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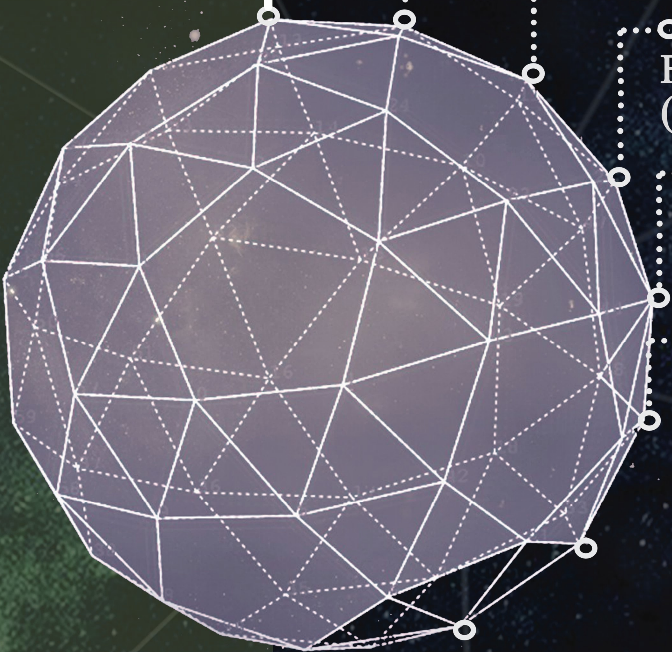

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Special Issue: Gyrogroups, Gyrovector Spaces and their Applications

Guest Editor: Abraham A. Ungar

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Abraham A. Ungar's Autobiography

Abraham A. Ungar⋆

Abstract

This autobiography presents the scientific living of Abraham Ungar and his role in Gyrogroups and Gyrovector spaces.

Keywords: Gyrogroup, Gyrovector space.

2010 Mathematics Subject Classification: Primary 20E07; Secondary 20N05, 20B35.

Figure 1: Abraham Ungar at 2016.

Abraham Ungar is professor in the Department of Mathematics at North Dakota State University. After gaining his B.Sc. (1965) and M.Sc. (1967) from

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the Hebrew University in Pure Mathematics and Ph.D. (1973) from Tel-Aviv University in Applied Mathematics, he held a postdoctoral position at the University of Toronto (1974). Ungar moved from Toronto to Pretoria where he held the position of a senior research officer at the National Research Institute for Mathematical Sciences of the Council for Scientific and Industrial Research (CSIR) in 1975-1977. From Pretoria Ungar moved to Grahamstown, South Africa, where he held the position of a lecturer and a senior lecturer at Rhodes University (1978-1983). From Grahamstown Ungar moved to Vancouver, where he held the position of a visiting associate professor at Simon Fraser University (1983-1984). Finally, in 1984 Ungar has accepted the position of an associate professor at North Dakota State University in Fargo, North Dakota, where he presently holds the position of a professor.

Ungar's favored research areas are related to hyperbolic geometry and its applications in relativity physics. He currently serves on the editorial boards of *Mathematics Interdisciplinary Research*, of *Journal of Geometry and Symmetry in Physics*, and of *Communications in Applied Geometry.*

When Ungar was a young, undergraduate student he was fascinated by the bijective correspondence between the field of complex numbers and the Lorentz transformation group of special relativity theory in one time and one space dimensions. He was aware of the result that the field of complex numbers does not admit extension to a field of higher than two dimensions while, in contrast, the Lorentz group admits extensions to one time and several space dimensions. Ungar, therefore, felt that the transition of the Lorentz group from one time and one space dimensions, where it is closely related to the field of complex numbers, to one time and two space dimensions is a mystery to be conquered. Hence, later young student Ungar was not surprised to discover in the literature that a new phenomenon comes into play in the above mentioned transition of the Lorentz group. The new phenomenon, which deeply attracted Ungar's attention, turned out to be the peculiar space rotation known in special relativity theory as *Thomas precession.*

Naturally, many explorers were fascinated by the relativistic Thomas precession. However, the elegant structure that Thomas precession encodes could not be decoded for a long time, being locked by complexity. Indeed, the hopeless status of Thomas precession that existed before Ungar's 1988 discovery is well described by Herbert Goldstein in his book *Classical Mechanics* (Addison-Wesley, 1980, pp. 285-286):

> The decomposition process can be carried through on the product of two pure Lorentz transformations to obtain explicitly the [Thomas] rotation of the coordinate axes resulting from the two successive boosts. In general, the algebra involved [in calculating the Thomas rotation] is *quite forbidding,* more than enough, usually, to *discourage any actual demonstration* of the rotation matrix. There is, however, one specific situation where allowable approximations reduce the calculational com-

> plexity, while the result obtained has important applications in many areas of modern physics. What is involved is a phenomenon known as the *Thomas precession.*
>
> Herbert Goldstein, Classical Mechanics

In 1988 Ungar expressed the Lorentz group parametrically in terms of relativistically admissible velocities and orientations in an article titled *Thomas Rotation and the Parametrization of the Lorentz Transformation Group.* The group structure of the resulting parametric realization of the Lorentz group, along with Einstein velocity addition law, enabled Ungar to discover the rich structure that Thomas precession possesses in terms of Einstein addition.

Revealing its underlying structure, Ungar was able to extend Thomas precession by abstraction, calling the abstract Thomas precessions *gyrations.* It turned out that gyrations are automorphisms that regulate Einstein addition in the sense that the seemingly structureless, noncommutative, nonassociative Einstein addition is, in fact, a gyrocommutative, gyroassociative binary operation in a gyrocommutative gyrogroup and in a gyrovector space. The ugly duckling of relativity physics, Thomas precession, thus became the beautiful swan called, in *gyrolanguage*, gyration.

The resulting emergence of the gyrogroup and the gyrovector space structures, along with their application in the hyperbolic geometry of Lobachevsky and Bolyai and in the special relativity theory of Einstein, is unfolded by Ungar in the following expository article titled:

The Intrinsic Beauty, Harmony and
Interdisciplinarity
in Einstein Velocity Addition Law:
Gyrogroups and Gyrovector Spaces.

Yet, the rich structure of Einstein addition and the interdisciplinarity of gyrogroups and gyrovector spaces that Ungar has exposed is still far from being exhausted, as the articles of the present special issue of the Journal demonstrate.

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The Intrinsic Beauty, Harmony and Interdisciplinarity in Einstein Velocity Addition Law: Gyrogroups and Gyrovector Spaces

Abraham A. Ungar⋆

Abstract

The only justification for the Einstein velocity addition law appeared to be its empirical adequacy, so that the intrinsic beauty and harmony in Einstein addition remained for a long time a mystery to be conquered. Accordingly, the aim of this expository article is to present (i) the Einstein relativistic vector addition, (ii) the resulting Einstein scalar multiplication, (iii) the Einstein relativistic mass, and (iv) the Einstein relativistic kinetic energy, along with remarkable analogies with classical results in groups and vector spaces that these Einstein concepts capture in gyrogroups and gyrovector spaces. Making the unfamiliar familiar, these analogies uncover the intrinsic beauty and harmony in the underlying Einstein velocity addition law of relativistically admissible velocities, as well as its interdisciplinarity.

Keywords: Einstein addition, gyrogroup, gyrovector space, hyperbolic geometry, special relativity.

2010 Mathematics Subject Classification: 20N05, 51P05, 83A05.

1. Introduction

A major obstacle to the widespread adoption of hyperbolic geometry is its complexity, which contrasts the simplicity of Euclidean geometry. Hence, the mere mention of hyperbolic geometry is enough to strike fear in the heart of the undergraduate mathematics and physics student. Some regard themselves as excluded from the profound insights of hyperbolic geometry so that this enormous portion of

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human achievement is a closed door to them. However, Einstein velocity addition law of relativistically admissible velocities opens that door, making the hyperbolic geometry of Lobachevsky and Bolyai accessible to a wider audience in terms of novel analogies that the modern and unknown share with the classical and familiar.

Einstein velocity addition law gives rise to a binary operation in the ball of relativistically admissible velocities, called *Einstein addition*. The intrinsic beauty and harmony in Einstein addition has several features, one of which is the gyrogroup isomorphism relation it shares with Möbius addition that results from the Möbius transformation of the complex disk. Einstein introduced the relativistic velocity addition law in his 1905 paper [17] that founded the special relativity theory. The only justification for the Einstein velocity addition law appeared to be its empirical adequacy, so that the intrinsic beauty and harmony in Einstein addition remained for a long time a mystery to be conquered. The discovery of the intrinsic beauty and harmony in Einstein addition is an ongoing process initiated in 1988 by the discovery of the parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces of signature $(1, n)$, $n \in \mathbb{N}$ in [54], resulting in many articles as well as seven related books [60,64,67,69,71,72,79], [39,40,83]. Recently, the parametric realization of Lorentz groups has been extended to pseudo-Euclidean spaces of any signature (m, n), $m, n \in \mathbb{N}$ in [80].

Most texts on special relativity, with a few outstanding exceptions including [3], [24], and [41,42], present the Einstein velocity addition only for parallel velocities. In this simplified special case Einstein velocity addition is both commutative and associative. In general, however, Einstein addition of velocities that need not be parallel is neither commutative nor associative.

Einstein velocity addition law gives rise to a binary operation $\oplus$, called Einstein addition, in the ball of all relativistically admissible velocities. Einstein addition, in turn, gives rise to the Einstein scalar multiplication $\otimes$. Einstein addition and scalar multiplication give rise to hyperbolic vector spaces called *gyrovector spaces*. Applications of gyrogroups and gyrovector spaces are presented in many publications as, for instance, in [60, 64, 67, 69, 71, 72, 79] and in [4–7, 38, 44], [15, 16], [19], [20–23], [48–52], [27], [37], [46], [81], [29] and [1, 25]. Evidently, gyrovector spaces form the algebraic setting for analytic hyperbolic geometry, just as vector spaces form the algebraic setting for analytic Euclidean geometry.

One of the remarkable analogies that Einstein scalar multiplication captures in Einstein's special theory of relativity is the novel analogy that classical and relativistic kinetic energy share, presented in Section 11. This analogy, in turn, augments the standard analogies that the classical, Newtonian mass shares with the Einstein relativistic, velocity dependent mass [73]. Being noncommutative and nonassociative, initially Einstein addition was viewed as a structureless binary operation. The subsequent discovery of the rich gyrostructure and the interdisciplinarity that Einstein addition possesses results in the emergence of intrinsic beauty and harmony that underlies Einstein addition, as evidenced from this article.

2. Einstein Addition

Let $c > 0$ be an arbitrarily fixed positive constant and let $\mathbb{R}^n = (\mathbb{R}^n, +, \cdot)$ be the Euclidean n-space, $n \in \mathbb{N}$, equipped with the common vector addition, $+$, and inner product, $\cdot$. The home of all n-dimensional Einsteinian velocities is the c-ball

$$\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\} \tag{1}$$

of its ambient space $\mathbb{R}^n$. The c-ball $\mathbb{R}^n_c$ is the open ball of radius c, centered at the origin of $\mathbb{R}^n$, consisting of all vectors $\mathbf{v}$ in $\mathbb{R}^n$ with magnitude $\|\mathbf{v}\|$ smaller than c.

Einstein velocity addition is a binary operation, $\oplus$, in the c-ball $\mathbb{R}^n_c$ given by the equation [60], [42, Eq. 2.9.2], [34, p. 55], [24],

$$\mathbf{u}\oplus\mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{c^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}\cdot\mathbf{v})\mathbf{u} \right\}, \tag{2}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$. Here $\gamma_{\mathbf{v}}$ is the Lorentz gamma factor given by the equation

$$\gamma_{\mathbf{v}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{v}\|^2}{c^2}}}, \tag{3}$$

where $\mathbf{u}\cdot\mathbf{v}$ and $\|\mathbf{v}\|$ are the inner product and the norm in the ball, which the ball $\mathbb{R}^n_c$ inherits from its ambient space $\mathbb{R}^n$, $\|\mathbf{v}\|^2 = \mathbf{v}\cdot\mathbf{v}$. A nonempty set with a binary operation is called a *groupoid* so that, accordingly, the pair $(\mathbb{R}^n_c, \oplus)$ is an *Einstein groupoid*.

In the Newtonian limit of large c, $c \to \infty$, the ball $\mathbb{R}^n_c$ expands to the whole of its ambient space $\mathbb{R}^n$, as we see from (1), and Einstein addition $\oplus$ in $\mathbb{R}^n_c$ reduces to the ordinary vector addition $+$ in $\mathbb{R}^n$, as we see from (2) and (3).

When the nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ in the ball $\mathbb{R}^n_c$ of $\mathbb{R}^n$ are parallel in $\mathbb{R}^n$, $\mathbf{u}\|\mathbf{v}$, that is, $\mathbf{u} = \lambda\mathbf{v}$ for some $\lambda \in \mathbb{R}$, $\lambda \neq 0$, Einstein addition (2) specializes to the Einstein addition of parallel velocities,

$$\mathbf{u}\oplus\mathbf{v} = \frac{\mathbf{u}+\mathbf{v}}{1 + \frac{1}{c^2}\mathbf{u}\cdot\mathbf{v}}, \qquad \mathbf{u}\|\mathbf{v}, \tag{4}$$

which was partially confirmed experimentally by the Fizeau's 1851 experiment [33].

The *restricted Einstein addition* in (4) is both commutative and associative. Accordingly, the restricted Einstein addition is a commutative group operation, as Einstein noted in [17]; see [18, p. 142]. In contrast, Einstein made no remark about group properties of his addition (2) of velocities that need not be parallel. Indeed, the general Einstein addition is not a group operation but, rather, a gyrocommutative gyrogroup operation, a structure discovered more than 80 years later, in 1988 [54, 55, 58], formally defined in Section 4.

In physical applications, $\mathbb{R}^n = \mathbb{R}^3$ is the Euclidean 3-space, which is the space of all classical, Newtonian velocities, and $\mathbb{R}^n_c = \mathbb{R}^3_c \subset \mathbb{R}^3$ is the c-ball of $\mathbb{R}^3$ of all relativistically admissible, Einsteinian velocities. The constant c represents in physical applications the vacuum speed of light. Since we are interested in both physics and geometry, we allow n to be any positive integer.

Einstein addition (2) of relativistically admissible velocities, with $n = 3$, was introduced by Einstein in his 1905 paper [17] [18, p. 141] that founded the special theory of relativity, where the magnitudes of the two sides of Einstein addition (2) are presented. One has to remember here that the Euclidean 3-vector algebra was not so widely known in 1905 and, consequently, was not used by Einstein. Einstein calculated in [17] the behavior of the velocity components parallel and orthogonal to the relative velocity between inertial systems, which is as close as one can get without vectors to the vectorial version (2) of Einstein addition. Einstein was aware of the nonassociativity of his velocity addition law of relativistically admissible velocities that need not be collinear. He therefore emphasized in his 1905 paper that his velocity addition law of relativistically admissible *collinear velocities* forms a group operation [17, p. 907].

We naturally use the abbreviation $\mathbf{u}\ominus\mathbf{v} = \mathbf{u}\oplus(-\mathbf{v})$ for Einstein subtraction, so that, for instance, $\mathbf{v}\ominus\mathbf{v} = \mathbf{0}$ and

$$\ominus\mathbf{v} = \mathbf{0}\ominus\mathbf{v} = -\mathbf{v}\,. \tag{5}$$

Einstein addition and subtraction satisfy the equations

$$\ominus(\mathbf{u}\oplus\mathbf{v}) = \ominus\mathbf{u}\ominus\mathbf{v} \tag{6}$$

and

$$\ominus\mathbf{u}\oplus(\mathbf{u}\oplus\mathbf{v}) = \mathbf{v} \tag{7}$$

for all $\mathbf{u}, \mathbf{v}$ in the ball $\mathbb{R}^n_c$, in full analogy with vector addition and subtraction in $\mathbb{R}^n$. Identity (6) is called the *gyroautomorphic inverse property* of Einstein addition, and Identity (7) is called the *left cancellation law* of Einstein addition. We may note that Einstein addition does not obey the naive right counterpart of the left cancellation law (7) since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\ominus\mathbf{v} \neq \mathbf{u}\,. \tag{8}$$

However, this seemingly lack of a *right cancellation law* of Einstein addition is naturally remedied in (29) – (30), p. 14.

Einstein addition and the gamma factor are related by the *gamma identity*,

$$\gamma_{\mathbf{u}\oplus\mathbf{v}} = \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right) \tag{9}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$.

A frequently used identity that follows immediately from (3) is

$$\frac{\mathbf{v}^2}{c^2} = \frac{\|\mathbf{v}\|^2}{c^2} = \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2}\,. \tag{10}$$

Einstein addition is noncommutative. Indeed, while Einstein addition is commutative under the norm,

$$\|\mathbf{u}\oplus\mathbf{v}\| = \|\mathbf{v}\oplus\mathbf{u}\|\,, \tag{11}$$

in general,

$$\mathbf{u}\oplus\mathbf{v} \neq \mathbf{v}\oplus\mathbf{u}\,, \tag{12}$$

$\mathbf{u},\mathbf{v} \in \mathbb{R}^n_c$. Moreover, Einstein addition is also nonassociative since, in general,

$$(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w} \neq \mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\,, \tag{13}$$

$\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n_c$.

As an application of the gamma identity (9), we prove the Einstein gyrotriangle inequality (14).

Theorem 2.1. (The Gyrotriangle Inequality).

$$\|\mathbf{u}\oplus\mathbf{v}\| \leq \|\mathbf{u}\|\oplus\|\mathbf{v}\| \tag{14}$$

for all $\mathbf{u},\mathbf{v}$ *in an Einstein gyrogroup* $(\mathbb{R}^n_c,\oplus)$.

Proof. By the gamma identity (9) with $\mathbf{u}$ and $\mathbf{v}$ replaced by $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$, and by the Cauchy-Schwarz inequality [32], we have

$$\begin{aligned}
\gamma_{\|\mathbf{u}\|\oplus\|\mathbf{v}\|} &= \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\|\mathbf{u}\|\|\mathbf{v}\|}{c^2}\right)\\
&\geq \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}\left(1+\frac{\mathbf{u}\cdot\mathbf{v}}{c^2}\right)\\
&= \gamma_{\mathbf{u}\oplus\mathbf{v}}\\
&= \gamma_{\|\mathbf{u}\oplus\mathbf{v}\|}
\end{aligned} \tag{15}$$

for all $\mathbf{u},\mathbf{v}$ in an Einstein gyrogroup $(\mathbb{R}^n_c,\oplus)$. But $\gamma_{\mathbf{x}} = \gamma_{\|\mathbf{x}\|}$ is a monotonically increasing function of $\|\mathbf{x}\|$, $0 \leq \|\mathbf{x}\| < c$. Hence (15) implies (14). $\square$

3. Einstein Addition Vs. Vector Addition

Vector addition, $+$, in $\mathbb{R}^n$ is both commutative and associative, satisfying

$$\begin{aligned}
\mathbf{u}+\mathbf{v} &= \mathbf{v}+\mathbf{u} && \text{Commutative Law}\\
\mathbf{u}+(\mathbf{v}+\mathbf{w}) &= (\mathbf{u}+\mathbf{v})+\mathbf{w} && \text{Associative Law}
\end{aligned} \tag{16}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w} \in \mathbb{R}^n$. In contrast, Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ is neither commutative nor associative. Rather, Einstein addition is both gyrocommutative and gyroassociative, as stated in (19) below.

In order to measure the extent to which Einstein addition deviates from associativity we introduce *gyrations*, which are self maps of $\mathbb{R}^n$ that are *trivial* in the special cases when the application of $\oplus$ is associative. For any $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$ the gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$ is a map of the Einstein groupoid $(\mathbb{R}^n_c,\oplus)$ onto itself. Gyrations $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\in\mathrm{Aut}(\mathbb{R}^n_c,\oplus)$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$, are defined in terms of Einstein addition by the equation

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}=\ominus(\mathbf{u}\oplus\mathbf{v})\oplus\{\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})\} \tag{17}$$

for all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$, and they turn out to be automorphisms of the Einstein groupoid $(\mathbb{R}^n_c,\oplus)$, $\mathrm{gyr}[\mathbf{u},\mathbf{v}]:\mathbb{R}^n_c\to\mathbb{R}^n_c$.

We recall that an automorphism of a groupoid $(S,\oplus)$ is a one-to-one map f of S onto itself that respects the binary operation, that is, $f(a\oplus b)=f(a)\oplus f(b)$ for all $a,b\in S$. The set of all automorphisms of a groupoid $(S,\oplus)$ forms a group, under automorphism composition, denoted $\mathrm{Aut}(S,\oplus)$. To emphasize that the gyrations of an Einstein gyrogroup $(\mathbb{R}^n_c,\oplus)$ are automorphisms of the gyrogroup, gyrations are also called *gyroautomorphisms*.

A gyration $\mathrm{gyr}[\mathbf{u},\mathbf{v}]$, $\mathbf{u},\mathbf{v}\in\mathbb{R}^n_c$, is *trivial* if $\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w}=\mathbf{w}$ for all $\mathbf{w}\in\mathbb{R}^n_c$. Thus, for instance, the gyrations $\mathrm{gyr}[\mathbf{0},\mathbf{v}]$, $\mathrm{gyr}[\mathbf{v},\mathbf{v}]$ and $\mathrm{gyr}[\mathbf{v},\ominus\mathbf{v}]$ are trivial, that is,

$$\begin{aligned}\mathrm{gyr}[\mathbf{0},\mathbf{v}]&=\mathrm{gyr}[\mathbf{v},\mathbf{0}]=I\\ \mathrm{gyr}[\mathbf{v},\ominus\mathbf{v}]&=\mathrm{gyr}[\ominus\mathbf{v},\mathbf{v}]=I\\ \mathrm{gyr}[\mathbf{v},\mathbf{v}]&=I\end{aligned} \tag{18}$$

for all $\mathbf{v}\in\mathbb{R}^n_c$, I being the identity map, as we see from (17) and (7).

Einstein gyrations, which possess their own rich structure, measure the extent to which Einstein addition deviates from commutativity and from associativity. We see this from the gyrocommutative and the gyroassociative laws of Einstein addition in the following list of elegant identities that involve Einstein addition, $\oplus$, and gyrations [60, 64, 67]. For all $\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$,

$$\begin{aligned}
\mathbf{u}\oplus\mathbf{v}&=\mathrm{gyr}[\mathbf{u},\mathbf{v}](\mathbf{v}\oplus\mathbf{u}) && \text{Gyrocommutative Law}\\
\mathbf{u}\oplus(\mathbf{v}\oplus\mathbf{w})&=(\mathbf{u}\oplus\mathbf{v})\oplus\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{w} && \text{Left Gyroassociative Law}\\
(\mathbf{u}\oplus\mathbf{v})\oplus\mathbf{w}&=\mathbf{u}\oplus(\mathbf{v}\oplus\mathrm{gyr}[\mathbf{v},\mathbf{u}]\mathbf{w}) && \text{Right Gyroassociative Law}\\
\mathrm{gyr}[\mathbf{u}\oplus\mathbf{v},\mathbf{v}]&=\mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Left Reduction Property}\\
\mathrm{gyr}[\mathbf{u},\mathbf{v}\oplus\mathbf{u}]&=\mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Right Reduction Property}\\
\mathrm{gyr}[\ominus\mathbf{u},\ominus\mathbf{v}]&=\mathrm{gyr}[\mathbf{u},\mathbf{v}] && \text{Gyration Even Property}\\
(\mathrm{gyr}[\mathbf{u},\mathbf{v}])^{-1}&=\mathrm{gyr}[\mathbf{v},\mathbf{u}] && \text{Gyration Inversion Law}
\end{aligned} \tag{19}$$

Einstein addition is thus regulated by the gyrations to which it gives rise owing to its nonassociativity. As such, Einstein addition and its gyrations are inextricably linked. The resulting gyrocommutative gyrogroup structure of Einstein addition was discovered in 1988 [54]. Interestingly, gyrations are the mathematical abstraction of the relativistic phenomenon known as *Thomas precession* [67, Section 10.3] [74] [79, Chapter 13]. Thomas precession, in turn, is related to the *mixed state geometric phase*, as Lévay discovered in his work [30] which, according to [30], was motivated by the author work in [61, 62].

The left and right reduction properties in (19) present important gyration identities since they trigger a remarkable reduction in complexity, as Chatelin noted in [11]. These two gyration identities are, however, just the tip of a giant iceberg. The identities in (19) and many other useful gyration identities are studied, for instance, in [60, 64, 67, 69, 71, 72, 79].

4. From Einstein Addition to Gyrogroups

Taking the key features of Einstein groupoids $(\mathbb{R}^n_c, \oplus)$, $n \in \mathbb{N}$, as axioms, and guided by analogies with groups, we are led to the following formal gyrogroup definition in which gyrogroups turn out to form a most natural generalization of groups.

Definition 4.1. (Gyrogroups [67, p. 17]**).** *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms. In* G *there is at least one element,* 0*, called a left identity, satisfying*

$(G1)$ $\qquad 0 \oplus a = a$

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of* a*, satisfying*

$(G2)$ $\qquad \ominus a \oplus a = 0\,.$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

$(G3)$ $\qquad a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c\,.$

The map $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

$(G4)$ $\qquad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a, b]$ *of* G *is called the gyroautomorphism, or the gyration, of* G *generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of* G*. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left reduction property*

$(G5)$ $\qquad \mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]\,.$

The gyrogroup axioms $(G1)-(G5)$ in Definition 4.1 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.
2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.
3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ in gyrogroup theory as well. In full analogy with groups, gyrogroups split up into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 4.2. (Gyrocommutative Gyrogroups). *A gyrogroup* $(G,\oplus)$ *is gyrocommutative if its binary operation obeys the gyrocommutative law*
$(G6)$ $\qquad a \oplus b = \mathrm{gyr}[a,b](b \oplus a)$
for all $a, b \in G$.

First gyrogroup properties are studied in [72, Chapter 1], and more gyrogroup theorems are studied in [60, 64, 67]. Thus, for instance, as in group theory, any gyrogroup possesses a unique identity element, which is both left and right, and any element of a gyrogroup possesses a unique inverse, which is both left and right.

In order to illustrate the power and elegance of the gyrogroup structure, we solve below the two basic gyrogroup equations (20) and (27).

Let us consider the gyrogroup equation

$$\mathbf{a}\oplus\mathbf{x} = \mathbf{b} \tag{20}$$

in a gyrogroup $(G,\oplus)$ for the unknown $\mathbf{x}$. If $\mathbf{x}$ exists, then by the right gyroassociative law (19) we have

$$\begin{aligned}\mathbf{x} &= \mathbf{0}\oplus\mathbf{x}\\ &= (\ominus\mathbf{a}\oplus\mathbf{a})\oplus\mathbf{x}\\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{a}]\mathbf{x})\\ &= \ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{x})\\ &= \ominus\mathbf{a}\oplus\mathbf{b}\,,\end{aligned} \tag{21}$$

noting that $\mathrm{gyr}[\mathbf{a},\ominus\mathbf{a}]$ is trivial by (18).

Thus, if a solution to (20) exists, it must be given uniquely by

$$\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b}\,. \tag{22}$$

Conversely, if $\mathbf{x} = \ominus\mathbf{a}\oplus\mathbf{b}$, then $\mathbf{x}$ is indeed a solution to (20) since by the left gyroassociative law and (18) we have

$$\begin{aligned}\mathbf{a}\oplus\mathbf{x} &= \mathbf{a}\oplus(\ominus\mathbf{a}\oplus\mathbf{b})\\ &= (\mathbf{a}\oplus(\ominus\mathbf{a}))\oplus\mathrm{gyr}[\mathbf{a},\ominus\mathbf{a}]\mathbf{b}\\ &= \mathbf{0}\oplus\mathbf{b}\\ &= \mathbf{b}\,.\end{aligned} \tag{23}$$

Substituting the solution (22) into its equation (20) and replacing $\mathbf{a}$ by $\ominus\mathbf{a}$ we recover the left cancellation law (7) for Einstein addition,

$$\ominus\mathbf{a}\oplus(\mathbf{a}\oplus\mathbf{b}) = \mathbf{b}\,. \tag{24}$$

The gyrogroup operation (or, addition) of any gyrogroup has an associated dual operation, called the *gyrogroup cooperation* (or, *coaddition*), which is defined below.

Definition 4.3. (The Gyrogroup Cooperation (Coaddition)). *Let* $(G, \oplus)$ *be a gyrogroup with gyrogroup operation (or, addition)* $\oplus$. *The gyrogroup cooperation (or, coaddition)* $\boxplus$ *is a second binary operation in* G *given by the equation*

$$\mathbf{a} \boxplus \mathbf{b} = \mathbf{a} \oplus \mathrm{gyr}[\mathbf{a}, \ominus\mathbf{b}]\mathbf{b} \tag{25}$$

for all $\mathbf{a}, \mathbf{b} \in G$.

Replacing b by $\ominus b$ in (25) we have the *cosubtraction* identity

$$\mathbf{a} \boxminus \mathbf{b} := \mathbf{a} \boxplus (\ominus\mathbf{b}) = \mathbf{a} \ominus \mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{b} \tag{26}$$

for all $\mathbf{a}, \mathbf{b} \in G$, noting that $\mathrm{gyr}[\mathbf{a}, \mathbf{b}]$ is an automorphism of $(G, \oplus)$ so that $\mathrm{gyr}[\mathbf{a}, \mathbf{b}](\ominus\mathbf{b}) = \ominus\mathrm{gyr}[\mathbf{a}, \mathbf{b}]\mathbf{b}$.

To motivate the introduction of the gyrogroup cooperation and to illustrate the use of the left reduction property in (19), we solve the equation

$$\mathbf{x} \oplus \mathbf{a} = \mathbf{b} \tag{27}$$

for the unknown $\mathbf{x}$ in a gyrogroup $(G, \oplus)$.

Equation (27) results from (20) by interchanging $\mathbf{a}$ and $\mathbf{x}$. Surprisingly, however, the solution of (27) is quite different from the solution of (20), suggesting the introduction of the second binary operation, the cooperation $\boxplus$ in G. We will find that Einstein coaddition, $\boxplus$, proves crucially important (i) in the understanding of Einstein addition, $\oplus$, in $\mathbb{R}^n_c$ in terms of analogies with vector addition in $\mathbb{R}^n$, and (ii) in our mission to capture analogies with classical results.

Assuming that a solution $\mathbf{x}$ to (27) exists, we have the following obvious chain of equations

$$\begin{aligned}
\mathbf{x} &= \mathbf{x} \oplus \mathbf{0} \\
&= \mathbf{x} \oplus (\mathbf{a} \ominus \mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \oplus \mathrm{gyr}[\mathbf{x}, \mathbf{a}](\ominus\mathbf{a}) \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x}, \mathbf{a}]\mathbf{a} \\
&= (\mathbf{x} \oplus \mathbf{a}) \ominus \mathrm{gyr}[\mathbf{x} \oplus \mathbf{a}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \ominus \mathrm{gyr}[\mathbf{b}, \mathbf{a}]\mathbf{a} \\
&= \mathbf{b} \boxminus \mathbf{a}\,.
\end{aligned} \tag{28}$$

The gyrogroup cosubtraction, (26), comes into play in (28) in order to capture an analogy with the classical result $\mathbf{x} + \mathbf{a} = \mathbf{b} \Rightarrow \mathbf{x} = \mathbf{b} - \mathbf{a}$. Thus, if a solution $\mathbf{x}$ to the gyrogroup equation (27) exists, it must be given uniquely by (28). One can show that the latter is indeed a solution to (27) [67, Section 2.4]. The use of the gyration left reduction property in (28) indicates the remarkable reduction of complexity that this property offers.

The gyrogroup cooperation is introduced into gyrogroups in order to capture useful analogies between gyrogroups and groups, and it results in the emergence of duality symmetries that the two gyrogroup operations, $\oplus$ and $\boxplus$, share. Thus, for instance, the gyrogroup cooperation uncovers the seemingly missing right counterpart of the left cancellation law (7), giving rise to the right cancellation law,

$$(\mathbf{b} \boxminus \mathbf{a}) \oplus \mathbf{a} = \mathbf{b} \tag{29}$$

for all $\mathbf{a}, \mathbf{b} \in G$, which is obtained by substituting the result of (28) into (27).

Remarkably, the right cancellation law (29) can be dualized, giving rise to the dual right cancellation law

$$(\mathbf{b} \ominus \mathbf{a}) \boxplus \mathbf{a} = \mathbf{b}\,. \tag{30}$$

As an example, and for later reference, we note that it follows from the right cancellation law (29) that

$$\mathbf{d} = (\mathbf{b} \boxplus \mathbf{c}) \ominus \mathbf{a} \qquad \Longleftrightarrow \qquad \mathbf{b} \boxplus \mathbf{c} = \mathbf{d} \boxplus \mathbf{a} \tag{31}$$

for $\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}$ in any gyrocommutative gyrogroup $(G, \oplus)$.

An elegant gyrocommutative gyrogroup identity that involves the gyrogroup cooperation, verified in [67, Theorem 3.12], is

$$\mathbf{a} \oplus (\mathbf{b} \oplus \mathbf{a}) = \mathbf{a} \boxplus (\mathbf{a} \oplus \mathbf{b})\,. \tag{32}$$

5. Émile Borel's Dream Comes True

It is not well-known that the famous mathematician Émile Borel was interested in Einstein's special theory of relativity, particularly in the relativistic phenomenon known as Thomas precession [79, Chapter 13] and in Einstein addition. Being noncommutative, Émile Borel considered Einstein addition as "defective". He, therefore, proposed an alternative, commutative addition of relativistically admissible velocities.

The gyrocommutative law of Einstein velocity addition was already known to Silberstein in 1914 [43] in the following sense: According to his 1914 book, Silberstein knew that the Thomas precession generated by $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3_c$ is the unique rotation that takes $\mathbf{v} \oplus \mathbf{u}$ into $\mathbf{u} \oplus \mathbf{v}$ about an axis perpendicular to the plane of $\mathbf{u}$ and $\mathbf{v}$ through an angle $< \pi$ in $\mathbb{R}^3$, thus giving rise to the gyrocommutative law. However, obviously, Silberstein did not use the terms "Thomas precession" and "gyrocommutative law". These terms have been coined later, respectively, (i) following Thomas' 1926 paper [53], and (ii) in 1991 [58,59], following the discovery of the accompanying gyroassociative law of Einstein addition in 1988 [54, 55].

A description of the 3-space rotation, which since 1926 is named after Thomas, is found in Silberstein's 1914 book [43]. In 1914 Thomas precession did not have a name, and Silberstein called it in his 1914 book a "certain space-rotation" [43, p. 169]. An early study of Thomas precession, made by Émile Borel in 1913, is described in his 1914 book [10] and, more recently, in [47].

The almost forgotten attempt of Émile Borel to "repair" the seemingly "defective" Einstein's velocity addition law in the years following 1912 is described by Walter in [82, p. 117]:

> "Borel could construct a tetrahedron in kinematic space, and determined thereby both the direction and magnitude of relative [composite] velocity in a symmetric [commutative] manner."

Borel has, thus, "repaired" the breakdown of commutativity in Einstein addition by proposing an alternative, commutative addition. But he did not pay attention to the accompanying breakdown of associativity in Einstein addition. Accordingly, it seemed appropriate to consider the Lorentz transformation group, rather than the groupoid of Einstein addition, as a primitive notion of special relativity [63].

It turns out that Einstein coaddition is commutative. Hence. Émile Borel's dream to construct a viable commutative relativistic velocity addition comes true with the discovery of Einstein coaddition, $\boxplus$. Unlike Borel's commutative addition, the commutative Einstein coaddition does not replace Einstein addition. Rather, it captures analogies with classical results *jointly* with Einstein addition, as the study of the Einstein gyroparallelogram addition law in Section 6 reveals.

A gyrogroup cooperation is commutative if and only if its associated gyrogroup operation is gyrocommutative [64, Theorem 3.4] [67, Theorem 3.4]. Hence, in particular, Einstein coaddition is commutative. Indeed, Einstein coaddition, $\boxplus$, in an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$, abstractly defined in (25), can be manipulated in Einstein gyrogroups, obtaining the following chain of equations [67, Eq. (3.195)],

$$\begin{aligned}
\mathbf{u} \boxplus \mathbf{v} &= \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{\gamma_{\mathbf{u}}^2 + \gamma_{\mathbf{v}}^2 + \gamma_{\mathbf{u}}\gamma_{\mathbf{v}}(1 + \frac{\mathbf{u}\cdot\mathbf{v}}{s^2}) - 1}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \\
&= \frac{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}{(\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}})^2 - (\gamma_{\mathbf{u}\ominus\mathbf{v}} + 1)}(\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}) \\
&= 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}} \\
&= 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u} + \gamma_{\mathbf{v}}\mathbf{v}}{2 + (\gamma_{\mathbf{u}} - 1) + (\gamma_{\mathbf{v}} - 1)}
\end{aligned} \tag{33}$$

$\mathbf{u}, \mathbf{v} \in \mathbb{R}^n_c$, demonstrating that the cooperation $\boxplus$ in Einstein gyrogroups $(\mathbb{R}^n_c, \oplus)$ is commutative.

The symbol $\otimes$ in (33) represents the Einstein *scalar multiplication* so that, for instance, $2\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v}$, for all $\mathbf{v}$ in a gyrogroup $(G, \oplus)$, as explained in Section 9. It turns out that Einstein coaddition $\boxplus$ is more than just a commutative binary operation in the ball. Remarkably, jointly with Einstein addition, $\oplus$, Einstein coaddition, $\boxplus$, gives rise to the hyperbolic parallelogram addition law in the ball. The latter is explained in Section 6 and illustrated in Figure 2.

6. From Parallelograms to Gyroparallelograms

Elements of a real inner product space $\mathbb{V} = (\mathbb{V}, +, \cdot)$, called points and denoted by capital italic letters, A, B, P, Q, etc, give rise to vectors in $\mathbb{V}$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathbb{V}$ give rise to a unique rooted vector $\mathbf{v} \in \mathbb{V}$, rooted at the point P. It has a tail at the point P and a head at the point Q, and it has the value $-P + Q$,

$$\mathbf{v} = -P + Q\,. \tag{34}$$

The length of the rooted vector $\mathbf{v} = -P + Q$ is the distance between the points P and Q, given by the equation

$$\|\mathbf{v}\| = \| -P + Q\|\,. \tag{35}$$

Two rooted vectors $-P + Q$ and $-R + S$ are equivalent if they have the same value, that is,

$$-P + Q \quad \sim \quad -R + S \qquad \text{if and only if} \qquad -P + Q = -R + S \tag{36}$$

The relation $\sim$ in (36) between rooted vectors is reflexive, symmetric and transitive, so that it is an equivalence relation that gives rise to equivalence classes of rooted vectors. To liberate rooted vectors from their roots we define a *vector* to be an equivalence class of rooted vectors. The vector $-P + Q$ is thus a representative of all rooted vectors with value $-P + Q$.

A point $P \in \mathbb{V}$ is identified with the vector $-O + P$, O being the arbitrarily selected origin of the space $\mathbb{V}$. Hence, the algebra of vectors can be applied to points as well. Naturally, geometric and physical properties regulated by a vector space are *origin independent*, that is, independent of the choice of the origin.

Let $A, B, C \in \mathbb{V}$ be three non-collinear points, and let

$$\begin{aligned} \mathbf{u} &= -A + B \\ \mathbf{v} &= -A + C \end{aligned} \tag{37}$$

be two vectors in $\mathbb{V}$ that possess the same tail, A. Furthermore, let D be a point of $\mathbb{V}$ given by the *parallelogram condition*

$$D = B + C - A\,. \tag{38}$$

The quadrangle (also known as a quadrilateral; see [13, p. 52]) $ABDC$ turns out to be a parallelogram in Euclidean geometry, shown in Figure 1, since its two diagonals, AD and BC, intersect at their midpoints, that is,

$$\tfrac{1}{2}(A + D) = \tfrac{1}{2}(B + C)\,. \tag{39}$$

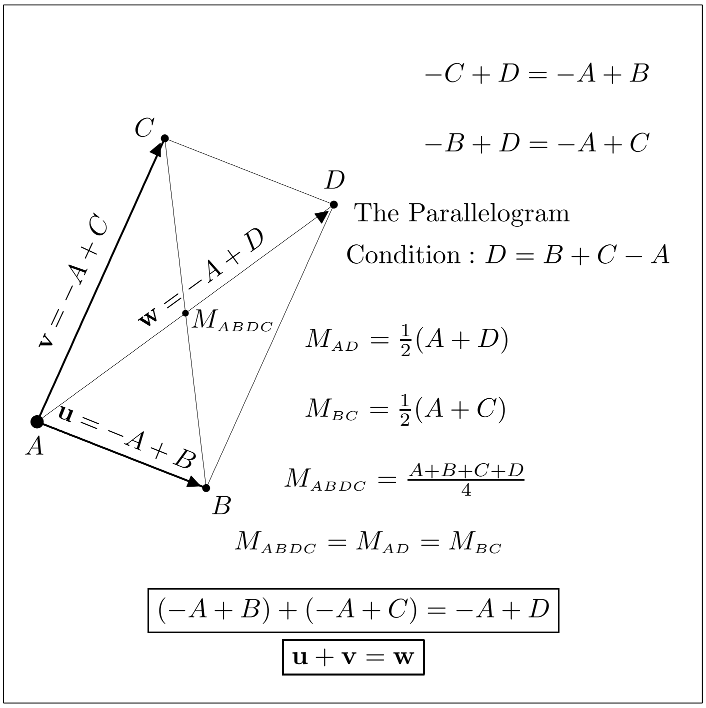

Figure 1: The Euclidean parallelogram and its addition law in a Euclidean vector plane $(\mathbb{R}^2, +, \cdot)$. The diagonals AD and BC of parallelogram $ABDC$ intersect each other at their midpoints. The midpoints of the diagonals AD and BC are, respectively, M_{AD} and M_{BC}, each of which coincides with the parallelogram center M_{ABDC}. This figure sets the stage for its hyperbolic counterpart in Figure 2.

Clearly, the midpoint equality (39) is equivalent to the parallelogram condition (38).

The vector addition of the vectors $\mathbf{u}$ and $\mathbf{v}$ that generate the parallelogram $ABDC$ according to (37), gives the vector $\mathbf{w}$ by the parallelogram addition law, shown in Figure 1,

$$\mathbf{w} := -A + D = (-A + B) + (-A + C) = \mathbf{u} + \mathbf{v}\,. \tag{40}$$

Here, by definition, $\mathbf{w}$ is the vector formed by the diagonal AD of the parallelogram $ABDC$, as shown in Figure 1.

Vectors in the space $\mathbb{V}$ are, thus, equivalence classes of ordered pairs of points, which add according to the parallelogram law, shown in Figure 1.

Gyrovectors emerge in an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ in a way fully analogous to the way vectors emerge in the space $\mathbb{V}$, where $\mathbb{V}_c$ is the c-ball of the space $\mathbb{V}$, $\mathbb{V}_c = \{\mathbf{v} \in \mathbb{V} : \|\mathbf{v}\| < c\}$.

