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Laplacian Sum-Eccentricity Energy of a Graph

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Abstract

We introduce the Laplacian sum-eccentricity matrix $\mathbf{LS_e}$ of a graph G, and its Laplacian sum-eccentricity energy $LS_eE = \sum_{i=1}^{n} |\eta_i|$, where $\eta_i = \zeta_i - \frac{2m}{n}$ and where $\zeta_1, \zeta_2, \ldots, \zeta_n$ are the eigenvalues of $\mathbf{LS_e}$. Upper bounds for LS_eE are obtained. A graph is said to be twinenergetic if $\sum_{i=1}^{n} |\eta_i| = \sum_{i=1}^{n} |\zeta_i|$. Conditions for the existence of such graphs are established.

Keywords: Sum-eccentricity eigenvalues, sum-eccentricity energy, Laplacian sum-eccentricity matrix, Laplacian sum-eccentricity energy.

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1. Introduction

Let G be a simple connected graph with vertex set $\mathbf{V}(G)$ and edge set $\mathbf{E}(G)$, of order $|\mathbf{V}(G)| = n$ and size $|\mathbf{E}(G)| = m$. Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of G. The eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $\mathbf{A}$ are the eigenvalues of the graph G [6]. Since $\mathbf{A}$ is a symmetric matrix with zero trace, these eigenvalues are real with sum equal to zero. The energy of the graph G is defined as the sum of the absolute values of its eigenvalues [10, 16]:

$$E(G) = \sum_{i=1}^{n} |\lambda_i| \,.$$

After the introduction of the graph–energy concept in the 1970s [10], several other "graph energies" have been put forward and their mathematical properties extensively studied; for details see the recent monograph [12] and the survey [11].

In the last few years, a whole class of graph energies was conceived, based on the eigenvalues of matrices associated with a particular topological index. Thus, let TI be a topological index is of the form

$$TI = TI(G) = \sum_{v_i v_j \in \mathbf{E}(G)} F(v_i, v_j)$$

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where F is a pertinently chosen function with the property $F(x,y) = F(y,x)$. Then a matrix **TI** can be associated to TI, defined as

$$(\mathbf{TI})_{ij} = \begin{cases} F(v_i, v_j) & \text{if } v_i v_j \in \mathbf{E}(G) \\ 0 & \text{otherwise}. \end{cases}$$

If $\tau_1, \tau_2, \ldots, \tau_n$ are the eigenvalues of the matrix **TI**, then an "*energy*" can be defined as

$$E_{TI} = E_{TI}(G) = \sum_{i=1}^{n} |\tau_i| \,. \tag{1}$$

The most extensively studied such graph energy is the *Randić energy* [2,3,7,12], based on the eigenvalues of the Randić matrix **R**, where

$$(\mathbf{R})_{ij} = \begin{cases} \frac{1}{\sqrt{d_i\, d_j}} & \text{if } v_i v_j \in \mathbf{E}(G) \\ 0 & \text{otherwise} \end{cases}$$

and where d_i is the degree of the i-th vertex of G. In an analogous manner the harmonic energy [14], ABC energy [9], geometric–arithmetic energy [23], Zagreb energy [15], and sum-eccentricity energy [22, 26] were put forward.

To any energy E_{TI} of the form (1), a "Laplacian energy" LE_{TI} can be associated, defined as

$$LE_{TI} = LE_{TI}(G) = \sum_{i=1}^{n} \left| \theta_i - \frac{2m}{n} \right| \tag{2}$$

where $\theta_1, \theta_2, \ldots, \theta_n$ are the eigenvalues of the matrix $\mathbf{LTI} = \mathbf{D} - \mathbf{TI}$, and where $\mathbf{D} = \mathbf{D}(G)$ is the diagonal matrix of vertex degrees.

The first such Laplacian energy, based on the adjacency matrix **A**, was introduced in 2006 [13] and its theory is nowadays elaborated in full detail, see [12]. It is worth noting that this Laplacian energy found interesting engineering applications in image processing [18, 25, 27]. Bearing this in mind, it is purposeful to study other Laplacian graph energies. Some recent studies along these lines are [1, 5, 8, 20, 21].

In this paper we study the Laplacian version of the sum-eccentricity energy. In order to define it, we need some preparations.

