley graphs $Cay(V_{8n}, S)$ are $\{\pm(2n+4), \pm(2n-4)\}$. Moreover these graphs have eigenvalues 0, with multiplicities $4n$ and $4Cos(\frac{\pi j}{n})$, $1 \leq j \leq n-1$, with multiplicities four. By our calculations, the energy of V_{8n} can be evaluated as follows:

$$E(Cay(V_{8n}, S)) \approx \begin{cases} 16 & n = 1 \\ 8n + \frac{32n}{\pi} - 16 & \textit{otherwise.} \end{cases}$$

A second minimal is $T = S \cup \{b^2\}$. The simple eigenvalues of $Cay(V_{8n}, T)$ are $5 \pm 2n$ and $-3 \pm 2n$. If n is even then other eigenvalues are -1 with multiplicity 4, 1 with multiplicity $4(n-1)$ and $4Cos(\frac{\pi j}{n}) + 1$, $1 \leq j \leq n-1$, each of which with multiplicity four. If n is odd then other eigenvalues are -1 with multiplicity $4n$ and $4Cos(\frac{\pi j}{n}) + 1$, $1 \leq j \leq n-1$, each of which with multiplicity four.

The group SD_{8n} is presented by $SD_{8n} = \langle a, b \mid a^{4n} = b^2 = 1 \ \& \ bab = a^{2n-1} \rangle$. In the following result the energy of the Cayley graph of this group with respect to its unique MNS is approximately computed.

Proposition 2.10. *Suppose S is an MNS of SD_{8n}. The energy of $E(Cay(SD_{8n}), S)$ can be evaluated as follows:*

$$E(Cay(SD_{8n}), S) \approx \begin{cases} \frac{32n}{\pi} + 4n & n = 1, 3 \\ 8n + \frac{32n}{\pi} - 16 & \textit{otherwise.} \end{cases}$$

Proof. All $8n$ elements of SD_{8n} may be given by $\{1, a, \ldots, a^{4n-1}, b, ba, \ldots, ba^{4n-1}\}$. Following Hormozi and Rodtes [11], we define $C^{even} = C_1 \cup C_2^{even} \cup C_3^{even}$ and $C^{odd} = C_1 \cup C_2^{odd} \cup C_3^{odd}$, where $C_1 = \{0, 2, 4, \ldots, 2n\}$, $C_2^{even} = \{1, 3, 5, \ldots, n-1\}$,

$C_3^{even} = \{2n+1, 2n+3, 2n+5, \ldots, 3n-1\}$, $C_2^{odd} = \{1, 3, 5, \ldots, n\}$, $C_3^{odd} = \{2n+1, 2n+3, 2n+5, \ldots, 3n\}$, $C_{even}^{\dagger} = C_1 \setminus \{0, 2n\}$ and $C_{odd}^{\dagger} = C_2^{even} \cup C_3^{even}$. Moreover, we assume that $C_{\star}^{even} = C^{even} \setminus \{0, 2n\}$ and $C_{\star}^{odd} = C^{odd} \setminus \{0, n, 2n, 3n\}$. Then by [11, Proposition 2.2], the conjugacy classes of SD_{8n}, $n \geq 2$, can be computed as follows:

- If n is even, there are $2n+3$ conjugacy classes as follows:
  - 2 classes of size one being $[1] = \{1\}$ and $[a^{2n}] = \{a^{2n}\}$,
  - $2n-1$ conjugacy classes of size two being $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_{\star}^{even}$,
  - 2 classes of size $2n$ being $[b] = \{ba^{2t} \mid 0 \leq t \leq 2n-1\}$ and $[ba] = \{ba^{2t+1} \mid 0 \leq t \leq 2n-1\}$.
- If n is odd, then there are $2n+6$ conjugacy classes as follows:
  - 4 classes of size one being $[1] = \{1\}$, $[a^n] = \{a^n\}$, $[a^{2n}] = \{a^{2n}\}$ and $[a^{3n}] = \{a^{3n}\}$,
  - $2n-2$ classes of size two being $[a^r] = \{a^r, a^{(2n-1)r}\}$, where $r \in C_{\star}^{odd}$,
  - 4 classes of size n being $[b] = \{ba^{4t} \mid 0 \leq t \leq n-1\}$, $[ba] = \{ba^{4t+1} \mid 0 \leq t \leq n-1\}$, $[ba^2] = \{ba^{4t+2} \mid 0 \leq t \leq n-1\}$ and $[ba^3] = \{ba^{4t+3} \mid 0 \leq t \leq n-1\}$.