Elements of $\mathbb{V}_c$, called points and denoted by capital italic letters, A, B, P, Q, etc, give rise to gyrovectors in $\mathbb{V}_c$, denoted by bold roman lowercase letters $\mathbf{u}, \mathbf{v}$, etc. Any two ordered points $P, Q \in \mathbb{V}_c$ give rise to a unique rooted gyrovector $\mathbf{v} \in \mathbb{V}_c$, rooted at the point P. It has a tail at the point P and a head at the point Q, and it has the value $\ominus P \oplus Q$,

$$\mathbf{v} = \ominus P \oplus Q\,. \tag{41}$$

The gyrolength of the rooted gyrovector $\mathbf{v} = \ominus P \oplus Q$ is the gyrodistance between the points P and Q, given by the equation

$$\|\mathbf{v}\| = \|\ominus P \oplus Q\|\,. \tag{42}$$

Two rooted gyrovectors $\ominus P \oplus Q$ and $\ominus R \oplus S$ are equivalent if they have the same value, that is,

$$\ominus P \oplus Q \quad \sim \quad \ominus R \oplus S \qquad \text{if and only if} \qquad \ominus P \oplus Q = \ominus R \oplus S \tag{43}$$

The relation $\sim$ in (43) between rooted gyrovectors is reflexive, symmetric and transitive, so that it is an equivalence relation that gives rise to equivalence classes of rooted gyrovectors. To liberate rooted gyrovectors from their roots we define a *gyrovector* to be an equivalence class of rooted gyrovectors. The gyrovector $\ominus P \oplus Q$ is thus a representative of all rooted gyrovectors with value $\ominus P \oplus Q$.

A point P of a gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$ is identified with the gyrovector $\ominus O \oplus P$, O being the arbitrarily selected origin of the space $\mathbb{V}_c$. Hence, the algebra of gyrovectors can be applied to points as well. Naturally, geometric and physical properties regulated by a gyrovector space are expected to be independent of the choice of the origin of the gyrovector space.

Let $A, B, C \in \mathbb{V}_c$ be three non-gyrocollinear points of an Einstein gyrovector space $(\mathbb{V}_c, \oplus, \otimes)$, and let

$$\begin{aligned} \mathbf{u} &= \ominus A \oplus B \\ \mathbf{v} &= \ominus A \oplus C \end{aligned} \tag{44}$$

be two gyrovectors in $\mathbb{V}$ that possess the same tail, A. Furthermore, let D be a point of $\mathbb{V}_c$ given by the gyroparallelogram condition

$$D = (B \boxplus C) \ominus A\,. \tag{45}$$

Then, the gyroquadrangle $ABDC$ is a gyroparallelogram in the Beltrami-Klein ball model of hyperbolic geometry in the sense that its two gyrodiagonals, AD and BC, intersect at their gyromidpoints, that is,

$$\tfrac{1}{2} \otimes (A \boxplus D) = \tfrac{1}{2} \otimes (B \boxplus C) \tag{46}$$

as illustrated in Figure 2. Clearly by (31), the gyromidpoint equality (46) is equivalent to the gyroparallelogram condition (45).

The gyrovector addition of the gyrovectors **u** and **v** that generate the gyroparallelogram $ABDC$ gives the gyrovector **w** by the gyroparallelogram addition law, shown in Figure 2,

$$\mathbf{w} := \ominus A \oplus D = (\ominus A \oplus B) \boxplus (\ominus A \oplus C) =: \mathbf{u} \boxplus \mathbf{v}\,. \tag{47}$$

Here, by definition, **w** is the gyrovector formed by the gyrodiagonal AD of the gyroparallelogram $ABDC$. The gyrovector identity in (47), where D is given by (45), is explained in (50) below.

Gyrovectors in the ball $\mathbb{V}_c$ are, thus, equivalence classes of ordered pairs of points, which add according to the gyroparallelogram law shown in Figure 2.

The equivalence relation in vectors is origin independent. Hence, expressions appropriately derived from vectors are origin independent as well. Thus, in particular, (i) the length of a vector, (ii) the angle between to vectors with a common tail, and (iii) the parallelogram addition of to vectors with a common tail, are origin independent.

In contrast, the equivalence relation in gyrovectors is origin dependent. Fortunately, however, some important expressions derived from gyrovectors are origin independent. Thus, for instance, (i) the gyrolength of a gyrovector, (ii) the gyroangle between to gyrovectors with a common tail, and (iii) the gyroparallelogram addition of two gyrovectors with a common tail, are origin independent. A deep study of origin independence involves the study of *gyroisometries*, found in [79, Sections 3.11-3.12].

7. The Gyroparallelogram Addition Law

In Euclidean geometry a parallelogram is a quadrangle the two diagonals of which intersect at their midpoints. In full analogy, in hyperbolic geometry a gyroparallelogram is a gyroquadrangle the two gyrodiagonals of which intersect at their gyromidpoints, as shown in Figure 2. Accordingly, if A, B and C are any three non-gyrocollinear points (that is, they do not lie on a gyroline) in an Einstein gyrovector space, and if a fourth point D is given by the *gyroparallelogram condition*

$$D = (B \boxplus C) \ominus A\,, \tag{48}$$

then the gyroquadrangle $ABDC$ is a gyroparallelogram, shown in Figure 2.

Indeed, the two gyrodiagonals of gyroquadrangle $ABDC$ are the gyrosegments AD and BC, shown in Figure 2, the gyromidpoints of which coincide, that is,

$$\tfrac{1}{2}\otimes(A \boxplus D) = \tfrac{1}{2}\otimes(B \boxplus C) \tag{49}$$

where, by (31), the result in (49) is equivalent to the gyroparallelogram condition (48).

The analogies that equations (48) – (49) in gyrovector spaces share with their counterpart equations (38) – (39) in vector spaces indicate that both the gyrogroup

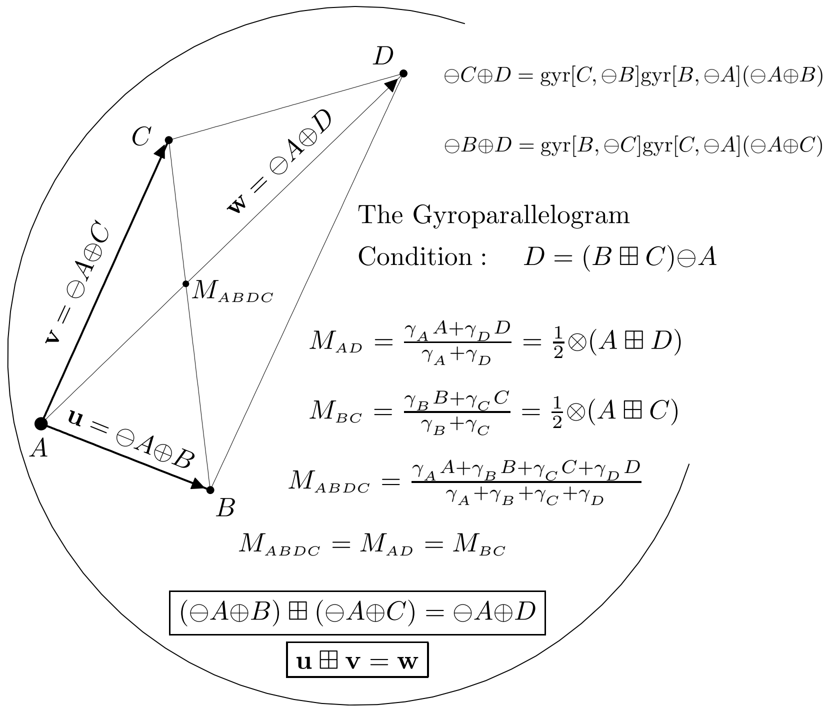

Figure 2: The Einstein gyroparallelogram and its addition law in an Einstein gyrovector plane $(\mathbb{R}^2_c, \oplus, \otimes)$. The gyrodiagonals AD and BC of gyroparallelogram $ABDC$ intersect each other at their gyromidpoints. The gyromidpoints of the gyrodiagonals AD and BC are, respectively, M_{AD} and M_{BC}, each of which coincides with the gyroparallelogram gyrocenter M_{ABDC}. The analogies that this figure shares with Figure 1 are obvious.

operation and cooperation, $\oplus$ and $\boxplus$, are necessary for our mission to capture analogies between vector and gyrovector spaces.

Let ABC be a gyrotriangle in an Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ and let D be the point that augments gyrotriangle ABC into the gyroparallelogram $ABDC$, as shown in Figure 2. Then, D is determined uniquely by the gyroparallelogram condition (48), obeying the gyroparallelogram addition law [72, Theorem 5.5]

$$(\ominus A \oplus B) \boxplus (\ominus A \oplus C) = (\ominus A \oplus D) \tag{50}$$

shown in Figure 2. In full analogy with the parallelogram addition law of vectors in Euclidean geometry, (40), the gyroparallelogram addition law (50) of gyrovectors in hyperbolic geometry can be written as

$$\mathbf{u} \boxplus \mathbf{v} = \mathbf{w} \tag{51}$$

where $\mathbf{u}, \mathbf{v}$ and $\mathbf{w}$ are the *gyrovectors*

$$\begin{aligned} \mathbf{u} &= \ominus A \oplus B \\ \mathbf{v} &= \ominus A \oplus C \\ \mathbf{w} &= \ominus A \oplus D \end{aligned} \tag{52}$$

which emanate from the point A [67, Chapter 5].

In his 1905 paper that founded the special theory of relativity [17], Einstein noted that his velocity addition does not satisfy the Euclidean parallelogram law:

> "Das Gesetz vom Parallelogramm der Geschwindigkeiten gilt also nach unserer Theorie nur in erster Annäherung."
>
> A. Einstein [17]

[English translation: Thus the law of velocity parallelogram is valid according to our theory only to a first approximation.]

Indeed, Einstein velocity addition, $\oplus$, is noncommutative and does not give rise to an exact "velocity parallelogram" in Euclidean geometry. However, as illustrated in Figure 2, Einstein velocity *coaddition*, $\boxplus$, which is commutative, does give rise to an exact "velocity gyroparallelogram" in hyperbolic geometry.

The breakdown of commutativity in Einstein velocity addition law seemed undesirable to the famous mathematician Émile Borel. Borel's resulting attempt to "repair" the seemingly "defective" Einstein velocity addition in the years following 1912 is described by Walter in [82, p. 117]. Here, however, we see that there is no need to repair Einstein velocity addition law for being noncommutative since, despite of being noncommutative, it gives rise to the gyroparallelogram law of gyrovector addition, which turns out to be commutative. The compatibility of the gyroparallelogram addition law of Einsteinian velocities with cosmological observations of stellar aberration is studied in [67, Chapter 13] and [72, Section 10.2]. The extension of the gyroparallelogram addition law of $k = 2$ summands into a higher dimensional gyroparallelotope addition law of $k > 2$ summands is presented in (54) − (55) below and studied in [67, Section 10.12] and [79, Section 6.4].

8. Gyroparallelotopes

The extreme sides of (33) give the equation

$$\mathbf{u} \boxplus_2 \mathbf{v} = 2 \otimes \frac{\gamma_{\mathbf{u}} \mathbf{u} + \gamma_{\mathbf{v}} \mathbf{v}}{2 + (\gamma_{\mathbf{u}} - 1) + (\gamma_{\mathbf{v}} - 1)} \tag{53}$$

where we replace $\boxplus$ by $\boxplus_2$ to emphasize that the binary operation $\boxplus = \boxplus_2$ is valid only for two summands.

Equation (53) is written in a form that suggests that the extension of the gyroparallelogram addition law (53), which involves two summands, to the gyroparallelepiped addition law, which involves three summands, is given by the following *gyroparallelepiped law*

$$\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 \mathbf{w} := 2\otimes\frac{\gamma_{\mathbf{u}}\mathbf{u}+\gamma_{\mathbf{v}}\mathbf{v}+\gamma_{\mathbf{w}}\mathbf{w}}{2+(\gamma_{\mathbf{u}}-1)+(\gamma_{\mathbf{v}}-1)+(\gamma_{\mathbf{w}}-1)} \tag{54}$$

$\mathbf{u},\mathbf{v},\mathbf{w}\in\mathbb{R}^n_c$.

Einstein coaddition (54) of three summands is commutative and associative in the generalized sense that it is a symmetric function of the summands. The gyroparallelepiped that results from the gyroparallelepiped law (54) is studied in detail in [67, Secions 10.9–10.12].

We may note that by (53) − (54) we have $\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 \mathbf{0} = \mathbf{u} \boxplus_2 \mathbf{v}$, as expected. However, unexpectedly we have $\mathbf{u} \boxplus_3 \mathbf{v} \boxplus_3 (\ominus\mathbf{v}) \neq \mathbf{u}$.

The extension of (54) to the Einstein coaddition of k summands, $k > 3$, is now straightforward, giving rise to the higher dimensional *gyroparallelotope law* in $\mathbb{R}^n_c$,

$$\mathbf{v}_1 \boxplus_k \mathbf{v}_2 \boxplus_k \ \ldots \ \boxplus_k \mathbf{v}_k := 2\otimes\frac{\sum_{i=1}^k \gamma_{\mathbf{v}_i}\mathbf{v}_i}{2+\sum_{i=1}^k(\gamma_{\mathbf{v}_i}-1)} \tag{55}$$

$\mathbf{v}_k \in G$, $k \in \mathbb{N}$. As expected, the gyroparallelotope law (55) is origin independent. An interesting study of parallelotopes in Euclidean geometry is found in [12].

In the Euclidean limit $c \to \infty$, (i) gamma factors tend to 1, and (ii) the hyperbolic scalar multiplication, $\otimes$, of a gyrovector (see Section 9) by 2 tends to the common scalar multiplication of a vector by 2. Hence, in the Euclidean limit, the right-hand side of (55) tends to the vector sum $\sum_{i=1}^k \mathbf{v}_i$ in $\mathbb{R}^n$, as expected.

9. Einstein Scalar Multiplication

The rich structure of Einstein addition is not limited to its gyrocommutative gyrogroup structure. Indeed, Einstein addition admits scalar multiplication (*gyromultiplication*), giving rise to the Einstein gyrovector space. Remarkably, the resulting Einstein gyrovector spaces form the setting for the Cartesian-Beltrami-Klein ball model of hyperbolic geometry just as vector spaces form the setting for the standard Cartesian model of Euclidean geometry, as shown in [60, 64, 67, 69, 71, 72, 79].

Let $k\otimes\mathbf{v}$ be the Einstein addition of k copies of $\mathbf{v} \in \mathbb{R}^n_c$, that is $k\otimes\mathbf{v} = \mathbf{v}\oplus\mathbf{v}\ldots\oplus\mathbf{v}$ (k terms). Then,

$$k\otimes\mathbf{v} = \frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^k - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^k}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^k + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^k}\frac{c\mathbf{v}}{\|\mathbf{v}\|} \tag{56}$$

for $\mathbf{v} \neq \mathbf{0}$, and $k\otimes\mathbf{0} = \mathbf{0}$.

The definition of scalar gyromultiplication in an Einstein gyrovector space requires analytically continuing k off the positive integers, thus obtaining the following definition.

Definition 9.1. (Einstein Scalar Multiplication). *An Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$ is an Einstein gyrogroup $(\mathbb{R}^n_c, \oplus)$ with scalar gyromultiplication $\otimes$ given by*

$$r\otimes\mathbf{v} = \frac{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r - \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r}{\left(1+\frac{\|\mathbf{v}\|}{c}\right)^r + \left(1-\frac{\|\mathbf{v}\|}{c}\right)^r} \frac{c\mathbf{v}}{\|\mathbf{v}\|} = \tanh\left(r \tanh^{-1}\frac{\|\mathbf{v}\|}{c}\right)\frac{c\mathbf{v}}{\|\mathbf{v}\|} \tag{57}$$

where r is any real number, $r \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}^n_c$, $\mathbf{v} \neq \mathbf{0}$, and $r\otimes\mathbf{0} = \mathbf{0}$, and with which we use the notation $\mathbf{v}\otimes r = r\otimes\mathbf{v}$.

As an example, it follows from Def. 9.1 that *Einstein half* is given by the equation

$$\tfrac{1}{2}\otimes\mathbf{v} = \frac{\gamma_{\mathbf{v}}\mathbf{v}}{1+\gamma_{\mathbf{v}}}\,, \tag{58}$$

so that, as expected, $\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v}\oplus\frac{\gamma_{\mathbf{v}}}{1+\gamma_{\mathbf{v}}}\mathbf{v} = \mathbf{v}$.

Einstein gyrovector spaces are studied in [60, 64, 67, 69, 71, 72, 79]. Einstein scalar multiplication does not distribute over Einstein addition, but it possesses other properties of vector spaces. For any positive integer k, and for all real numbers $r_1, r_2 \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^n_c$, we have

$$\begin{aligned} k\otimes\mathbf{v} &= \mathbf{v}\oplus\ldots\oplus\mathbf{v} && k \text{ terms} \\ (r_1+r_2)\otimes\mathbf{v} &= r_1\otimes\mathbf{v}\oplus r_2\otimes\mathbf{v} && \text{Scalar Distributive Law} \\ (r_1 r_2)\otimes\mathbf{v} &= r_1\otimes(r_2\otimes\mathbf{v}) && \text{Scalar Associative Law} \end{aligned} \tag{59}$$

in any Einstein gyrovector space $(\mathbb{R}^n_c, \oplus, \otimes)$.

Additionally, Einstein gyrovector spaces possess the *scaling property*

$$\frac{|r|\otimes\mathbf{a}}{\|r\otimes\mathbf{a}\|} = \frac{\mathbf{a}}{\|\mathbf{a}\|} \tag{60}$$

$\mathbf{a} \in \mathbb{R}^n_c,\ \mathbf{a} \neq \mathbf{0},\ r \in \mathbb{R},\ r \neq 0$, the *gyroautomorphism property*

$$\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a} \tag{61}$$

$\mathbf{a},\mathbf{u},\mathbf{v} \in \mathbb{R}^n_c$, $r \in \mathbb{R}$, and the identity gyroautomorphism

$$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I \tag{62}$$

$r_1, r_2 \in \mathbb{R}$, $\mathbf{v} \in \mathbb{R}_c^n$.

Any Einstein gyrovector space $(\mathbb{R}_c^n, \oplus, \otimes)$ inherits an inner product and a norm from its ambient vector space $\mathbb{R}^n$. These turn out to be invariant under gyrations,

$$\begin{aligned} \mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{u}\cdot\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v} &= \mathbf{u}\cdot\mathbf{v} \\ \|\mathrm{gyr}[\mathbf{a},\mathbf{b}]\mathbf{v}\| &= \|\mathbf{v}\| \end{aligned} \tag{63}$$

for all $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in \mathbb{R}_c^n$.

10. Gyrovector Spaces

Taking the key features of Einstein scalar multiplication as axioms, and guided by analogies with vector spaces, we are led to the following formal gyrovector space definition in which gyrovector spaces turn out to form a most natural generalization of vector spaces.

Definition 10.1. (Real Inner Product Gyrovector Spaces [67, p. 154]**).** A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:

(1) G is a subset of a real inner product vector space $\mathbb{V}$ called the ambient space of G, $G \subset \mathbb{V}$, from which it inherits its inner product, $\cdot$, and norm, $\|\cdot\|$, which are invariant under gyroautomorphisms, that is,

(V1)	$\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}\cdot\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{b} = \mathbf{a}\cdot\mathbf{b}$	Inner Product Gyroinvariance

for all points $\mathbf{a}, \mathbf{b}, \mathbf{u}, \mathbf{v} \in G$.

(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $\mathbf{a} \in G$:

(V2)	$1\otimes\mathbf{a} = \mathbf{a}$	Identity Scalar Multiplication
(V3)	$(r_1 + r_2)\otimes\mathbf{a} = r_1\otimes\mathbf{a}\oplus r_2\otimes\mathbf{a}$	Scalar Distributive Law
(V4)	$(r_1 r_2)\otimes\mathbf{a} = r_1\otimes(r_2\otimes\mathbf{a})$	Scalar Associative Law
(V5)	$\dfrac{\|r\|\otimes\mathbf{a}}{\\|r\otimes\mathbf{a}\\|} = \dfrac{\mathbf{a}}{\\|\mathbf{a}\\|}, \quad \mathbf{a} \neq \mathbf{0},\ r \neq 0$	Scaling Property
(V6)	$\mathrm{gyr}[\mathbf{u},\mathbf{v}](r\otimes\mathbf{a}) = r\otimes\mathrm{gyr}[\mathbf{u},\mathbf{v}]\mathbf{a}$	Gyroautomorphism Property
(V7)	$\mathrm{gyr}[r_1\otimes\mathbf{v}, r_2\otimes\mathbf{v}] = I$	Identity Gyroautomorphism.

(3) Real, one-dimensional vector space structure $(\|G\|, \oplus, \otimes)$ for the set $\|G\|$ of one-dimensional "vectors".

(V8)	$\\|G\\| = \{\pm\\|\mathbf{a}\\| : \mathbf{a} \in G\} \subset \mathbb{R}$	Vector Space

with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $\mathbf{a}, \mathbf{b} \in G$,

(V9) $\|r\otimes\mathbf{a}\| = |r|\otimes\|\mathbf{a}\|$ Homogeneity Property

(V10) $\|\mathbf{a}\oplus\mathbf{b}\| \le \|\mathbf{a}\|\oplus\|\mathbf{b}\|$ Gyrotriangle Inequality.

Einstein (gyro)addition and scalar (gyro)multiplication in $\mathbb{R}_c^n$ thus give rise to the Einstein gyrovector spaces $(\mathbb{R}_c^n, \oplus, \otimes)$, $n \ge 2$.

11. Relativistic and Classical Kinetic Energy

Kinetic energy depends on mass and relative velocity. The relativistic mass of an object with Newtonian mass m (also called *relativistically invariant mass*) moving with velocity $\mathbf{v} \in \mathbb{R}_c^3$ relative to an inertial frame Σ_0 is $m\gamma_{\mathbf{v}}$. Having Einstein half (58) in hand, we can recast the relativistic kinetic energy of moving objects into a form that shares analogies with its classical counterpart. The relativistic kinetic energy K_{rel} of an object with rest (Newtonian) mass m that moves uniformly with velocity $\mathbf{v}$ relative to an inertial frame Σ_0 is given by the equation [66]

$$K_{rel} = c^2 m(\gamma_{\mathbf{v}} - 1)\,, \tag{64}$$

where c is the vacuum speed of light. We manipulate (64) in the following chain of equations, some of which are numbered for subsequent explanation.

$$\begin{aligned} K_{rel} &\overset{(1)}{\widehat{=\!=}} c^2 m(\gamma_{\mathbf{v}} - 1) = \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} c^2 m(\gamma_{\mathbf{v}} - 1)\frac{\gamma_{\mathbf{v}}+1}{\gamma_{\mathbf{v}}^2} \\ &=\!= \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} c^2 m \frac{\gamma_{\mathbf{v}}^2 - 1}{\gamma_{\mathbf{v}}^2} \\ &\overset{(2)}{\widehat{=\!=}} \frac{\gamma_{\mathbf{v}}^2}{\gamma_{\mathbf{v}}+1} m\mathbf{v}^2 = \frac{\gamma_{\mathbf{v}}\mathbf{v}}{\gamma_{\mathbf{v}}+1} \cdot m\gamma_{\mathbf{v}}\mathbf{v} \\ &\overset{(3)}{\widehat{=\!=}} (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v})\,, \end{aligned} \tag{65}$$

$m\gamma_{\mathbf{v}}$ being the velocity dependent relativistic mass [73] of the moving object relative to Σ_0.

Derivation of the numbered equalities in (65) follows:

1. Follows from (64).
2. Follows from (10), p. 8.

3. Follows from Einstein half (58).

The relativistic kinetic energy K_{rel} in (65),

$$K_{rel} = (\tfrac{1}{2}\otimes\mathbf{v})\cdot(m\gamma_{\mathbf{v}}\mathbf{v})\,, \tag{66}$$

is given by the inner product of a "relativistic half-velocity" and a corresponding relativistic momentum, in full analogy with the classical kinetic energy K_{cls},

$$K_{cls} = \tfrac{1}{2}m\mathbf{v}^2 = (\tfrac{1}{2}\mathbf{v})\cdot(m\mathbf{v})\,, \tag{67}$$

which is given by the inner product of a "classical half-velocity" and a corresponding classical momentum. The ability of Einstein scalar multiplication to capture analogies between modern and classical results thus emerges.

The analogies that (66) and (67) share demonstrate that the relativistic counterpart of the Newtonian mass m is the relativistic, velocity dependent mass $m\gamma_{\mathbf{v}}$. The controversy around the relativistic mass is described in [73]. It is owing to analogies that Newtonian mass and Einsteinian relativistic mass share that the notion of barycentric coordinates in Euclidean geometry can be translated into the notion of gyrobarycentric coordinates in hyperbolic geometry, as shown in [71, 72] and [70, 75, 76], and in Sections 16-18.

12. Einstein Gyrolines – The Hyperbolic Lines

In applications to geometry, where the letters a, b, c are frequently used, it is convenient to replace the notation $\mathbb{R}_c^n$ for the c-ball of an Einstein gyrovector space by the s-ball, $\mathbb{R}_s^n$. We thus switch from c to s to avoid notational confusion. Moreover, it is understood that $n \geq 2$, unless specified otherwise.

Let $A, B \in \mathbb{R}_s^n$ be two distinct points of the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, and let $t \in \mathbb{R}$ be a real parameter. Then, the graph L_{AB} of the set of all points

$$L_{AB} = A\oplus(\ominus A\oplus B)\otimes t\,, \tag{68}$$

$t \in \mathbb{R}$, in the Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is a chord of the ball $\mathbb{R}_s^n$. As such, it is a geodesic line of the Beltrami-Klein ball model of hyperbolic geometry, shown in Figure 3 for $n = 2$. The geodesic line (68) is the unique gyroline that passes through the points A and B. It passes through the point A when $t = 0$ and, owing to the left cancellation law, (7), it passes through the point B when $t = 1$. Furthermore, it passes through the midpoint $m_{A,B}$ of A and B when $t = 1/2$. Accordingly, the *gyrosegment* AB that joins the points A and B in Figure 3 is obtained from gyroline (68) with $0 \leq t \leq 1$.

Gyrolines (68) are the geodesics of the Beltrami-Klein ball model of hyperbolic geometry. Similarly, gyrolines (68) with Einstein addition $\oplus$ replaced by Möbius addition $\oplus_{\mathrm{M}}$ are the geodesics of the Poincaré ball model of hyperbolic geometry.

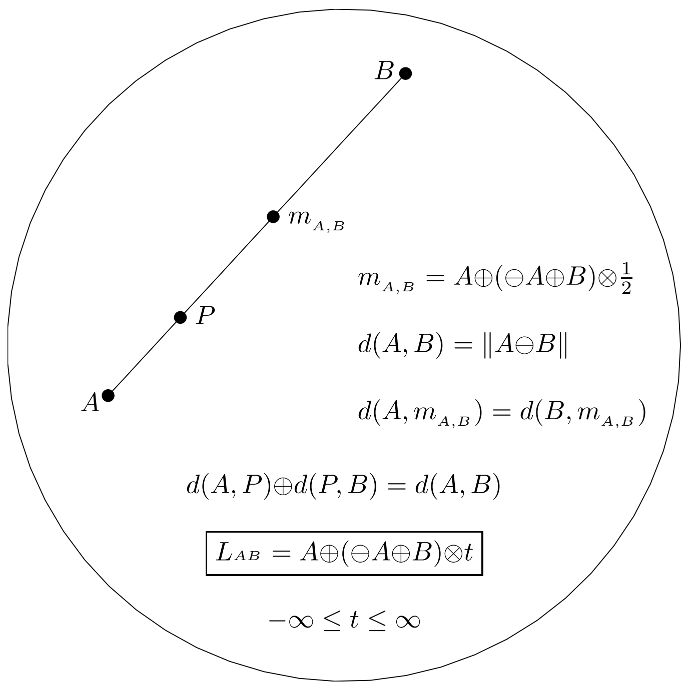

Figure 3: Gyrolines, the hyperbolic lines L_{AB} in Einstein gyrovector spaces, are fully analogous to the straight line $A + (-A + B)t$, $t \in \mathbb{R}$, in the Cartesian model of the Euclidean geometry of $\mathbb{R}^n$. Here $\oplus = \oplus_{\mathrm{E}}$ is Einstein addition, as opposed to Figure 4 where $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition. The figure shows that Einstein gyrolines in the hyperbolic plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are Euclidean segments in the disc $\mathbb{R}^2_s$.

These interesting results are established by methods of differential geometry in [65], and are illustrated in Figures 3 and 4.

Each point of (68) with $0 < t < 1$ is said to lie *between* A and B. Thus, for instance, the point P in Figure 3 lies between the points A and B. As such, the points A, P and B obey the *gyrotriangle equality* according to which

$$d(A,P) \oplus d(P,B) = d(A,B)\,, \tag{69}$$

in full analogy with Euclidean geometry. Here

$$d(A,B) = \|\ominus A \oplus B\|\,, \tag{70}$$

$A, B \in \mathbb{R}^n_s$, is the Einstein *gyrodistance function*, also called the Einstein *gyrometric*. This gyrodistance function in Einstein gyrovector spaces corresponds bijectively to a standard hyperbolic distance function, as demonstrated in [67, Section 6.19].

A contact between Einstein gyrodistance function and differential geometry is provided by the Riemannian gyroline element of Einstein gyrovector spaces, studied in [64, Section 7.5] and [65]. It turns out that the Riemannian gyroline element of Einstein gyrovector spaces, given by

$$ds^2 = \|(\mathbf{v} + d\mathbf{v}) \ominus \mathbf{v}\|^2 \tag{71}$$

is identical with the well-known Riemannian line element of the Beltrami-Klein disc model of hyperbolic geometry.

13. Möbius Addition

The most general Möbius transformation of the complex open unit disc

$$\mathbb{D} = \{z \in \mathbb{C} : \, |z| < 1\} \tag{72}$$

in the complex plane $\mathbb{C}$ is given by the polar decomposition [2, 28],

$$z \mapsto e^{i\theta} \frac{a + z}{1 + \overline{a}z} = e^{i\theta}(a \oplus_{\mathrm{M}} z) \,. \tag{73}$$

It induces the Möbius addition $\oplus_{\mathrm{M}}$ in the disc, allowing the Möbius transformation of the disc to be viewed as a Möbius left *gyrotranslation*

$$z \mapsto a \oplus_{\mathrm{M}} z = \frac{a + z}{1 + \overline{a}z} \tag{74}$$

followed by a rotation. Here $\theta \in \mathbb{R}$ is a real number, $a, z \in \mathbb{D}$, and $\overline{a}$ is the complex conjugate of a.

In order to extend Möbius addition from the disk to the ball, let us identify complex numbers of the complex plane $\mathbb{C}$ with vectors of the Euclidean plane $\mathbb{R}^2$ in the usual way,

$$\mathbb{C} \ni u = u_1 + iu_2 = (u_1, u_2) = \mathbf{u} \in \mathbb{R}^2 \,. \tag{75}$$

Then

$$\begin{aligned} \bar{u}v + u\bar{v} &= 2\mathbf{u}\cdot\mathbf{v} \\ |u| &= \|\mathbf{u}\| \end{aligned} \tag{76}$$

give the inner product and the norm in $\mathbb{R}^2$, so that Möbius addition in the disc $\mathbb{D}$ of $\mathbb{C}$ becomes Möbius addition in the disc

$$\mathbb{R}^2_{s=1} = \{\mathbf{v} \in \mathbb{R}^2 : \|\mathbf{v}\| < s = 1\} \tag{77}$$

of $\mathbb{R}^2$. Indeed,

$$\begin{aligned}
\mathbb{D} \ni u\oplus v &= \frac{u+v}{1+\bar{u}v} \\
&= \frac{(1+u\bar{v})(u+v)}{(1+\bar{u}v)(1+u\bar{v})} \\
&= \frac{(1+\bar{u}v+u\bar{v}+|v|^2)u+(1-|u|^2)v}{1+\bar{u}v+u\bar{v}+|u|^2|v|^2} \\
&= \frac{(1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{v}\|^2)\mathbf{u}+(1-\|\mathbf{u}\|^2)\mathbf{v}}{1+2\mathbf{u}\cdot\mathbf{v}+\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \\
&= \mathbf{u}\oplus\mathbf{v} \in \mathbb{R}^2_{s=1}
\end{aligned} \tag{78}$$

for all $u, v \in \mathbb{D}$ and all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^2_{s=1}$. The last equation in (78) is a vector equation, so that its restriction to the ball of the Euclidean two-dimensional space is a mere artifact. Suggestively, we thus arrive at the following definition of Möbius addition in the ball $\mathbb{R}^n_s$,

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \frac{(1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^2}\|\mathbf{v}\|^2)\mathbf{u}+(1-\frac{1}{s^2}\|\mathbf{u}\|^2)\mathbf{v}}{1+\frac{2}{s^2}\mathbf{u}\cdot\mathbf{v}+\frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2}\,. \tag{79}$$

Like Einstein groupoids $(\mathbb{R}^n_s, \oplus_{\mathrm{E}})$, Möbius groupoids $(\mathbb{R}^n_s, \oplus_{\mathrm{M}})$ are gyrocommutative gyrogroups. The gyrogroup isomorphism between Einstein addition $\oplus = \oplus_{\mathrm{E}}$ and Möbius addition $\oplus_{\mathrm{M}}$ is given by the equations [67, p. 227]

$$\begin{aligned}
\tfrac{1}{2}\otimes_{\mathrm{E}}(\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}) &= \tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes_{\mathrm{M}}\mathbf{v} \\
2\otimes_{\mathrm{M}}(\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v}) &= 2\otimes_{\mathrm{E}}\mathbf{u}\oplus_{\mathrm{E}}2\otimes_{\mathrm{E}}\mathbf{v}
\end{aligned} \tag{80}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$.

The operations $\otimes_{\mathrm{E}}$ and $\otimes_{\mathrm{M}}$ are identical to each other, $\otimes_{\mathrm{E}} = \otimes_{\mathrm{M}} =: \otimes$. Hence, Identities (80) can be written equivalently as

$$\begin{aligned}
\mathbf{u}\oplus_{\mathrm{E}}\mathbf{v} &= 2\otimes(\tfrac{1}{2}\otimes\mathbf{u}\oplus_{\mathrm{M}}\tfrac{1}{2}\otimes\mathbf{v}) \\
\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} &= \tfrac{1}{2}\otimes(2\otimes\mathbf{u}\oplus_{\mathrm{E}}2\otimes\mathbf{v})
\end{aligned} \tag{81}$$

for all $\mathbf{u}, \mathbf{v} \in \mathbb{V}_s$.

The related connection between Möbius transformation and Lorentz transformation of Einstein's special theory of relativity was recognized by H. Liebmann in 1905 [36, pp. 122–123].

When $\mathbf{u}$ and $\mathbf{v}$ are parallel in $\mathbb{R}^n_s \subset \mathbb{R}^n$, scalar gyromultiplication is distributive over gyroaddition [27]. Hence, in the special case when $\mathbf{u}\|\mathbf{v}$ in $\mathbb{R}^n$ the two equations in (81) degenerate to the single equation

$$\mathbf{u}\oplus_{\mathrm{M}}\mathbf{v} = \mathbf{u}\oplus_{\mathrm{E}}\mathbf{v}\,, \qquad \mathbf{u}\|\mathbf{v} \tag{82}$$

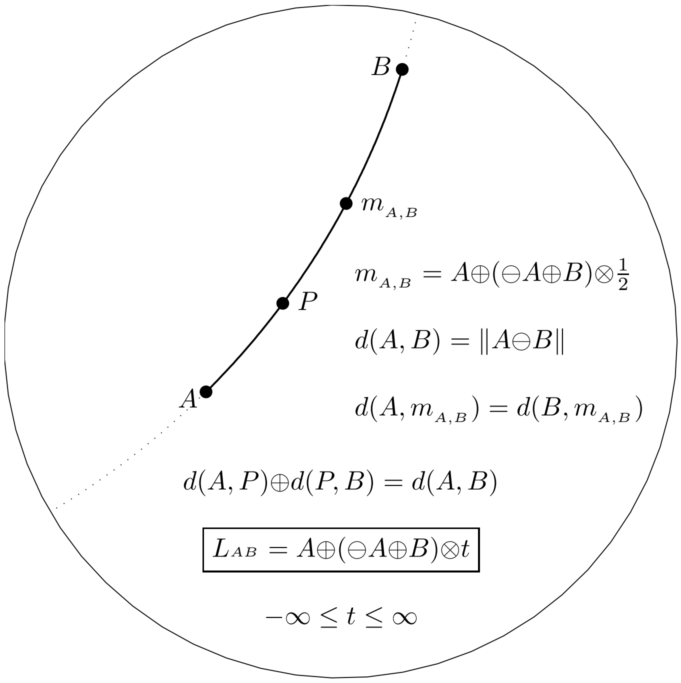

Figure 4: Gyrolines, the hyperbolic lines L_{AB} in Möbius gyrovector spaces, are fully analogous to lines in Euclidean spaces. The gyroline $L_{AB} = A\oplus(\ominus A\oplus B)\otimes t$, $t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ is a geodesic line in the Cartesian-Poincaré ball model of hyperbolic geometry. Here $\oplus = \oplus_{\mathrm{M}}$ is Möbius addition, as opposed to Figure 3 where $\oplus = \oplus_{\mathrm{E}}$ is Einstein addition. The figure indicates that Möbius gyrolines in the hyperbolic plane $(\mathbb{R}_s^2, \oplus, \otimes)$ are Euclidean circular arcs in the disc $\mathbb{R}_s^2$ that approach the boundary of the disc orthogonally.

Accordingly, Einstein scalar multiplication, $\otimes_{\mathrm{E}}$, and Möbius scalar multiplication, $\otimes_{\mathrm{M}}$, share the same formula, $\otimes_{\mathrm{E}} = \otimes_{\mathrm{M}} =: \otimes$, where $\otimes$ is given by (57),

Einstein and Möbius addition are originated from totally two different disciplines. Accordingly, the elegant relationship (81) between Einstein and Möbius addition indicates, once again, the intrinsic beauty, harmony and interdisciplinarity in Einstein addition.

14. Möbius Gyrolines

Replacing Einstein addition $\oplus = \oplus_{\mathrm{E}}$ in Section 12 by Möbius addition $\oplus = \oplus_{\mathrm{M}}$ in this section, we obtain the Möbius gyrolines

$$L_{AB} = A\oplus(\ominus A\oplus B)\otimes t\,, \tag{83}$$

$t \in \mathbb{R}$, in a Möbius gyrovector space $(\mathbb{R}^n_s, \oplus, \otimes)$, shown in Figure 4 for $n = 2$. As we see from Figure 4, Möbius gyrolines in the Möbius gyrovector plane $(\mathbb{R}^2_s, \oplus, \otimes)$ are circular arcs that approach the boundary of the disc $\mathbb{R}^2_s$ orthogonally. These are the well-known geodesics of the Poincaré disc model of hyperbolic geometry.

Along with Möbius gyrolines we have the Möbius gyrodistance function

$$d(A, B) = \|\ominus A \oplus B\| \tag{84}$$

and the gyrotriangle equality

$$d(A, P) \oplus d(P, B) = d(A, B) \tag{85}$$

for any $A, B, P \in (\mathbb{R}^n_s, \oplus)$, where P lies between A and B, as shown in Figure 4.

A contact between Möbius gyrodistance function and differential geometry is provided by the Riemannian gyroline element of Möbius gyrovector spaces, studied in [64, Section 7.3] and [65]. It turns out that the Riemannian gyroline element of Möbius gyrovector spaces, given by

$$ds^2 = \|(\mathbf{v} + d\mathbf{v}) \ominus \mathbf{v}\|^2 , \tag{86}$$

is identical with the well-known Riemannian line element of the Poincaré disc model of hyperbolic geometry.

It should be emphasized that Equations (83) – (86) of this section are identical in form with Equations (68) – (71) of Section 12. However, $\oplus = \oplus_{\text{E}}$ in Section 12, while $\oplus = \oplus_{\text{M}}$ in this section, where Einstein addition $\oplus = \oplus_{\text{E}}$ is given by (2), p. 7, and Möbius addition $\oplus_{\text{M}}$ is given by (79), p. 29.

15. Gyrotrigonometry

Hyperbolic trigonometry is called *gyrotrigonometry* and, similarly, hyperbolic angles are called *gyroangles*. Graphically, gyrotrigonometry is best illustrated in the Poincaré disc model of hyperbolic geometry since the Poincaré ball model is *conformal* in the following sense. A gyroangle between two intersecting Möbius gyrolines equals the angle between corresponding intersecting tangent lines, as shown in Figure 5. The equations in this section are valid in any gyrovector space. In particular, they are valid in Einstein gyrovector spaces, when $\oplus = \oplus_{\text{E}}$, and in Möbius gyrovector spaces, when $\oplus = \oplus_{\text{M}}$. Graphical illustrations are presented for Möbius gyrovector planes in Figures 5 and 6.

The gyroangle included by the gyrosegments AB and AC that emanate from the point A, denoted $\angle BAC$, has the measure α given by the equation [64, 67, 69, 71, 72, 79]

$$\cos \alpha = \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} , \tag{87}$$

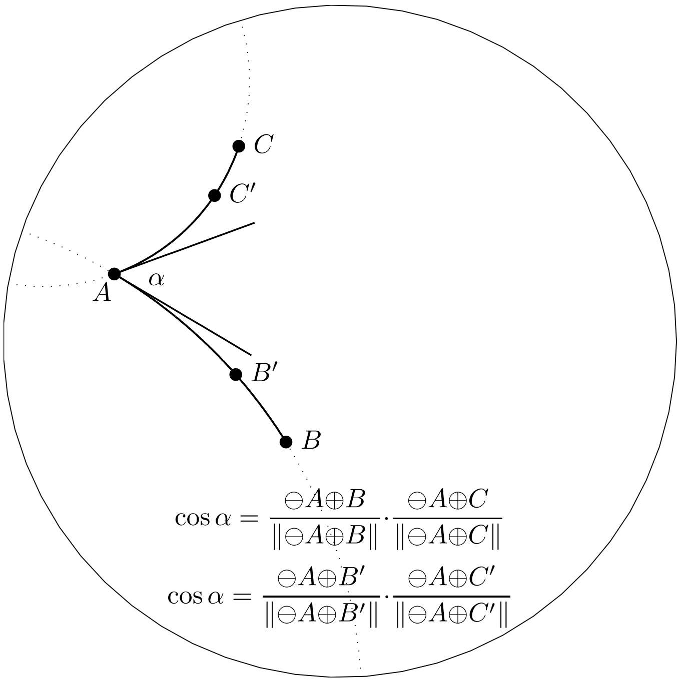

Figure 5: A Möbius gyroangle α generated by two intersecting Möbius geodesic rays (gyrorays). Its measure equals the measure of the Euclidean angle generated by corresponding intersecting tangent lines.