The distance $d(u,v)$ between two vertices u and v in a (connected) graph G is the length of a shortest path connecting u and v [4]. The eccentricity of a vertex $v \in \mathbf{V}(G)$ is $e(v) = max\{d(u,v) : u \in \mathbf{V}(G)\}$. The radius of G is $r(G) = min\{e(v) : v \in \mathbf{V}(G)\}$, whereas the diameter of G is $d(G) = max\{e(v) : v \in \mathbf{V}(G)\}$. Hence $r(G) \leq e(v) \leq d(G)$, for every $v \in \mathbf{V}(G)$.

In this paper, we denote by K_n, $K_{a,b}$, $K_{1,a}$, C_n, and P_n the complete graph, complete bipartite graph, star, cycle, and path, respectively.

The sum-eccentricity matrix of a graph G is denoted by $\mathbf{S_e}(G)$ and defined as $\mathbf{S_e}(G) = (s_{ij})$ [22, 26], where

$$s_{ij} = \begin{cases} e(v_i) + e(v_j) & if\ v_iv_j \in E \\ 0 & \text{otherwise.} \end{cases}$$

If $\mu_1, \mu_2, \ldots, \mu_n$, are the eigenvalues of $\mathbf{S_e}(G)$, then the sum-eccentricity energy is

$$ES_e(G) = \sum_{i=1}^{n} |\mu_i| .$$

Definition 1.1. Let G be a graph of order n and size m. The Laplacian sum-eccentricity matrix of G, denoted by $\mathbf{LS_e}(G) = (\ell_{ij})$, is defined as

$$\mathbf{LS_e}(G) = \mathbf{D}(G) - \mathbf{S_e}(G) .$$

The Laplacian sum-eccentricity spectrum of G, consisting of $\zeta_1, \zeta_2, \ldots, \zeta_n$, is the spectrum of the Laplacian sum-eccentricity matrix. This leads us to define the Laplacian sum-eccentricity energy of a graph G as

$$LS_e(G) = \sum_{i=1}^{n} \left| \zeta_i - \frac{2m}{n} \right| . \tag{3}$$

If, in addition, we define the auxiliary quantity η_i as

$$\eta_i = \zeta_i - \frac{2m}{n}$$

then

$$LS_eE(G) = \sum_{i=1}^{n} |\eta_i| .$$

Lemma 1.2. *Let G be an (n, m)-graph. Then*

$$\sum_{i=1}^{n} \zeta_i = 2m .$$

Proof.

$$\sum_{i=1}^{n} \zeta_i = trace(\mathbf{LS_e}(G)) = \sum_{i=1}^{n} \ell_{ii} = \sum_{i=1}^{n} d_i = 2m .$$

□

Theorem 1.3. *The Laplacian sum-eccentricity energy of the complete graph K_n is*

$$LS_eE(K_n) = 4(n-1) .$$

Proof. Recalling that the eccentricity of any vertex of K_n is unity, directly from the definition of the Laplacian sum-eccentricity matrix, we calculate that

$$Spec(\mathbf{LS_e}(K_n)) = \begin{bmatrix} -d & d+2 \\ 1 & n-1 \end{bmatrix}$$

where $d = n-1$ is the degree of any vertex of K_n. Using the fact that $\frac{2m}{n} = n-1 = d$, we get by Eq. (3)

$$\begin{aligned} LS_eE(K_n) &= |-d-d| + |d+2-d| + \cdots + |d+2-d| \\ &= 2d + 2(n-1) = 4(n-1)\,. \end{aligned}$$

□

2. Bounds for Laplacian Sum-Eccentricity Energy

Theorem 2.1. *Let G be an (n,m)-graph. Then*

$$LS_eE(G) \leq \sqrt{n\left(\sum_{i=1}^{n}\sum_{j=1}^{n}\ell_{ij}^2 - \frac{4m^2}{n}\right)}\,. \tag{4}$$

Proof. We have

$$\begin{aligned} \sum_{i=1}^{n}\eta_i^2 &= \sum_{i=1}^{n}\left(\zeta_i - \frac{2m}{n}\right)^2 = \sum_{i=1}^{n}\left(\zeta_i^2 - \frac{4m}{n}\zeta_i + \frac{4m^2}{n^2}\right) \\ &= \sum_{i=1}^{n}\zeta_i^2 - \frac{4m}{n}\sum_{i=1}^{n}\zeta_i + \frac{4m^2}{n}\,. \end{aligned}$$