An MNS of SD_{8n} can be computed as $S = (a)^{SD_{8n}} \cup (a^{-1})^{SD_{8n}} \cup b^{SD_{8n}}$. This graph has simple eigenvalues $\pm(4+2n)$ and $\pm(4-2n)$. If n is even then other eigenvalues are $4Cos(\frac{h\pi}{2n})$, where $h \in C_{even}^{\dagger}$, each of which with multiplicity four and 0 with multiplicity 4n. If n is odd, other eigenvalues are $4Cos(\frac{h\pi}{2n})$, where $h \in C_{even}^{\dagger}$, each of which with multiplicity four, $\pm n$ and 0 with multiplicities two and $4n-8$ respectively. By [19, Theorem 1], if n is even then $\lambda_{\chi_i} = \frac{2}{\chi_i(e)}(\chi_i(a) + \chi_i(a^{-1}) + n\chi_i(b))$. If n is odd, then $\lambda_{\chi_i} = \frac{1}{\chi_i(e)}(2\chi_i(a) + 2\chi_i(a^{-1}) + n\chi_i(b))$. By these calculations and a similar argument as Example 2.8, the energy of SD_{8n} can be computed as follows:

$$E(Cay(SD_{8n}), S) \approx \begin{cases} \frac{32n}{\pi} + 4n & n = 1, 3 \\ 8n + \frac{32n}{\pi} - 16 & otherwise. \end{cases}$$

which proves the result. $\square$

Example 2.11. In this example, the eigenvalues of $Cay(G, S)$ are computed, where G is a Mathieu group and S is an MNS or $SMNS$ for G. Since G is a simple group, $\langle A \rangle = G$, if A is a conjugacy class of G. So, if a real conjugacy class S of G has the minimum size between all conjugacy classes of G then S is an MNS for G. It is easy to see that for each Mathieu group, the conjugacy class $2A$ is real and it has the minimum size between all real conjugacy classes of G. The conjugacy class $2B$ in M_{12} and M_{24} is also real and has the second size between all real conjugacy classes of these groups and their union. The conjugacy class $3A$ in M_{11}, M_{22} and M_{23} is again real and has the second size between all real conjugacy classes of these groups and their union. These conjugacy classes are $SMNS$ of G and all eigenvalues with respect to these classes are recorded in Tables 2−6.

For the sake of completeness, we present here some details on the Suzuki group. Following Suzuki [16], a group G is called a $ZT-$group if G acts on a set Ω in such a way that, (1) G is a doubly transitive group on $1+N$ symbols, (2) the identity is the only element which leaves three distinct symbols invariant, (3) G contains no normal subgroup of order $1+N$, and (4) N is even. Suzuki [16] proved that for each prime power $q = 2^{2s+1}$, there is a unique $ZT-$group $Sz(q)$ of order $q^2(q-1)(q^2+1)$ which is called later the Suzuki group. This group is simple, when $q > 2$. Suppose that a is a symbol on which G acts and $H = G_a$. By [16], it follows from the conditions (1) and (2) that H is a Frobenius group on $\Omega \setminus \{a\}$. Apply a well-known result of Frobenius to deduce that H contains a regular normal subgroup Q of order N such that every non-identity element of Q leaves only the symbol a invariant. Suppose $b \in \Omega \setminus \{a\}$ and $K = H_b$. Suppose $x \in N_G(K)$ is an involution. Then it is well-known that the Suzuki group are containing two elements y and z such that y is an involution and $xyx = z^{-1}xz$, and three cyclic subgroups A_0, A_1 and A_2 of orders $q-1$, $q+r+1$ and $q-r+1$, respectively. By [16], the conjugacy classes of $Sz(q)$ can be computed as follows: $\{e\}$, $y^{Sz(q)}$, $z^{Sz(q)}$, $(z^{-1})^{Sz(q)}$, $b_0^{Sz(q)}$, $b_1^{Sz(q)}$ and $b_2^{Sz(q)}$ of lengths 1, $(q-1)(q^2+1)$, $\frac{1}{2}q(q-1)(q^2+1)$, $\frac{1}{2}q(q-1)(q^2+1)$, $q^2(q-1)(q+r+1)$, $q^2(q+r+1)(q-r+1)$ and $q^2(q-1)(q-r+1)$, respectively. Here, b_0, b_1 and b_2 are non-identity elements of A_i, $i = 0, 1, 2$, respectively. Note that there are $\frac{q-r}{4}$, $\frac{q}{2}-1$ and $\frac{q+r}{4}$ conjugacy classes of types $b_0^{Sz(q)}$, $b_1^{Sz(q)}$ and $b_2^{Sz(q)}$, respectively.