$A, B, C \in \mathbb{R}_s^n$, where "cos" is the common cosine function of trigonometry. Accordingly,

$$\alpha = \cos^{-1} \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|}, \tag{88}$$

$0 \le \alpha < \pi$. The point A is the vertex of the gyroangle $\angle BAC$. A gyroangle with vertex at the origin, $O = \mathbf{0}$, of the ball coincides with its Euclidean counterpart,

$$\cos\alpha = \frac{\ominus O \oplus B}{\|\ominus O \oplus B\|} \cdot \frac{\ominus O \oplus C}{\|\ominus O \oplus C\|} = \frac{B}{\|B\|} \cdot \frac{C}{\|C\|}. \tag{89}$$

The measure of a gyroangle is invariant under the motions of hyperbolic geometry, which are left gyrotranslations and rotations. In particular, any gyroangle with vertex A can be moved by a hyperbolic motion (gyromotion) to a gyroangle with vertex O while keeping the gyroangle measure invariant. Having vertex O, the resulting gyroangle behaves like an angle. Hence, trigonometric identities for angles as, for instance, $\cos^2\alpha + \sin^2\alpha = 1$, remain valid for gyroangles as well. Gyrotrigonometry and its application in analytic hyperbolic geometry are studied in [64, 67, 69, 71, 72, 79]. An elegant application of gyrotriangle gyrotrigonometry, which has no Euclidean counterpart, is presented in Figure 6.

A Möbius gyrotriangle along with its standard notation and some basic identities is presented in Figure 6. Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with vertices $A, B, C \in \mathbb{R}_s^n$, sides $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}_s^n$ and side gyrolengths $a, b, c \in (-s, s)$,

$$\begin{aligned} \mathbf{a} &= \ominus B \oplus C, & a &= \|\mathbf{a}\|, \\ \mathbf{b} &= \ominus C \oplus A, & b &= \|\mathbf{b}\|, \\ \mathbf{c} &= \ominus A \oplus B, & c &= \|\mathbf{c}\|. \end{aligned} \tag{90}$$

The gyroangle measures α, β and γ of the gyroangles at the vertices A, B and C are given by the gyrotrigonometric identities

$$\begin{aligned} \cos\alpha &= \frac{\ominus A \oplus B}{\|\ominus A \oplus B\|} \cdot \frac{\ominus A \oplus C}{\|\ominus A \oplus C\|} \\ \cos\beta &= \frac{\ominus B \oplus A}{\|\ominus B \oplus C\|} \cdot \frac{\ominus B \oplus A}{\|\ominus B \oplus C\|} \\ \cos\gamma &= \frac{\ominus C \oplus A}{\|\ominus C \oplus A\|} \cdot \frac{\ominus C \oplus B}{\|\ominus C \oplus B\|} \end{aligned} \tag{91}$$

in full analogy with corresponding trigonometric identities.

In Euclidean geometry the triangle angles do not determine its side lengths. In contrast, in hyperbolic geometry the gyrotriangle gyroangles determine uniquely its side gyrolengths according to the gyrotriangle gyrotrigonometric identities (92) of the following theorem 15.1 [64, Theorem 8.48].

Theorem 15.1. (AAA to SSS Conversion Law). *Let ABC be a gyrotriangle in a Möbius gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with vertices A, B, C, corresponding gyroangles α, β, γ, $0 < \alpha + \beta + \gamma < \pi$, and side gyrolengths a, b, c, as shown in Figure 6. The side gyrolengths of the gyrotriangle ABC are determined by its gyroangles according to the AAA to SSS conversion equations*

$$\begin{aligned} \frac{a^2}{s^2} &= \frac{\cos\alpha + \cos(\beta + \gamma)}{\cos\alpha + \cos(\beta - \gamma)} \\[2ex] \frac{b^2}{s^2} &= \frac{\cos\beta + \cos(\alpha + \gamma)}{\cos\beta + \cos(\alpha - \gamma)} \\[2ex] \frac{c^2}{s^2} &= \frac{\cos\gamma + \cos(\alpha + \beta)}{\cos\gamma + \cos(\alpha - \beta)} . \end{aligned} \tag{92}$$

In the Euclidean limit $s \to \infty$, the equations in (92) reduce, respectively, to

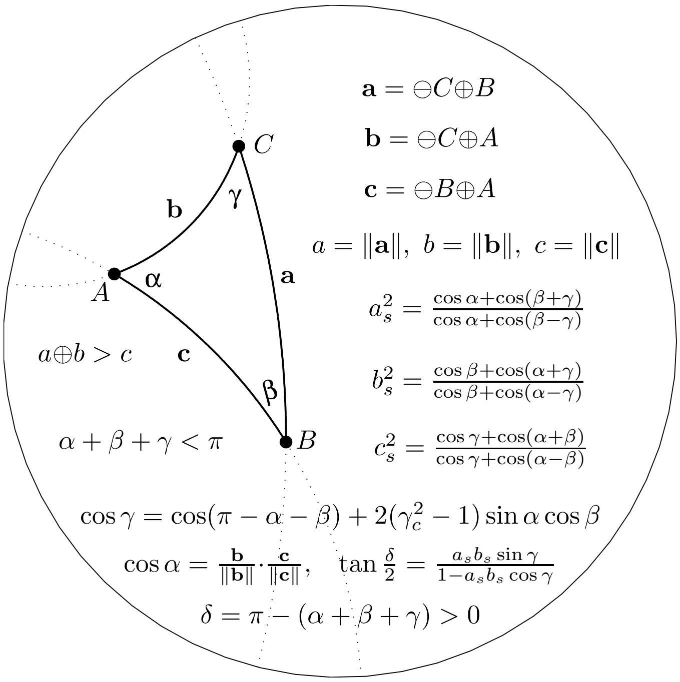

Figure 6: A Möbius gyrotriangle ABC in the Möbius gyrovector plane $\mathbb{D} = (\mathbb{R}_s^2, \oplus, \otimes)$ is shown. Its sides are formed by gyrovectors that link its vertices, in full analogy with Euclidean triangles. Its hyperbolic side lengths, a, b, c, are uniquely determined in (93) by its gyroangles. The gyrotriangle gyroangle sum is less than π. Here, $a_s = a/s$, etc. Note that in the limit of large s, $s \to \infty$, the $\cos\gamma$ equation reduces to $\cos\gamma = \cos(\pi - \alpha - \beta)$ so that $\alpha + \beta + \gamma = \pi$, implying that both sides of each of the squared side gyrolength equations, shown in the figure and listed in (93), vanish.

the equations

$$\begin{aligned} 0 &= \cos\alpha + \cos(\beta + \gamma) \\ 0 &= \cos\beta + \cos(\alpha + \gamma) \\ 0 &= \cos\gamma + \cos(\alpha + \beta) \end{aligned} \tag{93}$$

each of which is equivalent to the Euclidean identity

$$\alpha + \beta + \gamma = \pi\,. \tag{94}$$

Hence, the AAA (gyroAngle gyroAngle gyroAngle) to SSS (gyroSide gyroSide gyroSide) Conversion Law (92) in Theorem 15.1 is valid in hyperbolic geometry, where $\alpha+\beta+\gamma<\pi$, and it is invalid in Euclidean geometry, where the triangle angle identity (94) holds.

16. Resultant Relativistically Invariant Mass

The relativistic mass $m\gamma_{\mathbf{v}}$, already encountered in Section 11, plays an important role in Einstein's special relativity and in analytic hyperbolic geometry [73]. Einstein velocity addition admits the following theorem about relativistic mass.

Theorem 16.1. (Resultant Relativistically Invariant Mass Theorem). *Let $(\mathbb{R}_s^n, \oplus)$ be an Einstein gyrogroup, and let $m_k \in \mathbb{R}$ and $\mathbf{v}_k \in \mathbb{R}_s^n$, $k=1,2,\ldots,N$, be N real numbers and N elements of $\mathbb{R}_s^n$ satisfying*

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \neq 0\,. \tag{95}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{\mathbf{v}_k} \\ \gamma_{\mathbf{v}_k}\mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_{\mathbf{v}_0} \\ \gamma_{\mathbf{v}_0}\mathbf{v}_0 \end{pmatrix} \tag{96}$$

be an $(n+1)$-vector equation for the two unknowns $m_0 \in \mathbb{R}$ and $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (96) possesses a unique solution $(m_0, \mathbf{v}_0)$, $m_0 \neq 0$, $\mathbf{v}_0 \in \mathbb{R}_s^n$, satisfying the following three identities for all $\mathbf{w} \in \mathbb{R}_s^n$ (including, in particular, the interesting special case of $\mathbf{w}=\mathbf{0}$):

$$\mathbf{w}\oplus\mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}} \tag{97}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0} = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}}{m_0} \tag{98}$$

$$\gamma_{\mathbf{w}\oplus\mathbf{v}_0}(\mathbf{w}\oplus\mathbf{v}_0) = \frac{\sum_{k=1}^{N} m_k \gamma_{\mathbf{w}\oplus\mathbf{v}_k}(\mathbf{w}\oplus\mathbf{v}_k)}{m_0} \tag{99}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus(\mathbf{w}\oplus\mathbf{v}_j)\oplus(\mathbf{w}\oplus\mathbf{v}_k)} - 1)}\,. \tag{100}$$

The proof of Theorem 16.1 is found in [71, Theorem 3.7] and in [72, Theorem 3.2].

It follows from (96) that (i) $m_0\gamma_{\mathbf{v}_0}$ is the resultant relativistic mass of a system of N particles with relativistic masses $m_k\gamma_{\mathbf{v}_k}$, and (ii) $m_0\gamma_{\mathbf{v}_0}\mathbf{v}_0$ is the resultant relativistic momentum of a system of N particles with relativistic momenta $m_k\gamma_{\mathbf{v}_k}\mathbf{v}_k$,

$k = 1, \ldots, N$. In physical applications $n = 3$, and $m_k > 0$, $k = 0, 1, \ldots, N$, are positive real numbers that represent relativistically invariant (Newtonian) masses. In geometry, however, $n \geq 1$ and m_k are any real numbers that need not be positive.

Identities (97) − (99) of Theorem 16.1 are *covariant* in the sense the $\mathbf{v}_0$ and $\mathbf{v}_k$ vary together under left gyrotranslations by any $\mathbf{w} \in \mathbb{R}_s^n$. The constant m_0 in (100) in *invariant* in the sense that it remains invariant under left gyrotranslations of $\mathbf{v}_k$ by any $\mathbf{w} \in \mathbb{R}_s^n$.

It follows from (100) that the relativistically invariant mass m_0 of a particle system of N particles is greater than the sum $\sum_{k=1}^{N} m_k$ of the Newtonian Masses of its constituents. The excessive mass, $m_0 - \sum_{k=1}^{N} m_k$, is *dark* in the sense that (i) it is generated by internal relative velocities between the constituents of the particle system, and that (ii) it reveals its presence only gravitationally, since it emits no radiation and it involves no collisions [68, 73]. Interestingly, the relativistically invariant mass m_0 of a particle system in (100) is precisely what we need in order to adapt the Euclidean notion of barycentric coordinates for use in hyperbolic geometry without losing covariance.

To appreciate the power and elegance of Theorem 16.1 in relativistic mechanics in terms of novel analogies that it shares with familiar results in classical mechanics, we present below the classical counterpart, Theorem 16.2, of Theorem 16.1. Theorem 16.2 is derived from Theorem 16.1 by approaching the Newtonian/Euclidean limit when $s = c$ tends to infinity. The resulting Theorem 16.2 is immediate, and its importance in classical mechanics is well-known. Like Theorem 16.1, Theorem 16.2 involves an expression, (103) below, which is covariant under translations and, as such, fully analogous to (97), which is covariant under left gyrotranslations.

Theorem 16.2. (Resultant Newtonian Invariant Mass Theorem). *Let* $(\mathbb{R}^n, +)$ *be a Euclidean* n*-space, and let* $m_k \in \mathbb{R}$ *and* $\mathbf{v}_k \in \mathbb{R}^n$, $k = 1, 2, \ldots, N$, *be* N *real numbers and* N *elements of* $\mathbb{R}^n$ *satisfying*

$$\sum_{k=1}^{N} m_k \neq 0 \tag{101}$$

Furthermore, let

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ \mathbf{v}_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ \mathbf{v}_0 \end{pmatrix} \tag{102}$$

be an $(n+1)$*-vector equation for the two unknowns* $m_0 \in \mathbb{R}$ *and* $\mathbf{v}_0 \in \mathbb{R}^n$.

Then (102) *possesses a unique solution* $(m_0, \mathbf{v}_0)$, $m_0 \neq 0$, *satisfying the following equations for all* $\mathbf{w} \in \mathbb{R}^n$ *(including, in particular, the interesting special case of* $\mathbf{w} = \mathbf{0}$*):*

$$\mathbf{w} + \mathbf{v}_0 = \frac{\sum_{k=1}^{N} m_k(\mathbf{w} + \mathbf{v}_k)}{\sum_{k=1}^{N} m_k} \tag{103}$$

and

$$m_0 = \sum_{k=1}^{N} m_k \,. \tag{104}$$

The proof of Theorem 16.2 is immediate.

Unlike Identity (103) of Theorem 16.2, which is immediate, its counterpart in Theorem 16.1, Identity (97), is not immediate and, hence, unexpected. Yet, in full analogy with Theorem 16.2, the validity of Identity (97) in Theorem 16.1 for all $\mathbf{w} \in \mathbb{R}_c^n$ is geometrically important. This geometric importance of Identity (97) stems from its following implication: The velocity $\mathbf{v}_0$ of the center of momentum frame of a particle system relative to a given inertial rest frame in relativistic mechanics is independent of the choice of the origin of the relativistic velocity space $\mathbb{R}_s^n$ with its underlying Cartesian-Beltrami-Klein ball model of hyperbolic geometry.

Not unexpectedly, the Newtonian mass m_0 in (104) of a particle system plays an important role in Theorem 17.3, p. 39, on the covariance of barycentric coordinates under the motions of Euclidean geometry, which are translations and rotations. Remarkably, the relativistic invariant mass m_0 in (100) of a particle system plays an analogous important role in Theorem 18.3, p. 42, on the gyrocovariance of gyrobarycentric coordinates under the gyromotions of hyperbolic geometry, which are left gyrotranslations and rotations. Left gyrotranslations, in turn, play an important role in the application of gyrobarycentric coordinates for determining analytically various gyrotriangle gyrocenters in [71, 72, 79].

17. Barycentric Coordinates

The notion of barycentric coordinates dates back to Möbius. The use of barycentric coordinates in Euclidean geometry is described in [84], and the historical contribution of Möbius' barycentric coordinates to vector analysis is described in [14, pp. 48–50].

In this section we set the stage for the introduction in Section 18 of barycentric coordinates into hyperbolic geometry by illustrating the way Theorem 16.2, p. 36, suggests the introduction of barycentric coordinates into Euclidean geometry.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers such that

$$\sum_{k=1}^{N} m_k \neq 0 \tag{105}$$

and let $A_k \in \mathbb{R}^n$ be N given points in the Euclidean n-space $\mathbb{R}^n$, $k = 1, \ldots, N$. Theorem 16.2, p. 36, states the trivial, but geometrically significant, result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} 1 \\ A_k \end{pmatrix} = m_0 \begin{pmatrix} 1 \\ P \end{pmatrix} \tag{106}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}^n$ possesses the unique solution given by

$$m_0 = \sum_{k=1}^{N} m_k \tag{107}$$

and

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{108}$$

satisfying for all $X \in \mathbb{R}^n$,

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k}\,. \tag{109}$$

We view (108) as the representation of a point $P \in \mathbb{R}^n$ in terms of its *barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of points $S = \{A_1, \ldots, A_N\}$. Identity (109), then, insures that the barycentric coordinate representation (108) of P with respect to the set S is *covariant* (or, *invariant in form*) in the following sense. The point P and the points of the set S of its barycentric coordinate representation vary together under translations. Indeed, a translation $X + A_k$ of A_k by X, $k = 1, \ldots, N$, in (109) results in the translation $X + P$ of P by X.

In order to insure that barycentric coordinate representations with respect to a set S are unique, we require S to be pointwise independent.

Definition 17.1. (Pointwise Independence). A set S of N points $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^n$, $n \ge 2$, is *pointwise independent* if the $N-1$ vectors $-A_1 + A_k$, $k = 2, \ldots, N$, are linearly independent.

Definition 17.2. (Barycentric Coordinates). Let

$$S = \{A_1, \ldots, A_N\} \tag{110}$$

be a pointwise independent set of N points in $\mathbb{R}^n$. The real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \ne 0 \tag{111}$$

are barycentric coordinates of a point $P \in \mathbb{R}^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k}\,. \tag{112}$$

Barycentric coordinates are homogeneous in the sense that the barycentric coordinates $(m_1, \ldots, m_N)$ of the point P in (112) are equivalent to the barycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \ne 0$. Since in barycentric coordinates only ratios of coordinates are relevant, the barycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1 \colon \ldots \colon m_N)$.

Barycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{113}$$

are called *special barycentric coordinates.*

Equation (112) is said to be the (unique) barycentric coordinate representation of P with respect to the set S.

Theorem 17.3. (Covariance of Barycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k A_k}{\sum_{k=1}^{N} m_k} \tag{114}$$

be the barycentric coordinate representation of a point $P \in \mathbb{R}^n$ in a Euclidean n-space $\mathbb{R}^n$ with respect to a pointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}^n$. The barycentric coordinate representation (114) *is covariant, that is,*

$$X + P = \frac{\sum_{k=1}^{N} m_k (X + A_k)}{\sum_{k=1}^{N} m_k} \tag{115}$$

for all $X \in \mathbb{R}^n$, and

$$RP = \frac{\sum_{k=1}^{N} m_k R A_k}{\sum_{k=1}^{N} m_k} \tag{116}$$

for all $R \in SO(n)$.

Proof. The proof is immediate, noting that rotations $R \in SO(n)$ of $\mathbb{R}^n$ about its origin are linear maps of $\mathbb{R}^n$. □

Following the vision of Felix Klein in his *Erlangen Program* [8, 35], it is owing to the covariance with respect to translations and rotations that barycentric coordinate representations possess geometric significance. Indeed, translations and rotations in Euclidean geometry form the *group of motions* of the geometry, studied in [79], and according to Felix Klein's Erlangen Program [8], a geometric property is a property that remains invariant in form under the group of motions of the geometry.

18. Gyrobarycentric Coordinates

Guided by analogies with Section 17, in this section we introduce barycentric coordinates into hyperbolic geometry where, naturally, they are called *gyrobarycentric coordinates* [70–72, 75, 77, 78]. Gyrobarycentric coordinates prove useful in the analytic determination of various gyrotriangle gyrocenters, just as barycentric coordinates prove useful in the analytic determination of various triangle centers.

For any positive integer N, let $m_k \in \mathbb{R}$ be N given real numbers, and let $A_k \in \mathbb{R}_s^n$ be N given gyropoints in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$, $k = 1, \ldots, N$, satisfying,

$$\sum_{k=1}^{N} m_k \gamma_{\mathbf{v}_k} \; > \; 0 \tag{117}$$

Theorem 16.1, p. 35 presents the result that the equation

$$\sum_{k=1}^{N} m_k \begin{pmatrix} \gamma_{A_k} \\ \gamma_{A_k} A_k \end{pmatrix} = m_0 \begin{pmatrix} \gamma_P \\ \gamma_P P \end{pmatrix} \tag{118}$$

for the unknowns $m_0 \in \mathbb{R}$ and $P \in \mathbb{R}_s^n$ possesses the unique solution given by

$$m_0 \; = \; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)} \tag{119}$$

$m_0 > 0$, satisfying

$$m_0 \; = \; \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus(X \oplus A_j) \oplus (X \oplus A_k)} - 1)} \tag{120}$$

for all $X \in \mathbb{R}_s^n$, and

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{121}$$

satisfying

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{122}$$

for all $X \in \mathbb{R}_s^n$.

Furthermore, Theorem 16.1, p. 35, also asserts that P and m_0 satisfy the two identities

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{123}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{124}$$

and, more generally,

$$\gamma_{X\oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}}{m_0} \tag{125}$$

and

$$\gamma_{X\oplus P}(X\oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X\oplus A_k}(X\oplus A_k)}{m_0} \tag{126}$$

for all $X \in \mathbb{R}_s^n$.

We view (121) as the representation of a gyropoint $P \in \mathbb{R}_s^n$ in terms of its *hyperbolic barycentric coordinates* m_k, $k = 1, \ldots, N$, with respect to the set of gyropoints $S = \{A_1, \ldots, A_N\}$. Naturally in gyrolanguage, hyperbolic barycentric coordinates are called *gyrobarycentric coordinates*. Identity (122) insures that the *gyrobarycentric coordinate representation* (121) of P with respect to the set S is *gyrocovariant* as stated in Theorem 18.3 below. The gyropoint P and the gyropoints of the set S of its gyrobarycentric coordinate representation vary together under left gyrotranslations. Indeed, a left gyrotranslation $X\oplus A_k$ of A_k by X, $k = 1, \ldots, N$ in (122) results in the left gyrotranslation $X\oplus P$ of P by X.

In order to insure that gyrobarycentric coordinate representations with respect to a set S are unique, we require S to be hyperbolically pointwise independent or, in gyrolanguage, *gyropointwise Independent*.

Definition 18.1. (Gyropointwise Independence). A set S of N gyropoints $S = \{A_1, \ldots, A_N\}$ in $\mathbb{R}_s^n$, $n \geq 2$, is *gyropointwise independent* if the $N-1$ gyrovectors in $\mathbb{R}_s^n$, $\ominus A_1 \oplus A_k$, $k = 2, \ldots, N$, considered as vectors in $\mathbb{R}^n$, are linearly independent.

We are now in the position to present the formal definition of gyrobarycentric coordinates, as motivated by mass and center of momentum velocity of Einsteinian particle systems and by analogies with barycentric coordinates.

Definition 18.2. (Gyrobarycentric Coordinates). Let

$$S = \{A_1, \ldots, A_N\} \tag{127}$$

be a gyropointwise independent set of N gyropoints in $\mathbb{R}_s^n$. The real numbers $m_1, \ldots, m_N$, satisfying

$$\sum_{k=1}^{N} m_k \gamma_{A_k} > 0 \tag{128}$$

are gyrobarycentric coordinates of a gyropoint $P \in \mathbb{R}_s^n$ with respect to the set S if

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \,. \tag{129}$$

Gyrobarycentric coordinates are homogeneous in the sense that the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ of the gyropoint P in (129) are equivalent to the gyrobarycentric coordinates $(\lambda m_1, \ldots, \lambda m_N)$ for any real nonzero number $\lambda \in \mathbb{R}$, $\lambda \neq 0$. Since in gyrobarycentric coordinates only ratios of coordinates are relevant, the gyrobarycentric coordinates $(m_1, \ldots, m_N)$ are also written as $(m_1 \colon \ldots \colon m_N)$.

Gyrobarycentric coordinates that are normalized by the condition

$$\sum_{k=1}^{N} m_k = 1 \tag{130}$$

are called *special gyrobarycentric coordinates*.

Equation (129) is said to be the gyrobarycentric coordinate representation of P with respect to the set S.

Finally, the constant of the gyrobarycentric coordinate representation of P in (129) is $m_0 > 0$, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ . \tag{131}$$

Theorem 18.3. (Gyrocovariance of Gyrobarycentric Coordinate Representations). *Let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{132a}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}_s^n$ in an Einstein gyrovector space $(\mathbb{R}_s^n, \oplus, \otimes)$ with respect to a gyropointwise independent set $S = \{A_1, \ldots, A_N\} \subset \mathbb{R}_s^n$.

Then

$$\gamma_P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k}}{m_0} \tag{132b}$$

and

$$\gamma_P P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{m_0} \tag{132c}$$

where m_0, given by

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k\right)^2 + 2\sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)}\ , \tag{132d}$$

$m_0 > 0$, is the constant of the gyrobarycentric coordinate representation (132a).

Furthermore, the gyrobarycentric coordinate representation (132a) *and its associated identities in* (132b) – (132d) *are gyrocovariant, that is,*

$$X \oplus P = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}} \tag{133a}$$

$$\gamma_{X \oplus P} = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k}}{m_0} \tag{133b}$$

$$\gamma_{X \oplus P} (X \oplus P) = \frac{\sum_{k=1}^{N} m_k \gamma_{X \oplus A_k} (X \oplus A_k)}{m_0} \tag{133c}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus (X \oplus A_j) \oplus (X \oplus A_k)} - 1)} \tag{133d}$$

for all $X \in \mathbb{R}_s^n$, *and*

$$RP = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} RA_k}{\sum_{k=1}^{N} m_k \gamma_{RA_k}} \tag{134a}$$

$$\gamma_{RP} = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k}}{m_0} \tag{134b}$$

$$\gamma_{RP} (RP) = \frac{\sum_{k=1}^{N} m_k \gamma_{RA_k} (RA_k)}{m_0} \tag{134c}$$

where

$$m_0 = \sqrt{\left(\sum_{k=1}^{N} m_k \right)^2 + 2 \sum_{\substack{j,k=1 \\ j<k}}^{N} m_j m_k (\gamma_{\ominus (RA_j) \oplus (RA_k)} - 1)} \tag{134d}$$

for all $R \in SO(n)$.

The proof of Theorem 18.3 is found in [72, Theorem 4.6].

Following the vision of Felix Klein in his *Erlangen Program* [8, 35], it is owing to the gyrocovariance, that is, covariance with respect to left gyrotranslations and rotations, that gyrobarycentric coordinate representations are geometrically significant. Indeed, left gyrotranslations and rotations in hyperbolic geometry form the group of motions of the geometry, studied in [79, Section 3.12] and, according to Felix Klein's Erlangen Program, a geometric property is a property that remains invariant in form under the motions of the geometry.

The following two corollaries of Theorem 18.3 prove useful.

Corollary 18.4. *Let $S = \{A_1, \dots, A_N\} \subset \mathbb{R}^n_s$ be a gyropointwise independent set of N gyropoints in $\mathbb{R}^n_s$, and let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{135}$$

be a gyrobarycentric coordinate representation of a gyropoint $P \in \mathbb{R}^n$ with respect to the set S. Furthermore, let m_0 be the representation constant, given by

$$m_0^2 \;=\; \left(\sum_{k=1}^{N} m_k\right)^2 + 2 \sum_{\substack{j,k=1\\ j<k}}^{N} m_j m_k (\gamma_{\ominus A_j \oplus A_k} - 1)\,. \tag{136}$$

Then, the point P lies in the ball $\mathbb{R}^n_s$, $P \in \mathbb{R}^n_s$, if and only if $m_0^2 > 0$ (In other words, the point P is a gyropoint if and only if $m_0^2 > 0$).

The proof of Corollary 18.4 is found in [72, Corollary 4.9].

Corollary 18.5. *Let $S = \{A_1, \dots, A_N\} \subset \mathbb{R}^n_s$ be a gyropointwise independent set of N gyropoints in $\mathbb{R}^n_s$, and let*

$$P = \frac{\sum_{k=1}^{N} m_k \gamma_{A_k} A_k}{\sum_{k=1}^{N} m_k \gamma_{A_k}} \tag{137}$$

be a gyrobarycentric coordinate representation of a point $P \in \mathbb{R}^n$ with respect to the set S, with positive gyrobarycentric coordinates $m_k > 0$, $k = 1, \dots, N$. Then, $P \in \mathbb{R}^n_s$. Moreover, P lies on the the convex span of S if and only if $m_k > 0$, $k = 1, \dots, N$.

The proof of Corollary 18.5 is found in [72, Corollary 4.10]

19. Gyrolanguage

The checkered history of gyrolanguage begins in 1988 [54] with the discovery of the parametric realization of the Lorentz transformation group of special relativity theory in terms of relativistically admissible velocities. It turned out that the group structure of Lorentz transformations induces the gyrocommutative gyrogroup structure of the space $\mathbb{R}^3_c$ of all relativistically admissible velocities with the binary operation $\oplus$ given by Einstein's velocity addition law.

The gyrocommutative gyrogroup structure $(\mathbb{R}^3_c, \oplus)$ that regulates Einstein addition was initially called a *nonassociative group* [55]. In the initial study of the concrete example $(\mathbb{R}^3_c, \oplus)$, the gyrocommutative and gyroassociative laws of Einstein addition were called weakly commutative and weakly associative laws and, accordingly, gyrocommutative gyrogroups were called *weakly associative-commutative groups* (WACGs, in short) [57]. Furthermore, in this initial study of gyrocommutative gyrogroups the rich algebra of the gyrations that are associated with Einstein addition was discovered. Gyrations were called Thomas rotations for being related to the special relativistic phenomenon known as Thomas precession [55]. The term *K-loop* with "K" after Karzel, which refers to the gyrocommutative gyrogroup, was coined by the author in [56] as evidenced from [26, pp. 169-170]. The term *K-loop* is in use by some authors, and its prehistory is unfolded in [42, p. 142] and in [60, Remark 6.12].

Prior to its introduction by the author, the term "K-loop" has already been in use by Soĭkis, in 1970 [45] and later, but independently, by Basarab, in 1992 [9]. Unlike the term "K-loop" that Ungar coined, the "K" in each of the terms "K-loop" coined by Soĭkis and by Basarab does not refer to "Karzel".

Finally, in 1991 [58] the author has realized that a most appropriate term for the abstract Thomas precession is *Thomas gyration* (or gyration, in short) so that, accordingly, the weakly commutative and weakly associative laws of Einstein addition became the gyrocommutative and the gyroassociative laws. Hence, consistently, the extension by abstraction of the Einstein groupoid $(\mathbb{R}^3_c, \oplus)$ is now called a gyrocommutative gyrogroup.

Merging gyroterminology with terminology [59], the emergence of gyrolanguage is thus natural. It is a language in which we prefix a gyro to terms that describe concepts in algebra and geometry to mean the analogous concepts in gyroalgebra and gyrogeometry. An interesting example is provided by the term *gyrolayout*, which has been coined by D. K. Urribarri, S. M. Castro and S. R. Martig in the title of their paper [81], where the 3-dimensional Einstein gyrovector space is employed for the generation of computer hyperbolic visualization.

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Special Subgroups of Gyrogroups: Commutators, Nuclei and Radical

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Abstract

A gyrogroup is a nonassociative group-like structure modelled on the space of relativistically admissible velocities with a binary operation given by Einstein's velocity addition law. In this article, we present a few of groups sitting inside a gyrogroup G, including the commutator subgyrogroup, the left nucleus, and the radical of G. The normal closure of the commutator subgyrogroup, the left nucleus, and the radical of G are in particular normal subgroups of G. We then give a criterion to determine when a subgyrogroup H of a finite gyrogroup G, where the index $[G\colon H]$ is the smallest prime dividing $|G|$, is normal in G.

Keywords: Gyrogroup, commutator subgyrogroup, nucleus of gyrogroup, subgyrogroup of prime index, radical of gyrogroup.

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1. Introduction

A gyrogroup, discovered by Abraham A. Ungar [16], is a nonassociative group-like structure modelled on the space of relativistically admissible velocities, together with Einstein's velocity addition [18]. It is remarkable that the gyrogroup structure appears in various fields such as mathematical physics [10, 17], non-Euclidean geometry [19, 20], group theory [6, 7], loop theory [8, 14], harmonic analysis [3, 4], abstract algebra [13, 15], and analysis [1, 2].

This article explores an algebraic aspect of gyrogroups. Recall that in abstract algebra the following theme recurs: given an object X and a subobject Y,

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determine whether the quotient object X/Y has the same algebraic structure as X. It is known, for instance, that a subgroup Ξ of a group Γ gives rise to the quotient group Γ/Ξ if and only if Ξ is normal in Γ. Sometimes, it is possible to use information on a normal subgroup Ξ and on the quotient Γ/Ξ to obtain information about Γ. Therefore, determining the normal subgroups of Γ is useful for studying properties of Γ itself. The situation in gyrogroup theory is analogous. For example, the Lagrange theorem for finite gyrogroups follows from the fact that every gyrogroup G has a normal subgroup Ξ such that G/Ξ is a gyrocommutative gyrogroup [6, Theorem 4.11]. For more details, see Section 5 of [13]. From this point of view, we examine some normal subgyrogroups of a gyrogroup that form groups under the gyrogroup operation.

For basic knowledge of gyrogroup theory, the reader is referred to [13, 15, 19]. Here is the formal definition of a gyrogroup.

Definition 1.1 (Gyrogroup). A groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.

(G3) For all $a, b \in G$, there is an automorphism $\mathrm{gyr}\,[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}\,[a, b]c \qquad \text{(left gyroassociative law)}$$

for all $c \in G$.

(G4) For all $a, b \in G$, $\mathrm{gyr}\,[a \oplus b, b] = \mathrm{gyr}\,[a, b]$. (left loop property)

2. The Commutator Subgyrogroup

Throughout this section, G is an arbitrary gyrogroup unless otherwise stated.

2.1 The Direct Product and Normal Closure

Recall that the intersection of normal subgroups of a group Γ is again a normal subgroup of Γ. This result continues to hold for gyrogroups, as we will see shortly. Because of the missing of associativity in gyrogroups, it is not straightforward to determine whether a given subgyrogroup H of a gyrogroup G is normal in G. However, according to Theorem 2.3, the smallest (by inclusion) normal subgyrogroup of G that contains H, called the normal closure of H, always exists. The normal closure of H and H have some common features, and sometimes it is possible to obtain information about H from the normal closure of H. See for instance Corollary 2.12.

Given an indexed family of gyrogroups $\{G_i : i \in I\}$, the *direct product of G_i, $i \in I$*, denoted by $\prod_{i\in I} G_i$, consists of all functions $f: I \to \bigcup_{i\in I} G_i$ with the property that $f(i) \in G_i$ for all $i \in I$. For $f, g \in \prod_{i\in I} G_i$, define a function $f \oplus g$ by the equation

$$(f \oplus g)(i) = f(i) \oplus g(i), \qquad i \in I. \tag{1}$$

Theorem 2.1. *Let $\{G_i : i \in I\}$ be an indexed family of gyrogroups. The direct product $\prod_{i\in I} G_i$ with operation defined by $(f, g) \mapsto f \oplus g$ is a gyrogroup.*

Proof. Set $G = \prod_{i\in I} G_i$. The zero function, $i \mapsto 0$, $i \in I$, is a left identity of G. For each $f \in G$, the function $i \mapsto \ominus f(i)$, $i \in I$, is a left inverse of f. The gyroautomorphisms of G are given by

$$(\mathrm{gyr}\,[f, g]h)(i) = \mathrm{gyr}\,[f(i), g(i)]h(i), \qquad i \in I,$$

for all $f, g, h \in G$. It is straightforward to check that the axioms of a gyrogroup are satisfied. □

Theorem 2.2. *Let $\{N_i : i \in I\}$ be an indexed family of normal subgyrogroups of G. Then the intersection $\bigcap_{i\in I} N_i$ is a normal subgyrogroup of G.*

Proof. For each $i \in I$, there exists a gyrogroup homomorphism φ_i of G to a gyrogroup G_i such that $\ker \varphi_i = N_i$. Set $H = \prod_{i\in I} G_i$. For each $a \in G$, define a function $\varphi(a)$ by $\varphi(a)(i) = \varphi_i(a)$ for all $i \in I$. Then $a \mapsto \varphi(a)$, $a \in G$, defines a gyrogroup homomorphism from G to H. Direct computation shows that $\ker \varphi = \bigcap_{i\in I} \ker \varphi_i$. Hence, $\bigcap_{i\in I} N_i = \bigcap_{i\in I} \ker \varphi_i = \ker \varphi \trianglelefteq G$. □

Theorem 2.3. *Let A be a nonempty subset of G. Then there exists a unique normal subgyrogroup of G, denoted by $\langle\overline{A}\rangle$, such that*

1. *$A \subseteq \langle\overline{A}\rangle$, and*
2. *if $N \trianglelefteq G$ and $A \subseteq N$, then $\langle\overline{A}\rangle \subseteq N$.*

Proof. Set $\mathcal{A} = \{K \subseteq G : K \trianglelefteq G \text{ and } A \subseteq K\}$. By Theorem 2.2, $\langle\overline{A}\rangle := \bigcap_{K\in\mathcal{A}} K$ forms a normal subgyrogroup of G satisfying the two conditions. The uniqueness of $\langle\overline{A}\rangle$ follows from condition (2). □

Definition 2.4 (Normal closure). Let A be a nonempty subset of a gyrogroup G. The normal subgyrogroup $\langle\overline{A}\rangle$ in Theorem 2.3 is called the *normal closure of* A or *normal subgyrogroup of* G *generated by* A.

According to Theorem 2.3, the normal closure of A is the smallest (by inclusion) normal subgyrogroup of G that contains A. Note that if A itself is a normal subgyrogroup of G, then $\langle\overline{A}\rangle = A$. In other words, any normal subgyrogroup of G equals its normal closure. The concept of normal closures is needed in studying the commutator subgyrogroup of a gyrogroup in the next section.

2.2 Commutators

In this section, we extend the notion of commutators, which is defined for groups, to gyrogroups. Recall that if Γ is a group, then the commutator subgroup of Γ, denoted by Γ', is the smallest normal subgroup of Γ such that the quotient Γ/Γ' is an abelian group. Unlike the situation in group theory, it is still an open problem whether the commutator subgyrogroup of a gyrogroup G, denoted by G', is normal in G. However, it is true that if G' is normal in G, then the quotient G/G' forms a gyrocommutative gyrogroup. Therefore, we focus attention on the normal closure of G' instead of G'. It turns out that the normal closure of G' is the smallest normal subgyrogroup of G such that the quotient $G/\langle\overline{G'}\rangle$ is gyrocommutative. Further, the normal closure of G' (and hence G') forms a subgroup of G, as we will see shortly.

Let G be a gyrogroup. Given $a, b \in G$, define the *commutator of* a *and* b, denoted by $[a, b]$, by the equation

$$[a, b] = \ominus(a \oplus b) \oplus \operatorname{gyr}[a, b](b \oplus a). \tag{2}$$

Define

$$G' = \langle [a, b] \colon a, b \in G\rangle, \tag{3}$$

the subgyrogroup of G generated by commutators of elements from G, called the *commutator subgyrogroup* of G. Note that if G is a gyrogroup with trivial gyroautomorphisms, then G becomes a group, $[a, b]$ becomes the group-theoretic commutator of a and b, and G' becomes the familiar commutator subgroup of G.

Theorem 2.5. *Let G be a gyrogroup. Then the following hold.*

1. *For all $a, b \in G$, $[a, b] = 0$ if and only if $a \oplus b = \operatorname{gyr}[a, b](b \oplus a)$.*
2. *For all $a, b \in G$, $\ominus(a \oplus b) = (\ominus a \ominus b) \oplus [\ominus a, \ominus b]$.*
3. *If φ is a gyrogroup homomorphism of G, then $\varphi([a, b]) = [\varphi(a), \varphi(b)]$ for all $a, b \in G$.*
4. *If $\tau \in \operatorname{Aut}(G)$, then $\tau(G') = G'$.*
5. *$G' = \{0\}$ if and only if G is gyrocommutative.*

6. *If $G' \trianglelefteq G$, then G/G' is gyrocommutative.*

Proof. Item (1) follows from the left cancellation law.

To verify item (2), we compute

$$\begin{aligned} (\ominus a \ominus b) \oplus [\ominus a, \ominus b] &= \operatorname{gyr}[\ominus a, \ominus b](\ominus b \ominus a) \\ &= \operatorname{gyr}[a, b](\ominus b \ominus a) \\ &= \ominus(a \oplus b). \end{aligned}$$

We have the first equation from the definition of a commutator; the second equation from Theorem 2.34 of [19]; and the last equation from Theorem 2.11 of [19].

(3) By Proposition 23 of [15],

$$\begin{aligned} \varphi([a, b]) &= \varphi(\ominus(a \oplus b) \oplus \operatorname{gyr}[a, b](b \oplus a)) \\ &= \ominus(\varphi(a) \oplus \varphi(b)) \oplus \operatorname{gyr}[\varphi(a), \varphi(b)](\varphi(b) \oplus \varphi(a)) \\ &= [\varphi(a), \varphi(b)]. \end{aligned}$$

(4) Let $\tau \in \operatorname{Aut}(G)$. First, we prove that $G' \subseteq \tau(G')$. For all $a, b \in G$, we have $[a, b] = \tau([\tau^{-1}(a), \tau^{-1}(b)])$ belongs to $\tau(G')$. Hence, $\tau(G')$ contains all the commutators of G. Since G' is the smallest subgyrogroup of G containing the commutators of G and $\tau(G') \leqslant G$, it follows that $G' \subseteq \tau(G')$. Since τ^{-1} is also in $\operatorname{Aut}(G)$, $G' \subseteq \tau^{-1}(G')$. This implies $\tau(G') \subseteq \tau(\tau^{-1}(G')) = G'$ since τ is a bijection. Hence, $\tau(G') = G'$.

Item (5) follows immediately from item (1).

(6) Suppose that $G' \trianglelefteq G$. Then G/G' has the quotient gyrogroup structure. Let $a, b \in G$. According to Theorem 27 of [15], we have

$$\begin{aligned} \ominus((a \oplus G') \oplus (b \oplus G')) &= \ominus((a \oplus b) \oplus G') \\ &= (\ominus(a \oplus b)) \oplus G' \\ &= ((\ominus a \ominus b) \oplus [\ominus a, \ominus b]) \oplus G' \\ &= ((\ominus a \ominus b) \oplus G') \oplus ([\ominus a, \ominus b] \oplus G') \\ &= (\ominus a \ominus b) \oplus G' \\ &= (\ominus a \oplus G') \oplus (\ominus b \oplus G') \\ &= \ominus(a \oplus G') \ominus (b \oplus G'). \end{aligned}$$

This proves that G/G' satisfies the automorphic inverse property and so G/G' is gyrocommutative by Theorem 3.2 of [19]. □

A subgyrogroup H of G is called an *L-subgyrogroup* of G, denoted by $H \leqslant_L G$, if $\operatorname{gyr}[a, h](H) = H$ for all $a \in G$ and $h \in H$. For more information about L-subgyrogroups, see Section 4 of [15].