By Lemma 1.2,

$$\sum_{i=1}^{n}\eta_i^2 = \sum_{i=1}^{n}\zeta_i^2 - \frac{8m^2}{n} + \frac{4m^2}{n} = \sum_{i=1}^{n}\zeta_i^2 - \frac{4m^2}{n}\,.$$

Using the Cauchy–Schwarz inequality

$$\sum_{i=1}^{n}|\eta_i| \leq \sqrt{n\sum_{i=1}^{n}\eta_i^2}$$

we get

$$\sum_{i=1}^{n}|\eta_i| \leq \sqrt{n\left(\sum_{i=1}^{n}\zeta_i^2 - \frac{4m^2}{n}\right)}.$$

On the other hand,

$$\sum_{i=1}^{n} \zeta_i^2 = trace(\mathbf{LS_e^2}(G)) = \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_{ij}^2$$

and inequality (4) follows. □

It should be noted that inequality (4) is just a variant of the classical McClelland's upper bound for ordinary graph energy [16, 17].

Corollary 2.2. *Let G be an r-regular graph. Then*

$$\sum_{i=1}^{n} \eta_i^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \ell_{ij}^2 - nr^2 \,.$$

Example 2.3. If $G \cong K_n$, then

$$\sum_{i=1}^{n} \eta_i^2 = 4n(n-1)\,.$$

If $G \cong K_{a,b}$. Then

$$\sum_{i=1}^{n} \eta_i^2 = ab\left(a+b+32-\frac{4ab}{a+b}\right)\,.$$

In particular, for $G \cong K_{1,a}$, with $n = a+1$:

$$\sum_{i=1}^{n} \eta_i^2 = a^2\left(1-\frac{4}{a+1}\right) + 19a\,.$$

In what follows we derive another upper bound for the Laplacian sum-eccentricity energy using Weyl's inequality for matrices.

Theorem 2.4. *(Weyl's inequality)* [19] *Let $\mathbf{X}$ and $\mathbf{Y}$ be Hermitian $n \times n$ matrices. If for $1 \le i \le n$, $\lambda_i(\mathbf{X}), \lambda_i(\mathbf{Y}), \lambda_i(\mathbf{X}+\mathbf{Y})$ are the eigenvalues of $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{X}+\mathbf{Y}$, respectively, then*

$$\lambda_i(\mathbf{X}) + \lambda_n(\mathbf{Y}) \le \lambda_i(\mathbf{X}+\mathbf{Y}) \le \lambda_i(\mathbf{X}) + \lambda_1(\mathbf{Y})\,.$$

The matrices $\mathbf{LS_e}(G)$, $\mathbf{S_e}(G)$, and $\mathbf{D}(G)$ are all Hermitian $n \times n$ matrices. In addition, we use the facts that the eigenvalues of the diagonal matrix are the entries in the diagonal, and that the energy of a matrix $\mathbf{X}$ is equal to the energy of $-\mathbf{X}$. We thus arrive at:

Theorem 2.5. *Let G be an (n,m)-graph with maximal vertex degree Δ. Then*

$$LS_eE(G) \le ES_e(G) + k\,\Delta + \frac{2m}{n}(n-k) \tag{5}$$

where $k = |\{\zeta_i : \zeta_i \ge 2m/n\}|$.

Proof. Let $\zeta_1 \geq \zeta_2 \geq \cdots \geq \zeta_n$ be the Laplacian sum-eccentricity eigenvalues, $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n$ be the sum-eccentricity eigenvalues and $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_n$ be the eigenvalues of the degree matrix. We assume that $1 \leq k \leq r \leq n$. Using Theorem 2.4 we get

$$\mu_i + \rho_n \leq \zeta_i \leq \mu_i + \rho_1 \,.$$

Since $\rho_n \geq 0$,

$$\mu_i \leq \zeta_i \leq \mu_i + \rho_1 \,.$$

Since $2m/n \geq 0$,

$$\mu_i - \frac{2m}{n} \leq \zeta_i - \frac{2m}{n} \leq \mu_i + \rho_1 \,.$$

Now we have to distinguish between two cases.