One can also find the character table of this group in [16]. Hence, by applying above information on Suzuki groups and Proposition 2.1 we have the following proposition:

Proposition 2.12. *Consider the Suzuki group $Sz(q)$ with $q = 2^{2s+1}$, $r = 2^{s+1}$ and $s \geq 1$. The conjugacy class $S = y^{Sz(q)}$ and the normal subset $T = z^{Sz(q)} \cup (z^{-1})^{Sz(q)}$ are the MNS and $SMNS$ of $Sz(q)$, respectively. Moreover, $|S| = (q-1)(q^2+1)$, $|T| = q(q-1)(q^2+1)$ and the simple eigenvalues of $Cay(Sz(q), S)$ and $Cay(Sz(q), T)$ are $|S|$ and $|T|$, respectively. The Cayley graph $Cay(Sz(q), S)$ has eigenvalues 0, $-(q^2+1)$, $(q-1)$, $\frac{(1+q^2)(r-1)}{q-r+1}$ and $\frac{-(1+q^2)(r+1)}{q+r+1}$ with multiplicities q^4, $\frac{(q-1)^2(r^2)}{2}$, $\frac{q-2}{2}(q^2+1)^2$, $\frac{q+r}{4}((q-r+1)(q-1))^2$ and $\frac{q-r}{4}((q+r+1)(q-1))^2$, respectively. The energy of $Cay(Sz(q), S)$ is as follows:*

$$
\begin{aligned}
E(Cay(Sz(q), S)) &= -\frac{\sqrt{2q}}{2} + \frac{\sqrt{2q^3}}{2} - \sqrt{2q^7} + \frac{\sqrt{2q^5}}{2} + \frac{3\sqrt{2q^9}}{2} - \frac{3\sqrt{2q^{11}}}{2} \\
&+ \frac{\sqrt{2q^{13}}}{2} - q^5 + \frac{\sqrt{q^2}}{2} - q^3 + q^4 + \frac{\sqrt{q^6}}{2}.
\end{aligned}
$$

The Cayley graph $Cay(Sz(q), S)$ *has eigenvalues* 0, $q(q-1)$, $-\frac{q(q^2+1)}{q-r+1}$ *and* $-\frac{q(q^2+1)}{q+r+1}$ *with multiplicities* $\frac{1}{4}(4q^4+r^2(q-1)^2)$, $\frac{1}{2}(q-2)(q^2+1)^2$, $\frac{1}{4}(q-r+1)^2(q-1)^2(q+r)$ *and* $\frac{1}{4}(q+r+1)^2(q-1)^2(q-r)$, *respectively.*

We end our paper by computing eigenvalues of a group of type Ree of characteristic q. We refer to [15, 16, 18] for our notations and known results concerning this important class of simple groups. A finite group G has Ree type if G satisfies the following conditions:

- The Sylow $2-$subgroups of G are elementary abelian of order 8.
- The group G has no normal subgroup of index 2.
- There is an involution J in G such that the centralizer $C_G(J) \cong \langle J\rangle \times LF(2,q)$, where $L = LF(2,q)$ denotes the linear fractional group on $GF(q)$.
- If $\langle R\rangle$ denotes a cyclic subgroup of order $\frac{q+e}{2}$ in L, then for any subgroup $1 \neq \langle R_0\rangle$, we have $N_G(\langle R_0\rangle) \leq C_G(J)$.
- Let J' be an involution of L and S an element of L of order $\frac{q-e}{4}$ which centralizes J'. Then an element of G of order 3 which normalizes $\langle J, J'\rangle$ does not centralize S. We call q the characteristic of G.