Theorem 2.6. *The commutator subgyrogroup of G is an L-subgyrogroup of G.*

Proof. By Theorem 2.5 (4), G' is invariant under the gyroautomorphisms of G. Hence, $G' \leqslant_L G$. $\square$

Proposition 2.7. *Let N be a normal subgyrogroup of G. The following are equivalent:*

1. *G/N is gyrocommutative.*

2. *$G' \subseteq N$.*

3. *$[a, b] \in N$ for all $a, b \in G$.*

Proof. (1) $\Rightarrow$ (2) Let $a, b \in G$. Set $X = a \oplus N$ and $Y = b \oplus N$. Since G/N is gyrocommutative, $X \oplus Y = \mathrm{gyr}\,[X, Y](Y \oplus X)$. From Theorem 27 of [15], we have

$$(a \oplus b) \oplus N = (\mathrm{gyr}\,[a, b](b \oplus a)) \oplus N.$$

It follows that

$$\begin{aligned} [a, b] \oplus N &= (\ominus(a \oplus b) \oplus \mathrm{gyr}\,[a, b](b \oplus a)) \oplus N \\ &= \ominus((a \oplus b) \oplus N) \oplus (\mathrm{gyr}\,[a, b](b \oplus a) \oplus N) \\ &= \ominus((a \oplus b) \oplus N) \oplus ((a \oplus b) \oplus N) \\ &= 0 \oplus N. \end{aligned}$$

Hence, $[a, b] \in N$ for all $a, b \in G$ and so $G' \subseteq N$ by the minimality of G'.

The implication (2) $\Rightarrow$ (3) is trivial.

(3) $\Rightarrow$ (1) Since $N \trianglelefteq G$, G/N admits the quotient gyrogroup structure. The proof that G/N is gyrocommutative follows the same steps as in the proof of Theorem 2.5 (6). $\square$

Theorem 2.8. *The normal closure of G' is the unique normal subgyrogroup of G such that*

1. *$G/\langle\overline{G'}\rangle$ is gyrocommutative, and*

2. *if $\varphi\colon G \to A$ is a gyrogroup homomorphism into a gyrocommutative gyrogroup A, then φ factors through $\langle\overline{G'}\rangle$ in the sense that $\langle\overline{G'}\rangle \subseteq \ker\varphi$.*

Proof. By Theorem 2.3, $\langle\overline{G'}\rangle \trianglelefteq G$ and $G' \subseteq \langle\overline{G'}\rangle$. Hence, by Proposition 2.7, $G/\langle\overline{G'}\rangle$ is gyrocommutative. Suppose that $\varphi\colon G \to A$ is a gyrogroup homomorphism of G, where A is a gyrocommutative gyrogroup. For $a, b \in G$, we have

$$\varphi([a, b]) = [\varphi(a), \varphi(b)] = 0$$

since $\varphi(a), \varphi(b) \in A$ and A is gyrocommutative. Thus, $[a, b] \in \ker\varphi$ for all $a, b \in G$, which implies $G' \subseteq \ker\varphi$. Since $\ker\varphi \trianglelefteq G$, it follows from the minimality of $\langle\overline{G'}\rangle$ that $\langle\overline{G'}\rangle \subseteq \ker\varphi$.

(Uniqueness) Assume that K_1 and K_2 are normal subgyrogroups of G that satisfy the two conditions. Let $\Pi_1 : G \to G/K_1$ and $\Pi_2 : G \to G/K_2$ be the canonical projections. As K_1 satisfies the second condition and Π_2 is a gyrogroup homomorphism, we have $K_1 \subseteq \ker \Pi_2 = K_2$. Interchanging the roles of K_1 and K_2, one obtains that $K_2 \subseteq \ker \Pi_1 = K_1$. Hence, $K_1 = K_2$. □

Theorem 2.8 implies the *universal property* of the normal closure of G': given any gyrogroup homomorphism φ from G to a gyrocommutative gyrogroup A, there is a unique gyrogroup homomorphism $\Phi : G/\langle\overline{G'}\rangle \to A$ such that $\Phi \circ \Pi = \varphi$, that is, the following diagram commutes.

$$\begin{tikzcd}
G \arrow[rr, "\varphi"] \arrow[dr, "\Pi"'] & & A \\
& G/\langle\overline{G'}\rangle \arrow[ur, dotted, "\Phi"'] &
\end{tikzcd}$$

Here, Π denotes the canonical projection given by $\Pi(a) = a \oplus \langle\overline{G'}\rangle$ for all $a \in G$, and Φ is given by

$$\Phi(a \oplus \langle\overline{G'}\rangle) = \varphi(a) \tag{4}$$

for all $a \in G$.

Theorem 2.9. *Let N be a normal subgyrogroup of G. Then G/N is gyrocommutative if and only if $\langle\overline{G'}\rangle \subseteq N$.*

Proof. Suppose that G/N is gyrocommutative. Then the canonical projection $\Pi : G \to G/N$ fits item (2) of Theorem 2.8. Hence, $\langle\overline{G'}\rangle \subseteq \ker \Pi = N$. Conversely, if $\langle\overline{G'}\rangle \subseteq N$, then $G' \subseteq N$ and so G/N is gyrocommutative by Proposition 2.7. □

Proposition 2.10. $\langle\overline{G'}\rangle = \{0\}$ *if and only if G is gyrocommutative.*

Proof. If $\langle\overline{G'}\rangle = \{0\}$, then $G \cong G/\langle\overline{G'}\rangle$ via the canonical projection. Hence, G is gyrocommutative. Conversely, if G is gyrocommutative, then so is $G/\{0\}$. Hence, $\langle\overline{G'}\rangle \subseteq \{0\}$ by Theorem 2.9. This implies $\langle\overline{G'}\rangle = \{0\}$. □

By a *subgroup* of a gyrogroup G we mean a subgyrogroup of G that forms a group under the operation of G [13, Proposition 3.3]. One of the remarkable consequences of Theorem 2.8 is that the normal closure of G' (and hence G') is a subgroup of G.

Theorem 2.11. *The normal closure of G' is a subgroup of G.*

Proof. By Theorem 4.11 of [6], G has a normal subgroup Ξ such that G/Ξ is a gyrocommutative gyrogroup. By Theorem 2.9, $\langle\overline{G'}\rangle \subseteq \Xi$. Since Ξ is a subgroup of G, so is $\langle\overline{G'}\rangle$. □

Corollary 2.12. *The commutator subgyrogroup of G is a subgroup of G.*

Proof. The corollary follows from the fact that $G' \subseteq \langle\overline{G'}\rangle$. □

3. Nuclei and the Radical of a Gyrogroup

Throughout this section, G is an arbitrary gyrogroup. We follow [5] in presenting a few normal subgroups sitting inside a gyrogroup. The main goal of this section is to prove that the left nucleus and radical of G are normal subgyrogroups of G that form groups under the gyrogroup operation. The key idea is as follows. Every gyrogroup can be embedded into its left multiplication group, and normality of the subgyrogroup under consideration follows from normality of the corresponding subgroup of the left multiplication group. This in particular shows a remarkable connection between groups and gyrogroups.

As in loop theory, the *left nucleus*, *middle nucleus*, and *right nucleus of* G are defined, respectively, by

$$\begin{aligned}
N_l(G) &= \{a \in G \colon \forall b, c \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\},\\
N_m(G) &= \{b \in G \colon \forall a, c \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\},\\
N_r(G) &= \{c \in G \colon \forall a, b \in G,\ a \oplus (b \oplus c) = (a \oplus b) \oplus c\}.
\end{aligned}$$

Since G satisfies the left gyroassociative law and the general left cancellation law, the left nucleus, middle nucleus, and right nucleus of G can be restated in terms of gyroautomorphisms as follows:

$$\begin{aligned}
N_l(G) &= \{a \in G \colon \forall b \in G,\ \mathrm{gyr}\,[a, b] = \mathrm{id}_G\},\\
N_m(G) &= \{b \in G \colon \forall a \in G,\ \mathrm{gyr}\,[a, b] = \mathrm{id}_G\},\\
N_r(G) &= \{c \in G \colon \forall a, b \in G,\ \mathrm{gyr}\,[a, b]c = c\}.
\end{aligned}$$

By Theorem 2.34 of [19], $\mathrm{gyr}^{-1}[a, b] = \mathrm{gyr}\,[b, a]$ for all $a, b \in G$. It follows that the left nucleus and middle nucleus of G are identical.

Theorem 3.1. *The left nucleus, middle nucleus, and right nucleus of G are L-subgyrogroups of G. Furthermore, they are subgroups of G.*

Proof. Because $\mathrm{gyr}\,[0, a] = \mathrm{id}_G$ for all $a \in G$, $0 \in N_l(G)$. Let $a \in N_l(G)$ and let $b \in G$. By Theorem 2.34 of [19], $\mathrm{gyr}\,[\ominus a, b] = \mathrm{gyr}\,[\ominus a, \ominus(\ominus b)] = \mathrm{gyr}\,[a, \ominus b] = \mathrm{id}_G$. Hence, $\ominus a$ is in $N_l(G)$. Let $a, b \in N_l(G)$ and let $c, x \in G$. According to the gyrator identity [19, Theorem 2.10], we compute

$$\begin{aligned}
\mathrm{gyr}\,[a \oplus b, c]x &= \ominus((a \oplus b) \oplus c) \oplus ((a \oplus b) \oplus (c \oplus x))\\
&= \ominus((a \oplus b) \oplus c) \oplus (a \oplus (b \oplus \mathrm{gyr}\,[b, a](c \oplus x)))\\
&= \ominus((a \oplus b) \oplus c) \oplus (a \oplus (b \oplus (c \oplus x)))\\
&= \ominus((a \oplus b) \oplus c) \oplus (a \oplus ((b \oplus c) \oplus x))\\
&= \ominus((a \oplus b) \oplus c) \oplus ((a \oplus (b \oplus c)) \oplus x)\\
&= \ominus((a \oplus b) \oplus c) \oplus (((a \oplus b) \oplus c) \oplus x)\\
&= x.
\end{aligned}$$

We have the second equation from the right gyroassociative law; the third and forth equations since $b \in N_l(G)$; the fifth and sixth equations since $a \in N_l(G)$; the last equation from the left cancellation law. Since x is arbitrary, $\mathrm{gyr}\,[a \oplus b, c] = \mathrm{id}_G$ and so $a \oplus b \in N_l(G)$. By the subgyrogroup criterion [15, Proposition 14], $N_l(G) \leqslant G$. By definition of $N_l(G)$, $N_l(G) \leqslant_L G$. Since $\mathrm{gyr}\,[a, b]|_{N_l(G)} = \mathrm{id}_{N_l(G)}$ for all $a, b \in N_l(G)$, $N_l(G)$ is a subgroup of G. Since $N_m(G) = N_l(G)$, we have $N_m(G) \leqslant_L G$ and $N_m(G)$ is a subgroup of G as well. The proof that $N_r(G)$ is an L-subgyrogroup and a subgroup of G is straightforward. □

Let a be an arbitrary element of G. Recall that the *left gyrotranslation by* a, L_a, is a permutation of G defined by

$$L_a(x) = a \oplus x, \qquad x \in G.$$

For a given subgyrogroup H of G, define $L(H) = \{L_a \colon a \in H\}$. In the case $H = G$, we have $L(G) = \{L_a \colon a \in G\}$. The *left multiplication group of* G, $\mathrm{LMlt}\,(G)$, is the subgroup of the symmetric group on G generated by $L(G)$. In other words,

$$\mathrm{LMlt}\,(G) = \langle L_a \colon a \in G\rangle.$$

A subset X of a group Γ is a *twisted subgroup* [5, p. 187] of Γ if $1 \in X$, 1 being the identity element of Γ; $x \in X$ implies $x^{-1} \in X$; and $x, y \in X$ implies $xyx \in X$.

Theorem 3.2. *$L(G)$ is a twisted subgroup of* $\mathrm{LMlt}\,(G)$.

Proof. The theorem follows from the fact that $L_a^{-1} = L_{\ominus a}$ and

$$L_a \circ L_b \circ L_a = L_{(a \oplus b) \boxplus a}$$

for all $a, b \in G$. Here, the coaddition $\boxplus$ of G is defined by $a \boxplus b = a \oplus \mathrm{gyr}\,[a, \ominus b]b$ for all $a, b \in G$. □

In light of Theorem 3.2, $L(G)$ is a *generating* twisted subgroup of $\mathrm{LMlt}\,(G)$. This leads to the following theorem.

Theorem 3.3. *Define*

$$L(G)^{\#} = \bigcap_{a \in G} L_a L(G).$$

Then $L(G)^{\#}$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ *contained in $L(G)$.*

Proof. The theorem is an application of Theorem 3.8 of [5]. □

Theorem 3.4. *$L(N_l(G))$ is a normal subgroup of* $\mathrm{LMlt}\,(G)$.

Proof. From Theorem 5.7 of [5], we have $L(N_l(G)) = L(G)^{\#}$. Hence, $L(N_l(G))$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ by Theorem 3.3. □

Following [5], we define

$$L(G)' = \{L_{a_1} \circ L_{a_2} \circ \cdots \circ L_{a_n} : a_i \in G \text{ and } L_{a_n} \circ L_{a_{n-1}} \circ \cdots \circ L_{a_1} = \mathrm{id}_G\}. \quad (5)$$

Since $L(G)$ is a generating twisted subgroup of $\mathrm{LMlt}\,(G)$, it follows from a result of Foguel, Kinyon, and Phillips [5, p. 189] that $L(G)'$ is a normal subgroup of $\mathrm{LMlt}\,(G)$. In fact, we have the following theorem.

Theorem 3.5. *$L(G)'$ is a normal subgroup of $\mathrm{LMlt}\,(G)$ such that $L(G)' \subseteq L(G)^{\#}$.*

Proof. The theorem follows directly from Proposition 3.10 of [5]. □

Set $\mathrm{Sym}_0(G) = \{\sigma \in \mathrm{Sym}\,(G) : \sigma(0) = 0\}$. Note that $L(G) \cap \mathrm{Sym}_0(G) = \{\mathrm{id}_G\}$. This implies that if G is a gyrogroup with a nonidentity gyroautomorphism, say $\mathrm{gyr}\,[a, b]$, then $L(G)$ is a proper twisted subgroup of $\mathrm{LMlt}\,(G)$. In fact, $\mathrm{gyr}\,[a, b] = L^{-1}_{a\oplus b} \circ L_a \circ L_b$ belongs to $\mathrm{LMlt}\,(G)$, but does not belong to $L(G)$ since otherwise $\mathrm{gyr}\,[a, b]$ would belong to $L(G) \cap \mathrm{Sym}_0(G) = \{\mathrm{id}_G\}$. In this case, $L(G)'$ and $L(G)^{\#}$ form *proper* normal subgroups of $\mathrm{LMlt}\,(G)$ for they are contained in $L(G)$.

The following proposition provides a sufficient condition for normality of a subgyrogroup. As an application of this proposition, we prove that the left nucleus and radical of G are normal subgyrogroups of G.

Proposition 3.6 ([11]). *If H is a subgyrogroup of G such that*

1. $\mathrm{gyr}\,[h, a] = \mathrm{id}_G$ *for all* $h \in H, a \in G$,
2. $\mathrm{gyr}\,[a, b](H) \subseteq H$ *for all* $a, b \in G$, *and*
3. $a \oplus H = H \oplus a$ *for all* $a \in G$,

then H is a normal subgyrogroup of G.

Lemma 3.7. *Let G be a gyrogroup. Then*

1. $\mathrm{gyr}\,[a, b](N_l(G)) \subseteq N_l(G)$ *for all* $a, b \in G$, *and*
2. $N_l(G) \oplus a = a \oplus N_l(G)$ *for all* $a \in G$.

Proof. (1) Set $N = N_l(G)$ and let n be an arbitrary element of N. Let $a, b \in G$. According to the commutation relation [15, Equation (14)], we have

$$L_{\mathrm{gyr}\,[a,b]n} = \mathrm{gyr}\,[a, b] \circ L_n \circ \mathrm{gyr}^{-1}[a, b].$$

Since $\mathrm{gyr}\,[a, b] \in \mathrm{LMlt}\,(G)$ and $L(N) \trianglelefteq \mathrm{LMlt}\,(G)$, it follows that $L_{\mathrm{gyr}\,[a,b]n}$ belongs to $L(N)$. Hence, $L_{\mathrm{gyr}\,[a,b]n} = L_{\tilde{n}}$ for some $\tilde{n} \in N$, which implies $\mathrm{gyr}\,[a, b]n = \tilde{n} \in N$. Since n is arbitrary, we obtain $\mathrm{gyr}\,[a, b](N) \subseteq N$.

(2) Let $a \in G$ and let $n \in N$. By the left cancellation law, $x = \ominus a \oplus (n \oplus a)$ is such that $n \oplus a = a \oplus x$. We compute

$$\begin{aligned}
L_x &= L_{\ominus a \oplus (n \oplus a)} \\
&= L_{(\ominus a \oplus n) \oplus a} \\
&= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, a] \\
&= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, a \oplus (\ominus a \oplus n)] \\
&= L_{\ominus a \oplus n} \circ L_a \circ \mathrm{gyr}^{-1}[\ominus a \oplus n, n] \\
&= L_{\ominus a \oplus n} \circ L_a \\
&= L_{\ominus a} \circ L_n \circ \mathrm{gyr}^{-1}[\ominus a, n] \circ L_a \\
&= L_a^{-1} \circ L_n \circ L_a.
\end{aligned}$$

We obtain the second equation since $n \in N = N_m(G)$; the third and seventh equations from the identity $L_{a \oplus b} = L_a \circ L_b \circ \mathrm{gyr}^{-1}[a, b]$; the forth equation from the right loop property; the sixth and last equations since $n \in N$. Since $L(N) \trianglelefteq \mathrm{LMlt}\,(G)$, we have $L_x \in L(N)$, which implies $x \in N$. Thus, $N \oplus a \subseteq a \oplus N$.

From Lemma 2.19 of [19], we can let $y \in G$ be such that $a \oplus n = y \oplus a$. To conclude that $a \oplus N \subseteq N \oplus a$, we have to show that y belongs to N. In fact, one obtains similarly that $L_n = L_a^{-1} \circ L_y \circ L_a$, which implies $L_y = L_a \circ L_n \circ L_a^{-1} \in L(N)$. Hence, $y \in N$, as desired. □

Theorem 3.8. *The left nucleus of G is a normal subgroup of G.*

Proof. The theorem follows immediately from Theorem 3.1, Proposition 3.6, the defining property of $N_l(G)$, and Lemma 3.7. □

Corollary 3.9. *The middle nucleus of G is a normal subgroup of G.*

Proof. This is because the left nucleus and middle nucleus of G are the same. □

Following [5], the *radical of G*, denoted by $\mathrm{Rad}\,(G)$, is defined by

$$\mathrm{Rad}\,(G) = \{a \in G \colon L_a \in L(G)'\}. \tag{6}$$

Theorem 3.10. *The radical of G is a subgroup of G contained in the left nucleus of G.*

Proof. First, we prove that $\mathrm{Rad}\,(G) \subseteq N_l(G)$. Let $a \in \mathrm{Rad}\,(G)$. Then $L_a \in L(G)'$. By Theorem 3.5, $L(G)' \subseteq L(G)^{\#}$ and by Theorem 5.7 of [5], $L(G)^{\#} = L(N_l(G))$. It follows that $L_a \in L(N_l(G))$, which implies $a \in N_l(G)$.

Let $a \in \mathrm{Rad}\,(G)$. Then $L_{\ominus a} = L_a^{-1} \in L(G)'$ for $L(G)' \leqslant \mathrm{LMlt}\,(G)$. Hence, $\ominus a \in \mathrm{Rad}\,(G)$. Let $a, b \in \mathrm{Rad}\,(G)$. Since $\mathrm{Rad}\,(G) \subseteq N_l(G)$, $\mathrm{gyr}\,[a, b] = \mathrm{id}_G$. Thus, $L_{a \oplus b} = L_a \circ L_b \circ \mathrm{gyr}^{-1}[a, b] = L_a \circ L_b \in L(G)'$. This proves $a \oplus b \in \mathrm{Rad}\,(G)$ and by the subgyrogroup criterion, $\mathrm{Rad}\,(G) \leqslant G$. Since $N_l(G)$ is a subgroup of G, so is $\mathrm{Rad}\,(G)$. □

Lemma 3.11. *Let G be a gyrogroup. Then*

1. $\mathrm{gyr}\,[a,b](\mathrm{Rad}\,(G)) \subseteq \mathrm{Rad}\,(G)$ *for all* $a,b \in G$, *and*
2. $\mathrm{Rad}\,(G) \oplus a = a \oplus \mathrm{Rad}\,(G)$ *for all* $a \in G$.

Proof. The proof of this lemma follows the same steps as in the proof of Lemma 3.7 with appropriate modifications. □

Theorem 3.12. *The radical of G is a normal subgroup of G.*

Proof. The theorem follows directly from Proposition 3.6, Theorem 3.10, and Lemma 3.11. □

Note that a gyrogroup G is a group if and only if G equals its left nucleus. Hence, if G is a gyrogroup that is not a group, then $N_l(G)$ and $\mathrm{Rad}\,(G)$ are *proper* normal subgroups of G. Note also that normality of $N_l(G)$ and $\mathrm{Rad}\,(G)$ in G follows from normality of $L(N_l(G))$ and $L(L(G)')$ in the left multiplication group of G, see the proof of Lemma 3.7.

4. Subgyrogroups of Prime Index

Motivated by the study of subgroups of prime index in [9], we study subgyrogroups of prime index. Specifically, we are going to prove a gyrogroup version of the following well-known result in abstract algebra: if Ξ is a subgroup of a finite group Γ such that the index $[\Gamma\colon \Xi]$ is the smallest prime dividing the order of Γ, then Ξ is normal in Γ [9, Theorem 1]. It is notable that normality of a subgyrogroup H of a finite gyrogroup G, where $[G\colon H]$ is the smallest prime dividing the order of G, depends on the invariance of the left cosets of H in G under the gyroautomorphisms of G, see Theorem 4.4.

Unless stated otherwise, G is an arbitrary finite gyrogroup.

Let G be a gyrogroup, let $a \in G$, and let $m \in \mathbb{Z}$. Define recursively the following notation:

$$0a = 0, \quad ma = a \oplus ((m-1)a),\, m \geq 1, \quad ma = (-m)(\ominus a),\, m < 0. \tag{7}$$

By induction, one can verify the following usual rules of integral multiples:

1. $(-m)a = \ominus(ma) = m(\ominus a)$,
2. $(m+k)a = (ma) \oplus (ka)$, and
3. $(mk)a = m(ka)$

for all $a \in G$ and $m,k \in \mathbb{Z}$.

Theorem 4.1. *Suppose that H is a subgyrogroup of a gyrogroup G such that $[G\colon H] = p$, p being a prime. The following are equivalent:*

1. *For any $a \in G - H$, $pa \in H$.*

2. *For any $a \in G - H$, $na \in H$ for some positive integer n, depending on a, with no prime divisor less than p.*

3. *For any $a \in G - H$, $a, 2a, \ldots, (p-1)a \notin H$.*

Proof. (1) $\Rightarrow$ (2) Choosing $n = p$ gives item (2).

(2) $\Rightarrow$ (3) Let $a \in G - H$ and let n be as in item (2). By the well-ordering principle, we can let s be the smallest positive integer such that $sa \in H$. Note that $s > 1$. Write $n = st + r$ with $0 \leq r < s$. Then $ra = (n - st)a = (na) \oplus (-st)a = (na) \ominus (t(sa))$. Thus, $ra \in H$ for $na, sa \in H$. The minimality of s forces $r = 0$, so $n = st$. If $s < p$, then s (and hence n) would have a prime divisor less than p. Hence, $s \geq p$, which implies $a, 2a, \ldots, (p-1)a \notin H$.

(3) $\Rightarrow$ (1) First, we prove that $0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H$ are all distinct. Assume to the contrary that $ra \oplus H = sa \oplus H$ for some integers r and s such that $0 \leq r < s \leq p - 1$. Then $sa = (ra) \oplus h$ for some $h \in H$. It follows that $(-r + s)a = \ominus(ra) \oplus (sa) = h \in H$. This contradicts the assumption because $0 < s - r < p$.

Since $|G/H| = [G\colon H] = p$, we have $G/H = \{0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H\}$. Hence, $pa \oplus H = ta \oplus H$ for some t with $0 \leq t \leq p - 1$. As before, the equality gives $(p - t)a \in H$. Since $0 \leq t \leq p - 1$, we have $1 \leq p - t \leq p$. By assumption, $p - t = p$, which implies $t = 0$. Hence, $pa \oplus H = 0 \oplus H = H$ and so $pa \in H$. □

Proposition 4.2. *Let H be a subgyrogroup of a gyrogroup G such that $[G\colon H] = p$, p being a prime. If H satisfies one of the conditions in Theorem 4.1, then*

$$G/H = \{0 \oplus H, a \oplus H, \ldots, (p-1)a \oplus H\}$$

for any $a \in G - H$.

Proof. There is no loss in assuming that H satisfies condition (3) of Theorem 4.1. As proved in Theorem 4.1, $G/H = \{0 \oplus H, \ldots, (p-1)a \oplus H\}$ for any $a \notin H$. □

Proposition 4.3. *Let H be a subgyrogroup of G. If $[G\colon H]$ is the smallest prime dividing the order of G, then H satisfies condition (2) of Theorem 4.1.*

Proof. By Proposition 6.1 of [13], $|G|a = 0 \in H$. Since $|G|$ has no prime divisors less than $[G\colon H]$, condition (2) of Theorem 4.1 holds. □

Theorem 4.4. *Let H be a subgyrogroup of G such that $[G\colon H]$ is the smallest prime dividing the order of G. Then $H \trianglelefteq G$ if and only if there is an element $y \in G - H$ such that*

$$\operatorname{gyr}[a, b](iy \oplus H) \subseteq iy \oplus H$$

for all $a, b \in G$ and $i \in \{0, 1, \ldots, p-1\}$.

Proof. Set $[G\colon H] = p$.

($\Rightarrow$) Suppose that $H \trianglelefteq G$. Then G/H admits the gyrogroup structure and becomes a gyrogroup of order p. By Theorem 6.2 of [13], G/H forms a cyclic group. In particular, $\mathrm{gyr}\,[X,Y]Z = Z$ for all $X,Y,Z \in G/H$. Let a,b,c be arbitrary elements of G. Set $X = a \oplus H$ and $Y = b \oplus H$. From Theorem 27 of [15], we have $c \oplus H = \mathrm{gyr}\,[X,Y](c \oplus H) = (\mathrm{gyr}\,[a,b]c) \oplus H$. Since $H \trianglelefteq G$, $\mathrm{gyr}\,[a,b](H) = H$, which implies $(\mathrm{gyr}\,[a,b]c) \oplus H = \mathrm{gyr}\,[a,b](c \oplus H)$. Hence, $\mathrm{gyr}\,[a,b](c \oplus H) = c \oplus H$.

($\Leftarrow$) Let y be as in the assumption. By Propositions 4.2 and 4.3,

$$G/H = \{0 \oplus H, y \oplus H, \ldots, (p-1)y \oplus H\}.$$

For each $x \in G$, $x \oplus H = iy \oplus H$ for some $i \in \{0,1,\ldots,p-1\}$. By assumption,

$$\mathrm{gyr}\,[a,b](x \oplus H) = \mathrm{gyr}\,[a,b](iy \oplus H) \subseteq iy \oplus H = x \oplus H.$$

By Theorem 4.5 of [12], G acts on G/H by left gyroaddition. By Proposition 3.5 (2) and Theorem 4.6 of [12], $\ker\dot{\varphi} \subseteq H$, where $\dot{\varphi}$ is the associated permutation representation of G. By the first isomorphism theorem [15, Theorem 28],

$$G/\ker\dot{\varphi} \cong \mathrm{Im}\,\dot{\varphi} \leqslant \mathrm{Sym}\,(G/H).$$

Hence, $[G\colon \ker\dot{\varphi}]$ divides $p!$. Since $\ker\dot{\varphi} \leqslant_L G$ and $H \leqslant_L G$, we have

$$[G\colon \ker\dot{\varphi}] = [G\colon H][H\colon \ker\dot{\varphi}] = p[H\colon \ker\dot{\varphi}],$$

which implies $[H\colon \ker\dot{\varphi}]$ divides $(p-1)!$. If $[H\colon \ker\dot{\varphi}] > 1$, one would find a prime q dividing $[H\colon \ker\dot{\varphi}]$ and would have $q|(p-1)!$. Thus, $q < p$ and q divides $|H|$. Since $|H|$ divides $|G|$, we have q divides $|G|$, a contradiction. Hence, $[H\colon \ker\dot{\varphi}] = 1$ and so $H = \ker\dot{\varphi} \trianglelefteq G$. □

Recall from abstract algebra that a subgroup of a group Γ of index two is normal in Γ. This result can be generalized to the case of gyrogroups as follows.

Theorem 4.5. *If H is a subgyrogroup of G such that* $\mathrm{gyr}\,[a,b](H) \subseteq H$ *for all $a,b \in G$ and $[G\colon H] = 2$, then $H \trianglelefteq G$.*

Proof. Let $y \in G - H$ be fixed. By Propositions 4.2 and 4.3, $G/H = \{H, y \oplus H\}$. To complete the proof, we show that $\mathrm{gyr}\,[a,b](y \oplus H) \subseteq y \oplus H$ for all $a,b \in G$. If $z \in \mathrm{gyr}\,[a,b](y \oplus H)$, then $z = \mathrm{gyr}\,[a,b](y \oplus h) = (\mathrm{gyr}\,[a,b]y) \oplus (\mathrm{gyr}\,[a,b]h)$ for some $h \in H$. By assumption, $z \in (\mathrm{gyr}\,[a,b]y) \oplus H$. Note that $\mathrm{gyr}\,[a,b]y \notin H$ since otherwise $\mathrm{gyr}\,[a,b]y = \tilde{h} \in H$ would imply $y = \mathrm{gyr}^{-1}[a,b]\tilde{h} = \mathrm{gyr}\,[b,a]\tilde{h} \in H$, a contradiction. Hence, $(\mathrm{gyr}\,[a,b]y) \oplus H = y \oplus H$ and so $z \in y \oplus H$. This proves $\mathrm{gyr}\,[a,b](x \oplus H) \subseteq x \oplus H$ for all $a,b,x \in G$. By Theorem 4.4, $H \trianglelefteq G$. □

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Gyroharmonic Analysis on Relativistic Gyrogroups

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Abstract

Einstein, Möbius, and Proper Velocity gyrogroups are relativistic gyrogroups that appear as three different realizations of the proper Lorentz group in the real Minkowski space-time $\mathbb{R}^{n,1}$. Using the gyrolanguage we study their gyroharmonic analysis. Although there is an algebraic gyroisomorphism between the three models we show that there are some differences between them. Our study focus on the translation and convolution operators, eigenfunctions of the Laplace-Beltrami operator, Poisson transform, Fourier-Helgason transform, its inverse, and Plancherel's Theorem. We show that in the limit of large t, $t \to +\infty$, the resulting gyroharmonic analysis tends to the standard Euclidean harmonic analysis on $\mathbb{R}^n$, thus unifying hyperbolic and Euclidean harmonic analysis.

Keywords: Gyrogroups, gyroharmonic analysis, Laplace Beltrami operator, eigenfunctions, generalized Helgason-Fourier transform, Plancherel's theorem.

2010 Mathematics Subject Classification: Primary 43A85; Secondary 43A30, 43A90, 44A35, 20N05.

1. Introduction

Harmonic analysis is the branch of mathematics that studies the representation of functions or signals as the superposition of basic waves called harmonics. Closely related is the study of Fourier series and Fourier transforms. Its applications are of major importance and can be found in diverse areas such as signal processing, quantum mechanics, and neuroscience (see [23] for an overview). The classical Fourier transform on $\mathbb{R}^n$ is still an area of research, particularly concerning Fourier

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transformation on more general objects such as tempered distributions. Some of its properties can be translated in terms of the Fourier transform. For instance, the Paley-Wiener theorem states that if a function is a nonzero distribution of compact support then its Fourier transform is never compactly supported [22]. This is a very elementary form of an uncertainty principle in the harmonic analysis setting. Fourier series can be conveniently studied in the context of Hilbert spaces, which provides a connection between harmonic analysis and functional analysis.

In the last century the Fourier transform was generalised to compact groups, abelian locally compact groups, symmetric spaces, etc.. For compact groups, the Peter-Weyl theorem establish the relationship between harmonics and irreducible representations. This choice of harmonics enjoys some of the useful properties of the classical Fourier transform in terms of carrying convolutions to pointwise products, or otherwise showing a certain understanding of the underlying group structure. For general nonabelian locally compact groups, harmonic analysis is closely related to the theory of unitary group representations. Noncommutative harmonic analysis appeared mainly in the context of symmetric spaces where many Lie groups are locally compact and noncommutative. These examples are of interest and frequently applied in mathematical physics, and contemporary number theory, particularly automorphic representations. The development of noncommutative harmonic analysis was done by many mathematicians like John von Neumann, Harisch-Chandra and Sigurdur Helgason [13, 14].

It is well-known that Fourier analysis is intimately connected with the action of the group of translations on Euclidean space. The group structure enters into the study of harmonic analysis by allowing the consideration of the translates of the object under study (functions, measures, etc.). First we study the spectral analysis finding the elementary components for the decomposition and second we perform the harmonic or spectral synthesis, finding a way in which the object can be construed as a combination of its elementary components [16]. Harmonic analysis in Euclidean spaces is rich because of its connection with several classes of transformations: the dilations and the rotations as well as the translations. The Fourier transform in $\mathbb{R}^n$ has a very simple transformation law under dilations and it commutes with the action of rotations.

The real hyperbolic space is commonly viewed as a homogeneous space obtained from the quotient $\mathrm{SO}_0(n,1)/\mathrm{SO}(n)$ where $\mathrm{SO}_0(n,1)$ is the proper Lorentz group in the Minkowski space $\mathbb{R}^{n,1}$ and $\mathrm{SO}(n)$ is the special orthogonal group. It is well known that pure Lorentz transformations (the translations in hyperbolic space) do not form a group since the composition of two is no longer a pure Lorentz transformation. However, by incorporating the gyration operator it is possible to obtain a gyroassociative law. The resulting algebraic structure called gyrogroup by A.A. Ungar [25] repairs the breakdown of associativity and commutativity of the relativistic additions. The gyrogroup structure is a natural extension of the group structure, discovered in 1988 by A. A. Ungar in the context of Einstein's velocity addition law [24, 25]. It has been studied by A. A. Ungar and others see, for instance, [6, 8, 26, 27, 29, 30]. Gyrogroups provide a fruitful bridge between nonassociative algebra and hyperbolic geometry, just as groups lay the bridge between associative algebra and Euclidean geometry.

In this survey paper we show the similarities and differences between gyroharmonic analysis on three relativistic gyrogroups: Möbius, Einstein, and Proper Velocity gyrogroups. For the Möbius and Eintein cases we provide a generalization of the results in [9,10] by replacing the real parameter σ by a complex parameter z, under the identification $2z = n+\sigma-2$, where n is the dimension of the hyperbolic space.

The paper is organized as follows. In Section 2 we review harmonic analysis on $\mathbb{R}^n$ as spectral theory of the Laplace operator. In Sections 3, 4, and 5 we present the results concerning gyroharmonic analysis for the Einstein, Möbius, and Proper Velocity gyrogroups, respectively. Each of these sections focus the following aspects: the relativistic addition and its properties, the generalised translation operator and the associated convolution operator, the eigenfunctions of the generalised Laplace-Beltrami operator, the generalized spherical functions, the generalized Poisson transform, the generalized Helgason Fourier transform, its inverse and Plancherel's Theorem. We show that in the limit $t \to +\infty$ we recover the well-known results in Euclidean harmonic analysis. Two appendices, A and B, concerning all necessary facts on spherical harmonics and Jacobi functions, are found at the end of the paper.

2. Euclidean Harmonic Analysis Revisited

Euclidean harmonic analysis in $\mathbb{R}^n$ is associated to the translation group $(\mathbb{R}^n, +)$ and the spectral theory of the Laplace operator Δ. The Fourier transform of $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ is defined by

$$(\mathfrak{F}f)(\xi) = \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)\, dx.$$

Since $\mathfrak{F}$ is a unitary operator on $L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ which is dense in $L^2(\mathbb{R}^n)$ then the Fourier transform can be uniquely extended to a unitary operator in $L^2(\mathbb{R}^n)$, denoted by the same symbol. Denoting $(\mathfrak{F}f)(\xi) = \widehat{f}(\xi)$ we can write the Fourier inverse formula in polar coordinates

$$f(x) = \frac{1}{(2\pi)^n} \int_0^\infty \left(\int_{S^{n-1}} \widehat{f}(\lambda u)\; e^{i\lambda\langle x,u\rangle} du \right) \lambda^{n-1}\; \mathrm{d}\lambda.$$

The expression in parenthesis is an eigenfunction of the Laplace operator with eigenvalue $-\lambda^2$ ($\Delta f_\lambda = -\lambda^2 f_\lambda$). Thus, the function f can be represented by an integral of such eigenfunctions. Defining the spectral projection operator

$$\mathcal{P}_\lambda f(x) = \frac{1}{(2\pi)^n} \lambda^{n-1} \int_{S^{n-1}} \widehat{f}(\lambda u)\; e^{i\lambda\langle x,u\rangle}\; \mathrm{d}u$$

we obtain the spectral representation formula

$$f(x) = \int_0^\infty \mathcal{P}_\lambda f(x)\ \mathrm{d}\lambda.$$

We can also write

$$\mathcal{P}_\lambda f(x) = \int_{\mathbb{R}^n} \varphi_\lambda(|x-y|) f(y)\ dy, \tag{1}$$

where

$$\varphi_\lambda(r) = (2\pi)^{-\frac{n}{2}} \lambda^{\frac{n}{2}} r^{1-\frac{n}{2}} J_{\frac{n}{2}-1}(r\lambda)$$

is a multiple of the usual spherical function, because $\varphi_\lambda(0) = (2\pi)^{-n}\lambda^{n-1}\omega_{n-1}$ instead of one. Formula (1) involves only the distance $|x-y|$ between points in $\mathbb{R}^n$ and the Euclidean measure, which are both invariants of the Euclidean motion group. The following characterisation of $\mathcal{P}_\lambda f$ for $f \in L^2(\mathbb{R}^n)$ was given in [21]

Theorem 2.1. *[21] Let $f_\lambda(x)$ be a measurable function on $(0,\infty)\times\mathbb{R}^n$ such that $\Delta f_\lambda = -\lambda^2 f_\lambda$ for almost every λ. Then there exists $f \in L^2(\mathbb{R}^n)$ with $\mathcal{P}_\lambda f = f_\lambda$ a.e. if and only if one of the following equivalent conditions holds:*

(*i*) $$\int_0^\infty \left(\sup_{z,t} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2\ \mathrm{d}x \right) \mathrm{d}\lambda < \infty$$

(*ii*) $$\sup_{z,t} \int_0^\infty \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2\ \mathrm{d}x\ \mathrm{d}\lambda < \infty$$

(*iii*) $$\int_0^\infty \left(\lim_{t\to\infty} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2\ \mathrm{d}x \right) \mathrm{d}\lambda < \infty \quad \text{for some } z$$

(*iv*) $$\lim_{t\to\infty} \int_0^\infty \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2\ \mathrm{d}x\mathrm{d}\lambda < \infty, \quad \text{for some } z.$$

Furthermore, we have

$$||f||_2^2 = \pi \int_0^\infty \left(\lim_{t\to\infty} \frac{1}{t} \int_{B_t(z)} |f_\lambda(x)|^2\ \mathrm{d}x \right) \mathrm{d}\lambda.$$

3. Gyroharmonic Analysis on the Einstein Gyrogroup

3.1 Einstein Addition in the Ball

The Beltrami-Klein model of the $n-$dimensional real hyperbolic geometry can be realised as the open ball $\mathbb{B}_t^n = \{x \in \mathbb{R}^n : \|x\| < t\}$ of $\mathbb{R}^n$, endowed with the Riemannian metric

$$ds^2 = \frac{\|dx\|^2}{1-\frac{\|x\|^2}{t^2}} + \frac{(\langle x, dx\rangle)^2}{t^2\left(1-\frac{\|x\|^2}{t^2}\right)^2}.$$

This metric corresponds to the metric tensor

$$g_{ij}(x) = \frac{\delta_{ij}}{1-\frac{\|x\|^2}{t^2}} + \frac{x_i x_j}{t^2\left(1-\frac{\|x\|^2}{t^2}\right)^2}, \quad i,j \in \{1,\ldots,n\}$$

and its inverse is given by

$$g^{ij}(x) = \left(1-\frac{\|x\|^2}{t^2}\right)\left(\delta_{ij} - \frac{x_i x_j}{t^2}\right), \quad i,j \in \{1,\ldots,n\}.$$

The group of all isometries of the Klein model [34] consists of the elements of the group $\mathrm{O}(n)$ and the mappings given by

$$T_a(x) = \frac{a + P_a(x) + \mu_a Q_a(x)}{1 + \frac{1}{t^2}\langle a, x\rangle} \tag{2}$$

where

$$P_a(x) = \begin{cases} \langle a, x\rangle \frac{a}{\|a\|^2} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}, \quad Q_a(x) = x - P_a(x), \text{ and } \mu_a = \sqrt{1-\frac{\|a\|^2}{t^2}}.$$

Some properties are listed in the next proposition.

Proposition 3.1. *Let $a \in \mathbb{B}_t^n$. Then*

(i) $P_a^2 = P_a$, $\quad Q_a^2 = Q_a$, $\quad \langle a, P_a(x)\rangle = \langle a, x\rangle$, *and* $\langle a, Q_a(x)\rangle = 0$.

(ii) $T_a(0) = a$ *and* $T_a(-a) = 0$.

(iii) $T_a(T_{-a}(x)) = T_{-a}(T_a(x)) = x, \quad \forall\, x \in \mathbb{B}_t^n$.

(iv) $T_a\left(\pm t\frac{a}{\|a\|}\right) = \pm t\frac{a}{\|a\|}$. *Moreover, T_a fixes two points on $\partial\mathbb{B}_t^n$ and no point of $\mathbb{B}_t^n$.*

(v) The identity

$$1 - \frac{\langle T_a(x), T_a(y)\rangle}{t^2} = \frac{\left(1-\frac{\|a\|^2}{t^2}\right)\left(1-\frac{\langle x,y\rangle}{t^2}\right)}{\left(1+\frac{\langle x,a\rangle}{t^2}\right)\left(1+\frac{\langle y,a\rangle}{t^2}\right)} \tag{3}$$

holds for all $x, y \in \mathbb{B}_t^n$. In particular, when $x = y$ we have

$$1 - \frac{\|T_a(x)\|^2}{t^2} = \frac{\left(1-\frac{\|a\|^2}{t^2}\right)\left(1-\frac{\|x\|^2}{t^2}\right)}{\left(1+\frac{\langle x,a\rangle}{t^2}\right)^2} \tag{4}$$

and when $x = 0$ in (3) we obtain

$$1 - \frac{\langle a, T_a(y)\rangle}{t^2} = \frac{1-\frac{\|a\|^2}{t^2}}{1+\frac{\langle y,a\rangle}{t^2}}. \tag{5}$$

(vi) For $R \in O(n)$

$$R \circ T_a = T_{Ra} \circ R. \tag{6}$$

To endow the ball $\mathbb{B}_t^n$ with a binary operation, closely related to vector addition in $\mathbb{R}^n$, we define the Einstein addition on $\mathbb{B}_t^n$ by

$$a \oplus x := T_a(x), \quad a, x \in \mathbb{B}_t^n. \tag{7}$$

This definition agrees with Ungar's definition for the Einstein addition since we can write (2) as

$$a \oplus x = \frac{1}{1 + \frac{\langle a,x \rangle}{t^2}} \left(a + \frac{1}{\gamma_a} x + \frac{1}{t^2} \frac{\gamma_a}{1+\gamma_a} \langle a, x \rangle \, a \right) \tag{8}$$

where $\gamma_a = \left(\sqrt{1 - \frac{\|a\|^2}{t^2}} \right)^{-1}$ is the relativistic gamma factor.