Case 1: If $\zeta_i - \frac{2m}{n} \geq 0$, then

$$\left|\zeta_i - \frac{2m}{n}\right| \leq \mu_i + \rho_1 \,.$$

If there are k ζ_i's, satisfy this condition, then

$$\sum_{i=1}^{k} \left|\zeta_i - \frac{2m}{n}\right| \leq \sum_{i=1}^{k} (\mu_i + \rho_1) \leq \sum_{i=1}^{k} |\mu_i| + k\rho_1 \,. \tag{6}$$

Case 2: If $\zeta_i - \frac{2m}{n} \leq 0$, then

$$\left|\zeta_i - \frac{2m}{n}\right| \leq \left|\mu_i - \frac{2m}{n}\right| .$$

If we have, $\mu_i \leq 0$ for $i = k+1, \ldots, r$ and $\mu_i \geq 0$ for $i = r+1, \ldots, n$. Then

$$\begin{aligned} \sum_{i=k+1}^{n} \left|\zeta_i - \frac{2m}{n}\right| &\leq \sum_{i=k+1}^{r} \left|\mu_i - \frac{2m}{n}\right| + \sum_{i=r+1}^{n} \left|\mu_i - \frac{2m}{n}\right| \\ &= \sum_{i=k+1}^{r} \frac{2m}{n} + \sum_{i=k+1}^{r} |\mu_i| + \sum_{i=r+1}^{n} \frac{2m}{n} - \sum_{i=r+1}^{n} |\mu_i| \,. \end{aligned} \tag{7}$$

Combining the relations (6) and (7), we get

$$\begin{aligned} \sum_{i=1}^{n} \left|\zeta_i - \frac{2m}{n}\right| &\leq \sum_{i=1}^{k} |\mu_i| + k\rho_1 + \sum_{i=k+1}^{r} |\mu_i| - \sum_{i=r+1}^{n} |\mu_i| + \frac{2m}{n}(n-k) \\ &\leq ES_e(G) + k\rho_1 + \frac{2m}{n}(n-k) \end{aligned}$$

from which (5) follows straightforwardly. □

Corollary 2.6. *If the graph G is r-regular, then*

$$LS_eE(G) \leq ES_e(G) + nr\,. \tag{8}$$

Proof. From Theorem 2.5, we have

$$LS_eE(G) \leq ES_e(G) + kr + \frac{2m}{n}(n-k)\,.$$

Since, in addition, for an r-regular graph, $2m/n = r$,

$$LS_eE(G) \leq ES_e(G) + r(k+n-k)$$

and inequality (8) follows. □

Lemma 2.7. [24] *For the complete bipartite graph $K_{a,b}$, the sum-eccentricity energy is $ES_e(K_{a,b}) = 8\sqrt{ab}$.*

Corollary 2.8. *For the complete bipartite graph $K_{a,b}$,*

$$LS_eE(K_{a,b}) \leq 8\sqrt{ab} + k\,\max\{a,b\} + \frac{2ab}{a+b}(a+b-k)\,. \tag{9}$$

Proof. For $K_{a,b}$, $2m/n = 2ab/(a+b)$. Using Lemma 2.7, we get (9) from (5). □

3. Twinenergetic Graphs

In this section, we point out a remarkable feature of Laplacian sum-eccentricity energy.

Definition 3.1. Let G be a graph of order n, and let ζ_i , $i = 1, 2, \ldots, n$, be its Laplacian sum-eccentricity eigenvalues. We say that G is twinenergetic if

$$LS_eE(G) = \sum_{i=1}^{n} |\zeta_i|\,.$$

The above definition means that

$$\sum_{i=1}^{n} \left| \zeta_i - \frac{2m}{n} \right| = \sum_{i=1}^{n} |\zeta_i|\,. \tag{10}$$

The number of positive eigenvalues and negative eigenvalues (including their multiplicities) are denoted by $\zeta^+(G)$ and $\zeta^-(G)$, respectively. For the sake of simplicity, we assume that there are no zero Laplacian sum-eccentricity eigenvalues, i.e., that $\zeta^+(G) + \zeta^-(G) = n$.