Note that the (I) implies that $q \equiv 4 + e \ (mod\ 8)$ where $e = \pm 1$. In the end of this paper, we consider the simple Ree group $G_2(q)$ of characteristic q and order $q^3(q-1)(q^3+1)$, where $q = 3^{2k+1}$ and $k \geq 1$. This group has exactly $q+8$ conjugacy classes [18]. Suppose $m = 3^k$. Then we have:

Proposition 2.13. *The conjugacy class* $S = J^{G_2(q)}$ *is the unique* MNS *of* $G_2(q)$ *with size* $\frac{q(q^2-q+1)}{q^2+1}$. *The energy of* $Cay(G_2(q), S)$ *can be computed by the following formula:*

$$\begin{aligned} E(Cay(G_2(q),S)) &= \frac{1}{4(1+3^{4k+2})}(19\cdot 3^{22k} + 4e3^{8k} + 3^{4k+3} + 3^{2k+1} + (2e-11)3^{21k} \\ &+ 2e3^{2k+1} + 3^{25k} + 280\cdot 3^{18k} + 3^{13k} - 4\cdot 3^{10k}) \\ &+ \frac{4}{(3^{2k+1}+3^{k+1}+1)(1+3^{4k+2})}(3^{22k} + 2\cdot 3^{19k} + 20\cdot 3^{16k} + 3^{8k} \\ &- 11\cdot 3^{10k} + 14\cdot 3^{13k}). \end{aligned}$$

Proof. Since $G_2(q)$ has exactly one conjugacy class of involutions, $S = S^{-1}$, and since $G_2(q)$ is simple, $G_2(q) = \langle S\rangle$. On the other hand, the lengths of non trivial conjugacy classes of $G_2(q)$ are as follows:

$$q^3(q^2-q+1), \frac{q^3(q-1)(q^3+1)}{q^2+1}, \frac{q^3(q-1)(q^3+1)}{q^2-3^{\frac{q+2}{2}}},$$
$$\frac{q^3(q-1)(q^3+1)}{q^2+3^{\frac{q+2}{2}}}, \frac{(q-1)(q^3+1)}{q^2}, \frac{q(q-1)(q^3+1)}{2},$$
$$\frac{(q-1)(q^3+1)}{2q}, \frac{q(q-1)(q^3+1)}{3}, \frac{q(q^2-q+1)}{q^2+1}.$$

By above information on conjugacy lengths of $G_2(q)$ and some tedious calculations, one can see that $S = J^{G_2(q)}$ is the unique MNS of $G_2(q)$. The simple eigenvalues of $Cay(G_2(q), S)$ is $|S|$. This graph has eigenvalues

$$0, -\tfrac{q}{q^2+1}, \tfrac{3q}{q^2+1}, \tfrac{q}{q^2+1},$$
$$\tfrac{q(q^2-q+1)}{m(q^2+1)(q+3m+1)}, \tfrac{q(q^2-q+1)}{m(q^2+1)(q-3m+1)}$$

with multiplicities, $3^{14k+6} - 3^{12k+5} - 5\cdot 3^{10k+4} - 7\cdot 3^{6k+2} + 2\cdot 3^{8k+3} - 3^{4k+1} - 3^{2k+1}$, $\frac{1}{8}\cdot 3^{14k+8} - \frac{11}{8}\cdot 3^{12k+6} + \frac{e}{4}\cdot 3^{12k+6} + \frac{5}{2}\cdot 3^{10k+5} - \frac{11}{4}\cdot 3^{8k+4} + \frac{7}{4}\cdot 3^{6k+3} - \frac{1}{2}\cdot 3^{4k+2} - \frac{1}{8}\cdot 3^{2k+1} + \frac{1}{8} + \frac{e}{2}\cdot 3^{6k+3} + \frac{e}{4}$, $\frac{1}{8}\cdot 3^{14k+6} - \frac{7}{8}\cdot 3^{12k+5} + \frac{5}{2}\cdot 3^{10k+4} - \frac{17}{4}\cdot 3^{8k+3} + \frac{19}{4}\cdot 3^{6k+2} - \frac{7}{2}\cdot 3^{4k+1} + \frac{13}{8}\cdot 3^{2k} - \frac{1}{8}$, $\frac{1}{4}3^{14k+7} - \frac{1}{2}\cdot 3^{12k+6} + \frac{e}{4}\cdot 3^{12k+6} + \frac{1}{2}\cdot 3^{8k+4} - 3^{6k+3} + \frac{e}{2}\cdot 3^{6k+3} + \frac{1}{4}\cdot 3^{2k+1} - \frac{1}{2} + \frac{e}{4}$, $2\cdot 3^{10k+4} + 4\cdot 3^{9k+4} + 2\cdot 3^{8k+4} - 4\cdot 3^{7k+3} - 16\cdot 3^{6k+2} - 4\cdot 3^{5k+2} + 2\cdot 3^{4k+2} + 4\cdot 3^{3k+1} + 2\cdot 3^{2k}$, $2\cdot 3^{10k+4} - 4\cdot 3^{9k+4} + 2\cdot 3^{8k+4} + 4\cdot 3^{7k+3} - 16\cdot 3^{6k+2} + 4\cdot 3^{5k+2} + 2\cdot 3^{4k+2} - 4\cdot 3^{3k+1} - 2\cdot 3^{2k}$, respectively. Therefore, the energy of $Cay(G_2(q), S)$ is as follows:

$$\begin{aligned} E(Cay(G_2(q),S)) &= \frac{1}{4(1+3^{4k+2})}(19\cdot 3^{22k} + 4e3^{8k} + 3^{4k+3} + 3^{2k+1} + (2e-11)3^{21k} \\ &+ 2e3^{2k+1} + 3^{25k} + 280\cdot 3^{18k} + 3^{13k} - 4\cdot 3^{10k}) \\ &+ \frac{4}{(3^{2k+1}+3^{k+1}+1)(1+3^{4k+2})}(3^{22k} + 2\cdot 3^{19k} + 20\cdot 3^{16k} + 3^{8k} \\ &- 11\cdot 3^{10k} + 14\cdot 3^{13k}). \end{aligned}$$

This completes the proof. $\square$

By the notation of [18] we have the following proposition.

Proposition 2.14. *The conjugacy class $T = X^{G_2(q)}$ is the unique $SMNS$ of $G_2(q)$ with size $\frac{(q-1)(q^3+1)}{q^2}$. The simple eigenvalues of $Cay(G_2(q), T)$ is $|T|$ and the graph has the eigenvalues*

$$0, \tfrac{-(q-1)^2(q^3+1)}{q^2(q^2-q+1)}, \tfrac{(q-1)(q^3+1)}{q^2(q^2-q+1)}, \tfrac{-(q+m)(q^3+1)}{mq^2(q+3m+1)}, \tfrac{(q+m)(q^3+1)}{mq^2(q-3m+1)},$$
$$\tfrac{-(q^3+1)}{q^2(q+1)}, \tfrac{q-1}{q^2}, \tfrac{(2q-1)(q^3+1)}{q^2(q^2-q+1)}, \tfrac{-(q+1+3m)(q^3+1)}{(q+1)(q+1+3m)q^2}, \tfrac{-(q+1-3m)(q^3+1)}{(q+1)(q+1-3m)q^2},$$