It is known that $(\mathbb{B}_t^n, \oplus)$ is a gyrogroup (see [24, 27]), i.e., it satisfies the following axioms:

(G1) There is at least one element 0 satisfying $0 \oplus a = a$, for all $a \in \mathbb{B}_t^n$;

(G2) For each $a \in B$ there is an element $\ominus a \in \mathbb{B}_t^n$ such that $\ominus a \oplus a = 0$;

(G3) For any $a, b, c \in \mathbb{B}_t^n$ there exists a unique element $gyr[a, b]c \in \mathbb{B}_t^n$ such that the binary operation satisfies the *left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus gyr[a, b]c; \tag{9}$$

(G4) *The map $gyr[a, b] : \mathbb{B}_t^n \to \mathbb{B}_t^n$ given by $c \mapsto gyr[a, b]c$ is an automorphism of $(\mathbb{B}_t^n, \oplus)$;*

(G5) *The gyroautomorphism $gyr[a, b]$ possesses the left loop property*

$$gyr[a, b] = gyr[a \oplus b, b] \tag{10}$$

for all $a, b \in B$.

The gyration operator can be given in terms of the Einstein addition $\oplus$ by the equation (see [27])

$$\mathrm{gyr}\,[a, b]c = \ominus(a \oplus b) \oplus (a \oplus (b \oplus c)).$$

The Einstein gyrogroup is gyrocommutative since Einstein addition satisfies

$$a \oplus b = \mathrm{gyr}\,[a, b](b \oplus a). \tag{11}$$

In the limit $t \to +\infty$, the ball $\mathbb{B}_t^n$ expands to the whole of the space $\mathbb{R}^n$, Einstein addition reduces to vector addition in $\mathbb{R}^n$ and, therefore, the gyrogroup $(\mathbb{B}_t^n, \oplus)$

reduces to the translation group $(\mathbb{R}^n,+)$. Some useful gyrogroup identities ([27], pp. 48 and 68) that will be used in this paper are

$$\ominus(a\oplus b)=(\ominus a)\oplus(\ominus b) \tag{12}$$
$$a\oplus(\ominus a\oplus b)=b \tag{13}$$
$$(\mathrm{gyr}\,[a,b])^{-1}=\mathrm{gyr}\,[b,a] \tag{14}$$
$$\mathrm{gyr}\,[a\oplus b,\ominus a]=\mathrm{gyr}\,[a,b] \tag{15}$$
$$\mathrm{gyr}\,[\ominus a,\ominus b]=\mathrm{gyr}\,[a,b] \tag{16}$$
$$\mathrm{gyr}\,[a,\ominus a]=I \tag{17}$$
$$\mathrm{gyr}\,[a,b](b\oplus(a\oplus c))=(a\oplus b)\oplus c \tag{18}$$

Properties (14) and (15) are valid for general gyrogroups while properties (12) and (18) are valid only for gyrocommutative gyrogroups. Combining formulas (15) and (18) with (14) we obtain the identities

$$\mathrm{gyr}\,[\ominus a,a\oplus b]=\mathrm{gyr}\,[b,a] \tag{19}$$
$$b\oplus(a\oplus c)=\mathrm{gyr}\,[b,a]((a\oplus b)\oplus c). \tag{20}$$

In the special case when $n=1$, the Einstein gyrogroup becomes a group since gyrations are trivial (a trivial map being the identity map). For $n\geq 2$ the gyrosemidirect product of $(\mathbb{B}_t^n,\oplus)$ and $\mathrm{O}(n)$ (see [27]) gives the group $\mathbb{B}_t^n\rtimes_{\mathrm{gyr}}\mathrm{O}(n)$ for the operation

$$(a,R)(b,S)=(a\oplus Rb,\mathrm{gyr}\,[a,Rb]RS)\,.$$

This group is a realisation of the Lorentz group $\mathrm{O}(n,1)$. In the limit $t\to+\infty$ the group $\mathbb{B}_t^n\rtimes_{\mathrm{gyr}}\mathrm{O}(n)$ reduces to the Euclidean group $\mathrm{E}(n)=\mathbb{R}^n\rtimes\mathrm{O}(n)$. In [9] we developed the harmonic analysis on the Einstein gyrogroup depending on a real parameter σ. We provide here a generalization of these results considering a complex parameter z, under the identification $2z=n+\sigma-2$. Most of the proofs are analogous as in [9] and therefore will be omitted.

3.2 The Generalised Translation and Convolution

Definition 3.2. For a complex valued function f defined on B_t^n, $a\in\mathbb{B}_t^n$ and $z\in\mathbb{C}$ we define the generalised translation operator $\tau_a f$ by

$$\tau_a f(x)=j_a(x)f((-a)\oplus x) \tag{21}$$

with the automorphic factor $j_a(x)$ given by

$$j_a(x)=\left(\frac{\sqrt{1-\frac{\|a\|^2}{t^2}}}{1-\frac{\langle a,x\rangle}{t^2}}\right)^z. \tag{22}$$

For $z=n+1$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $T_{-a}(x)=(-a)\oplus x$. For any $z\in\mathbb{C}$, we obtain in the limit $t\to+\infty$ the Euclidean translation operator $\tau_a f(x)=f(-a+x)=f(x-a)$.

Lemma 3.3. *For any $a, b, x, y \in \mathbb{B}_t^n$ the following relations hold*

$$(i) \quad j_{-a}(-x) = j_a(x) \tag{23}$$

$$(ii) \quad j_a(a)j_a(0) = 1 \tag{24}$$

$$(iii) \quad j_a(x) = j_x(a)j_a(0)j_x(x) \tag{25}$$

$$(iv) \quad j_a(a \oplus x) = (j_{-a}(x))^{-1} \tag{26}$$

$$(v) \quad j_{(-a)\oplus x}(0) = j_{x\oplus(-a)}(0) = j_x(a)j_a(0) = j_a(x)j_x(0) \tag{27}$$

$$(vi) \quad j_{(-a)\oplus x}((-a) \oplus x) = (j_a(x))^{-1}j_x(x) \tag{28}$$

$$(vii) \quad \tau_a j_y(x) = [\tau_{-a} j_x(y)]j_x(x)j_y(0) \tag{29}$$

$$(viii) \quad \tau_{-a} j_a(x) = 1 \tag{30}$$

$$(ix) \quad \tau_a j_y(x) = j_{a\oplus y}(x) \tag{31}$$

$$(x) \quad \tau_a f(x) = [\tau_x f(-\mathrm{gyr}\,[x, a]a)]j_a(0)j_x(x) \tag{32}$$

$$(xi) \quad \tau_b \tau_a f(x) = \tau_{b\oplus a} f(\mathrm{gyr}\,[a, b]\,x) \tag{33}$$

$$(xii) \quad \tau_{-a} \tau_a f(x) = f(x) \tag{34}$$

$$(xiii) \quad \tau_b \tau_a f(x) = [\tau_{-b} \tau_x f(-\mathrm{gyr}\,[-b, x \oplus a]\,\mathrm{gyr}\,[x, a]\,a)]\,j_a(0)j_x(x). \tag{35}$$

For the translation operator to be an unitary operator we have to properly define a Hilbert space. We consider the complex weighted Hilbert space $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ with

$$\mathrm{d}\mu_{z,t}(x) = \left(1 - \frac{\|x\|^2}{t^2}\right)^{z-\frac{n+1}{2}} \mathrm{d}x,$$

where $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. For the special case $z = 0$ we recover the invariant measure associated to the transformations $T_a(x)$.

Proposition 3.4. *For $f, g \in L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $a \in \mathbb{B}_t^n$ we have*

$$\int_{\mathbb{B}_t^n} \tau_a f(x)\ \overline{g(x)}\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x)\ \overline{\tau_{-a} g(x)}\ \mathrm{d}\mu_{z,t}(x). \tag{36}$$

Corollary 3.5. *For $f, g \in L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $a \in \mathbb{B}_t^n$ we have*

$$(i) \quad \int_{\mathbb{B}_t^n} \tau_a f(x)\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x) j_{-a}(x)\ \mathrm{d}\mu_{z,t}(x); \tag{37}$$

$$(ii) \quad \textit{If } z = 0 \textit{ then} \int_{\mathbb{B}_t^n} \tau_a f(x)\ \mathrm{d}\mu_{z,t}(x) = \int_{\mathbb{B}_t^n} f(x)\ \mathrm{d}\mu_{z,t}(x); \tag{38}$$

$$(iii) \quad ||\tau_a f||_2 = ||f||_2. \tag{39}$$

From Corollary 3.5 we see that the generalised translation τ_a is an unitary operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and the measure $\mathrm{d}\mu_{z,t}$ is translation invariant only for the case $z = 0$. Now we define the generalised convolution of two functions in $\mathbb{B}_t^n$.

Definition 3.6. The generalised convolution of two measurable functions f and g is given by

$$(f * g)(x) = \int_{\mathbb{B}_t^n} f(y)\ \tau_x g(-y)\ j_x(x)\ \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{B}_t^n. \tag{40}$$

By Proposition 3.4 and the change of variables $-y \mapsto z$ we can see that the generalised convolution is commutative, i.e., $f * g = g * f$.

Before we prove that it is well defined for $\mathrm{Re}(z) < \frac{n-1}{2}$ we need the following lemma.

Lemma 3.7. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$. *Then*

$$\int_{\mathbb{S}^{n-1}} |j_x(r\xi)\ j_x(x)|\ \mathrm{d}\sigma(\xi) \le C_z$$

with

$$C_z = \begin{cases} 1, & \text{if } \mathrm{Re}(z) \in]-1, 0[\\ \dfrac{\Gamma\left(\frac{n}{2}\right)\left(\frac{n-2\,\mathrm{Re}(z)-1}{2}\right)}{\Gamma\left(\frac{n-\mathrm{Re}(z)}{2}\right)\Gamma\left(\frac{n-\mathrm{Re}(z)-1}{2}\right)}, & \text{if } \mathrm{Re}(z) \in]-\infty, -1] \cup [0, \frac{n-1}{2}[\end{cases}. \tag{41}$$

Proof. Using (A.2) in Appendix A we obtain

$$\int_{\mathbb{S}^{n-1}} |j_x(r\xi)\ j_x(x)|\ \mathrm{d}\sigma(\xi) \quad = \quad {}_2F_1\left(\frac{\mathrm{Re}(z)}{2}, \frac{\mathrm{Re}(z)+1}{2}; \frac{n}{2}; \frac{r^2\|x\|^2}{t^4}\right).$$

Considering the function $g(s) = {}_2F_1\left(\frac{\mathrm{Re}(z)}{2}, \frac{\mathrm{Re}(z)+1}{2}; \frac{n}{2}; s\right)$ and applying (A.8) and (A.6) in Appendix A we get

$$\begin{aligned} g'(s) &= \frac{\mathrm{Re}(z)(\mathrm{Re}(z)+1)}{2n}\,{}_2F_1\left(\frac{\mathrm{Re}(z)+2}{2}, \frac{\mathrm{Re}(z)+3}{2}; \frac{n}{2}+1; s\right). \\ &= \underbrace{\frac{\mathrm{Re}(z)(\mathrm{Re}(z)+1)}{2n}}_{(I)}(1-s)^{\frac{n-2\,\mathrm{Re}(z)-3}{2}}\ \underbrace{{}_2F_1\left(\frac{n-\mathrm{Re}(z)}{2}, \frac{n-\mathrm{Re}(z)-1}{2}; \frac{n}{2}+1; s\right)}_{(II)}. \end{aligned}$$

Since $\mathrm{Re}(z) < \frac{n-1}{2}$ then the hypergeometric function (II) is positive for $s > 0$, and therefore, positive on the interval $[0, 1[$. Studying the sign of (I) we conclude that the function g is strictly increasing when $\mathrm{Re}(z) \in]-\infty, -1] \cup [0, \frac{n-1}{2}[$ and strictly decreasing when $\mathrm{Re}(z) \in]-1, 0[$. Since $\mathrm{Re}(z) < \frac{n-1}{2}$, then it exists the limit $\lim_{s\to 1^-} g(s)$ and by (A.5) it is given by

$$g(1) = \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\frac{n-2\,\mathrm{Re}(z)-1}{2}\right)}{\Gamma\left(\frac{n-\mathrm{Re}(z)}{2}\right)\Gamma\left(\frac{n-\mathrm{Re}(z)-1}{2}\right)}.$$

Thus,

$$g(s) \leq \max\{g(0), g(1)\} = C_z$$

with $g(0) = 1$. $\square$

Proposition 3.8. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_1 \leq C_z \, ||f||_1 \, ||\widetilde{g}||_1 \tag{42}$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi \in \mathbb{S}^{n-1} \\ y \in \mathbb{B}_t^n}} g(\mathrm{gyr}\,[y, r\xi]r\xi)$ *for any* $r \in [0, t[$.

In the special case when g is a radial function we obtain as a corollary that $||f * g||_1 \leq C_z ||f||_1 ||g||_1$ since $\widetilde{g} = g$. We can also prove that for $f \in L^\infty(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ we have the inequality

$$||f * g||_\infty \leq C_z \, ||\widetilde{g}||_1 \, ||f||_\infty. \tag{43}$$

By (42), (43), and the Riesz-Thorin interpolation Theorem we further obtain for $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ the inequality

$$||f * g||_p \leq C_z \, ||\widetilde{g}||_1 \, ||f||_p.$$

To obtain a Young's inequality for the generalised convolution we restrict ourselves to the case $\mathrm{Re}(z) \leq 0$.

Theorem 3.9. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)} ||\widetilde{g}||_q^{1-s} \, ||g||_q^s \, ||f||_p \tag{44}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y \in \mathbb{B}_t^n} g(\mathrm{gyr}\,[y, x]x)$, *for any* $x \in \mathbb{B}_t^n$.

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y) j_x(x)| \leq 2^{-\mathrm{Re}(z)}, \; \forall x, y \in \mathbb{B}_t^n, \; \forall\, \mathrm{Re}(z) \leq 0. \tag{45}$$

Corollary 3.10. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)} ||g||_q \, ||f||_p. \tag{46}$$

Remark 1. For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$.

Another important property of the Euclidean convolution is its translation invariance. Next theorem shows that the generalised convolution is gyro-translation invariant.

Theorem 3.11. *The generalised convolution is gyro-translation invariant, i.e.,*

$$\tau_a(f * g)(x) = (\tau_a f(\cdot) * g(\mathrm{gyr}\,[-a, x]\ \cdot\ ))(x). \tag{47}$$

In Theorem 3.11 if g is a radial function then we obtain the translation invariant property $\tau_a(f * g) = (\tau_a f) * g$. The next theorem shows that the generalised convolution is gyroassociative.

Theorem 3.12. *If $f, g, h \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ then*

$$(f *_a (g *_x h))(a) = (((f(x) *_y g(\mathrm{gyr}\,[a, -(y \oplus x)]\mathrm{gyr}\,[y, x]x))(y)) *_a h(y))(a) \tag{48}$$

Corollary 3.13. *If $f, g, h \in L^1(\mathbb{B}_t^n, d\mu_{z,t})$ and g is a radial function then the generalised convolution is associative. i.e.,*

$$f * (g * h) = (f * g) * h.$$

From Theorem 3.12 we see that the generalised convolution is associative up to a gyration of the argument of the function g. However, if g is a radial function then the corresponding gyration is trivial (that is, it is the identity map) and therefore the convolution becomes associative. Moreover, in the limit $t \to +\infty$ gyrations reduce to the identity, so that formula (48) becomes associative in the Euclidean case. If we denote by $L^1_R(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ the subspace of $L^1(\mathbb{B}_t^n, d\mu_{z,t})$ consisting of radial functions then, for $\mathrm{Re}(z) < \frac{n-1}{2}$, $L^1_R(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ is a commutative associative Banach algebra under the generalised convolution.

3.3 Laplace Beltrami Operator $\Delta_{z,t}$ and its Eigenfunctions

The gyroharmonic analysis on the Einstein gyrogroup is based on the generalised Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \left(1 - \frac{\|x\|^2}{t^2}\right)\left(\Delta - \sum_{i,j=1}^{n} \frac{x_i x_j}{t^2}\frac{\partial^2}{\partial x_i \partial x_j} - \frac{2(z+1)}{t^2}\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} - \frac{z(z+1)}{t^2}\right).$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator τ_a.

Proposition 3.14. *For each $f \in C^2(\mathbb{B}_t^n)$ and $a \in \mathbb{B}_t^n$*

$$(\Delta_{z,t} f)(a) = (j_a(0))^{-1}\Delta(\tau_{-a} f)(0) - \frac{z(z+1)}{t^2}(\tau_{-a} f)(0) \tag{49}$$

A very important property is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 3.15. *The operator $\Delta_{z,t}$ commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall\, f \in C^2(\mathbb{B}_t^n),\, \forall\, b \in \mathbb{B}_t^n.$$

There is an important relation between the operator $\Delta_{z,t}$ and the measure $\mathrm{d}\mu_{z,t}$. Up to a constant the Laplace-Beltrami operator $\Delta_{z,t}$ corresponds to a weighted Laplace operator on $\mathbb{B}_t^n$ for the weighted measure $\mathrm{d}\mu_{\sigma,t}$ in the sense defined in [12], Section 3.6. From Theorem 11.5 in [12] we know that the Laplace operator on a weighted manifold is essentially self-adjoint if all geodesics balls are relatively compact. Therefore, $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 3.16. *The operator $\Delta_{z,t}$ is essentially self-adjoint in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.*

Definition 3.17. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{B}_t^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{\left(\sqrt{1-\frac{\|x\|^2}{t^2}}\right)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}}{\left(1-\frac{\langle x,\xi\rangle}{t}\right)^{\frac{n-1}{2}+\mathrm{i}\lambda t}}. \tag{50}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x,\lambda\xi\rangle}$. Since

$$e_{\lambda,\xi;t}(x) = \left(1-\frac{\langle x,\xi\rangle}{t}\right)^{-\frac{n-1}{2}-\mathrm{i}\lambda t}\left(\sqrt{1-\frac{\|x\|^2}{t^2}}\right)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}$$

then we obtain

$$\lim_{t\to+\infty} e_{\lambda,\xi;t}(x) = \lim_{t\to+\infty}\left[\left(1-\frac{\langle x,\xi\rangle}{t}\right)^t\right]^{-i\lambda} = \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}. \tag{51}$$

Proposition 3.18. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue $-\lambda^2 - \dfrac{(n-1-2z)^2}{4t^2}$.*

In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. In the Euclidean case given two eigenfunctions $\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}$ and $\mathrm{e}^{\mathrm{i}\langle x,\gamma\omega\rangle}$, $\lambda,\gamma \in \mathbb{R}$, $\xi,\omega \in \mathbb{S}^{n-1}$ of the Laplace operator with eigenvalues $-\lambda^2$ and $-\gamma^2$ respectively, the product of the two eigenfunctions is again an eigenfunction of the Laplace operator with eigenvalue $-(\lambda^2+\gamma^2+2\lambda\gamma\langle\xi,\omega\rangle)$. Indeed,

$$\Delta(\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}\mathrm{e}^{\mathrm{i}\langle x,\gamma\omega\rangle}) = -\|\lambda\xi+\gamma\omega\|^2\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi+\gamma\omega\rangle} = -(\lambda^2+\gamma^2+2\lambda\gamma\langle\xi,\omega\rangle)\mathrm{e}^{\mathrm{i}\langle x,\lambda\xi+\gamma\omega\rangle}. \tag{52}$$

Unfortunately, in the hyperbolic case this is no longer true in general. The only exception is the case $n = 1$ and $z = 0$ as the next proposition shows.

Proposition 3.19. *For $n \geq 2$ the product of two eigenfunctions of $\Delta_{z,t}$ is not an eigenfunction of $\Delta_{z,t}$ and for $n = 1$ the product of two eigenfunctions of $\Delta_{z,t}$ is an eigenfunction of $\Delta_{z,t}$ only in the case $z = 0$.*

In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (50) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\frac{1+\frac{x}{t}}{1-\frac{x}{t}}\right)^{\frac{i\lambda t}{2}}$$

and, therefore, the exponential law is valid, i.e., $e_{\lambda;t}(x)e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x)$. In the Euclidean case the translation of the Euclidean plane waves $e^{i\langle x,\lambda\xi\rangle}$ decomposes into the product of two plane waves one being a modulation. In the hyperbolic case, the generalised translation of (50) factorises also in a modulation and the hyperbolic plane wave but it appears an Einstein transformation acting on $\mathbb{S}^{n-1}$ as the next proposition shows.

Proposition 3.20. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,a\oplus\xi;t}(x). \tag{53}$$

Remark 2. The fractional linear mappings $T_a(\xi) = a \oplus \xi, a \in \mathbb{B}_t^n, \xi \in \mathbb{S}^{n-1}$ are obtained from (2) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{T_a(x)}{t} = T_a(\xi)$ and are given by

$$T_a(\xi) = \frac{\frac{a}{t} + P_a(\xi) + \mu_a Q_a(\xi)}{1 + \frac{\langle \xi, a\rangle}{t}}.$$

They map $\mathbb{S}^{n-1}$ onto itself for any $t > 0$ and $a \in \mathbb{B}_t^n$, and in the limit $t \to +\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. Therefore, formula (53) converges in the limit to the well-known formula in the Euclidean case

$$\mathrm{e}^{\mathrm{i}\langle -a+x,\lambda\xi\rangle} = \mathrm{e}^{\mathrm{i}\langle -a,\lambda\xi\rangle} e^{\mathrm{i}\langle x,\lambda\xi\rangle}, \quad a, x, \lambda\xi \in \mathbb{R}^n.$$

Now we study the radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions.

Definition 3.21. For each $\lambda \in \mathbb{C}$, we define the generalised spherical function $\phi_{\lambda;t}$ by

$$\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{54}$$

Using (A.2) in Appendix A and then (A.6) in Appendix A we can write this function as

$$\begin{aligned}
\phi_{\lambda;t}(x) &= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4}, \frac{n+1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right) \qquad (55)\\
&= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-z+\frac{n-1}{2}-\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n+1-2\mathrm{i}\lambda t}{4}, \frac{n-1-2\mathrm{i}\lambda t}{4}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right).
\end{aligned}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t} = \phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda \in \mathbb{C}$. Putting $\|x\| = t\tanh s$, with $s \in \mathbb{R}^+$, and using (A.7) in

Appendix A we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\begin{aligned}\phi_{\lambda;t}(t\,\tanh s) &= (\cosh s)^z\,{}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4},\frac{n-1-2\mathrm{i}\lambda t}{4};\frac{n}{2};-\sinh^2(s)\right)\\ &= (\cosh s)^z\varphi_{\lambda t}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(s).\end{aligned}\tag{56}$$

The following theorem characterises all generalised spherical functions.

Theorem 3.22. *The function $\phi_{\lambda;t}$ is a generalised spherical function with eigenvalue $-\lambda^2-\frac{(n-1-2z)^2}{4t^2}$. Moreover, if we normalize spherical functions such that $\phi_{\lambda;t}(0)=1$, then all generalised spherical functions are given by $\phi_{\lambda;t}$.*

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 3.23. *For $\mathrm{Im}(\lambda)<0$ we have*

$$\lim_{s\to+\infty}\phi_{\lambda;t}(t\,\tanh s)e^{\left(\frac{n-1-2z}{2}-\mathrm{i}\lambda t\right)s}=c(\lambda t)$$

where $c(\lambda t)$ is the Harish-Chandra c-function given by

$$c(\lambda t)=\frac{2^{\frac{n-1-2z}{2}-\mathrm{i}\lambda t}\Gamma\left(\frac{n}{2}\right)\Gamma(\mathrm{i}\lambda t)}{\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4}\right)\Gamma\left(\frac{n+1+2\mathrm{i}\lambda t}{4}\right)}.\tag{57}$$

Remark 3. Using the relation $\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)=2^{1-2z}\sqrt{\pi}\,\Gamma(2z)$ we can write

$$\Gamma\left(\frac{n+1+2\mathrm{i}\lambda t}{4}\right)=\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4}+\frac{1}{2}\right)=\frac{2^{1-\frac{n-1+2\mathrm{i}\lambda t}{2}}\sqrt{\pi}\,\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{2}\right)}{\Gamma\left(\frac{n-1+2\mathrm{i}\lambda t}{4}\right)}$$

and, therefore, (57) simplifies to

$$c(\lambda t)=\frac{2^{n-2-z}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\mathrm{i}\lambda t\right)}{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}\tag{58}$$

Finally, we have the addition formula for the generalised spherical functions.

Proposition 3.24. *For every $\lambda\in\mathbb{C}$, $t\in\mathbb{R}^+$, and $x,y\in\mathbb{B}_t^n$*

$$\begin{aligned}\tau_a\phi_{\lambda;t}(x) &= j_a(0)\int_{\mathbb{S}^{n-1}}e_{-\lambda,\xi;t}(a)\;e_{\lambda,\xi;t}(x)\;\mathrm{d}\sigma(\xi)\\ &= j_a(0)\int_{\mathbb{S}^{n-1}}e_{\lambda,\xi;t}(a)\;e_{-\lambda,\xi;t}(x)\;\mathrm{d}\sigma(\xi).\end{aligned}\tag{59}$$

3.4 The Generalised Poisson Transform

Definition 3.25. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t}f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{60}$$

For a spherical harmonic Y_k of degree k we have by (A.1)

$$(P_{\lambda,t}Y_k)(x) = C_{k,\nu}\left(1-\frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right) \tag{61}$$

with $\nu = \frac{n-1+2\mathrm{i}\lambda t}{2}$, $\mu = \frac{1-\sigma+2\mathrm{i}\lambda t}{4}$, and $C_{k,\nu} = 2^{-k}\frac{(\nu)_k}{(n/2)_k}$. For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ then is given by

$$(P_{\lambda,t}f)(x) = \sum_{k=0}^{\infty} a_k C_{k,\nu}\left(1-\frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right). \tag{62}$$

Proposition 3.26. *The Poisson transform $P_{\lambda,t}$ is injective in $L^2(\mathbb{S}^{n-1})$ if and only if $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right)$ for all $k \in \mathbb{Z}_+$.*

Corollary 3.27. *Let $\lambda \neq \mathrm{i}\left(\frac{2k_0+n-1}{2t}\right), k_0 \in \mathbb{Z}^+$. Then the space of functions $\widehat{f}(\lambda,\xi)$ as f ranges over $C_0^\infty(\mathbb{B}_t^n)$ is dense in $L^2(\mathbb{S}^{n-1})$.*

3.5 The Generalised Helgason Fourier Transform

Definition 3.28. For $f \in C_0^\infty(\mathbb{B}_t^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda,\xi;t) = \int_{\mathbb{B}_t^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{63}$$

Remark 4. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda,\xi;t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda;t) = \int_{\mathbb{B}_t^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{64}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 3.16 and 3.18 we obtain the following result.

Proposition 3.29. *If $f \in C_0^\infty(\mathbb{B}_t^n)$ then*

$$\widehat{\Delta_{z,t}f}(\lambda,\xi;t) = -\left(\lambda^2 + \frac{(n-1-2z)^2}{4t^2}\right)\widehat{f}(\lambda,\xi;t). \tag{65}$$

Now we study the hyperbolic convolution theorem with respect to the generalised Helgason Fourier transform. We begin with the following lemma.

Lemma 3.30. *For $a \in \mathbb{B}_t^n$ and $f \in C_0^\infty(\mathbb{B}_t^n)$ we have*

$$\widehat{\tau_a f}(\lambda, \xi; t) = j_a(0)\; e_{-\lambda,\xi;t}(a)\; \widehat{f}(\lambda, (-a) \oplus \xi; t). \tag{66}$$

Theorem 3.31 (Generalised Hyperbolic convolution theorem)**.** *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$. Then*

$$\widehat{f * g}(\lambda, \xi) = \int_{\mathbb{B}_t^n} f(y)\; e_{-\lambda,\xi;t}(y)\; \widehat{\widetilde{g}_y}(\lambda, (-y) \oplus \xi; t)\; \mathrm{d}\mu_{z,t}(y) \tag{67}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y, x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $(-y) \oplus \xi$ reduces to ξ, formula (67) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 4 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 3.32. *Let $f, g \in C_0^\infty(\mathbb{B}_t^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda, \xi; t) = \widehat{f}(\lambda, \xi; t)\; \widehat{g}(\lambda; t). \tag{68}$$

3.6 Inversion of the Generalised Helgason Fourier Transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 3.33. *The generalised spherical transform $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}}$ is the Jacobi transform (B.1) in Appendix B with parameters $\alpha = \frac{n}{2} - 1$ and $\beta = -\frac{1}{2}$ and

$$(M_z f)(s) := 2^{1-n} A_{n-1} t^n (\cosh s)^{-z} f(t \tanh s). \tag{69}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C_{0,R}^\infty(\mathbb{B}_t^n)$ denotes the space of all radial C^∞ functions on $\mathbb{B}_t^n$ with compact support and $\mathcal{E}(\mathbb{C} \times S^{n-1})$ the space of functions $g(\lambda, \xi)$ on $\mathbb{C} \times \mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n \in \mathbb{N}$

$$\sup_{(\lambda,\xi) \in \mathbb{C} \times \mathbb{S}^{n-1}} |g(\lambda, \xi)|(1 + |\lambda|)^n\, e^{A_g |\mathrm{Im}(\lambda)|} < \infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 3.34. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C^\infty_{0,R}(\mathbb{B}^n_t)$ onto $\mathcal{E}(\mathcal{C}\times\mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z} = \dfrac{1}{2^{2z+2-n}t^{n-1}\pi A_{n-1}}$.

Theorem 3.35. *For all $f\in C^\infty_{0,R}(\mathbb{B}^n_t)$ we have for the radial case the inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{70}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{71}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 3.36. *For $f\in C^\infty_0(\mathbb{B}^n_t)$ and $\lambda\in\mathbb{C}$,*

$$f*\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{72}$$

Theorem 3.37. *(Inversion formula) If $f\in C^\infty_0(\mathbb{B}^n_t)$ then we have the general inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda \tag{73}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{74}$$

Theorem 3.38. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from $L^2(\mathbb{B}^n_t,\mathrm{d}\mu_{z,t})$ onto $L^2(\mathbb{R}^+\times\mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}$ $\mathrm{d}\lambda\ \mathrm{d}\sigma)$, i.e.,*

$$\int_{\mathbb{B}^n_t} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi;t)|^2\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda. \tag{75}$$

Having obtained the main results we now study the limit $t\to+\infty$ of the previous results. It is anticipated that in the Euclidean limit we recover the usual inversion formula for the Fourier transform and Plancherel's Theorem on $\mathbb{R}^n$. To see that this is indeed the case, we observe that from (58)

$$\frac{1}{|c(\lambda t)|^2} = \frac{(A_{n-1})^2}{\pi^{n-1}2^{2n-2-2z}}\left|\frac{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}{\Gamma(\mathrm{i}\lambda t)}\right|^2, \tag{76}$$

with $A_{n-1} = \dfrac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)}$ being the surface area of $\mathbb{S}^{n-1}$. Finally, using (76) the generalised Helgason inverse Fourier transform (73) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x) \left| \frac{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)}{\Gamma\left(\mathrm{i}\lambda t\right)} \right|^2 \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \end{aligned} \tag{77}$$

with

$$N^{(n)}(\lambda t) = \left| \frac{\Gamma(i\lambda t)}{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)} \right|^2 (\lambda t)^{n-1}\,. \tag{78}$$

Some particular values are $N^{(1)}(\lambda t) = 1$, $N^{(2)}(\lambda t) = \coth(\lambda t)$, $N^{(3)} = 1$, and $N^{(4)}(\lambda t) = \frac{(2\lambda t)^2 \coth(\pi\lambda t)}{1+(2\lambda t)^2}$. Since $\lim\limits_{t\to+\infty} N^{(n)}(\lambda t) = 1$, for any $n \in \mathbb{N}$ and $\lambda \in \mathbb{R}^+$ (see [3]), we conclude that in the Euclidean limit the generalised Helgason inverse Fourier transform (77) converges to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi)\ \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda, \quad x, \lambda\xi \in \mathbb{R}^n.$$

Finally, Plancherel's Theorem (75) can be written as

$$\int_{\mathbb{B}_t^n} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi)|^2\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \tag{79}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)}\ \mathrm{d}\lambda\ \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (79) we recover Plancherel's Theorem in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |f(x)|^2\ \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda.$$

4. Gyroharmonic Analysis on the Möbius Gyrogroup

The Möbius gyrogroup appears in the study of the Poincaré ball model of hyperbolic geometry. Considering again the open ball $\mathbb{B}_t^n = \{x \in \mathbb{R}^n : \|x\| < t\}$ of $\mathbb{R}^n$, we now endow it with the Poincaré metric

$$ds^2 = \frac{dx_1^2 + \ldots + dx_n^2}{\left(1 - \frac{\|x\|^2}{t^2}\right)^2}.$$

The group of all conformal orientation preserving transformations of $\mathbb{B}_t^n$ is given by the mappings $K\varphi_a$, where $K \in \mathrm{SO}(n)$ and φ_a are Möbius transformations on $\mathbb{B}_t^n$ given by (see [1, 2, 8])

$$\varphi_a(x) \quad = \quad \frac{(1 + \frac{2}{t^2}\langle a, x\rangle + \frac{1}{t^2}\|x\|^2)a + (1 - \frac{1}{t^2}\|a\|^2)x}{1 + \frac{2}{t^2}\langle a, x\rangle + \frac{1}{t^4}\|a\|^2\|x\|^2}. \tag{80}$$

Möbius addition $\oplus_M$ on the ball appears considering the identification

$$a \oplus_M x := \varphi_a(x), \quad a, x \in \mathbb{B}_t^n. \tag{81}$$

Möbius addition satisfies the "gamma identity"

$$\gamma_{a\oplus_M v} = \gamma_a\gamma_b\sqrt{1 + \frac{2}{c^2}\langle a, b\rangle + \frac{1}{t^4}||a||^2||b||^2} \tag{82}$$

for all $a, b \in \mathbb{B}_t^n$ where γ_a is the Lorentz factor. The gyrogroup $(\mathbb{B}_t^n, \oplus_M)$ is gyrocommutative. In [10] we developed harmonic analysis on the Möbius gyrogroup depending on a real parameter σ. We provide here a generalization of these results considering a complex parameter z under the identification $2z = n + \sigma - 2$. Most of the proofs are analogous as in [10] and therefore will be omitted.

4.7 The Generalised Translation and Convolution

For the Möbius gyrogroup the generalised translation operator is defined by

$$\tau_a f(x) = j_a(x) f((-a) \oplus_M x) \tag{83}$$

where $a \in \mathbb{B}_t^n$, f is a function defined on $\mathbb{B}_t^n$, and the automorphic factor $j_a(x)$ is given by

$$j_a(x) = \left(\frac{1 - \frac{\|a\|^2}{t^2}}{1 - \frac{2}{t^2}\langle a, x\rangle + \frac{\|a\|^2\|x\|^2}{t^4}}\right)^z \tag{84}$$

with $z \in \mathbb{C}$. For $z = n$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $\varphi_{-a}(x) = (-a) \oplus x$ and for $z = n$ the translation operator reduces to $\tau_a f(x) = f((-a) \oplus x)$. For any $z \in \mathbb{C}$, we obtain in the limit $t \to +\infty$ the Euclidean translation operator $\tau_a f(x) = f(-a + x) = f(x - a)$. The relations in Lemma 3.3 are also true in this case. We define the complex weighted Hilbert space $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$, where

$$\mathrm{d}\mu_{z,t}(x) = \left(1 - \frac{\|x\|^2}{t^2}\right)^{2z-n} \mathrm{d}x,$$

and $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. Proposition 3.4 and Corollary 3.5 remains the same in this case. For two measurable functions f and g the generalised convolution is defined by

$$(f * g)(x) = \int_{\mathbb{B}_t^n} f(y)\ \tau_x g(-y)\ j_x(x)\ \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{B}_t^n. \tag{85}$$

Proposition 4.1. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_1 \leq C_z \, ||f||_1 \, ||\widetilde{g}||_1 \tag{86}$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi\in\mathbb{S}^{n-1}\\ y\in\mathbb{B}_t^n}} g(\mathrm{gyr}\,[y, r\xi]r\xi)$ *for any* $r \in [0, t[$ *and*

$$C_z = \begin{cases} 1, & \text{if} \quad \mathrm{Re}(z) \in \left]2, \frac{n-2}{2}\right[\\ \dfrac{\Gamma\left(\frac{n}{2}\right)(n - 2\,\mathrm{Re}(z) - 1)}{\Gamma\left(\frac{n-2\,\mathrm{Re}(z)}{2}\right)\Gamma\left(n - \mathrm{Re}(z) - 1\right)}, & \text{if} \quad \mathrm{Re}(z) \in]-\infty, 2] \cup [\frac{n-2}{2}, \frac{n-1}{2}[\end{cases}. \tag{87}$$

For the case of the Möbius gyrogroup Young's inequality for the generalised convolution is given by the next theorem.

Theorem 4.2. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$*. Then*

$$||f * g||_r \leq 2^{-\frac{\mathrm{Re}(z)}{2}} ||\widetilde{g}||_q^{1-s} \, ||g||_q^s \, ||f||_p \tag{88}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y\in\mathbb{B}_t^n} g(\mathrm{gyr}\,[y, x]x)$*, for any* $x \in \mathbb{B}_t^n$*.*

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y) j_x(x)| \leq 2^{-\frac{\mathrm{Re}(z)}{2}}, \; \forall x, y \in \mathbb{B}_t^n, \; \forall\, \mathrm{Re}(z) \leq 0. \tag{89}$$

Corollary 4.3. *Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\frac{\mathrm{Re}(z)}{2}} ||g||_q \, ||f||_p. \tag{90}$$

For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$. The generalised convolution (85) is gyro-translation invariant and gyroassociative in a similar way as expressed in Theorems 3.11 and 3.12.

4.8 Laplace Beltrami Operator and Eigenfunctions

The gyroharmonic analysis on the Möbius gyrogroup is based on the Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \left(1 - \frac{\|x\|^2}{t^2}\right)\left(\left(1 - \frac{\|x\|^2}{t^2}\right)\Delta - \frac{2(2z + 2 - n)}{t^2}\sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} + \frac{2z(2z - n + 2)}{t^2}\right).$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator τ_a.

Proposition 4.4. *For each $f \in C^2(\mathbb{B}_t^n)$ and $a \in \mathbb{B}_t^n$*

$$(\Delta_{z,t} f)(a) = (j_a(0))^{-1} \Delta(\tau_{-a} f)(0) - \frac{2z(2z+2-n)}{t^2} f(a) \tag{91}$$

An important fact is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 4.5. *The operator $\Delta_{z,t}$ commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall\, f \in C^2(\mathbb{B}_t^n),\, \forall\, b \in \mathbb{B}_t^n.$$

The operator $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 4.6. *The operator $\Delta_{z,t}$ is essentially self-adjoint in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.*

Definition 4.7. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{B}_t^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{\left(1 - \frac{\|x\|^2}{t^2}\right)^{-z+\frac{n-1}{2}+\frac{\mathrm{i}\lambda t}{2}}}{\left(\left\|\xi - \frac{x}{t}\right\|^2\right)^{\frac{n-1}{2}+\frac{\mathrm{i}\lambda t}{2}}}. \tag{92}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x, \lambda\xi\rangle}$.

Proposition 4.8. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue $-\lambda^2 - \dfrac{(n-1-2z)^2}{t^2}$.*

In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. Proposition 3.19 holds also in the Möbius case. In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (92) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\frac{1 + \frac{x}{t}}{1 - \frac{x}{t}}\right)^{\frac{\mathrm{i}\lambda t}{2}}$$

and, therefore, the exponential law is valid in this particular case, i.e.

$$e_{\lambda;t}(x) e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x).$$

Proposition 4.9. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,a\oplus_M \xi;t}(x). \tag{93}$$

Remark 5. The fractional linear mappings $a \oplus_M \xi, a \in \mathbb{B}_t^n, \xi \in \mathbb{S}^{n-1}$ are obtained from (80) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{\varphi_a(x)}{t} = \varphi_a(\xi)$ and are given by

$$a \oplus_M \xi = \frac{2\left(1 + \frac{1}{t}\langle a, \xi\rangle\right)\frac{a}{t} + \left(1 - \frac{\|a\|^2}{t^2}\right)\xi}{1 + \frac{2}{t}\langle a, \xi\rangle + \frac{\|a\|^2}{t^2}}.$$

They map $\mathbb{S}^{n-1}$ onto itself for any $t > 0$ and $a \in \mathbb{B}_t^n$, and in the limit $t \to +\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. Therefore, formula (93) converges in the limit to the well-known formula in the Euclidean case

$$\mathrm{e}^{\mathrm{i}\langle -a+x, \lambda\xi\rangle} = \mathrm{e}^{\mathrm{i}\langle -a, \lambda\xi\rangle} e^{\mathrm{i}\langle x, \lambda\xi\rangle}, \quad a, x, \lambda\xi \in \mathbb{R}^n.$$

The radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions, are defined by

$$\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\, \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{94}$$

Using (A.4) in Appendix A and then (A.6) in Appendix A we have

$$\begin{aligned} \phi_{\lambda;t}(x) &= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-2z+n-1+\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n-1+\mathrm{i}\lambda t}{2}, \frac{1+\mathrm{i}\lambda t}{2}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right) \qquad (95)\\ &= \left(1 - \frac{\|x\|^2}{t^2}\right)^{\frac{-2z+n-1-\mathrm{i}\lambda t}{2}} {}_2F_1\left(\frac{n-1-\mathrm{i}\lambda t}{2}, \frac{1-\mathrm{i}\lambda t}{2}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right). \end{aligned}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t} = \phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda \in \mathbb{C}$. Putting $\|x\| = t\tanh s$, with $s \in \mathbb{R}^+$, and using (A.7) in Appendix A we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\begin{aligned} \phi_{\lambda;t}(t\tanh s) &= (\cosh s)^{2z}\, {}_2F_1\left(\frac{n-1-\mathrm{i}\lambda t}{2}, \frac{n-1+\mathrm{i}\lambda t}{2}; \frac{n}{2}; -\sinh^2(s)\right)\\ &= (\cosh s)^{2z} \varphi_{\lambda t}^{\left(\frac{n}{2}-1, \frac{n}{2}-1\right)}(s). \qquad (96) \end{aligned}$$

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 4.10. *For* $\mathrm{Im}(\lambda) < 0$ *we have*

$$\lim_{s \to +\infty} \phi_{\lambda;t}(t\tanh s) e^{(n-1-2z-\mathrm{i}\lambda t)s} = c(\lambda t)$$

where $c(\lambda t)$ *is the Harish-Chandra c-function given by*

$$c(\lambda t) = \frac{2^{n-1-2z-\mathrm{i}\lambda t}\Gamma\left(\frac{n}{2}\right)\Gamma(\mathrm{i}\lambda t)}{\Gamma\left(\frac{n-1+\mathrm{i}\lambda t}{2}\right)\Gamma\left(\frac{1+\mathrm{i}\lambda t}{2}\right)}. \tag{97}$$

The addition formula for the generalised spherical functions is given in the next theorem.