Theorem 3.2. *A graph G is a Laplacian sum-eccentricity twinenergetic if it satisfies the following two conditions:*

$$\zeta_i(G) \geq \frac{2m}{n}\ ,\ i=1,2,\ldots,\zeta_i^+(G)\,. \tag{11}$$

$$\zeta^+(G) = \zeta^-(G)\,. \tag{12}$$

Proof. Let $\zeta^+(G)=r$, where $1 \leq r \leq n$. Then

$$\begin{aligned}\sum_{i=1}^{n}\left|\zeta_i-\frac{2m}{n}\right| &= \sum_{i=1}^{r}\left|\zeta_i-\frac{2m}{n}\right|+\sum_{i=r+1}^{n}\left|\zeta_i-\frac{2m}{n}\right| \\ &= \sum_{i=1}^{r}\left|\zeta_i-\frac{2m}{n}\right|+\sum_{i=r+1}^{n}|\zeta_i|+\frac{2m}{n}(n-r)\,. \end{aligned}\tag{13}$$

Let k be the number of eigenvalues satisfying the condition $\zeta_i(G) \geq \frac{2m}{n}$. Then, $1 \leq k \leq r$, and

$$\begin{aligned}\sum_{i=1}^{r}\left|\zeta_i-\frac{2m}{n}\right| &= \sum_{i=1}^{k}\left(\zeta_i-\frac{2m}{n}\right)+\sum_{i=k+1}^{r}\left(\frac{2m}{n}-\zeta_i\right) \\ &= \sum_{i=1}^{k}|\zeta_i|-\frac{2m}{n}\,k-\sum_{i=k+1}^{r}|\zeta_i|+\frac{2m}{n}(r-k)\,. \end{aligned}\tag{14}$$

Substituting (14) back into (13) yields

$$\begin{aligned}\sum_{i=1}^{n}\left|\zeta_i-\frac{2m}{n}\right| &= \sum_{i=1}^{k}|\zeta_i|+\sum_{i=r+1}^{n}|\zeta_i|-\sum_{i=k+1}^{r}|\zeta_i|+\frac{2m}{n}(n-2k) \\ &= \sum_{i=1}^{n}|\zeta_i|-2\sum_{i=k+1}^{r}|\zeta_i|+\frac{2m}{n}(n-2k)\,. \end{aligned}$$

If the condition (11) is obeyed, i.e., if $k=r$, then

$$\sum_{i=k+1}^{r}|\zeta_i|=0\,.$$

If, in addition, also the condition (12) is obeyed, i.e., $2r=n$, then

$$\frac{2m}{n}(n-2k)=0\,.$$

Thus, if both conditions (11) and (12) are satisfied, then the relation (10) holds, i.e., the graph G is twinenergetic. $\square$

It now remains to see if twinenergetic graphs exist at all. That such graphs do exist is verified by the following examples.

Example 3.3. The paths P_2, P_4, P_8, and P_{16} are twinenergetic graphs.

For P_2, direct calculations gives $\zeta_1 = 3$, $\zeta_2 = -1$, and $2m/n = 1$. Therefore,

$$\sum_{i=1}^{2} |\zeta_i| = 4 \qquad \text{and} \qquad \sum_{i=1}^{2} \left| \zeta_i - \frac{2m}{n} \right| = 4\,.$$

For P_4 we get $\zeta_1 = 9.0902$, $\zeta_2 = 4.7202$, $\zeta_3 = -2.0902$, $\zeta_4 = -5.7202$, and $2m/n = 6/4$. Therefore,

$$\sum_{i=1}^{4} |\zeta_i| = \sum_{i=1}^{4} \left| \zeta_i - \frac{2m}{n} \right| = 21.6208\,.$$

The cases P_8 and P_{16} are verified analogously.

Finding more examples of twinenergetic graphs, as well as their complete structural characterization remains a task for the future.

References

[1] P. G. Bhat, S. D'Souza, Color signless Laplacian energy of graphs, *AKCE Int. J. Graphs Comb.* **14** (2017) 142–148.

[2] Ş. B. Bozkurt, A. D. Güngör, I. Gutman, Randić spectral radius and Randić energy, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 321–334.

[3] Ş. B. Bozkurt, A. D. Güngör, I. Gutman, A. S Çevik, Randić matrix and Randić energy, *MATCH Commun. Math. Comput. Chem.* **64** (2010) 239–250.

[4] F. Buckley, F. Harary, *Distance in Graphs*, Addison–Wesley, Redwood, 1990.

[5] A. Cafure, D. A. Jaume, L. N. Grippo, A. Pastine, M. D. Safe, V. Trevisan, I. Gutman, Some results for the (signless) Laplacian resolvent, *MATCH Commun. Math. Comput. Chem.* **77** (2017) 105–114.