with multiplicities, q^6, $3^{8k+4}-2\cdot 3^{6k+3}+3^{4k+3}-2\cdot 3^{2k+1}+1$, $3^{12k+6}-2\cdot 3^{10k+5}+3^{8k+5}-2\cdot 3^{6k+3}$, $\frac{1}{2}\cdot 3^{10k+4}+\frac{1}{2}\cdot 3^{8k+4}+3^{9k+4}-4\cdot 3^{6k+4}-3^{7k+3}+\frac{1}{2}\cdot 3^{4k+2}-3^{5k+2}+\frac{1}{2}\cdot 3^{2k}+3^{3k+1}+3^{3k+1}$, $\frac{1}{2}\cdot 3^{10k+4}+3^{9k+4}+\frac{1}{2}\cdot 3^{8k+4}+3^{7k+3}-4\cdot 3^{6k+2}+3^{5k+2}+\frac{1}{2}\cdot 3^{4k+2}-3^{3k+1}+\frac{1}{2}\cdot 3^{2k}$, $2\cdot 3^{10k+4}-4\cdot 3^{6k+2}+2\cdot 3^{2k}$, $\frac{1}{2}\cdot 3^{14k+7}-\frac{1}{2}\cdot(1+\frac{e}{2})\cdot 3^{12k+6}+3^{8k+4}+(e-2)\cdot 3^{6k+3}+\frac{1}{2}\cdot 3^{2k+1}-1+\frac{e}{2}$, $\frac{1}{2}(3^{14k+6}-7\cdot 3^{12k+5}+20\cdot 3^{10k+4}-34\cdot 3^{8k+3}+38\cdot 3^{6k+2}-28\cdot 3^{4k+1}+13\cdot 3^{2k}-1)$, $\frac{1}{2}\cdot 3^{14k+6}-\frac{7}{2}\cdot 3^{12k+5}+10\cdot 3^{10k+4}-17\cdot 3^{8k+3}+19\cdot 3^{6k+2}-14\cdot 3^{4k+1}\frac{13}{2}\cdot 3^{2k}-\frac{1}{2}$, $\frac{1}{2}\cdot 3^{14k+6}+\frac{1}{2}\cdot 3^{13k+7}+\frac{11}{2}\cdot 3^{12k+5}+\frac{7}{2}\cdot 3^{11k+5}+\frac{5}{2}\cdot 3^{10k+4}-\frac{5}{2}\cdot 3^{9k+4}-11\cdot 3^{8k+3}-7\cdot 3^{3^{7k+3}}-\frac{13}{2}\cdot 3^{6k+2}+\frac{1}{2}\cdot 3^{5k+2}+\frac{11}{2}\cdot 3^{4k+1}+\frac{7}{2}\cdot 3^{3k+1}+\frac{7}{2}\cdot 3^{2k}+\frac{1}{2}\cdot 3^{k}$, $\frac{1}{2}\cdot 3^{14k+6}-\frac{1}{2}\cdot 3^{13k+6}-\frac{11}{2}\cdot 3^{12k+5}+\frac{1}{2}\cdot 3^{11k+6}-\frac{7}{2}\cdot 3^{10k+4}+\frac{1}{2}\cdot 3^{9k+5}+3^{8k+3}-3^{7k+4}+\frac{11}{2}\cdot 3^{6k+2}-\frac{1}{2}\cdot 3^{5k+3}-\frac{1}{2}\cdot 3^{4k+1}+\frac{1}{2}\cdot 3^{3k+2}-\frac{5}{2}\cdot 3^{2k}+\frac{1}{2}\cdot 3^{k}$, *respectively.*

Table 1: The Character Table of T_{4n}.

g_i	e	a^n	a^r $(1 \le r \le n-1)$	b	ab
$\|C_G(g_i)\|$	$4n$	$4n$	$2n$	4	4
Non-Linear Characters					
ψ_j $(1 \le j \le n-1)$	2	$2(-1)^j$	$w^{rj}+w^{-rj}$	0	0
Linear Characters			*n is odd*		
χ_1	1	1	1	1	1
χ_2	1	-1	$(-1)^r$	i	$-i$
χ_3	1	1	1	-1	-1
χ_4	1	-1	$(-1)^r$	$-i$	i
Linear Characters			*n is even*		
λ_1	1	1	1	1	1
λ_2	1	1	1	-1	-1
λ_3	1	1	$(-1)^r$	1	-1
λ_4	1	1	$(-1)^r$	-1	1

Table 2: The Eigenvalues of $Cay(M_{11}, 2A)$ and $Cay(M_{11}, 3A)$

$2A-Eigenvalues$	$Multiplicities$	$3A-Eigenvalues$	$Multiplicities$
-33	$2^3\cdot 5^2$	-55	2^9
-11	$3^4\cdot 5^2$	-10	$2^4\cdot 11^2$
-3	$5^2\cdot 11^2$	0	$3^4\cdot 5^2$
0	2^9	8	$5^2\cdot 11^2$
15	$2^4\cdot 11^2$	44	$2^2\cdot 3\cdot 5^2$
33	$2^2\cdot 5^2$	80	11^2
45	11^2	440	1
165	1	–	–

Table 3: The Eigenvalues of $Cay(M_{12}, 2A)$ and $Cay(M_{12}, 2B)$.

$2A - Eigenvalues$	$Multiplicities$	$2B - Eigenvalues$	$Multiplicities$
-36	$7 \cdot 11^3$	-33	$3^2 \cdot 5^2 \cdot 73$
-9	$2^8 \cdot 11^2$	-9	$2 \cdot 5^2 \cdot 11^2$
-4	$3^4 \cdot 11^2$	0	$3^4 \cdot 5^2$
0	$2^6 \cdot 3^2 \cdot 5^2$	15	$3^2 \cdot 11^2 \cdot 13$
11	$2^8 \cdot 3^4$	55	$2^2 \cdot 3^6$
36	$2^2 \cdot 3^2 \cdot 11^2$	63	$5^2 \cdot 11^2$
44	$3^4 \cdot 61$	135	$2 \cdot 11^2$
99	2^9	495	1
396	1	$-$	$-$

Table 4: The Eigenvalues of $Cay(M_{22}, 2A)$ and $Cay(M_{22}, 3A)$.