Proposition 4.11. *For every* $\lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, *and* $x, y \in \mathbb{B}_t^n$

$$\begin{aligned} \tau_a \phi_{\lambda;t}(x) &= j_a(0) \int_{\mathbb{S}^{n-1}} e_{-\lambda,\xi;t}(a)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi) \\ &= j_a(0) \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(a)\ e_{-\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \end{aligned} \tag{98}$$

4.9 The Generalised Poisson Transform

Definition 4.12. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t} f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{B}_t^n. \tag{99}$$

For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ we have by (A.3)

$$(P_{\lambda,t} f)(x) = \sum_{k=0}^{\infty} a_k c_{k,\nu} \left(1 - \frac{|x|^2}{t^2}\right)^{\mu} {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right). \tag{100}$$

with $c_{k,\nu} = \frac{(\nu)_k}{(n/2)_k}$, $\nu = \frac{n-1+\mathrm{i}\lambda t}{2}$, and $\mu = -z + \frac{n-1}{2} + \frac{\mathrm{i}\lambda t}{2}$.

Proposition 4.13. *The Poisson transform* $P_{\lambda,t}$ *is injective in* $L^2(\mathbb{S}^{n-1})$ *if and only if* $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{t}\right)$ *for all* $k \in \mathbb{Z}_+$.

Corollary 4.14. *Let* $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{t}\right), k \in \mathbb{Z}^+$. *Then the space of functions* $\widehat{f}(\lambda, \xi)$ *as* f *ranges over* $C_0^\infty(\mathbb{B}_t^n)$ *is dense in* $L^2(\mathbb{S}^{n-1})$.

4.10 The Generalised Helgason Fourier Transform

Definition 4.15. For $f \in C_0^\infty(\mathbb{B}_t^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda, \xi; t) = \int_{\mathbb{B}_t^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{101}$$

Remark 6. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda, \xi; t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda; t) = \int_{\mathbb{B}_t^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{102}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 4.6 and 4.8 we obtain the following result.

Proposition 4.16. *If $f \in C_0^\infty(\mathbb{B}_t^n)$ then*

$$\widehat{\Delta_{z,t} f}(\lambda,\xi;t) = -\left(\lambda^2 + \frac{(n-1-2z)^2}{t^2}\right)\widehat{f}(\lambda,\xi;t). \tag{103}$$

The hyperbolic convolution theorem remains the same in the Möbius case.

Lemma 4.17. *For $a \in \mathbb{B}_t^n$ and $f \in C_0^\infty(\mathbb{B}_t^n)$*

$$\widehat{\tau_a f}(\lambda,\xi;t) = j_a(0)\; e_{-\lambda,\xi;t}(a)\; \widehat{f}(\lambda,(-a)\oplus\xi;t). \tag{104}$$

Theorem 4.18 (Generalised Hyperbolic convolution theorem)**.** *Let $f,g \in C_0^\infty(\mathbb{B}_t^n)$. Then*

$$\widehat{f * g}(\lambda,\xi) = \int_{\mathbb{B}_t^n} f(y)\; e_{-\lambda,\xi;t}(y)\; \widehat{\widetilde{g}_y}(\lambda,(-y)\oplus\xi;t)\; \mathrm{d}\mu_{z,t}(y) \tag{105}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y,x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $(-y)\oplus\xi$ reduces to ξ, formula (105) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 6 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 4.19. *Let $f,g \in C_0^\infty(\mathbb{B}_t^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda,\xi;t) = \widehat{f}(\lambda,\xi;t)\; \widehat{g}(\lambda;t). \tag{106}$$

4.11 Inversion of the Generalised Helgason Fourier Transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 4.20. *The generalised spherical transform $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,\frac{n}{2}-1} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,\frac{n}{2}-1}$ is the Jacobi transform (B.1) in Appendix B with parameters $\alpha = \beta = \frac{n}{2} - 1$ and

$$(M_z f)(s) := 2^{2-2n} A_{n-1} t^n (\cosh s)^{-2z} f(t\,\tanh s). \tag{107}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C^\infty_{0,R}(\mathbb{B}^n_t)$ denotes the space of all radial C^∞ functions on $\mathbb{B}^n_t$ with compact support and $\mathcal{E}(\mathbb{C}\times S^{n-1})$ the space of functions $g(\lambda,\xi)$ on $\mathbb{C}\times\mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n\in\mathbb{N}$

$$\sup_{(\lambda,\xi)\in\mathbb{C}\times\mathbb{S}^{n-1}} |g(\lambda,\xi)|(1+|\lambda|)^n\, e^{A_g|\mathrm{Im}(\lambda)|}<\infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 4.21. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C^\infty_{0,R}(\mathbb{B}^n_t)$ onto $\mathcal{E}(\mathcal{C}\times\mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z}=\dfrac{1}{2^{4z+3-2n}t^{n-1}\pi A_{n-1}}$.

Theorem 4.22. *For all $f\in C^\infty_{0,R}(\mathbb{B}^n_t)$ we have for the radial case the inversion formulas*

$$f(x)=C_{n,t,z}\int_0^{+\infty}\widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{108}$$

or

$$f(x)=\frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{109}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 4.23. *For $f\in C^\infty_0(\mathbb{B}^n_t)$ and $\lambda\in\mathbb{C}$,*

$$f*\phi_{\lambda;t}(x)=\int_{\mathbb{S}^{n-1}}\widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{110}$$

Theorem 4.24. *(Inversion formula) If $f\in C^\infty_0(\mathbb{B}^n_t)$ then we have the general inversion formulas*

$$f(x)=C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}}\widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda \tag{111}$$

or

$$f(x)=\frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}}\widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{112}$$

Theorem 4.25. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from* $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ *onto* $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}\, \mathrm{d}\lambda \mathrm{d}\sigma)$, *i.e.,*

$$\int_{\mathbb{B}_t^n} |f(x)|^2 \, \mathrm{d}\mu_{z,t}(x) = C_{n,t,z} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \xi; t)|^2 \; |c(\lambda t)|^{-2} \, \mathrm{d}\sigma(\xi) \, \mathrm{d}\lambda. \tag{113}$$

By (76) the generalised Helgason inverse Fourier transform (111) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t) \; e_{\lambda,\xi;t}(x) \left| \frac{\Gamma\left(\frac{n-1}{2} + \mathrm{i}\lambda t\right)}{\Gamma(\mathrm{i}\lambda t)} \right|^2 \mathrm{d}\sigma(\xi) \, \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda, \xi; t) \; e_{\lambda,\xi;t}(x) \; \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)} \, \mathrm{d}\xi \, \mathrm{d}\lambda \end{aligned} \tag{114}$$

with $N^{(n)}(\lambda t)$ defined by (78). As in the Einstein case, the generalised Helgason inverse Fourier transform (114) converges, when $t \to +\infty$, to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi) \; \mathrm{e}^{\mathrm{i}\langle x, \lambda\xi \rangle} \; \lambda^{n-1} \, \mathrm{d}\xi \, \mathrm{d}\lambda, \quad x, \lambda\xi \in \mathbb{R}^n.$$

Finally, Plancherel's Theorem (113) can be written as

$$\int_{\mathbb{B}_t^n} |f(x)|^2 \, \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda, \xi)|^2 \; \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)} \, \mathrm{d}\xi \, \mathrm{d}\lambda \tag{115}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)} \, \mathrm{d}\lambda \, \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (115) we recover Plancherel's Theorem in $\mathbb{R}^n$:

$$\int_{\mathbb{R}^n} |f(x)|^2 \, \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2 \; \lambda^{n-1} \, \mathrm{d}\xi \, \mathrm{d}\lambda.$$

5. Gyroharmonic Analysis on the Proper Velocity Gyrogroup

In this section we present the main results about the gyroharmonic analysis on the proper velocity gyrogroup. Proper velocities in special relativity theory are velocities measured by proper time, that is, by traveler's time rather than by observer's time [6]. The addition of proper velocities was defined by A.A. Ungar in [6] giving rise to the proper velocity gyrogroup.

Definition 5.1. Let $(V, +, \langle\, ,\rangle)$ be a real inner product space with addition $+$, and inner product $\langle\, ,\rangle$. The PV (Proper Velocity) gyrogroup $(V, \oplus)$ is the real inner product space V equipped with addition $\oplus$ given by

$$a \oplus x = x + \left(\frac{\beta_a}{1+\beta_a} \frac{\langle a, x\rangle}{t^2} + \frac{1}{\beta_x} \right) a \tag{116}$$

where $t \in \mathbb{R}^+$ and β_a, called the relativistic beta factor, is given by the equation

$$\beta_a = \frac{1}{\sqrt{1 + \frac{||a||^2}{t^2}}}. \tag{117}$$

PV addition is the relativistic addition of proper velocities rather than coordinate velocities as in Einstein addition. PV addition satisfies the beta identity

$$\beta_{a \oplus x} = \frac{\beta_a \beta_x}{1 + \beta_a \beta_x \frac{\langle a, x\rangle}{t^2}} \tag{118}$$

or, equivalently,

$$\frac{\beta_x}{\beta_{a\oplus x}} = \frac{1}{\beta_a} + \beta_x \frac{\langle a, x\rangle}{t^2}. \tag{119}$$

It is known that $(V, \oplus)$ is a gyrocommutative gyrogroup (see [27]). In the limit $t \to +\infty$, PV addition reduces to vector addition in $(V, +)$ and, therefore, the gyrogroup $(V, \oplus)$ reduces to the translation group $(V, +)$. To see the connection between proper velocity addition, proper Lorentz transformations, and real hyperbolic geometry let us consider the one sheeted hyperboloid $H^n_t = \{x \in \mathbb{R}^{n+1} : x^2_{n+1} - x^2_1 - \ldots - x^2_n = t^2 \wedge x_{n+1} > 0\}$ in $\mathbb{R}^{n+1}$ where $t \in \mathbb{R}^+$ is the radius of the hyperboloid. The $n-$dimensional real hyperbolic space is usually viewed as the rank one symmetric space G/K of noncompact type, where $G = \mathrm{SO}_0(n, 1)$ is the identity connected component of the group of orientation preserving isometries of H^n_t and $K = \mathrm{SO}(n)$ is the maximal compact subgroup of G which stabilizes the base point $O := (0, ..., 0, 1)$ in $\mathbb{R}^{n+1}$. Thus, $H^n_t \cong \mathrm{SO}_0(n, 1)/\mathrm{SO}(n)$ and it is one model for real hyperbolic geometry with constant negative curvature. Restricting the semi-Riemannian metric $dx^2_{n+1} - dx^2_1 - \ldots - dx^2_n$ on the ambient space we obtain the Riemannian metric on H^n_t which is given by

$$ds^2 = \frac{(\langle x, dx\rangle)^2}{t^2 + \|x\|^2} - \|dx\|^2$$

with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $dx = (dx_1, \ldots, dx_n)$. This metric corresponds to the metric tensor

$$g_{ij}(x) = \frac{x_i x_j}{t^2 + \|x\|^2} - \delta_{ij}, \quad i, j \in \{1, \ldots, n\}$$

whereas the inverse metric tensor is given by

$$g^{ij}(x) = -\delta_{ij} - \frac{x_i x_j}{t^2}, \quad i, j \in \{1, \ldots, n\}.$$

The group of all orientation preserving isometries of H_t^n consists of elements of the group $\mathrm{SO}(n)$ and proper Lorentz transformations acting on H_t^n. A simple way of working in H_t^n is to consider its projection into $\mathbb{R}^n$. Given an arbitray point $(x, \sqrt{t^2+||x||^2}) \in H_t^n$ we define the mapping $\Pi : H_t^n \to \mathbb{R}^n$, such that $\Pi(x, \sqrt{t^2+||x||^2}) = x$.

A proper Lorentz boost in the direction $\omega \in S^{n-1}$ and rapidity α acting in an arbitrary point $(x, \sqrt{t^2+||x||^2}) \in H_t^n$ yields a new point $(x, x_{n+1})_{\omega,\alpha} \in H_t^n$ given by (see [7])

$$\begin{aligned}(x, x_{n+1})_{\omega,\alpha} &= \Big(x + \Big((\cosh(\alpha)-1)\langle\omega, x\rangle - \sinh(\alpha)\sqrt{t^2+||x||^2}\Big)\,\omega, \\ &\quad \cosh(\alpha)\sqrt{t^2+||x||^2} - \sinh(\alpha)\langle\omega, x\rangle\Big). \end{aligned} \tag{120}$$

Since

$$\sqrt{t^2 + \left\| x + \Big((\cosh(\alpha)-1)\langle\omega, x\rangle - \sinh(\alpha)\sqrt{t^2+||x||^2}\Big)\,\omega \right\|^2} = x_{n+1}$$

the projection of (120) into $\mathbb{R}^n$ is given by

$$\Pi(x, x_{n+1})_{\omega,\alpha} = x + \Big((\cosh(\alpha)-1)\langle\omega, x\rangle - \sinh(\alpha)\sqrt{t^2+||x||^2}\Big)\,\omega. \tag{121}$$

Rewriting the parameters of the Lorentz boost to depend on a point $a \in \mathbb{R}^n$ as

$$\cosh(\alpha) = \sqrt{1 + \frac{||a||^2}{t^2}}, \quad \sinh(\alpha) = -\frac{||a||}{t}, \quad \text{and} \quad \omega = \frac{a}{||a||}. \tag{122}$$

and replacing (122) in (121) we finally obtain the relativistic addition of proper velocities in $\mathbb{R}^n$:

$$\begin{aligned} a \oplus x &= x + \left(\frac{\sqrt{1+\frac{||a||^2}{t^2}} - 1}{||a||^2}\langle a, x\rangle + \sqrt{1 + \frac{||x||^2}{t^2}} \right) a \\ &= x + \left(\frac{\beta_a}{1+\beta_a}\frac{\langle a, x\rangle}{t^2} + \frac{1}{\beta_x} \right) a \end{aligned} \tag{123}$$

The results presented for the Proper Velocity gyrogroup were obtained in [11]. The proofs are omitted here.

5.12 The Generalised Translation and Convolution

For the proper velocity gyrogroup the generalised translation operator is defined by

$$\tau_a f(x) = j_a(x) f((-a) \oplus_P x) \tag{124}$$

where $a \in \mathbb{R}$, f is a complex function defined on $\mathbb{R}^n$, and the automorphic factor $j_a(x)$ is given by

$$j_a(x) = \left(\frac{\beta_a}{1 - \beta_a \beta_x \frac{\langle a,x \rangle}{t^2}} \right)^z \tag{125}$$

with $z \in \mathbb{C}$. For $z = 1$ the multiplicative factor $j_a(x)$ agrees with the Jacobian of the transformation $(-a) \oplus_P x$ and for $z = 0$ the translation operator reduces to $\tau_a f(x) = f((-a) \oplus x)$. For any $z \in \mathbb{C}$, we obtain in the limit $t \to +\infty$ the Euclidean translation operator $\tau_a f(x) = f(-a + x) = f(x - a)$. The relations in Lemma 3.3 are also true in this case. We define the complex weighted Hilbert space $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$, where

$$d\mu_{z,t}(x) = \left(1 + \frac{\|x\|^2}{t^2}\right)^{-\frac{2z+1}{2}} dx,$$

and $\mathrm{d}x$ stands for the Lebesgue measure in $\mathbb{R}^n$. For the special case $z = 0$ we recover the invariant measure associated to $a \oplus x$. Proposition 3.4 and Corollary 3.5 remains the same in this case. For two measurable functions f and g the generalised convolution is defined by

$$(f * g)(x) = \int_{\mathbb{R}^n} f(y)\; \tau_x g(-y)\; j_x(x)\; \mathrm{d}\mu_{z,t}(y), \quad x \in \mathbb{R}^n. \tag{126}$$

Proposition 5.2. *Let* $\mathrm{Re}(z) < \frac{n-1}{2}$ *and* $f, g \in L^1(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$. *Then*

$$||f * g||_1 \leq C_z \, ||f||_1 \, ||\widetilde{g}||_1 \tag{127}$$

where $\widetilde{g}(r) = \operatorname*{ess\,sup}_{\substack{\xi \in \mathbb{S}^{n-1} \\ y \in \mathbb{R}^n}} g(\mathrm{gyr}\,[y, r\xi] r\xi)$ *for any* $r \in [0, t[$ *and*

$$C_z = \begin{cases} 1, & \textit{if} \;\; \mathrm{Re}(z) \in]-1, 0[\\ \dfrac{\Gamma\left(\frac{n}{2}\right) \Gamma\left(\frac{n - 2\,\mathrm{Re}(z) - 1}{2}\right)}{\Gamma\left(\frac{n - \mathrm{Re}(z)}{2}\right) \Gamma\left(\frac{n - \mathrm{Re}(z) - 1}{2}\right)}, & \textit{if} \;\; \mathrm{Re}(z) \in]-\infty, -1] \cup [0, \frac{n-1}{2}[\end{cases}. \tag{128}$$

For the case of the PV gyrogroup Young's inequality for the generalised convolution is given by the next theorem.

Theorem 5.3. *[11] Let* $\mathrm{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $s = 1 - \frac{q}{r}$, $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$. *Then*

$$||f * g||_r \leq 2^{-\mathrm{Re}(z)} ||\widetilde{g}||_q^{1-s} \, ||g||_q^s \, ||f||_p \tag{129}$$

where $\widetilde{g}(x) := \operatorname*{ess\,sup}_{y \in \mathbb{R}^n} g(\mathrm{gyr}\,[y, x]x)$, *for any* $x \in \mathbb{R}^n$.

The proof is analogous to the proof given in [9] and uses the following estimate:

$$|j_x(y)j_x(x)| \leq 2^{-\operatorname{Re}(z)},\ \forall x, y \in \mathbb{R}^n,\ \forall \operatorname{Re}(z) \leq 0. \tag{130}$$

Corollary 5.4. *Let* $\operatorname{Re}(z) \leq 0$, $1 \leq p, q, r \leq \infty$, $\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}$, $f \in L^p(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *and* $g \in L^q(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ *a radial function. Then,*

$$||f * g||_r \leq 2^{-\operatorname{Re}(z)} ||g||_q \, ||f||_p. \tag{131}$$

For $z = 0$ and taking the limit $t \to +\infty$ in (44) we recover Young's inequality for the Euclidean convolution in $\mathbb{R}^n$ since in the limit $\widetilde{g} = g$. The generalised convolution (126) is gyrotranslation invariant and gyroassociative in a similar way as expressed in Theorems 3.11 and 3.12.

5.13 Laplace Beltrami Operator and Eigenfunctions

The gyroharmonic analysis on the proper velocity gyrogroup is based on the Laplace Beltrami operator $\Delta_{z,t}$ defined by

$$\Delta_{z,t} = \Delta + \sum_{i,j=1}^{n} \frac{x_i x_j}{t^2} \frac{\partial^2}{\partial x_i \partial x_j} + (n - 2z) \sum_{i=1}^{n} \frac{x_i}{t^2} \frac{\partial}{\partial x_i} + \frac{z(z+1)}{t^2}(1 - \beta_x^2). \tag{132}$$

A simpler representation formula for $\Delta_{z,t}$ can be obtained using the Euclidean Laplace operator Δ and the generalised translation operator (124).

Proposition 5.5. *For each* $f \in C^2(\mathbb{R}^n)$ *and* $a \in \mathbb{R}^n$

$$\Delta_{z,t} f(a) = (j_a(0))^{-1} \Delta(\tau_{-a} f)(0). \tag{133}$$

An important fact is that the generalised Laplace-Beltrami operator $\Delta_{z,t}$ commutes with generalised translations.

Proposition 5.6. *The operator* $\Delta_{z,t}$ *commutes with generalised translations, i.e.*

$$\Delta_{z,t}(\tau_b f) = \tau_b(\Delta_{z,t} f) \qquad \forall\, f \in C^2(\mathbb{R}^n),\ \forall\, b \in \mathbb{R}^n.$$

There is an important relation between the operator $\Delta_{z,t}$ and the measure $\mathrm{d}\mu_{z,t}$. Up to a constant the Laplace-Beltrami operator $\Delta_{z,t}$ corresponds to a weighted Laplace operator on $\mathbb{B}_t^n$ for the weighted measure $\mathrm{d}\mu_{\sigma,t}$ in the sense defined in [12], Section 3.6. From Theorem 11.5 in [12] we know that the Laplace operator on a weighted manifold is essentially self-adjoint if all geodesics balls are relatively compact. Therefore, $\Delta_{z,t}$ can be extended to a self adjoint operator in $L^2(\mathbb{B}_t^n, \mathrm{d}\mu_{z,t})$.

Proposition 5.7. *The operator* $\Delta_{z,t}$ *is essentially self-adjoint in* $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$.

Definition 5.8. For $\lambda \in \mathbb{C}$, $\xi \in \mathbb{S}^{n-1}$, and $x \in \mathbb{R}^n$ we define the functions $e_{\lambda,\xi;t}$ by

$$e_{\lambda,\xi;t}(x) = \frac{(\beta_x)^{-z+\frac{n-1}{2}+\mathrm{i}\lambda t}}{\left(1 - \frac{\langle \beta_x\, x, \xi\rangle}{t}\right)^{\frac{n-1}{2}+\mathrm{i}\lambda t}}. \tag{134}$$

The hyperbolic plane waves $e_{\lambda,\xi;t}(x)$ converge in the limit $t \to +\infty$ to the Euclidean plane waves $\mathrm{e}^{i\langle x,\lambda\xi\rangle}$.

Proposition 5.9. *The function $e_{\lambda,\xi;t}$ is an eigenfunction of $\Delta_{z,t}$ with eigenvalue $-\lambda^2 - \dfrac{(n-1)^2}{4t^2} + \dfrac{nz}{t^2}$.*

As we can see the parametrization of the eigenvalues of the Laplace-Beltrami operator in the PV gyrogroup is different from the cases of Möbius and Einstein gyrogroups. In the limit $t \to +\infty$ the eigenvalues of $\Delta_{z,t}$ reduce to the eigenvalues of Δ in $\mathbb{R}^n$. Proposition 3.19 holds also in the PV case. In the case when $n = 1$ and $z = 0$ the hyperbolic plane waves (134) are independent of ξ since they reduce to

$$e_{\lambda;t}(x) = \left(\sqrt{1 + \frac{x^2}{t^2}} - \frac{x}{t}\right)^{-\mathrm{i}\lambda t}$$

and, therefore, the exponential law is valid in this particular case, i.e.

$$e_{\lambda;t}(x) e_{\gamma;t}(x) = e_{\lambda+\gamma;t}(x).$$

Proposition 5.10. *The generalised translation of $e_{\lambda,\xi;t}(x)$ admits the factorisation*

$$\tau_a e_{\lambda,\xi;t}(x) = j_a(0)\ e_{\lambda,\xi;t}(-a)\ e_{\lambda,T_a(\xi);t}(x). \tag{135}$$

where

$$T_a(\xi) = \frac{\xi + \frac{a}{t} + \frac{\beta_a}{1+\beta_a}\frac{\langle a,\xi\rangle a}{t^2}}{\frac{1}{\beta_a} + \frac{\langle a,\xi\rangle}{t}}. \tag{136}$$

Remark 7. The fractional linear mappings $T_a(\xi)$, with $a \in \mathbb{R}^n, \xi \in \mathbb{S}^{n-1}$ defined in (136) map the unit sphere S^{n-1} onto itself for any $t > 0$ and $a \in \mathbb{R}^n$. Moreover, in the limit $t \to +\infty$ they reduce to the identity mapping on $\mathbb{S}^{n-1}$. It is interesting to observe that the fractional linear mappings obtained from PV addition (123) making the formal substitutions $\frac{x}{t} = \xi$ and $\frac{a \oplus x}{t} = a \oplus \xi$ given by

$$a \oplus \xi = \xi + \left(\frac{\beta_a}{1+\beta_a}\frac{\langle a, \xi\rangle}{t} + \sqrt{2}\right)\frac{a}{t}$$

do not map S^{n-1} onto itself. This is different in comparison with the Möbius and Einstein gyrogroups. It can be explained by the fact that the hyperboloid is tangent to the null cone and therefore, the extension of PV addition to the the null cone is not possible by the formal substitutions above. Surprisingly, by Proposition 5.10 we obtained the induced PV addition on the sphere which is given by the fractional linear mappings $T_a(\xi)$.

The radial eigenfunctions of $\Delta_{z,t}$, the so called spherical functions, are defined by

$$\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{R}^n. \tag{137}$$

Using (A.4) in Appendix A and then (A.6) in Appendix A we have

$$\begin{aligned} \phi_{\lambda;t}(x) &= \left(1+\frac{\|x\|^2}{t^2}\right)^{\frac{2z-n+1-2\mathrm{i}\lambda t}{4}} {}_2F_1\left(\frac{n-1+2\mathrm{i}\lambda t}{4}, \frac{n+1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; 1-\beta_x^2\right) \qquad (138)\\ &= \left(1+\frac{\|x\|^2}{t^2}\right)^{\frac{2z-n+1+2\mathrm{i}\lambda t}{4}} {}_2F_1\left(\frac{n-1-2\mathrm{i}\lambda t}{4}, \frac{n+1-2\mathrm{i}\lambda t}{4}; \frac{n}{2}; 1-\beta_x^2\right). \end{aligned}$$

Therefore, $\phi_{\lambda;t}$ is a radial function that satisfies $\phi_{\lambda;t} = \phi_{-\lambda;t}$ i.e., $\phi_{\lambda;t}$ is an even function of $\lambda \in \mathbb{C}$. Applying (A.7) in Appendix A we obtain that

$$\phi_{\lambda;t}(x) = \left(1+\frac{\|x\|^2}{t^2}\right)^{\frac{z}{2}} {}_2F_1\left(\frac{n-1-2\mathrm{i}\lambda t}{4}, \frac{n-1+2\mathrm{i}\lambda t}{4}; \frac{n}{2}; -\frac{\|x\|^2}{t^2}\right).$$

Finally, considering $x = t\sinh(s)\,\xi$, with $s \in \mathbb{R}^+$ and $\xi \in S^{n-1}$ we have the following relation between $\phi_{\lambda;t}$ and the Jacobi functions $\varphi_{\lambda t}$ (see (B.2) in Appendix B):

$$\phi_{\lambda;t}(t\sinh(s)\,\xi) = (\cosh s)^z\ \varphi_{\lambda t}^{\left(\frac{n}{2}-1,-\frac{1}{2}\right)}(s). \tag{139}$$

Now we study the asymptotic behavior of $\phi_{\lambda;t}$ at infinity.

Lemma 5.11. *For* $\mathrm{Im}(\lambda) < 0$ *we have*

$$\lim_{s\to+\infty} \phi_{\lambda;t}(t\sinh s)\ e^{\left(\frac{n-1}{2}-z-\mathrm{i}\lambda t\right)s} = c(\lambda t)$$

where $c(\lambda t)$ *is the Harish-Chandra c-function given by*

$$c(\lambda t) = \frac{2^{n-2-z}}{\sqrt{\pi}} \frac{\Gamma\left(\frac{n}{2}\right)\Gamma\left(\mathrm{i}\lambda t\right)}{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}. \tag{140}$$

The addition formula for the generalised spherical functions is given in the next theorem.

Proposition 5.12. *For every* $\lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, *and* $a, x \in \mathbb{R}^n$

$$\begin{aligned} \tau_a\phi_{\lambda;t}(x) &= j_a(0)\int_{\mathbb{S}^{n-1}} e_{-\lambda,\xi;t}(a)\ e_{\lambda,\xi;t}(x)\ d\sigma(\xi)\\ &= j_a(0)\int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(a)\ e_{-\lambda,\xi;t}(x)\ d\sigma(\xi). \qquad (141) \end{aligned}$$

5.14 The Generalised Poisson Transform

Definition 5.13. Let $f \in L^2(\mathbb{S}^{n-1})$. Then the generalised Poisson transform is defined by

$$P_{\lambda,t}f(x) = \int_{\mathbb{S}^{n-1}} e_{\lambda,\xi;t}(x)\ f(\xi)\ \mathrm{d}\sigma(\xi), \quad x \in \mathbb{R}^n. \tag{142}$$

For $f = \sum_{k=0}^{\infty} a_k Y_k \in L^2(\mathbb{S}^{n-1})$ we have by (A.3)

$$P_{\lambda,t}f(x) = \sum_{k=0}^{\infty} a_k c_{k,\nu} (\beta_x)^{-z+\frac{n-1}{2}+i\lambda t}{}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; 1-\beta_x^2\right) Y_k\left(\beta_x \frac{x}{t}\right). \tag{143}$$

with $c_{k,\nu} = 2^{-k}\frac{(\nu)_k}{(n/2)_k}$ and $\nu = \frac{n-1}{2} + \mathrm{i}\lambda t$.

Proposition 5.14. *The Poisson transform $P_{\lambda,t}$ is injective in $L^2(\mathbb{S}^{n-1})$ if and only if $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right)$ for all $k \in \mathbb{Z}^+$.*

Corollary 5.15. *Let $\lambda \neq \mathrm{i}\left(\frac{2k+n-1}{2t}\right), k \in \mathbb{Z}^+$. Then for f in $C_0^\infty(\mathbb{R}^n)$ the space of functions $\widehat{f}(\lambda,\xi)$ is dense in $L^2(\mathbb{S}^{n-1})$.*

5.15 The Generalised Helgason Fourier Transform

Definition 5.16. For $f \in C_0^\infty(\mathbb{R}^n)$, $\lambda \in \mathbb{C}$ and $\xi \in \mathbb{S}^{n-1}$ we define the generalised Helgason Fourier transform of f as

$$\widehat{f}(\lambda,\xi;t) = \int_{\mathbb{R}^n} e_{-\lambda,\xi;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{144}$$

Remark 8. If f is a radial function i.e., $f(x) = f(\|x\|)$, then $\widehat{f}(\lambda,\xi;t)$ is independent of ξ and we obtain by (54) the generalised spherical transform of f defined by

$$\widehat{f}(\lambda;t) = \int_{\mathbb{R}^n} \phi_{-\lambda;t}(x)\ f(x)\ \mathrm{d}\mu_{z,t}(x). \tag{145}$$

Moreover, by (51) we recover in the Euclidean limit the usual Fourier transform in $\mathbb{R}^n$.

From Propositions 5.7 and 5.9 we obtain the following result.

Proposition 5.17. *If $f \in C_0^\infty(\mathbb{R}^n)$ then*

$$\widehat{\Delta_{z,t}f}(\lambda,\xi;t) = -\left(\lambda^2 + \frac{(n-1)^2}{4t^2} - \frac{nz}{t^2}\right)\widehat{f}(\lambda,\xi;t). \tag{146}$$

Now we study the hyperbolic convolution theorem with respect to the generalised Helgason Fourier transform. We begin with the following lemma.

Lemma 5.18. *For $a \in \mathbb{R}^n$ and $f \in C_0^\infty(\mathbb{R}^n)$*

$$\widehat{\tau_a f}(\lambda, \xi; t) = j_a(0)\ e_{-\lambda,\xi;t}(a)\ \widehat{f}(\lambda, (-a) \oplus \xi; t). \tag{147}$$

Theorem 5.19 (Generalised Hyperbolic convolution theorem)**.** *Let $f, g \in C_0^\infty(\mathbb{R}^n)$. Then*

$$\widehat{f * g}(\lambda, \xi) = \int_{\mathbb{R}^n} f(y)\ e_{-\lambda,\xi;t}(y)\ \widehat{\widetilde{g}_y}(\lambda, T_{-y}(\xi); t)\ d\mu_{z,t}(y) \tag{148}$$

where $\widetilde{g}_y(x) = g(\mathrm{gyr}\,[y, x]x)$.

Since in the limit $t \to +\infty$ gyrations reduce to the identity and $T_{-y}(\xi)$ reduces to ξ, formula (148) converges in the Euclidean limit to the well-know Convolution Theorem: $\widehat{f * g} = \widehat{f} \cdot \widehat{g}$. By Remark 8 if g is a radial function we obtain the pointwise product of the generalised Helgason Fourier transform.

Corollary 5.20. *Let $f, g \in C_0^\infty(\mathbb{R}^n)$ and g radial. Then*

$$\widehat{f * g}(\lambda, \xi; t) = \widehat{f}(\lambda, \xi; t)\ \widehat{g}(\lambda; t). \tag{149}$$

5.16 Inversion of the Generalised Helgason Fourier Transform and Plancherel's Theorem

We obtain first an inversion formula for the radial case, that is, for the generalised spherical transform.

Lemma 5.21. *The generalised spherical transform denoted by $\mathcal{H}$ can be written as*

$$\mathcal{H} = \mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}} \circ M_z$$

where $\mathcal{J}_{\frac{n}{2}-1,-\frac{1}{2}}$ is the Jacobi transform (see (B.1) in Appendix B) with parameters $\alpha = \frac{n}{2} - 1$ and $\beta = -\frac{1}{2}$ and

$$(M_{z,t} f)(s) := 2^{1-n} A_{n-1} t^n (\cosh s)^{-z} f(t \sinh s). \tag{150}$$

The previous lemma allow us to obtain a Paley-Wiener Theorem for the generalised Helgason Fourier transform by using the Paley-Wiener Theorem for the Jacobi transform (Theorem B.1 in Appendix B). Let $C_{0,R}^\infty(\mathbb{R}^n)$ denotes the space of all radial C^∞ functions on $\mathbb{R}^n$ with compact support and $\mathcal{E}(\mathbb{C} \times S^{n-1})$ the space of functions $g(\lambda, \xi)$ on $\mathbb{C} \times \mathbb{S}^{n-1}$, even and holomorphic in λ and of uniform exponential type, i.e., there is a positive constant A_g such that for all $n \in \mathbb{N}$

$$\sup_{(\lambda,\xi) \in \mathbb{C} \times \mathbb{S}^{n-1}} |g(\lambda, \xi)|(1 + |\lambda|)^n\, e^{A_g |\mathrm{Im}(\lambda)|} < \infty$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Corollary 5.22. *(Paley-Wiener Theorem) The generalised Helgason Fourier transform is bijective from $C^\infty_{0,R}(\mathbb{R}^n)$ onto $\mathcal{E}(\mathcal{C}\times\mathbb{S}^{n-1})$.*

In the sequel we denote $C_{n,t,z} = \dfrac{1}{2^{2z-n+2}t^{n-1}\pi A_{n-1}}$.

Theorem 5.23. *For all $f\in C^\infty_{0,R}(\mathbb{R}^n)$ we have for the radial case the inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda \tag{151}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}} \widehat{f}(\lambda;t)\ \phi_{\lambda;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\lambda. \tag{152}$$

Now that we have an inversion formula for the radial case we present our main results, the inversion formula for the generalised Helgason Fourier transform and the associated Plancherel's Theorem.

Proposition 5.24. *For $f\in C^\infty_0(\mathbb{R}^n)$ and $\lambda\in\mathbb{C}$,*

$$f*\phi_{\lambda;t}(x) = \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \mathrm{d}\sigma(\xi). \tag{153}$$

Theorem 5.25. *(Inversion formula) If $f\in C^\infty_0(\mathbb{R}^n)$ then we have the general inversion formulas*

$$f(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda \tag{154}$$

or

$$f(x) = \frac{C_{n,t,z}}{2}\int_{\mathbb{R}}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\,\mathrm{d}\lambda. \tag{155}$$

Theorem 5.26. *(Plancherel's Theorem) The generalised Helgason Fourier transform extends to an isometry from $L^2(\mathbb{R}^n,\mathrm{d}\mu_{z,t})$ onto $L^2(\mathbb{R}^+\times\mathbb{S}^{n-1}, C_{n,t,z}|c(\lambda t)|^{-2}$ $\mathrm{d}\lambda\ \mathrm{d}\sigma)$, i.e.,*

$$\int_{\mathbb{R}^n} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = C_{n,t,z}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi;t)|^2\ |c(\lambda t)|^{-2}\ \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda. \tag{156}$$

By (76) the generalised Helgason inverse Fourier transform (154) simplifies to

$$\begin{aligned} f(x) &= \frac{A_{n-1}}{(2\pi)^n t^{n-1}}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\left|\frac{\Gamma\left(\frac{n-1}{2}+\mathrm{i}\lambda t\right)}{\Gamma\left(\mathrm{i}\lambda t\right)}\right|^2 \mathrm{d}\sigma(\xi)\ \mathrm{d}\lambda \\ &= \frac{1}{(2\pi)^n}\int_0^{+\infty}\int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda,\xi;t)\ e_{\lambda,\xi;t}(x)\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \end{aligned} \tag{157}$$

with $N^{(n)}(\lambda t)$ defined by (78). As in the Einstein case, the generalised Helgason inverse Fourier transform (157) converges, when $t \to +\infty$, to the usual inverse Fourier transform in $\mathbb{R}^n$ written in polar coordinates:

$$f(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} \widehat{f}(\lambda\xi)\ \mathrm{e}^{\mathrm{i}\langle x,\lambda\xi\rangle}\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda, \quad x, \lambda\xi \in \mathbb{R}^n.$$

Finally, Plancherel's Theorem (156) can be written as

$$\int_{\mathbb{R}^n} |f(x)|^2\ \mathrm{d}\mu_{z,t}(x) = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda,\xi)|^2\ \frac{\lambda^{n-1}}{N^{(n)}(\lambda t)}\ \mathrm{d}\xi\ \mathrm{d}\lambda \tag{158}$$

and, therefore, we have an isometry between the spaces $L^2(\mathbb{R}^n, \mathrm{d}\mu_{z,t})$ and $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, \frac{\lambda^{n-1}}{(2\pi)^n N^{(n)}(\lambda t)}\ \mathrm{d}\lambda\ \mathrm{d}\xi)$. Applying the limit $t \to +\infty$ to (158) we recover Plancherel's Theorem in the Euclidean setting:

$$\int_{\mathbb{R}^n} |f(x)|^2\ \mathrm{d}x = \frac{1}{(2\pi)^n} \int_0^{+\infty} \int_{\mathbb{S}^{n-1}} |\widehat{f}(\lambda\xi)|^2\ \lambda^{n-1}\ \mathrm{d}\xi\ \mathrm{d}\lambda.$$

6. Appendices

A Spherical Harmonics

A spherical harmonic of degree $k \geq 0$ denoted by Y_k is the restriction to $\mathbb{S}^{n-1}$ of a homogeneous harmonic polynomial in $\mathbb{R}^n$. The set of all spherical harmonics of degree k is denoted by $\mathcal{H}_k(\mathbb{S}^{n-1})$. This space is a finite dimensional subspace of $L^2(\mathbb{S}^{n-1})$ and we have the direct sum decomposition

$$L^2(\mathbb{S}^{n-1}) = \bigoplus_{k=0}^{\infty} \mathcal{H}_k(\mathbb{S}^{n-1}).$$

The following integrals are obtained from the generalisation of Proposition 5.2 in [34].

Lemma A.1. *Let $\nu \in \mathbb{C}, k \in \mathbb{N}_0, t \in \mathbb{R}^+$, and $Y_k \in \mathcal{H}_k(\mathbb{S}^{n-1})$. Then*

$$\int_{\mathbb{S}^{n-1}} \frac{Y_k(\xi)}{\left(1 - \frac{\langle x,\xi\rangle}{t}\right)^{\nu}}\ \mathrm{d}\sigma(\xi) = 2^{-k}\, \frac{(\nu)_k}{(n/2)_k}\, {}_2F_1\left(\frac{\nu+k}{2}, \frac{\nu+k+1}{2}; k+\frac{n}{2}; \frac{\|x\|^2}{t^2}\right) Y_k\left(\frac{x}{t}\right) \tag{A.1}$$

where $x \in \mathbb{B}_t^n$, $(\nu)_k$, denotes the Pochhammer symbol, and $\mathrm{d}\sigma$ is the normalised surface measure on $\mathbb{S}^{n-1}$. In particular, when $k = 0$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\left(1 - \frac{\langle x,\xi\rangle}{t}\right)^{\nu}}\ \mathrm{d}\sigma(\xi) = {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{n}{2}; \frac{\|x\|^2}{t^2}\right). \tag{A.2}$$

For the Möbius case we need a generalization of Lemma 2.4 in [19].