[6] D. Cvetković, P. Rowlinson, S. Simić, *An Introduction to the Theory of Graph Spectra*, Cambridge Univ. Press, Cambridge, 2010.

[7] K. C. Das, S. Sorgun, K. Xu, On Randić energy of graphs, in: I. Gutman, X. Li (Eds.), *Graph Energies – Theory and Applications*, Univ. Kragujevac, Kragujevac, 2016, pp. 111–122.

[8] N. De, On eccentricity version of Laplacian energy of a graph, *Math. Interdisc. Res.* **2** (2017) 131–139.

[9] E. Estrada, The ABC matrix, *J. Math. Chem.* **55** (2017) 1021–1033.

[10] I. Gutman, The energy of a graph, *Ber. Math. Stat. Sekt. Forschungsz. Graz* **103** (1978) 1–22.

[11] I. Gutman, B. Furtula, Survey of graph energies, *Math. Interdisc. Res.* **2** (2017) 85–129.

[12] I. Gutman, X. Li (Eds.), *Graph Energies – Theory and Applications*, Univ. Kragujevac, KragujevaC, 2016.

[13] I. Gutman, B. Zhou, Laplacian energy of a graph, *Lin. Algebra Appl.* **414** (2006) 29–37.

[14] S. M. Hosamani, B. B. Kulkarni, R. G. Boli, V. M. Gadag, QSPR analysis of certain graph theoretical matrices and their corresponding energy, *Appl. Math. Nonlin. Sci.* **2** (2017) 131–150.

[15] N. Jafari Rad, A. Jahanbani, I. Gutman, Zagreb energy and Zagreb Estrada index of graphs, *MATCH Commun. Math. Comput. Chem.* **79** (2018) 371–386.

[16] X. Li, Y. Shi, I. Gutman, *Graph Energy*, Springer, New York, 2012.

[17] B. J. McClelland, Properties of the latent roots of a matrix: The estimation of π-electron energies, *J. Chem. Phys.* **54** (1971) 640–643.

[18] Z. Meng, B. Xiao, High–resolution satellite image classification and segmentation using Laplacian graph energy, Proceeding of Geoscience and Remote Sensing Symposium (IGARSS), Vancouver, 2011, pp. 605–608.

[19] J. K. Merikoski and R. Kumar, Inequalities for spreads of matrix sums and products, *Appl. Maths. E-notes* **4** (2004) 150–159.

[20] H. S. Ramane, I. Gutman, J. B. Patil, R. B. Jummannaver, Seidel signless Laplacian energy of graphs, *Math. Interdisc. Res.* **2** (2017) 181–191.

[21] H. S. Ramane, R. B. Jummannaver, I. Gutman, Seidel Laplacian energy of graphs, *Int. J. Appl. Graph Theory* **1**(2) (2017) 74–82.

[22] D. S. Revankar, M. M. Patil, H. S.Ramane, On eccentricity sum eigenvalue and eccentricity sum energy of a graph, *Ann. Pure Appl. Math.* **13** (2017) 125–130.

[23] J. M. Rodríguez, J. M. Sigarreta, Spectral properties of geometric–arithmetic index, *Appl. Math. Comput.* **277** (2016) 142–153.

[24] B. Sharada, M. Sowaity, On the sum-eccentricity energy of a graph, *TWMS J. Appl. Eng. Math.*, in press.

[25] Y. Z. Song, P. Arbelaez, P. Hall, C. Li, A. Balikai, Finding semantic structures in image hierarchies using Laplacian graph energy, in: K. Daniilidis, P. Maragos, N. Paragios (Eds.), *Computer Vision – CECV 2010 (European Conference on Computer Vision, 2010), Part IV*, Springer, Berlin, 2010, pp. 694–707.

[26] M. Sowaity and B. Sharada, The sum-eccentricity energy of a graph, *Int. J. Rec. Innovat. Trends Comput. Commun.* **5** (2017) 293–304.

[27] H. Zhang, X. Bai, H. Zheng, H. Zhao, J, Zhou, J. Cheng, H. Lu, Hierarchical remote sensing image analysis via graph Laplacian energy, *IEEE Geosci. Remote Sensing Lett.* **10** (2013) 396–400.

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