$2A - Eigenvalues$	$Multiplicities$	$3A - Eigenvalues$	$Multiplicities$
-77	$2 \cdot 3^4 \cdot 5^2$	-160	$3^2 \cdot 7^2 \cdot 11^2$
-33	$2^7 \cdot 5^2 \cdot 7^2$	-64	$5^2 \cdot 7^2 \cdot 11^2$
3	$5^2 \cdot 7^2 \cdot 11^2$	0	$3^6 \cdot 19$
11	$2^2 \cdot 3^2 \cdot 5^2 \cdot 7^2$	44	$2^7 \cdot 5^2 \cdot 7^2$
35	$3^4 \cdot 11^2$	80	$2^2 \cdot 7^2 \cdot 11^2$
75	$2^2 \cdot 7^2 \cdot 11^2$	224	$5^2 \cdot 11^2$
147	$5^2 \cdot 11^2$	1760	$3^2 \cdot 7^2$
275	$3^2 \cdot 7^2$	12320	1
1155	1	$-$	$-$

Table 5: The Eigenvalues of $Cay(M_{23}, 2A)$ and $Cay(M_{23}, 3A)$.

$2A - Eigenvalues$	$Multiplicities$	$3A - Eigenvalues$	$Multiplicities$
-253	$2 \cdot 3^4 \cdot 5^2$	-736	$2 \cdot 3^2 \cdot 7^2 \cdot 11^2$
-69	$2^4 \cdot 5^3 \cdot 11^2 \cdot 13$	-253	$2^{15} \cdot 7^2$
0	$2^1 5 \cdot 7^2$	-28	$2^6 \cdot 11^2 \cdot 23^2$
15	$2^6 \cdot 11^2 \cdot 23^2$	0	$3^4 \cdot 5^2 \cdot 1499$
99	$3^4 \cdot 5^2 \cdot 23^2$	224	$11^2 \cdot 23^2$
115	$3^3 \cdot 7^2 \cdot 11^2$	368	$2^3 \cdot 5^2 \cdot 7^2 \cdot 11^2$
195	$11^2 \cdot 23^2$	1232	$2^2 \cdot 5^2 \cdot 23^2$
363	$2^2 \cdot 5^2 \cdot 23^2$	1472	$3^2 \cdot 7^2 \cdot 11^2$
1035	$2^2 \cdot 11^2$	10304	$2^2 \cdot 11^2$
3795	1	56672	1

Table 6: The Eigenvalues of $Cay(M_{24}, 2A)$ and $Cay(M_{24}, 2B)$.

$2A - Eigenvalues$	$Multiplicities$	$2B - Eigenvalues$	$Multiplicities$
-759	$2 \cdot 3^4 \cdot 5^2$	-1386	$2 \cdot 23^2 \cdot 61$
-231	$2 \cdot 3^4 \cdot 5^2 \cdot 23^2$	-1242	$2 \cdot 3^2 \cdot 7^2 \cdot 11^2$
-207	$2^4 \cdot 5^3 \cdot 11^2 \cdot 13$	-378	$5^2 \cdot 11^2 \cdot 23^2$
-135	$7^2 \cdot 11^2 \cdot 23^2$	-322	$2^3 \cdot 3^4 \cdot 5^2 \cdot 11^2$
-115	$2^6 \cdot 3^4 \cdot 7^2 \cdot 11^2$	-266	$3^4 \cdot 11^2 \cdot 23^2$
-55	$2^4 \cdot 3^4 \cdot 7^2 \cdot 23^2$	-154	$2^5 \cdot 5^2 \cdot 23^2$
-23	$3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2$	-138	$3^6 \cdot 5^2 \cdot 7^2 \cdot 11^2$
45	$2^6 \cdot 11^2 \cdot 23^2$	0	$2^{12} \cdot 5^2 \cdot 11^2$
105	$2^3 \cdot 3^2 \cdot 11 \cdot 23 \cdot 29 \cdot 2783$	54	$3^2 \cdot 7^2 \cdot 11^2 \cdot 23^2$
165	$2^8 \cdot 3^4 \cdot 23^2$	138	$2^6 \cdot 3^4 \cdot 7^2 \cdot 11^2$
207	$2^{12} \cdot 5^2 \cdot 11^2$	154	$2^8 \cdot 3^4 \cdot 23^2$
345	$2 \cdot 3^2 \cdot 7^2 \cdot 11^2$	198	$2^2 \cdot 5^2 \cdot 23^2$
297	$3^4 \cdot 5^2 \cdot 23^2$	378	$2^6 \cdot 11^2 \cdot 23^2$
585	$11^2 \cdot 23^2$	414	$2^3 \cdot 5^2 \cdot 7^2 \cdot 11^2$
297	$3^4 \cdot 5^2 \cdot 23^2$	1078	$2^4 \cdot 5^2 \cdot 23^2$
441	$5^2 \cdot 11^2 \cdot 23^2$	1518	$2^4 \cdot 3^4 \cdot 7^2$
585	$11^2 \cdot 23^2$	3542	$3^4 \cdot 5^2$
1265	$2^4 \cdot 3^4 \cdot 7^2$	$-$	$-$
3465	23^2	$-$	$-$
11385	1	$-$	$-$