Lemma A.2. *Let $\nu \in \mathbb{C}, k \in \mathbb{N}_0, t \in \mathbb{R}^+$, and $Y_k \in \mathcal{H}_k(\mathbb{S}^{n-1})$. Then*

$$\int_{\mathbb{S}^{n-1}} \frac{Y_k(\xi)}{\left\|\frac{x}{t}-\xi\right\|^{2\nu}}\,\mathrm{d}\sigma(\xi) = \frac{(\nu)_k}{(n/2)_k}\,{}_2F_1\left(\nu+k,\nu-\frac{n}{2}+1;k+\frac{n}{2};\frac{\|x\|^2}{t^2}\right)Y_k\left(\frac{x}{t}\right) \quad \text{(A.3)}$$

where $x \in \mathbb{B}^n_t$, In particular, when $k=0$, we have

$$\int_{\mathbb{S}^{n-1}} \frac{1}{\left\|\frac{x}{t}-\xi\right\|^{2\nu}}\,\mathrm{d}\sigma(\xi) = {}_2F_1\left(\nu,\nu-\frac{n}{2}+1;\frac{n}{2};\frac{\|x\|^2}{t^2}\right). \quad \text{(A.4)}$$

The Gauss Hypergeometric function ${}_2F_1$ is an analytic function for $|z|<1$ defined by

$${}_2F_1(a,b;c;z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}$$

with $c \notin -\mathbb{N}_0$. If $\mathrm{Re}(c-a-b) > 0$ and $c \notin -\mathbb{N}_0$ then exists the limit $\lim\limits_{t\to 1^-} {}_2F_1(a,b;c;t)$ and equals

$${}_2F_1(a,b;c;1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \quad \text{(A.5)}$$

Some useful properties of this function are

$${}_2F_1(a,b;c;z) = (1-z)^{c-a-b}\,{}_2F_1(c-a,c-b;c;z) \quad \text{(A.6)}$$

$${}_2F_1(a,b;c;z) = (1-z)^{-b}\,{}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right) \quad \text{(A.7)}$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\,{}_2F_1(a,b;c;z) = \frac{ab}{c}\,{}_2F_1(a+1,b+1;c+1;z). \quad \text{(A.8)}$$

B Jacobi Functions

The classical theory of Jacobi functions involves the parameters $\alpha,\beta,\lambda \in \mathbb{C}$ (see [17, 18]). Here we introduce the additional parameter $t \in \mathbb{R}^+$ since we develop our hyperbolic harmonic analysis on a ball of arbitrary radius t and a hyperboloid of radius t. For $\alpha,\beta,\lambda \in \mathbb{C}$, $t \in \mathbb{R}^+$, and $\alpha \neq -1,-2,\ldots$, we define the Jacobi transform as

$$\mathcal{J}_{\alpha,\beta}\,g(\lambda t) = \int_0^{+\infty} g(r)\,\varphi^{(\alpha,\beta)}_{\lambda t}(r)\,\omega_{\alpha,\beta}(r)\,\mathrm{d}r \quad \text{(B.1)}$$

for all functions g defined on $\mathbb{R}^+$ for which the integral (B.1) is well defined. The weight function $\omega_{\alpha,\beta}$ is given by

$$\omega_{\alpha,\beta}(r) = (2\sinh(r))^{2\alpha+1}(2\cosh(r))^{2\beta+1}$$

and the function $\varphi_{\lambda t}^{(\alpha,\beta)}(r)$ denotes the Jacobi function which is defined as the even C^∞ function on $\mathbb{R}$ that equals 1 at 0 and satisfies the Jacobi differential equation

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + ((2\alpha+1)\coth(r) + (2\beta+1)\tanh(r))\frac{\mathrm{d}}{\mathrm{d}r} + (\lambda t)^2 + (\alpha+\beta+1)^2\right)\varphi_{\lambda t}^{(\alpha,\beta)}(r) = 0.$$

The function $\varphi_{\lambda t}^{(\alpha,\beta)}(r)$ can be expressed as an hypergeometric function

$$\varphi_{\lambda t}^{(\alpha,\beta)}(r) = {}_2F_1\left(\frac{\alpha+\beta+1+i\lambda t}{2}, \frac{\alpha+\beta+1-i\lambda t}{2}; \alpha+1; -\sinh^2(r)\right). \tag{B.2}$$

Since $\varphi_{\lambda t}^{(\alpha,\beta)}$ are even functions of $\lambda t \in \mathbb{C}$ then $\mathcal{J}_{\alpha,\beta}g(\lambda t)$ is an even function of λt. Inversion formulas for the Jacobi transform and a Paley-Wiener Theorem are found in [18]. We denote by $C_{0,R}^\infty(\mathbb{R})$ the space of even C^∞-functions with compact support on $\mathbb{R}$ and $\mathcal{E}$ the space of even and entire functions g for which there are positive constants A_g and $C_{g,n}, n = 0, 1, 2, \ldots$, such that for all $\lambda \in \mathbb{C}$ and all $n = 0, 1, 2, \ldots$

$$|g(\lambda)| \leq C_{g,n}(1+|\lambda|)^{-n}\, e^{A_g|\mathrm{Im}(\lambda)|}$$

where $\mathrm{Im}(\lambda)$ denotes the imaginary part of λ.

Theorem B.1. *([18],p.8) (Paley-Wiener Theorem) For all $\alpha, \beta \in \mathbb{C}$ with $\alpha \neq -1, -2, \ldots$ the Jacobi transform is bijective from $C_{0,R}^\infty(\mathbb{R})$ onto $\mathcal{E}$.*

The Jacobi transform can be inverted under some conditions [18]. Here we only refer to the case which is used in this paper.

Theorem B.2. *([18],p.9) Let $\alpha, \beta \in \mathbb{R}$ such that $\alpha > -1, \alpha \pm \beta + 1 \geq 0$. Then for every $g \in C_{0,R}^\infty(\mathbb{R})$ we have*

$$g(r) = \frac{1}{2\pi}\int_0^{+\infty} (\mathcal{J}_{\alpha,\beta}g)(\lambda t)\ \varphi_{\lambda t}^{(\alpha,\beta)}(r)\ |c_{\alpha,\beta}(\lambda t)|^{-2}\ t\ \mathrm{d}\lambda, \tag{B.3}$$

where $c_{\alpha,\beta}(\lambda t)$ is the Harish-Chandra c-function associated to $\mathcal{J}_{\alpha,\beta}(\lambda t)$ given by

$$c_{\alpha,\beta}(\lambda t) = \frac{2^{\alpha+\beta+1-i\lambda t}\Gamma(\alpha+1)\Gamma(i\lambda t)}{\Gamma\left(\frac{\alpha+\beta+1+i\lambda t}{2}\right)\Gamma\left(\frac{\alpha-\beta+1+i\lambda t}{2}\right)}. \tag{B.4}$$

This theorem provides a generalisation of Theorem 2.3 in [18] for arbitrary $t \in \mathbb{R}^+$. From [18] and considering $t \in \mathbb{R}^+$ arbitrary we have the following asymptotic behavior of $\phi_{\lambda t}^{\alpha,\beta}$ for $\mathrm{Im}(\lambda) < 0$:

$$\lim_{r\to+\infty} \varphi_{\lambda t}^{(\alpha,\beta)}(r) e^{(-i\lambda t+\alpha+\beta+1)r} = c_{\alpha,\beta}(\lambda t). \tag{B.5}$$

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Bi-Gyrogroup: The Group-Like Structure Induced by Bi-Decomposition of Groups

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Abstract

The decomposition $\Gamma = BH$ of a group Γ into a subset B and a subgroup H of Γ induces, under general conditions, a group-like structure for B, known as a gyrogroup. The famous concrete realization of a gyrogroup, which motivated the emergence of gyrogroups into the mainstream, is the space of all relativistically admissible velocities along with a binary operation given by the Einstein velocity addition law of special relativity theory. The latter leads to the Lorentz transformation group $\mathrm{SO}(1, n)$, $n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature $(1, n)$. The study in this article is motivated by generalized Lorentz groups $\mathrm{SO}(m, n)$, $m, n \in \mathbb{N}$, in pseudo-Euclidean spaces of signature (m, n). Accordingly, this article explores the bi-decomposition $\Gamma = H_L B H_R$ of a group Γ into a subset B and subgroups H_L and H_R of Γ, along with the novel bi-gyrogroup structure of B induced by the bi-decomposition of Γ. As an example, we show by methods of Clifford algebras that the quotient group of the spin group $\mathrm{Spin}\,(m, n)$ possesses the bi-decomposition structure.

Keywords: Bi-decomposition of group, bi-gyrogroup, gyrogroup, spin group, pseudo-orthogonal group.

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1. Introduction

Lorentz transformation groups $\Gamma = \mathrm{SO}(1, n)$, $n \in \mathbb{N}$, possess the decomposition structure $\Gamma = BH$, where B is a subset of Γ and H is a subgroup of Γ [26]. The decomposition structure of Γ induces a group-like structure for B. This group-like structure was discovered in 1988 [26] and became known as a *gyrogroup* [27, 28].

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Subsequently, gyrogroups turned out to play a universal computational role that extends far beyond the domain of Lorentz groups $\mathrm{SO}(1, n)$ [32, 33], as noted by Chatelin in [4, p. 523] and in references therein. In fact, gyrogroups are special loops that, according to [17], are placed centrally in loop theory.

The use of Clifford algebras to employ gyrogroups as a computational tool in harmonic analysis is presented by Ferreira in the seminal papers [9, 10]. The use of Clifford algebras to obtain a better understanding of gyrogroups is found, for instance, in [7, 8, 11, 20, 24].

Generalized Lorentz transformation groups $\Gamma = \mathrm{SO}(m, n)$, $m, n \in \mathbb{N}$, possess the so-called *bi-decomposition* structure $\Gamma = H_L B H_R$, where B is a subset of Γ and H_L and H_R are subgroups of Γ. The bi-decomposition structure of Γ induces a group-like structure for B, called a *bi-gyrogroup* [34]. The use of Clifford algebras that may improve our understanding of bi-gyrogroups is found in [12]. Clearly, the notion of bi-gyrogroups extends the notion of gyrogroups. Accordingly, "gyrolanguage", the algebraic language crafted for gyrogroup theory is extended to "bi-gyro-language" for bi-gyrogroup theory.

As a first step towards demonstrating that bi-gyrogroups play a universal computational role that extends far beyond the domain of generalized Lorentz groups $\mathrm{SO}(m, n)$, the aim of the present article is to approach the study of bi-gyrogroups from the abstract viewpoint.

The article is organized as follows. In Section 2 we give the definition of a bi-gyrogroupoid. In Section 3 we show that the bi-transversal decomposition of a group with additional properties yields a highly structured type of bi-gyrogroupoids. In Section 4 we introduce the notion of bi-gyrodecomposition of groups and prove that any bi-gyrodecomposition of a group gives rise to a bi-gyrogroup. Finally, in Sections 5 and 6 we demonstrate that the pseudo-orthogonal group $\mathrm{SO}(m, n)$ and the quotient group of the spin group $\mathrm{Spin}\,(m, n)$ possess the bi-gyrodecomposition structure.

2. Bi-Gyrogroupoids

We begin with the abstract definition of a bi-gyrogroupoid, which is modeled on the groupoid $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices with bi-gyroaddition studied in detail in [34]. We recall that a groupoid $(B, \oplus_b)$ is a non-empty set B with a binary operation $\oplus_b$. An automorphism of a groupoid $(B, \oplus_b)$ is a bijection from B to itself that preserves the groupoid operation. The group of all automorphisms of $(B, \oplus_b)$ is denoted by $\mathrm{Aut}\,(B, \oplus_b)$ or simply $\mathrm{Aut}\,(B)$.

Definition 2.1 (Bi-gyrogroupoid). A groupoid $(B, \oplus_b)$ is a *bi-gyrogroupoid* if its binary operation satisfies the following axioms.
(BG1) There is an element $0 \in B$ such that $0 \oplus_b a = a \oplus_b 0 = a$ for all $a \in B$.
(BG2) For each $a \in B$, there is an element $b \in B$ such that $b \oplus_b a = 0$.
(BG3) Each pair of a and b in B corresponds to a left automorphism $\mathrm{lgyr}[a, b]$ and

a right automorphism $\text{rgyr}[a, b]$ in $\text{Aut}\,(B, \oplus_b)$ such that for all $c \in B$,

$$(a \oplus_b b) \oplus_b \text{lgyr}[a, b]c = \text{rgyr}[b, c]a \oplus_b (b \oplus_b c). \tag{1}$$

(BG4) For all $a, b \in B$,

(a) $\text{rgyr}[a, b] = \text{rgyr}[\text{lgyr}[a, b]a, a \oplus_b b]$, and

(b) $\text{lgyr}[a, b] = \text{lgyr}[\text{lgyr}[a, b]a, a \oplus_b b]$.

(BG5) For all $a \in B$, $\text{lgyr}[a, 0]$ and $\text{rgyr}[a, 0]$ are the identity automorphism of B.

A concrete realization of Axioms (BG1) through (BG5) will be presented in Section 5.

Roughly speaking, any bi-gyrogroupoid is a groupoid that comes with two families of automorphisms, called left and right automorphisms or, collectively, bi-automorphisms. Note that if bi-automorphisms of a bi-gyrogroupoid $(B, \oplus_b)$ reduce to the identity automorphism of B, then $(B, \oplus_b)$ forms a group.

Let $\text{lgyr}^{-1}[a, b]$ and $\text{rgyr}^{-1}[a, b]$ be the inverse map of $\text{lgyr}[a, b]$ and $\text{rgyr}[a, b]$, respectively. Let $\circ$ denote *function composition* and let id_X denote the identity map on a non-empty set X. The following theorem asserts that bi-gyrogroupoids satisfy a generalized associative law.

Theorem 2.2. *Any bi-gyrogroupoid B satisfies the left bi-gyroassociative law*

$$a \oplus_b (b \oplus_b c) = (\text{rgyr}^{-1}[b, c]a \oplus_b b) \oplus_b \text{lgyr}[\text{rgyr}^{-1}[b, c]a, b]c \tag{2}$$

and the right bi-gyroassociative law

$$(a \oplus_b b) \oplus_b c = \text{rgyr}[b, \text{lgyr}^{-1}[a, b]c]a \oplus_b (b \oplus_b \text{lgyr}^{-1}[a, b]c) \tag{3}$$

for all $a, b, c \in B$.

Proof. Let $a, b, c \in B$ be arbitrary. Since $\text{rgyr}[b, c]$ is surjective, there is an element $d \in B$ for which $\text{rgyr}[b, c]d = a$. By (BG3),

$$a \oplus_b (b \oplus_b c) = \text{rgyr}[b, c]d \oplus_b (b \oplus_b c) = (d \oplus_b b) \oplus_b \text{lgyr}[d, b]c.$$

Since $d = \text{rgyr}^{-1}[b, c]a$, (2) is obtained. One obtains (3) in a similar way. □

Lemma 2.3. *Any bi-gyrogroupoid B has a unique two-sided identity element.*

Proof. By Definition 2.1, B has a two-sided identity element. Suppose that e and f are two-sided identity elements of B. As e is a left identity, $e \oplus_b f = f$. As f is a right identity, $e \oplus_b f = e$. Hence, $e = e \oplus_b f = f$. □

Following Lemma 2.3, the unique two-sided identity of a bi-gyrogroupoid will be denoted by 0. Let B be a bi-gyrogroupoid and let $a \in B$. We say that $b \in B$ is a *left inverse* of a if $b \oplus_b a = 0$ and that $c \in B$ is a *right inverse* of a if $a \oplus_b c = 0$. To see that each element of a bi-gyrogroupoid has a unique two-sided inverse, we investigate some basic properties of a bi-gyrogroupoid.

Theorem 2.4. *Let B be a bi-gyrogroupoid. The following properties are true.*

1. *For all $a, b \in B$,* $\mathrm{lgyr}[a, b]0 = 0$ *and* $\mathrm{rgyr}[a, b]0 = 0$.
2. *For all $a \in B$,* $\mathrm{lgyr}[a, a] = \mathrm{id}_B$ *and* $\mathrm{rgyr}[a, a] = \mathrm{id}_B$.
3. *If a is a left inverse of b, then* $\mathrm{lgyr}[a, b] = \mathrm{id}_B$ *and* $\mathrm{rgyr}[a, b] = \mathrm{id}_B$.
4. *For all $b, c \in B$, if a is a left inverse of b, then* $\mathrm{rgyr}[b, c]a \oplus_b (b \oplus_b c) = c$.
5. *For all $a \in B$, if b is a left inverse of a, then b is a right inverse of a.*

Proof. (1) Let $a, b \in B$. Let $c \in B$ be arbitrary. Since $\mathrm{lgyr}[a, b]$ is surjective, $c = \mathrm{lgyr}[a, b]d$ for some $d \in B$. Then

$$c \oplus_b \mathrm{lgyr}[a, b]0 = \mathrm{lgyr}[a, b]d \oplus_b \mathrm{lgyr}[a, b]0 = \mathrm{lgyr}[a, b](d \oplus_b 0) = \mathrm{lgyr}[a, b]d = c.$$

Similarly, $(\mathrm{lgyr}[a, b]0) \oplus_b c = c$. Hence, $\mathrm{lgyr}[a, b]0$ is a two-sided identity of B. By Lemma 2.3, $\mathrm{lgyr}[a, b]0 = 0$. Similarly, one can prove that $\mathrm{rgyr}[a, b]0 = 0$.

(2) Setting $b = 0$ in (BG4a) gives $\mathrm{rgyr}[a, a] = \mathrm{rgyr}[a, 0] = \mathrm{id}_B$ by (BG5). Similarly, setting $b = 0$ in (BG4b) gives $\mathrm{lgyr}[a, a] = \mathrm{id}_B$.

(3) Let $b \in B$ and let a be a left inverse of b. By (BG4a) and (BG5),

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[\mathrm{lgyr}[a, b]a, a \oplus_b b] = \mathrm{rgyr}[\mathrm{lgyr}[a, b]a, 0] = \mathrm{id}_B.$$

Similarly, $\mathrm{lgyr}[a, b] = \mathrm{id}_B$ by (BG4b) and (BG5).

(4) Let $b, c \in B$ and let a be a left inverse of b. From Identity (1) and Item (3), we have $\mathrm{rgyr}[b, c]a \oplus_b (b \oplus_b c) = (a \oplus_b b) \oplus_b \mathrm{lgyr}[a, b]c = 0 \oplus_b c = c$.

(5). Let $a \in B$ and let b be a left inverse of a. By (BG2), b has a left inverse, say $\tilde{b}$. From Items (4) and (3), we have

$$a = \mathrm{rgyr}[b, a]\tilde{b} \oplus_b (b \oplus_b a) = \mathrm{rgyr}[b, a]\tilde{b} \oplus_b 0 = \mathrm{rgyr}[b, a]\tilde{b} = \tilde{b}.$$

It follows that $a \oplus_b b = \tilde{b} \oplus_b b = 0$, which proves b is a right inverse of a. □

Theorem 2.5. *Any element of a bi-gyrogroupoid B has a unique two-sided inverse in B.*

Proof. Let $a \in B$. By (BG2), a has a left inverse b in B. By Theorem 2.4 (5), b is also a right inverse of a. Hence, b is a two-sided inverse of a. Suppose that c is a two-sided inverse of a. Then a is a left inverse of c. By Theorem 2.4 (3)–(4), $c = \mathrm{rgyr}[a, c]b \oplus_b (a \oplus_b c) = \mathrm{rgyr}[a, c]b \oplus_b 0 = \mathrm{rgyr}[a, c]b = b$, which proves the uniqueness of b. □

Following Theorem 2.5, if a is an element of a bi-gyrogroupoid, then the unique two-sided inverse of a will be denoted by $\ominus_b a$. We also write $a \ominus_b b$ instead of $a \oplus_b (\ominus_b b)$. As a consequence of Theorems 2.4 and 2.5, we derive the following theorem.

Theorem 2.6. *Let B be a bi-gyrogroupoid. The following properties are true for all $a, b, c \in B$:*

1. $\ominus_b(\ominus_b a) = a$;
2. $\mathrm{lgyr}[a, b](\ominus_b c) = \ominus_b \mathrm{lgyr}[a, b]c$ *and* $\mathrm{rgyr}[a, b](\ominus_b c) = \ominus_b \mathrm{rgyr}[a, b]c$;
3. $\mathrm{lgyr}[a, \ominus_b a] = \mathrm{lgyr}[\ominus_b a, a] = \mathrm{rgyr}[a, \ominus_b a] = \mathrm{rgyr}[\ominus_b a, a] = \mathrm{id}_B$.

Any bi-gyrogroupoid satisfies a generalized cancellation law, as shown in the following theorem.

Theorem 2.7. *Any bi-gyrogroupoid B satisfies the left cancellation law*

$$\ominus_b \mathrm{rgyr}[a, b]a \oplus_b (a \oplus_b b) = b \tag{4}$$

and the right cancellation law

$$(a \oplus_b b) \ominus_b \mathrm{lgyr}[a, b]b = a \tag{5}$$

for all $a, b \in B$.

Proof. Identity (4) follows from Theorem 2.4 (4) and Theorem 2.6 (2). Identity (5) follows from (BG3) with $c = \ominus_b b$. □

Definition 2.8 (Bi-gyrocommutative bi-gyrogroupoid). A bi-gyrogroupoid B is *bi-gyrocommutative* if it satisfies the bi-gyrocommutative law

$$a \oplus_b b = (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b])(b \oplus_b a) \tag{6}$$

for all $a, b \in B$.

Definition 2.9 (Automorphic inverse property). A bi-gyrogroupoid B has the *automorphic inverse property* if

$$\ominus_b(a \oplus_b b) = (\ominus_b a) \oplus_b (\ominus_b b)$$

for all $a, b \in B$.

Definition 2.10 (Bi-gyration inversion law). A bi-gyrogroupoid B satisfies the *bi-gyration inversion law* if

$$\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b, a] \quad \text{and} \quad \mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b, a]$$

for all $a, b \in B$.

Under certain conditions, the bi-gyrocommutative property and the automorphic inverse property are equivalent, as the following theorem asserts.

Theorem 2.11. *Let B be a bi-gyrogroupoid such that*

1. $\mathrm{lgyr}[a,b]\circ\mathrm{rgyr}[a,b]=\mathrm{rgyr}[a,b]\circ\mathrm{lgyr}[a,b]$;
2. $\mathrm{lgyr}^{-1}[a,b]=\mathrm{lgyr}[\ominus_b b,\ominus_b a]$ *and* $\mathrm{rgyr}^{-1}[a,b]=\mathrm{rgyr}[\ominus_b b,\ominus_b a]$;
3. $\ominus_b(a\oplus_b b)=(\mathrm{lgyr}[a,b]\circ\mathrm{rgyr}[a,b])(\ominus_b b\ominus_b a)$

for all $a,b\in B$. If B is bi-gyrocommutative, then B has the automorphic inverse property. The converse is true if B satisfies the bi-gyration inversion law.

Proof. Suppose that B is bi-gyrocommutative and let $a,b\in B$. Then $b\oplus_b a=(\mathrm{lgyr}[b,a]\circ\mathrm{rgyr}[b,a])(a\oplus_b b)$ and hence

$$\begin{aligned}a\oplus_b b&=(\mathrm{lgyr}[b,a]\circ\mathrm{rgyr}[b,a])^{-1}(b\oplus_b a)\\&=(\mathrm{rgyr}^{-1}[b,a]\circ\mathrm{lgyr}^{-1}[b,a])(b\oplus_b a)\\&=(\mathrm{rgyr}[\ominus_b a,\ominus_b b]\circ\mathrm{lgyr}[\ominus_b a,\ominus_b b])(b\oplus_b a)\\&=(\mathrm{lgyr}[\ominus_b a,\ominus_b b]\circ\mathrm{rgyr}[\ominus_b a,\ominus_b b])(b\oplus_b a)\\&=\ominus_b(\ominus_b a\ominus_b b).\end{aligned}\tag{7}$$

The extreme sides of (7) imply $\ominus_b(a\oplus_b b)=\ominus_b a\ominus_b b$ and so B has the automorphic inverse property. Suppose that B satisfies the bi-gyration inversion law and let $a,b\in B$. As in (7), we have

$$(\mathrm{lgyr}[a,b]\circ\mathrm{rgyr}[a,b])(b\oplus_b a)=\ominus_b(\ominus_b a\ominus_b b)=a\oplus_b b.$$

Hence, B is bi-gyrocommutative. □

3. Bi-Transversal Decomposition

In this section we study the bi-decomposition $\Gamma=H_LBH_R$ of a group Γ into a subset B and subgroups H_L and H_R of Γ. The bi-decomposition $\Gamma=H_LBH_R$ leads to a bi-gyrogroupoid B, and under certain conditions, a group-like structure for B, called a *bi-gyrogroup*. Further, in the special case when H_L is the trivial subgroup of Γ, the bi-decomposition $\Gamma=H_LBH_R$ descends to the decomposition studied in [14]. It turns out that the bi-gyrogroup B induced by the bi-decomposition of Γ forms a gyrogroup, a rich algebraic structure extensively studied, for instance, in [7, 9–11, 18, 22–25, 28–31].

Definition 3.1 (Bi-transversal). A subset B of a group Γ is said to be a *bi-transversal* of subgroups H_L and H_R of Γ if every element g of Γ can be written uniquely as $g=h_\ell bh_r$, where $h_\ell\in H_L$, $b\in B$, and $h_r\in H_R$.

Let B be a bi-transversal of subgroups H_L and H_R in a group Γ. For each pair of elements b_1 and b_2 in B, the product b_1b_2 gives unique elements $h_\ell(b_1,b_2)\in H_L$, $b_1\odot b_2\in B$, and $h_r(b_1,b_2)\in H_R$ such that

$$b_1b_2=h_\ell(b_1,b_2)(b_1\odot b_2)h_r(b_1,b_2).\tag{8}$$

Hence, any bi-transversal B of H_L and H_R gives rise to

1. a binary operation $\odot$ in B, called the *bi-transversal operation*;

2. a map $h_\ell\colon B \times B \to H_L$, called *the left transversal map*;

3. a map $h_r\colon B \times B \to H_R$, called *the right transversal map.*

The pair $(B, \odot)$ is called the *bi-transversal groupoid of H_L and H_R*.

We will see shortly that the left and right transversal maps of the bi-transversal groupoid $(B, \odot)$ generate automorphisms of $(B, \odot)$, called *left* and *right gyrations* or, collectively, *bi-gyrations*. Accordingly, left and right gyrations are also called *left* and *right gyroautomorphisms*.

Definition 3.2 (Bi-gyration). Let B be a bi-transversal of subgroups H_L and H_R in a group Γ. Let h_ℓ and h_r be the left and right transversal maps, respectively. The *left gyration* $\mathrm{lgyr}[b_1, b_2]$ *of B generated by* $b_1, b_2 \in B$ is defined by

$$\mathrm{lgyr}[b_1, b_2]b = h_r(b_1, b_2)bh_r(b_1, b_2)^{-1}, \quad b \in B. \tag{9}$$

The *right gyration* $\mathrm{rgyr}[b_1, b_2]$ *of B generated by* $b_1, b_2 \in B$ is defined by

$$\mathrm{rgyr}[b_1, b_2]b = h_\ell(b_1, b_2)^{-1}bh_\ell(b_1, b_2), \quad b \in B. \tag{10}$$

Remark 1. In Definition 3.2, left gyrations are associated with the right transversal map h_r, and right gyrations are associated with the left transversal map h_ℓ.

We use the convenient notation $x^h = hxh^{-1}$ and denote *conjugation by h* by α_h. That is, $\alpha_h(x) = x^h = hxh^{-1}$. With this notation, the left and right gyrations in Definition 3.2 read

$$\mathrm{lgyr}[a, b] = \alpha_{h_r(a,b)} \quad \text{and} \quad \mathrm{rgyr}[a, b] = \alpha_{h_\ell(a,b)^{-1}} \tag{11}$$

for all $a, b \in B$. Let B be a non-empty subset of a group Γ. We say that a subgroup H of Γ *normalizes* B if $hBh^{-1} \subseteq B$ for all $h \in H$.

Definition 3.3 (Bi-gyrotransversal). A bi-transversal B of subgroups H_L and H_R in a group Γ is a *bi-gyrotransversal* if

1. H_L and H_R normalize B, and

2. $h_\ell h_r = h_r h_\ell$ for all $h_\ell \in H_L$, $h_r \in H_R$.

Proposition 3.4. *If B is a bi-gyrotransversal of subgroups H_L and H_R in a group Γ, then $H_L H_R$ is a subgroup of Γ with normal subgroups H_L and H_R. If B contains the identity 1 of Γ, then $H_L \cap H_R = \{1\}$. In this case, $H_L H_R$ is isomorphic to the direct product $H_L \times H_R$ as groups.*

Proof. Since $H_L H_R = H_R H_L$, $H_L H_R$ forms a subgroup of Γ by Proposition 14 of [5, Chapter 3]. If $g \in H_L H_R$, then $g = h_\ell h_r$ for some $h_\ell \in H_L$ and $h_r \in H_R$. For any $h \in H_L$, $h_r h = h h_r$ implies $ghg^{-1} = h_\ell h h_\ell^{-1} \in H_L$. Hence, $gH_Lg^{-1} \subseteq H_L$. This proves $H_L \trianglelefteq H_LH_R$. Similarly, $H_R \trianglelefteq H_LH_R$.

Suppose that $1 \in B$ and let $h \in H_L \cap H_R$. The unique decomposition of 1, $1 = hh^{-1} = h1h^{-1}$, implies $h = 1$. Hence, $H_L \cap H_R = \{1\}$. It follows from Theorem 9 of [5, Chapter 5] that $H_LH_R \cong H_L \times H_R$ as groups. □

Theorem 3.5. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If $h \in H_LH_R$, then conjugation by h is an automorphism of $(B, \odot)$.*

Proof. Note first that H_LH_R normalizes B. In fact, if $h = h_\ell h_r$ with h_ℓ in H_L and h_r in H_R, then $hBh^{-1} = h_\ell(h_rBh_r^{-1})h_\ell^{-1} \subseteq B$ for H_R and H_L normalize B.

Let $h \in H_LH_R$. Since H_LH_R normalizes B, α_h is a bijection from B to itself. Next, we will show that $(x \odot y)^h = x^h \odot y^h$ for all $x, y \in B$. Employing (8), we have

$$(xy)^h = (h_\ell(x, y)(x \odot y)h_r(x, y))^h = h_\ell(x, y)^h(x \odot y)^h h_r(x, y)^h.$$

Since $x^h, y^h \in B$, we also have

$$x^h y^h = h_\ell(x^h, y^h)(x^h \odot y^h)h_r(x^h, y^h).$$

Note that $h_\ell(x, y)^h \in H_L$ and $h_r(x, y)^h \in H_R$ because H_L and H_R are normal in H_LH_R. Thus, $(xy)^h = x^h y^h$ implies

$$h_\ell(x, y)^h = h_\ell(x^h, y^h), \quad (x \odot y)^h = x^h \odot y^h, \quad \text{and} \quad h_r(x, y)^h = h_r(x^h, y^h),$$

which completes the proof. □

Corollary 3.6. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. Then $\mathrm{lgyr}[a, b]$ and $\mathrm{rgyr}[a, b]$ are automorphisms of $(B, \odot)$ for all $a, b \in B$.*

Proof. This is because $\mathrm{lgyr}[a, b] = \alpha_{h_r(a,b)}$ and $\mathrm{rgyr}[a, b] = \alpha_{h_\ell(a,b)^{-1}}$. □

The next theorem provides us with *commuting relations* between conjugation automorphisms of the bi-transversal groupoid $(B, \odot)$ and its bi-gyrations.

Theorem 3.7. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. The following commuting relations hold.*

1. $\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[c, d] = \mathrm{rgyr}[c, d] \circ \mathrm{lgyr}[a, b]$ *for all $a, b, c, d \in B$.*

2. $\alpha_h \circ \mathrm{lgyr}[a, b] = \mathrm{lgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h$ *for all $h \in H_LH_R$ and $a, b \in B$.*

3. $\alpha_h \circ \mathrm{rgyr}[a, b] = \mathrm{rgyr}[\alpha_h(a), \alpha_h(b)] \circ \alpha_h$ *for all $h \in H_LH_R$ and $a, b \in B$.*

Proof. Item (1) follows from the fact that $h_\ell h_r = h_r h_\ell$ for all $h_\ell \in H_L$ and $h_r \in H_R$ and that $\alpha_{gh} = \alpha_g \circ \alpha_h$ for all $g, h \in \Gamma$.

Let $h \in H_L H_R$ and let $a, b \in B$. As in the proof of Theorem 3.5, $h_r(a,b)^h = h_r(a^h, b^h)$. Hence, $\alpha_h \circ \mathrm{lgyr}[a,b] \circ \alpha_h^{-1} = \mathrm{lgyr}[a^h, b^h]$ and Item (2) follows. Similarly, $h_\ell(a,b)^h = h_\ell(a^h, b^h)$ implies Item (3). □

As a consequence of Theorem 3.7, left gyrations are invariant under right gyrations, and vice versa. In fact, we have the following two theorems.

Theorem 3.8. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If ρ is a finite composition of right gyrations of B, then*

$$\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\rho(a), \rho(b)] \tag{12}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\lambda(a), \lambda(b)] \tag{13}$$

for all $a, b \in B$.

Proof. By assumption, $\rho = \mathrm{rgyr}[a_1, b_1] \circ \mathrm{rgyr}[a_2, b_2] \circ \cdots \circ \mathrm{rgyr}[a_n, b_n]$ for some $a_i, b_i \in B$. Since $\mathrm{rgyr}[a_i, b_i] = \alpha_{h_\ell(a_i,b_i)^{-1}}$ for all i, it follows that $\rho = \alpha_h$, where $h = h_\ell(a_1,b_1)^{-1} h_\ell(a_2,b_2)^{-1} \cdots h_\ell(a_n,b_n)^{-1}$. As $\rho = \alpha_h$ and $h \in H_L$, Theorem 3.7 (2) implies $\rho \circ \mathrm{lgyr}[a,b] = \mathrm{lgyr}[\rho(a), \rho(b)] \circ \rho$. Since ρ and $\mathrm{lgyr}[a,b]$ commute, we have (12). One obtains similarly that $\lambda = \alpha_h$ for some $h \in H_R$, which implies (13) by Theorem 3.7 (3). □

Theorem 3.9. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If ρ is a finite composition of right gyrations of B, then*

$$\rho \circ \mathrm{rgyr}[a,b] = \mathrm{rgyr}[\rho(a), \rho(b)] \circ \rho \tag{14}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\lambda \circ \mathrm{lgyr}[a,b] = \mathrm{lgyr}[\lambda(a), \lambda(b)] \circ \lambda \tag{15}$$

for all $a, b \in B$.

Proof. As in the proof of Theorem 3.8, $\rho = \alpha_h$ for some $h \in H_L$. Hence, (14) is an application of Theorem 3.7 (3). Similarly, (15) is an application of Theorem 3.7 (2). □

The associativity of Γ is reflected in its bi-gyrotransversal decomposition $\Gamma = H_L B H_R$, as shown in the following theorem.

Theorem 3.10. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. For all $a, b, c \in B$,*

$$(a \odot b) \odot \mathrm{lgyr}[a,b]c = \mathrm{rgyr}[b,c]a \odot (b \odot c).$$

Proof. Let $a, b, c \in B$. Set $a_r = \text{rgyr}[b, c]a$ and $c_l = \text{lgyr}[a, b]c$. Then $a_r \in B$ and $c_l \in B$. By employing (8),

$$\begin{aligned}
a(bc) &= a(h_\ell(b,c)(b \odot c)h_r(b,c)) \\
&= h_\ell(b,c)(h_\ell(b,c)^{-1}ah_\ell(b,c))(b \odot c)h_r(b,c) \\
&= h_\ell(b,c)a_r(b \odot c)h_r(b,c) \\
&= [h_\ell(b,c)h_\ell(a_r, b \odot c)][a_r \odot (b \odot c)][h_r(a_r, b \odot c)h_r(b,c)]
\end{aligned}$$

and, similarly, $(ab)c = [h_\ell(a,b)h_\ell(a \odot b, c_l)][(a \odot b) \odot c_l][h_r(a \odot b, c_l)h_r(a,b)]$. Since $a(bc) = (ab)c$, it follows that $(a \odot b) \odot c_l = a_r \odot (b \odot c)$, which was to be proved. □

Proposition 3.11. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. For all $a, b, c \in B$,*

1. $\text{rgyr}[\text{rgyr}[b,c]a, b \odot c] \circ \text{rgyr}[b,c] = \text{rgyr}[a \odot b, \text{lgyr}[a,b]c] \circ \text{rgyr}[a,b]$*, and*
2. $\text{lgyr}[a \odot b, \text{lgyr}[a,b]c] \circ \text{lgyr}[a,b] = \text{lgyr}[\text{rgyr}[b,c]a, b \odot c] \circ \text{lgyr}[b,c]$.

Proof. As we have computed in the proof of Theorem 3.10,

$$h_\ell(b,c)h_\ell(a_r, b \odot c) = h_\ell(a,b)h_\ell(a \odot b, c_l),$$

where $a_r = \text{rgyr}[b,c]a$ and $c_l = \text{lgyr}[a,b]c$. Thus, Item (1) is obtained. Similarly, $h_r(a_r, b \odot c)h_r(b,c) = h_r(a \odot b, c_l)h_r(a,b)$ gives Item (2). □

Twisted Subgroups

Twisted subgroups abound in group theory, gyrogroup theory, and loop theory, as evidenced, for instance, from [1–3, 6, 13, 14, 18]. Here, we demonstrate that a bi-gyrotransversal decomposition $\Gamma = H_L B H_R$ in which B is a twisted subgroup gives rise to a highly structured type of bi-gyrogroupoids and, eventually, a bi-gyrogroup. We follow Aschbacher for the definition of a twisted subgroup.

Definition 3.12 (Twisted subgroup). A subset B of a group Γ is a *twisted subgroup* of Γ if the following conditions hold:

1. $1 \in B$, 1 being the identity of Γ;
2. if $b \in B$, then $b^{-1} \in B$;
3. if $a, b \in B$, then $aba \in B$.

Theorem 3.13. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ, then the following properties are true for all $a, b \in B$.*

1. $1 \odot b = b \odot 1 = b$.
2. $b^{-1} \in B$ *and* $b^{-1} \odot b = b \odot b^{-1} = 1$.
3. $\mathrm{lgyr}[1, b] = \mathrm{lgyr}[b, 1] = \mathrm{rgyr}[1, b] = \mathrm{rgyr}[b, 1] = \mathrm{id}_B$.
4. $\mathrm{lgyr}[b^{-1}, b] = \mathrm{lgyr}[b, b^{-1}] = \mathrm{rgyr}[b^{-1}, b] = \mathrm{rgyr}[b, b^{-1}] = \mathrm{id}_B$.
5. $\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b^{-1}, a^{-1}]$ *and* $\mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b^{-1}, a^{-1}]$.
6. $(a \odot b)^{-1} = (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b])(b^{-1} \odot a^{-1})$.

Proof. (1) As $b = 1b = h_\ell(1, b)(1 \odot b)h_r(1, b)$, we have $h_\ell(1, b) = 1$, $1 \odot b = b$, and $h_r(1, b) = 1$. Similarly, $b = b1$ implies $b \odot 1 = b$.

(2) Let $b \in B$. Since B is a twisted subgroup, $b^{-1} \in B$. Further,

$$1 = b^{-1}b = h_\ell(b^{-1}, b)(b^{-1} \odot b)h_r(b^{-1}, b)$$

implies $h_\ell(b^{-1}, b) = 1$, $b^{-1} \odot b = 1$, and $h_r(b^{-1}, b) = 1$. Similarly, $bb^{-1} = 1$ implies $b \odot b^{-1} = 1$.

(3) We have $h_\ell(1, b) = h_\ell(b, 1) = h_r(1, b) = h_r(b, 1) = 1$, as computed in Item (1). Hence, Item (3) follows.

(4) We have $h_\ell(b^{-1}, b) = h_\ell(b, b^{-1}) = h_r(b^{-1}, b) = h_r(b, b^{-1}) = 1$, as computed in Item (2). Hence, Item (4) follows.

(5) Let $a, b \in B$. Then $a^{-1}, b^{-1} \in B$. On the one hand, we have

$$(ab)^{-1} = (h_\ell(a, b)(a \odot b)h_r(a, b))^{-1} = h_r(a, b)^{-1}(a \odot b)^{-1}h_\ell(a, b)^{-1},$$

and on the other hand we have $b^{-1}a^{-1} = h_\ell(b^{-1}, a^{-1})(b^{-1} \odot a^{-1})h_r(b^{-1}, a^{-1})$. Since $(ab)^{-1} = b^{-1}a^{-1}$, it follows that

$$\begin{aligned}
(a \odot b)^{-1} &= h_r(a, b)h_\ell(b^{-1}, a^{-1})(b^{-1} \odot a^{-1})h_r(b^{-1}, a^{-1})h_\ell(a, b) \\
&= h_\ell(b^{-1}, a^{-1})h_r(a, b)(b^{-1} \odot a^{-1})h_\ell(a, b)h_r(b^{-1}, a^{-1}) \\
&= h_\ell(b^{-1}, a^{-1})h_\ell(a, b)\tilde{b}h_r(a, b)h_r(b^{-1}, a^{-1}),
\end{aligned} \tag{16}$$

where $\tilde{b} = \mathrm{lgyr}[a, b](\mathrm{rgyr}[a, b](b^{-1} \odot a^{-1}))$. Because $(a \odot b)^{-1}$ and $\tilde{b}$ belong to B, we have from the extreme sides of (16) that

$$h_r(a, b)h_r(b^{-1}, a^{-1}) = 1 \quad \text{and} \quad h_\ell(b^{-1}, a^{-1})h_\ell(a, b) = 1.$$

Hence, $h_r(a, b)^{-1} = h_r(b^{-1}, a^{-1})$, which implies $\mathrm{lgyr}^{-1}[a, b] = \mathrm{lgyr}[b^{-1}, a^{-1}]$. Likewise, $h_\ell(a, b) = h_\ell(b^{-1}, a^{-1})^{-1}$ implies $\mathrm{rgyr}^{-1}[a, b] = \mathrm{rgyr}[b^{-1}, a^{-1}]$.

(6) As in Item (5), $(a \odot b)^{-1} = \tilde{b} = \mathrm{lgyr}[a, b](\mathrm{rgyr}[a, b](b^{-1} \odot a^{-1}))$. □

Remark 2. Note that we do not invoke the third defining property of a twisted subgroup in proving Theorem 3.13.

At this point, we have shown that any bi-gyrotransversal decomposition $\Gamma = H_L B H_R$ in which B is a twisted subgroup of Γ gives the bi-transversal groupoid B that satisfies all the axioms of a bi-gyrogroupoid except for (BG4). In order to complete this, we have to impose additional conditions on the left and right transversal maps, as the following lemma indicates.

Lemma 3.14. *If B is a bi-transversal of subgroups H_L and H_R in a group Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a,b \in B$, then*

$$\mathrm{lgyr}^{-1}[a,b] = \mathrm{lgyr}[b,a] \quad \textit{and} \quad \mathrm{rgyr}^{-1}[a,b] = \mathrm{rgyr}[b,a]$$

for all $a,b \in B$.

Proof. Note first that $\alpha_h^{-1} = \alpha_{h^{-1}}$ for all $h \in \Gamma$. From this we have $\mathrm{lgyr}[b,a] = \alpha_{h_r(b,a)} = \alpha_{h_r(a,b)^{-1}} = \alpha^{-1}_{h_r(a,b)} = \mathrm{lgyr}^{-1}[a,b]$. One can prove in a similar way that $\mathrm{rgyr}^{-1}[a,b] = \mathrm{rgyr}[b,a]$. □

Theorem 3.15. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a,b \in B$, then the following relations hold for all $a,b \in B$:*

1. $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$;
2. $\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$;
3. $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{rgyr}[b,a]a, b \odot a]$;
4. $\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{rgyr}[b,a]a, b \odot a]$.