Acknowledgement. The author is indebted to professor Marty Isaacs for some discussion on this paper and for giving the counterexample given Example 2.6. This paper is a part of my PhD thesis supervised under the direction of Professor Ali Reza Ashrafi at the University of Kashan.

References

[1] R. C. Alperin, Rational subsets of finite groups, *Int. J. Group Theory* **3**(2) (2014) 53–55.

[2] R. C. Alperin, B. L. Peterson, Integral sets and Cayley graphs of finite groups, *Electron. J. Combin.* **19** (2012) #P44 1–12.

[3] H. Behravesh, Quasi−permutation representation of Suzuki group, *J. Sci. I. R. Iran* **10**(1) (1999) 53–56.

[4] N. Biggs, *Algebraic Graph Theory*, *Cambridge University Press*, Cambridge, 1974.

[5] J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, R. A. Wilson, *Atlas of Finite Groups*, Oxford Univ. Press, Clarendon, Oxford, 1985.

[6] M. R. Darafsheh, N. S. Poursalavati, On the existence of the orthogonal basis of the symmetry classes of tensors associated with certain groups, *SUT J. Math.* **37**(1) (2001) 1–17.

[7] P. Diaconis, M. Shahshahani, Generating a Random Permutation with Random Transpositions, *Z. Wahr. Verw. Gebiete* **57**(2) (1981) 159–179.

[8] M. Ghorbani, On the eigenvalues of normal edge-transitive Cayley graphs, *Bull. Iranian Math. Soc.* **41**(1) (2015) 101–107

[9] I. Gutman, A graph theoretical study of conjugated systems containing a linear polyence fragment, *Croat. Chem. Acta* **48**(2) (1976) 97–108.

[10] I. Gutman, The energy of a graph, 10. Steiermärkisches Mathematisches Symposium (Stift Rein, Graz, 1978). *Ber. Math.-Statist. Sekt. Forsch. Graz* **103** (1978) 22 pp.

[11] M. Hormozi, K. Rodtes, Symmetry classes of tensors associated with the semi-dihedral groups SD_{8n}, *Colloq. Math.* **131**(1) (2013) 59–67.

[12] I. M. Isaacs, *Character Theory of Finite Groups*, *Dover, New-York*, 1976.

[13] G. James, M. Liebeck, *Representations and Characters of Groups*, Cambridge Univ. Press, London-New York, 1993.

[14] M. Ram Murty, Ramanujan Graphs, *J. Ramanujan Math. Soc.* **18**(1) (2003) 1–20.

[15] R. Ree, A family of simple groups associated with the simple Lie algebra of type (G_2), *Amer. J. Math.* **83** (1961) 432–462.

[16] M. Suzuki, On a class of doubly transitive groups, *Annals Math.* **75**(1) (1962) 105–145.

[17] The GAP Team, *GAP-Groups, Algorithms and Programming*, Lehrstul D Für Mathematik, RWTH, Aachen, 1995.

[18] H. N. Ward, On Ree's series of simple groups, *Trans. Amer. Math. Soc.* **121**(1) (1966) 62–89.

[19] P. -H. Zieschang, Cayley graphs of finite groups, *J. Algebra* **118** (1988) 447–454.

Maryam Jalali-Rad
Department of Pure Mathematics,
Faculty of Mathematical Sciences,
University of Kashan,
Kashan 87317-53153, I. R. Iran