Proof. Let $a,b \in B$. Set $a_l = \mathrm{lgyr}[a,b]a$. Employing (8), we obtain

$$\begin{aligned} (ab)a &= (h_\ell(a,b)(a \odot b)h_r(a,b))a \\ &= h_\ell(a,b)(a \odot b)a_l h_r(a,b) \\ &= [h_\ell(a,b)h_\ell(a \odot b, a_l)][(a \odot b) \odot a_l][h_r(a \odot b, a_l)h_r(a,b)]. \end{aligned} \tag{17}$$

Since $(ab)a \in B$, the extreme sides of (17) imply

$$h_\ell(a,b)h_\ell(a \odot b, a_l) = 1 \quad \text{and} \quad h_r(a \odot b, a_l)h_r(a,b) = 1. \tag{18}$$

The first equation of (18) implies $h_\ell(a \odot b, \mathrm{lgyr}[a,b]a) = h_\ell(a,b)^{-1}$. Hence,

$$\mathrm{rgyr}^{-1}[a \odot b, \mathrm{lgyr}[a,b]a] = \mathrm{rgyr}[a,b].$$

From Lemma 3.14, we have $\mathrm{rgyr}[a,b] = \mathrm{rgyr}[\mathrm{lgyr}[a,b]a, a \odot b]$. The second equation of (18) implies $h_r(a,b) = h_r(a \odot b, \mathrm{lgyr}[a,b]a)^{-1}$. Hence,

$$\mathrm{lgyr}[a,b] = \mathrm{lgyr}[\mathrm{lgyr}[a,b]a, a \odot b].$$

This proves Items (1) and (2). Items (3) and (4) can be proved in a similar way by computing the product $a(ba)$. □

Theorem 3.16. *Let B be a bi-gyrotransversal of subgroups H_L and H_R in a group Γ. If B is a twisted subgroup of Γ such that $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a, b \in B$, then left and right gyrations of B are even in the sense that*

$$\mathrm{lgyr}[a^{-1}, b^{-1}] = \mathrm{lgyr}[a, b] \quad \text{and} \quad \mathrm{rgyr}[a^{-1}, b^{-1}] = \mathrm{rgyr}[a, b]$$

for all $a, b \in B$.

Proof. This theorem follows directly from Theorem 3.13 (5) and Lemma 3.14. □

4. Bi-Gyrodecomposition and Bi-Gyrogroups

Taking the key features of bi-gyrotransversal decomposition of a group given in Section 3, we formulate the definition of bi-gyrodecomposition and show that any bi-gyrodecomposition leads to a bi-gyrogroup, which in turn is a gyrogroup. Most of the results in Section 3 are directly translated into results in this section with appropriate modifications.

Definition 4.1 (Bi-gyrodecomposition). Let Γ be a group, let B be a subset of Γ, and let H_L and H_R be subgroups of Γ. A decomposition $\Gamma = H_L B H_R$ is a *bi-gyrodecomposition* if

1. B is a bi-gyrotransversal of H_L and H_R in Γ;
2. B is a twisted subgroup of Γ; and
3. $h_\ell(a,b)^{-1} = h_\ell(b,a)$ and $h_r(a,b)^{-1} = h_r(b,a)$ for all $a, b \in B$,

where h_ℓ and h_r are the bi-transversal maps given below Definition 3.1.

Theorem 4.2. *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then B equipped with the bi-transversal operation forms a bi-gyrogroupoid.*

Proof. Axiom (BG1) holds by Theorem 3.13 (1), where the identity 1 of Γ acts as the identity of B. Axiom (BG2) holds by Theorem 3.13 (2), where b^{-1} acts as a left inverse of $b \in B$ with respect to the bi-transversal operation. Axiom (BG3) holds by Corollary 3.6 and Theorem 3.10. Axiom (BG4) holds by Theorem 3.15. Axiom (BG5) holds by Theorem 3.13 (3). □

It is shown in Section 3 that any bi-transversal decomposition $\Gamma = H_L B H_R$ gives rise to a bi-transversal groupoid $(B, \odot)$. Theorem 4.2 asserts that in the special case when the decomposition is a bi-gyrodecomposition, the bi-transversal groupoid $(B, \odot)$ becomes the bi-gyrogroupoid $(B, \oplus_b)$ described in Definition 2.1. Hence, in particular, the binary operations $\oplus_b$ and $\odot$ share the same algebraic properties. Further, the identity of the bi-gyrogroupoid B coincides with the group identity of Γ and $\ominus_b b = b^{-1}$ for all $b \in B$.

Theorem 4.3 (Bi-gyration invariant relation). *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition. If ρ is a finite composition of right gyrations of B, then*

$$\operatorname{lgyr}[a, b] = \operatorname{lgyr}[\rho(a), \rho(b)] \tag{19}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\operatorname{rgyr}[a, b] = \operatorname{rgyr}[\lambda(a), \lambda(b)] \tag{20}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.8. ☐

Theorem 4.4 (Bi-gyration commuting relation). *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition. If ρ is a finite composition of right gyrations of B, then*

$$\rho \circ \operatorname{rgyr}[a, b] = \operatorname{rgyr}[\rho(a), \rho(b)] \circ \rho \tag{21}$$

for all $a, b \in B$. If λ is a finite composition of left gyrations of B, then

$$\lambda \circ \operatorname{lgyr}[a, b] = \operatorname{lgyr}[\lambda(a), \lambda(b)] \circ \lambda \tag{22}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.9. ☐

Theorem 4.5 (Trivial bi-gyration). *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then for all $a \in B$,*

$$\begin{aligned}
\operatorname{lgyr}[0, a] = \operatorname{lgyr}[a, 0] &= \operatorname{id}_B\\
\operatorname{lgyr}[a, \ominus_b a] = \operatorname{lgyr}[\ominus_b a, a] &= \operatorname{id}_B\\
\operatorname{rgyr}[0, a] = \operatorname{rgyr}[a, 0] &= \operatorname{id}_B\\
\operatorname{rgyr}[a, \ominus_b a] = \operatorname{rgyr}[\ominus_b a, a] &= \operatorname{id}_B\\
\operatorname{lgyr}[a, a] = \operatorname{rgyr}[a, a] &= \operatorname{id}_B.
\end{aligned} \tag{23}$$

Proof. The theorem follows from Theorem 2.4 (2) and Theorem 3.13 (3)–(4). ☐

Theorem 4.6 (Bi-gyration inversion law). *If $\Gamma = H_L B H_R$ is a bi-gyrodecomposition, then*

$$\operatorname{lgyr}^{-1}[a, b] = \operatorname{lgyr}[b, a] \quad \textit{and} \quad \operatorname{rgyr}^{-1}[a, b] = \operatorname{rgyr}[b, a]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Lemma 3.14. ☐

Theorem 4.7 (Even bi-gyration). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then left and right gyrations of* B *are even:*

$$\mathrm{lgyr}[\ominus_b a, \ominus_b b] = \mathrm{lgyr}[a, b] \quad \textit{and} \quad \mathrm{rgyr}[\ominus_b a, \ominus_b b] = \mathrm{rgyr}[a, b]$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 3.16. □

Theorem 4.8 (Left and right cancellation laws). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then* B *satisfies the left cancellation law*

$$\ominus_b \mathrm{rgyr}[a, b] a \oplus_b (a \oplus_b b) = b \tag{24}$$

and the right cancellation law

$$(a \oplus_b b) \ominus_b \mathrm{lgyr}[a, b] b = a \tag{25}$$

for all $a, b \in B$.

Proof. The theorem follows immediately from Theorem 2.7. □

Theorem 4.9 (Left and right bi-gyroassociative laws). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then* B *satisfies the left bi-gyroassociative law*

$$a \oplus_b (b \oplus_b c) = (\mathrm{rgyr}[c, b] a \oplus_b b) \oplus_b \mathrm{lgyr}[\mathrm{rgyr}[c, b] a, b] c \tag{26}$$

and the right bi-gyroassociative law

$$(a \oplus_b b) \oplus_b c = \mathrm{rgyr}[b, \mathrm{lgyr}[b, a] c] a \oplus_b (b \oplus_b \mathrm{lgyr}[b, a] c) \tag{27}$$

for all $a, b, c \in B$.

Proof. The theorem follows from Theorems 2.2 and 4.6. □

Theorem 4.10 (Left gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[\mathrm{rgyr}[b, a] a, b \oplus_b a] \tag{28}$$

and

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus_b b, \mathrm{rgyr}[a, b] b] \tag{29}$$

for all $a, b \in B$.

Proof. Identity (28) follows from Theorem 3.15 (4). Identity (29) is obtained from (28) by applying the bi-gyration inversion law (Theorem 4.6) followed by interchanging a and b. □

Theorem 4.11 (Right gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[\mathrm{lgyr}[a, b]a, a \oplus_b b] \tag{30}$$

and

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[b \oplus_b a, \mathrm{lgyr}[b, a]b] \tag{31}$$

for all $a, b \in B$.

Proof. Identity (30) follows from Theorem 3.15 (1). Identity (31) is obtained from (30) by applying the bi-gyration inversion law followed by interchanging a and b. □

Theorem 4.12 (Bi-gyration reduction property). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[\mathrm{lgyr}[a, b]a, a \oplus_b b] \tag{32}$$

and

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b] \tag{33}$$

for all $a, b \in B$.

Proof. Identity (32) follows from Theorem 3.15 (2). Identity (33) is obtained from Theorem 3.15 (3) by applying the bi-gyration inversion law followed by interchanging a and b. □

Theorem 4.13 (Left and right gyration reduction properties). *If* $\Gamma = H_L B H_R$ *is a bi-gyrodecomposition, then*

$$\begin{aligned} \mathrm{rgyr}[a, b] &= \mathrm{rgyr}[\ominus_b \mathrm{lgyr}[a, b]b, a \oplus_b b] \\ \mathrm{lgyr}[a, b] &= \mathrm{lgyr}[\ominus_b \mathrm{lgyr}[a, b]b, a \oplus_b b] \end{aligned} \tag{34}$$

and

$$\begin{aligned} \mathrm{rgyr}[a, b] &= \mathrm{rgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a] \\ \mathrm{lgyr}[a, b] &= \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a] \end{aligned} \tag{35}$$

for all $a, b \in B$.

Proof. Setting $c = \ominus_b b$ in Proposition 3.11 (1)–(2) followed by using the bi-gyration inversion law gives (34). Setting $a = \ominus_b b$ in the same proposition followed by using the bi-gyration inversion law gives

$$\begin{aligned} \mathrm{rgyr}[b, c] &= \mathrm{rgyr}[b \oplus_b c, \ominus_b \mathrm{rgyr}[b, c]b] \\ \mathrm{lgyr}[b, c] &= \mathrm{lgyr}[b \oplus_b c, \ominus_b \mathrm{rgyr}[b, c]b]. \end{aligned}$$

Replacing b by a and c by b, we obtain (35). □

Theorem 4.14 (Left and right gyration reduction properties). *If* $\Gamma = H_LBH_R$ *is a bi-gyrodecomposition, then*

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \ominus_b a]\\ \mathrm{rgyr}[a, b] &= \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \ominus_b a]\end{aligned} \tag{36}$$

for all $a, b \in B$.

Proof. From the second equation of (35), we have

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a].$$

Applying Theorem 4.3 to the previous equation with $\rho = \mathrm{rgyr}[b, a]$ gives

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a]\\ &= \mathrm{lgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\ominus_b \mathrm{rgyr}[a, b]a)]\\ &= \mathrm{lgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \ominus_b a].\end{aligned}$$

We obtain the last equation since $\mathrm{rgyr}[b, a] = \mathrm{rgyr}^{-1}[a, b]$. Similarly, the first equation of (35) and Identity (21) together imply

$$\begin{aligned}\mathrm{id}_B &= \mathrm{rgyr}^{-1}[a, b] \circ \mathrm{rgyr}[a \oplus_b, \ominus_b \mathrm{rgyr}[a, b]a]\\ &= \mathrm{rgyr}[b, a] \circ \mathrm{rgyr}[a \oplus_b b, \ominus_b \mathrm{rgyr}[a, b]a]\\ &= \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\ominus_b \mathrm{rgyr}[a, b]a)] \circ \mathrm{rgyr}[b, a]\\ &= \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \ominus_b a] \circ \mathrm{rgyr}[b, a].\end{aligned} \tag{37}$$

The extreme sides of (37) imply $\mathrm{rgyr}[a, b] = \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \ominus_b a]$. □

Bi-Gyrogroups

We are now in a position to present the formal definition of a bi-gyrogroup.

Definition 4.15 (Bi-gyrogroup). Let $\Gamma = H_LBH_R$ be a bi-gyrodecomposition. The *bi-gyrogroup operation* $\oplus$ in B is defined by

$$a \oplus b = \mathrm{rgyr}[b, a](a \oplus_b b), \quad a, b \in B. \tag{38}$$

Here, $\oplus_b$ is the bi-transversal operation induced by the decomposition $\Gamma = H_LBH_R$. The groupoid $(B, \oplus)$ consisting of the set B and the bi-gyrogroup operation $\oplus$ is called a *bi-gyrogroup*.

Throughout the remaining of this section, we assume that $\Gamma = H_LBH_R$ is a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup.

Proposition 4.16. *The unique two-sided identity element of* $(B, \oplus)$ *is* 0. *For each* $a \in B$, $\ominus_b a$ *is the unique two-sided inverse of* a *in* $(B, \oplus)$.

Proof. Let $a \in B$. Since $\mathrm{rgyr}[a,0] = \mathrm{rgyr}[0,a] = \mathrm{id}_B$, we have

$$a \oplus 0 = \mathrm{rgyr}[0,a](a \oplus_b 0) = a = \mathrm{rgyr}[a,0](0 \oplus_b a) = (0 \oplus a).$$

Hence, 0 is a two-sided identity of $(B, \oplus)$. The uniqueness of 0 follows, as in the proof of Lemma 2.3. Since $\mathrm{rgyr}[a, \ominus_b a] = \mathrm{rgyr}[\ominus_b a, a] = \mathrm{id}_B$, we have

$$a \oplus (\ominus_b a) = \mathrm{rgyr}[\ominus_b a, a](a \ominus_b a) = 0 = \mathrm{rgyr}[a, \ominus_b a](\ominus_b a \oplus_b a) = (\ominus_b a) \oplus a.$$

Hence, $\ominus_b a$ acts as a two-sided inverse of a with respect to $\oplus$. Suppose that b is a two-sided inverse of a with respect to $\oplus$. Then $0 = a \oplus b = \mathrm{rgyr}[b,a](a \oplus_b b)$, which implies $a \oplus_b b = 0$. Similarly, $b \oplus a = 0$ implies $b \oplus_b a = 0$. This proves that b is a two-sided inverse of a with respect to $\oplus_b$. Hence, $b = \ominus_b a$ by Theorem 2.5. ◻

Following Proposition 4.16, if a is an element of B, then the unique two-sided inverse of a with respect to $\oplus$ will be denoted by $\ominus a$. Further,

$$\ominus a = \ominus_b a$$

for all $a \in B$. We also write $a \ominus b$ instead of $a \oplus (\ominus b)$. The following theorem asserts that left and right gyrations of the bi-transversal groupoid $(B, \oplus_b)$ ascend to automorphisms of the bi-gyrogroup $(B, \oplus)$.

Theorem 4.17. *If λ is a finite composition of left gyrations of $(B, \oplus_b)$, then*

$$\lambda(a \oplus b) = \lambda(a) \oplus \lambda(b) \tag{39}$$

for all $a, b \in B$. If ρ is a finite composition of right gyrations of $(B, \oplus_b)$, then

$$\rho(a \oplus b) = \rho(a) \oplus \rho(b) \tag{40}$$

for all $a, b \in B$.

Proof. Let $a, b \in B$. By Theorem 3.7 (1), λ and $\mathrm{rgyr}[b,a]$ commute. Hence,

$$\begin{aligned}
\lambda(a \oplus b) &= (\lambda \circ \mathrm{rgyr}[b,a])(a \oplus_b b) \\
&= (\mathrm{rgyr}[b,a] \circ \lambda)(a \oplus_b b) \\
&= \mathrm{rgyr}[b,a](\lambda(a) \oplus_b \lambda(b)) \\
&= \mathrm{rgyr}[\lambda(b), \lambda(a)](\lambda(a) \oplus_b \lambda(b)) \\
&= \lambda(a) \oplus \lambda(b).
\end{aligned}$$

We have the third equation since λ is a finite composition of left gyrations; the forth equation from (20); and the last equation from Definition 4.15. Similarly, (40) is obtained from (21). ◻

Lemma 4.18. *In the bi-gyrogroup B,*

$$\operatorname{rgyr}[c, a \oplus b] \circ \operatorname{rgyr}[b, a] = \operatorname{rgyr}[b \oplus c, a] \circ \operatorname{rgyr}[c, b]$$

for all $a, b, c \in B$.

Proof. By Theorem 4.6 and Proposition 3.11 (1),

$$\begin{aligned}
&\operatorname{rgyr}[b, a] \circ \operatorname{rgyr}[\operatorname{lgyr}[a, b]c, a \oplus_b b] \\
&\quad = (\operatorname{rgyr}[a \oplus_b b, \operatorname{lgyr}[a, b]c] \circ \operatorname{rgyr}[a, b])^{-1} \\
&\quad = (\operatorname{rgyr}[\operatorname{rgyr}[b, c]a, b \oplus_b c] \circ \operatorname{rgyr}[b, c])^{-1} \\
&\quad = \operatorname{rgyr}[c, b] \circ \operatorname{rgyr}[b \oplus_b c, \operatorname{rgyr}[b, c]a].
\end{aligned} \tag{41}$$

By Identity (21) and Theorem 4.6, the extreme sides of (41) imply

$$\operatorname{rgyr}[c, \operatorname{rgyr}[b, a](a \oplus_b b)] \circ \operatorname{rgyr}[b, a] = \operatorname{rgyr}[\operatorname{rgyr}[c, b](b \oplus_b c), a] \circ \operatorname{rgyr}[c, b].$$

According to Definition 4.15, the previous equation reads

$$\operatorname{rgyr}[c, a \oplus b] \circ \operatorname{rgyr}[b, a] = \operatorname{rgyr}[b \oplus c, a] \circ \operatorname{rgyr}[c, b],$$

which completes the proof. $\square$

Theorem 4.19 (Bi-gyroassociative law in bi-gyrogroups). *The bi-gyrogroup B satisfies the left bi-gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus (\operatorname{lgyr}[a, b] \circ \operatorname{rgyr}[b, a])(c) \tag{42}$$

and the right bi-gyroassociative law

$$(a \oplus b) \oplus c = a \oplus (b \oplus (\operatorname{lgyr}[b, a] \circ \operatorname{rgyr}[a, b])(c)) \tag{43}$$

for all $a, b, c \in B$.

Proof. From Theorem 3.10, we have

$$(a \oplus_b b) \oplus_b \operatorname{lgyr}[a, b]c = \operatorname{rgyr}[b, c]a \oplus_b (b \oplus_b c).$$

Applying $\operatorname{rgyr}[c, b]$ followed by applying $\operatorname{rgyr}[b \oplus c, a]$ to the previous equation gives

$$(\operatorname{rgyr}[b \oplus c, a] \circ \operatorname{rgyr}[c, b])((a \oplus_b b) \oplus_b \operatorname{lgyr}[a, b]c) = a \oplus (b \oplus c). \tag{44}$$

On the other hand, we compute

$$\begin{aligned}
&(a \oplus b) \oplus (\operatorname{lgyr}[a, b] \circ \operatorname{rgyr}[b, a])(c) \\
&\quad = (a \oplus b) \oplus (\operatorname{rgyr}[b, a] \circ \operatorname{lgyr}[a, b])(c) \\
&\quad = [\operatorname{rgyr}[b, a](a \oplus_b b)] \oplus [\operatorname{rgyr}[b, a](\operatorname{lgyr}[a, b]c)] \\
&\quad = \operatorname{rgyr}[b, a]((a \oplus_b b) \oplus \operatorname{lgyr}[a, b]c) \\
&\quad = (\operatorname{rgyr}[b, a] \circ \operatorname{rgyr}[\operatorname{lgyr}[a, b]c, a \oplus_b b])((a \oplus_b b) \oplus_b \operatorname{lgyr}[a, b]c) \\
&\quad = (\operatorname{rgyr}[c, \operatorname{rgyr}[b, a](a \oplus_b b)] \circ \operatorname{rgyr}[b, a])((a \oplus_b b) \oplus_b \operatorname{lgyr}[a, b]c) \\
&\quad = (\operatorname{rgyr}[c, a \oplus b] \circ \operatorname{rgyr}[b, a])((a \oplus_b b) \oplus_b \operatorname{lgyr}[a, b]c).
\end{aligned} \tag{45}$$

We obtain the first equation from Theorem 3.7 (1); the third equation from (40); the fifth equation from Identity (21) and Theorem 4.6.

By the lemma, $\mathrm{rgyr}[b \oplus c, a] \circ \mathrm{rgyr}[c, b] = \mathrm{rgyr}[c, a \oplus b] \circ \mathrm{rgyr}[b, a]$. Hence, (44) and (45) together imply $a \oplus (b \oplus c) = (a \oplus b) \oplus (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a])(c)$. Replacing c by $(\mathrm{lgyr}[b, a] \circ \mathrm{rgyr}[a, b])(c)$ in (42) followed by commuting $\mathrm{lgyr}[b, a]$ and $\mathrm{rgyr}[a, b]$ gives (43). □

Theorem 4.20 (Left gyration reduction property of bi-gyrogroups). *The bi-gyrogroup B has the left gyration left reduction property*

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a \oplus b, b] \tag{46}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[a, b] = \mathrm{lgyr}[a, b \oplus a] \tag{47}$$

for all $a, b \in B$.

Proof. From (29), (19) with $\rho = \mathrm{rgyr}[b, a]$, and Theorem 4.6, we have the following series of equations

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b]\\ &= \mathrm{lgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\mathrm{rgyr}[a, b]b)]\\ &= \mathrm{lgyr}[a \oplus b, b],\end{aligned}$$

thus proving (46). One obtains similarly that

$$\begin{aligned}\mathrm{lgyr}[a, b] &= \mathrm{lgyr}[\mathrm{rgyr}[b, a]a, b \oplus_b a]\\ &= \mathrm{lgyr}[\mathrm{rgyr}[a, b](\mathrm{rgyr}[b, a]a), \mathrm{rgyr}[a, b](b \oplus_b a)]\\ &= \mathrm{lgyr}[a, b \oplus a].\end{aligned}$$ □

Theorem 4.21 (Right gyration reduction property of bi-gyrogroups). *The bi-gyrogroup B satisfies the right gyration left reduction property*

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus b, b] \tag{48}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a, b \oplus a] \tag{49}$$

for all $a, b \in B$.

Proof. From (33), (21) with $\rho = \mathrm{rgyr}[b, a]$, and Theorem 4.6, we have the following series of equations

$$\begin{aligned}\mathrm{id}_B &= \mathrm{rgyr}[b, a] \circ \mathrm{rgyr}[a \oplus_b b, \mathrm{rgyr}[a, b]b]\\ &= \mathrm{rgyr}[\mathrm{rgyr}[b, a](a \oplus_b b), \mathrm{rgyr}[b, a](\mathrm{rgyr}[a, b]b)] \circ \mathrm{rgyr}[b, a]\\ &= \mathrm{rgyr}[a \oplus b, b] \circ \mathrm{rgyr}[b, a].\end{aligned} \tag{50}$$

Hence, the extreme sides of (50) imply $\mathrm{rgyr}[a, b] = \mathrm{rgyr}[a \oplus b, b]$. Applying the bi-gyration inversion law to (48) followed by interchanging a and b gives (49). □

Let $(B, \oplus)$ be the corresponding bi-gyrogroup of a bi-gyrodecomposition $\Gamma = H_L B H_R$. By Theorem 4.17, left and right gyrations of $(B, \oplus_b)$ preserve the bi-gyrogroup operation. This result and Theorem 4.19 motivate the following definition.

Definition 4.22 (Gyration of bi-gyrogroups). Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup. The *gyrator* is the map

$$\mathrm{gyr}\colon B \times B \to \mathrm{Aut}\,(B, \oplus)$$

defined by

$$\mathrm{gyr}[a, b] = \mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a] \tag{51}$$

for all $a, b \in B$.

Theorem 4.23. *For all $a, b \in B$, $\mathrm{gyr}[a, b]$ is an automorphism of the bi-gyrogroup B.*

Proof. The theorem follows from Theorem 4.17. □

Theorem 4.24 (Gyroassociative law in bi-gyrogroups). *The bi-gyrogroup B satisfies the left gyroassociative law*

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c \tag{52}$$

and the right gyroassociative law

$$(a \oplus b) \oplus c = a \oplus (b \oplus \mathrm{gyr}[b, a]c) \tag{53}$$

for all $a, b, c \in B$.

Proof. The theorem follows directly from Theorem 4.19 and Definition 4.22. □

Theorem 4.25 (Gyration reduction property in bi-gyrogroups). *The bi-gyrogroup B has the left reduction property*

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b] \tag{54}$$

and the right reduction property

$$\mathrm{gyr}[a, b] = \mathrm{gyr}[a, b \oplus a] \tag{55}$$

for all $a, b \in B$.

Proof. From (46) and (49), we have the following series of equations

$$\begin{aligned}
\mathrm{gyr}[a \oplus b, b] &= \mathrm{lgyr}[a \oplus b, b] \circ \mathrm{rgyr}[b, a \oplus b] \\
&= \mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a] \\
&= \mathrm{gyr}[a, b],
\end{aligned}$$

thus proving (54). Similarly, (47) and (48) together imply (55). □

Theorems 4.24 and 4.25 indicate that any bi-gyrogroup is indeed a gyrogroup. Therefore, we recall the following definition of a gyrogroup.

Definition 4.26 (Gyrogroup, [29]). A groupoid $(G, \oplus)$ is a *gyrogroup* if its binary operation satisfies the following axioms.

(G1) There is an element $0 \in G$ such that $0 \oplus a = a$ for all $a \in G$.

(G2) For each $a \in G$, there is an element $b \in G$ such that $b \oplus a = 0$.

(G3) For all a, b in G, there is an automorphism $\mathrm{gyr}[a, b] \in \mathrm{Aut}\,(G, \oplus)$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c$$

for all $c \in G$.

(G4) For all a, b in G, $\mathrm{gyr}[a, b] = \mathrm{gyr}[a \oplus b, b]$.

Definition 4.27 (Gyrocommutative gyrogroup, [29]). A gyrogroup $(G, \oplus)$ is *gyrocommutative* if it satisfies the gyrocommutative law

$$a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$$

for all $a, b \in G$.

Theorem 4.28. *Let $\Gamma = H_L B H_R$ be a bi-gyrodecomposition and let $(B, \oplus)$ be the corresponding bi-gyrogroup. Then B equipped with the bi-gyrogroup operation is a gyrogroup.*

Proof. Axioms (G1) and (G2) are validated in Proposition 4.16. Axiom (G3) is validated in Theorems 4.23 and 4.24. Axiom (G4) is validated in Theorem 4.25. □

Definition 4.29. A bi-gyrodecomposition $\Gamma = H_L B H_R$ is *bi-gyrocommutative* if its bi-transversal groupoid is bi-gyrocommutative in the sense of Definition 2.8.

Theorem 4.30. *If $\Gamma = H_L B H_R$ is a bi-gyrocommutative bi-gyrodecomposition, then B equipped with the bi-gyrogroup operation is a gyrocommutative gyrogroup.*

Proof. Let $a, b \in B$. We compute

$$\begin{aligned} a \oplus b &= \mathrm{rgyr}[b, a](a \oplus_b b) \\ &= \mathrm{rgyr}[b, a](\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[a, b](b \oplus_b a)) \\ &= (\mathrm{lgyr}[a, b] \circ \mathrm{rgyr}[b, a])(\mathrm{rgyr}[a, b](b \oplus_b a)) \\ &= \mathrm{gyr}[a, b](b \oplus a), \end{aligned}$$

thus proving that B satisfies the gyrocommutative law. □

We close this section by proving that having a bi-gyrodecomposition is an invariant property of groups.

Theorem 4.31. *Let Γ_1 and Γ_2 be isomorphic groups via an isomorphism ϕ. If $\Gamma_1 = H_L B H_R$ is a bi-gyrodecomposition, then so is $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_r)$.*

Proof. The proof of this theorem is straightforward, using the fact that ϕ is a group isomorphism from Γ_1 to Γ_2. □

Theorem 4.32. *Let Γ_1 and Γ_2 be isomorphic groups via an isomorphism ϕ. If $\Gamma_1 = H_L B H_R$ is a bi-gyrocommutative bi-gyrodecomposition, then so is $\Gamma_2 = \phi(H_L)\phi(B)\phi(H_r)$.*

Proof. This theorem follows from the fact that

$$\begin{aligned} \mathrm{rgyr}[\phi(b_1), \phi(b_2)]\phi(b) &= \phi(\mathrm{rgyr}[b_1, b_2]b) \\ \mathrm{lgyr}[\phi(b_1), \phi(b_2)]\phi(b) &= \phi(\mathrm{lgyr}[b_1, b_2]b) \end{aligned}$$

for all $b_1, b_2 \in B$. □

Theorem 4.33. *Let Γ_1 and Γ_2 be isomorphic groups via an isomorphism ϕ and let $\Gamma_1 = H_L B H_R$ be a bi-gyrodecomposition. Then the bi-gyrogroups B and $\phi(B)$ are isomorphic as gyrogroups via ϕ.*

Proof. By Theorem 4.28, B forms a gyrogroup whose gyrogroup operation is given by $a \oplus b = \mathrm{rgyr}[b, a](a \odot_1 b)$ for all $a, b \in B$, and $\phi(B)$ forms a gyrogroup whose gyrogroup operation is given by $c \oplus d = \mathrm{rgyr}[d, c](c \odot_2 d)$ for all $c, d \in \phi(B)$. Let $a, b \in B$. We compute

$$\begin{aligned} \phi(a \oplus b) &= \phi(\mathrm{rgyr}[b, a](a \odot_1 b)) \\ &= \mathrm{rgyr}[\phi(b), \phi(a)]\phi(a \odot_1 b) \\ &= \mathrm{rgyr}[\phi(b), \phi(a)](\phi(a) \odot_2 \phi(b)) \\ &= \phi(a) \oplus \phi(b). \end{aligned}$$

Hence, the restriction of ϕ to B acts as a gyrogroup isomorphism from B to $\phi(B)$. □

5. Special Pseudo-Orthogonal Groups

In this section, we provide a concrete realization of a bi-gyrocommutative bi-gyrodecomposition.

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature $(m, n), m, n \in \mathbb{N}$, is an $(m + n)$-dimensional linear space with the pseudo-Euclidean inner product of signature (m, n). The *special pseudo-orthogonal group*, denoted by $\mathrm{SO}(m, n)$, consists of all the Lorentz transformations of order (m, n) that leave the pseudo-Euclidean inner product invariant and that can be reached continuously from the identity transformation in $\mathbb{R}^{m,n}$. Denote by $\mathrm{SO}(m)$ the group of $m \times m$ special orthogonal matrices and by $\mathrm{SO}(n)$ the group of $n \times n$ special orthogonal matrices.

Following [34], $\mathrm{SO}(m)$ and $\mathrm{SO}(n)$ can be embedded into $\mathrm{SO}(m,n)$ as subgroups by defining

$$\rho\colon O_m \mapsto \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix}, \quad O_m \in \mathrm{SO}(m), \tag{56}$$

$$\lambda\colon O_n \mapsto \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix}, \quad O_n \in \mathrm{SO}(n). \tag{57}$$

Let β be the map defined on the space $\mathbb{R}^{n\times m}$ of all $n\times m$ real matrices by

$$\beta\colon P \mapsto \begin{pmatrix} \sqrt{I_m + P^{\mathrm{t}}P} & P^{\mathrm{t}} \\ P & \sqrt{I_n + PP^{\mathrm{t}}} \end{pmatrix}, \quad P \in \mathbb{R}^{n\times m}. \tag{58}$$

It is easy to see that β is a bijection from $\mathbb{R}^{n\times m}$ to $\beta(\mathbb{R}^{n\times m})$.

Note that

$$\rho(\mathrm{SO}(m)) = \left\{ \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} : O_m \in \mathrm{SO}(m) \right\}$$

$$\lambda(\mathrm{SO}(n)) = \left\{ \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} : O_n \in \mathrm{SO}(n) \right\}$$

$$\beta(\mathbb{R}^{n\times m}) = \left\{ \begin{pmatrix} \sqrt{I_m + P^{\mathrm{t}}P} & P^{\mathrm{t}} \\ P & \sqrt{I_n + PP^{\mathrm{t}}} \end{pmatrix} : P \in \mathbb{R}^{n\times m} \right\}.$$

It follows from Examples 22 and 23 of [34] that $\lambda(\mathrm{SO}(n))$ and $\rho(\mathrm{SO}(m))$ are subgroups of $\mathrm{SO}(m,n)$. Further, $\mathrm{SO}(m)$ and $\rho(\mathrm{SO}(m))$ are isomorphic as groups via ρ, and $\mathrm{SO}(n)$ and $\lambda(\mathrm{SO}(n))$ are isomorphic as groups via λ.

We will see shortly that

$$\mathrm{SO}(m,n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n))$$

is a bi-gyrocommutative bi-gyrodecomposition.

By Theorem 8 of [34], $\beta(\mathbb{R}^{n\times m})$ is a bi-transversal of subgroups $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ in the pseudo-orthogonal group $\mathrm{SO}(m,n)$. From Lemma 6 of [34], we have

$$\begin{aligned} \rho(O_m)\beta(P)\rho(O_m)^{-1} &= \beta(PO_m^{-1}) \\ \lambda(O_n)\beta(P)\lambda(O_n)^{-1} &= \beta(O_nP) \end{aligned}$$

for all $O_m \in \mathrm{SO}(m)$, $O_n \in \mathrm{SO}(n)$, and $P \in \mathbb{R}^{n\times m}$. Hence, $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ normalize $\beta(\mathbb{R}^{n\times m})$. Setting $P = 0_{n,m}$ in the third identity of (77) of [34], we have

$$\lambda(O_n)\rho(O_m) = \rho(O_m)\lambda(O_n)$$

for all $O_m \in \mathrm{SO}(m), O_n \in \mathrm{SO}(n)$ because $\beta(P) = \beta(0_{n,m}) = I_{m+n}$. Thus, $\beta(\mathbb{R}^{n\times m})$ is a bi-gyrotransversal of $\rho(\mathrm{SO}(m))$ and $\lambda(\mathrm{SO}(n))$ in $\mathrm{SO}(m,n)$.

In Theorem 13 of [34], the bi-gyroaddition, $\oplus_U$, and bi-gyrations in the parameter bi-gyrogroupoid $\mathbb{R}^{n\times m}$ are given by

$$\begin{aligned}
P_1 \oplus_U P_2 &= P_1\sqrt{I_m + P_2^{\mathrm{t}}P_2} + \sqrt{I_n + P_1P_1^{\mathrm{t}}}P_2\\
\mathrm{lgyr}[P_1, P_2] &= \sqrt{I_n + P_{1,2}P_{1,2}^{\mathrm{t}}}^{\,-1}\left\{P_1P_2^{\mathrm{t}} + \sqrt{I_n + P_1P_1^{\mathrm{t}}}\sqrt{I_n + P_2P_2^{\mathrm{t}}}\right\}\\
\mathrm{rgyr}[P_1, P_2] &= \left\{P_1^{\mathrm{t}}P_2 + \sqrt{I_m + P_1^{\mathrm{t}}P_1}\sqrt{I_m + P_2^{\mathrm{t}}P_2}\right\}\sqrt{I_m + P_{1,2}^{\mathrm{t}}P_{1,2}}^{\,-1}
\end{aligned}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$ and $P_{1,2} = P_1 \oplus_U P_2$.

From (74) of [34], we have $I_{m+n} = B(0_{n,m}) \in \beta(\mathbb{R}^{n\times m})$. From Theorem 10 of [34], we have $\beta(P)^{-1} = \beta(-P) \in \beta(\mathbb{R}^{n\times m})$ for all $P \in \mathbb{R}^{n\times m}$. From Equations (179) and (184) of [34], we have

$$\beta(P_1)\beta(P_2)\beta(P_1) = \beta((P_1 \oplus_U P_2) \oplus_U \mathrm{lgyr}[P_1, P_2]P_1).$$

Hence, $\beta(P_1)\beta(P_2)\beta(P_1) \in \beta(\mathbb{R}^{n\times m})$ for all $P_1, P_2 \in \mathbb{R}^{n\times m}$. This proves that $\beta(\mathbb{R}^{n\times m})$ is a twisted subgroup of $\mathrm{SO}(m, n)$.

By (104) of [34],

$$\beta(P_1)\beta(P_2) = \rho(\mathrm{rgyr}[P_1, P_2])\beta(P_1 \oplus_U P_2)\lambda(\mathrm{lgyr}[P_1, P_2]) \tag{59}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$. Hence, the left and right transversal maps induced by the decomposition $\mathrm{SO}(m, n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times})\lambda(\mathrm{SO}(n))$ are given by

$$h_\ell(\beta(P_1), \beta(P_2)) = \rho(\mathrm{rgyr}[P_1, P_2]) \tag{60}$$

and

$$h_r(\beta(P_1), \beta(P_2)) = \lambda(\mathrm{lgyr}[P_1, P_2]) \tag{61}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

By (162b) of [34], $\mathrm{rgyr}^{-1}[P_1, P_2] = \mathrm{rgyr}[P_2, P_1]$. Hence,

$$h_\ell(\beta(P_1), \beta(P_2))^{-1} = \rho(\mathrm{rgyr}^{-1}[P_1, P_2]) = \rho(\mathrm{rgyr}[P_2, P_1]) = h_\ell(\beta(P_2), \beta(P_1)).$$

Similarly, (162a) of [34] implies $h_r(\beta(P_1), \beta(P_2))^{-1} = h_r(\beta(P_2), \beta(P_1))$. Combining these results gives

Theorem 5.1. *The decomposition*

$$\mathrm{SO}(m, n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n)) \tag{62}$$

is a bi-gyrodecomposition.

By (59), the bi-transversal operation induced by the decomposition (62) is given by

$$\beta(P_1) \oplus_b \beta(P_2) = \beta(P_1 \oplus_U P_2) \tag{63}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

Note that $\mathrm{rgyr}[P_1, P_2]$ is an $m \times m$ matrix and $\mathrm{lgyr}[P_1, P_2]$ is an $n \times n$ matrix, while $\mathrm{rgyr}[\beta(P_1), \beta(P_2)]$ and $\mathrm{lgyr}[\beta(P_1), \beta(P_2)]$ are maps. By (11), the action of left and right gyrations on $\beta(\mathbb{R}^{n\times n})$ is given by

$$\mathrm{lgyr}[\beta(P_1), \beta(P_2)]\beta(P) = \beta(\mathrm{lgyr}[P_1, P_2]P) \tag{64}$$

and

$$\mathrm{rgyr}[\beta(P_1), \beta(P_2)]\beta(P) = \beta(P\mathrm{rgyr}[P_1, P_2]) \tag{65}$$

for all $P_1, P_2, P \in \mathbb{R}^{n\times m}$. Using (64) and (65), together with Theorem 25 of [34], we have

Theorem 5.2. *The bi-gyrodecomposition*

$$\mathrm{SO}(m, n) = \rho(\mathrm{SO}(m))\beta(\mathbb{R}^{n\times m})\lambda(\mathrm{SO}(n))$$

is bi-gyrocommutative.

By Theorem 52 of [34], the space $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices forms a gyrocommutative gyrogroup under the operation $\oplus'_U$ given by

$$P_1 \oplus'_U P_2 = (P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1], \quad P_1, P_2 \in \mathbb{R}^{n\times m}. \tag{66}$$

Theorem 5.3. *The set*

$$\beta(\mathbb{R}^{n\times m}) = \left\{ \begin{pmatrix} \sqrt{I_m + P^tP} & P^t \\ P & \sqrt{I_n + PP^t} \end{pmatrix} : P \in \mathbb{R}^{n\times m} \right\}$$

together with the bi-gyrogroup operation $\oplus$ given by

$$\beta(P_1) \oplus \beta(P_2) = \beta((P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1])$$

is a gyrocommutative gyrogroup isomorphic to $(\mathbb{R}^{n\times m}, \oplus'_U)$.

Proof. The theorem follows from Theorems 5.1, 5.2, 4.28, and 4.30. Further, the bi-gyrogroup operation $\oplus$ is given by

$$\begin{aligned}
\beta(P_1) \oplus \beta(P_2) &= \mathrm{rgyr}[\beta(P_2), \beta(P_1)](\beta(P_1) \oplus_b \beta(P_2)) \\
&= \mathrm{rgyr}[\beta(P_2), \beta(P_1)]\beta(P_1 \oplus_U P_2) \\
&= \beta((P_1 \oplus_U P_2)\mathrm{rgyr}[P_2, P_1]).
\end{aligned}$$

From (66), we have $\beta(P_1) \oplus \beta(P_2) = \beta(P_1 \oplus'_U P_2)$. Hence, β acts as a gyrogroup isomorphism from $\mathbb{R}^{n\times m}$ to $\beta(\mathbb{R}^{n\times m})$. $\square$

6. Spin Groups

We establish that the spin group of the Clifford algebra of pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m,n) has a bi-gyrocommutative bi-gyrodecomposition. For basic knowledge of Clifford algebras, the reader is referred to [15, 16, 19, 21].

Let (V,B) be a real quadratic space. That is, V is a linear space over $\mathbb{R}$, together with a non-degenerate symmetric bilinear form B. Let Q be the associated quadratic form given by $Q(v) = B(v,v)$ for $v \in V$. Denote by $\mathrm{C}\ell(V,Q)$ the *Clifford algebra* of (V,B). Set

$$\Gamma(V,Q) = \{g \in \mathrm{C}\ell^{\times}(V,Q)\colon \forall v \in V,\, \hat{g}vg^{-1} \in V\}. \tag{67}$$

Here, $\hat{}$ stands for the unique involutive automorphism of $\mathrm{C}\ell(V,Q)$ such that $\hat{v} = -v$ for all $v \in V$, known as the *grade involution*. If V is *finite* dimensional, then $\Gamma(V,Q)$ is indeed a subgroup of the group of units of $\mathrm{C}\ell(V,Q)$, called the *Clifford group of* $\mathrm{C}\ell(V,Q)$. In this case, any element g of $\Gamma(V,Q)$ induces the linear automorphism T_g of V given by

$$T_g(v) = \hat{g}vg^{-1}, \quad v \in V. \tag{68}$$

Since $T_g \circ T_h = T_{gh}$ for all $g,h \in \Gamma(V,Q)$, the map $\pi\colon g \mapsto T_g$ defines a group homomorphism from $\Gamma(V,Q)$ to the general linear group $\mathrm{GL}\,(V)$, known as the *twisted adjoint representation of* $\Gamma(V,Q)$. The kernel of π equals $\mathbb{R}^{\times}1 := \{\lambda 1\colon \lambda \in \mathbb{R}, \lambda \neq 0\}$. By the Cartan-Dieudonné theorem, π maps $\Gamma(V,Q)$ onto the orthogonal group $\mathrm{O}(V,Q)$.

Recall that, in the Clifford algebra $\mathrm{C}\ell(V,Q)$, we have $v^2 = Q(v)$ for all $v \in V$. Hence, if $v \in V$ and $Q(v) \neq 0$, then v is invertible whose inverse is $v/Q(v)$. Further, we have an important identity $uv + vu = 2B(u,v)1$ for all $u,v \in V$. Using this identity, we obtain

$$-vuv^{-1} = u - (uv+vu)v^{-1} = u - (2B(u,v)1)\left(\frac{v}{Q(v)}\right) = u - \frac{2B(u,v)}{Q(v)}v,$$

which implies $\hat{v}uv^{-1} = -vuv^{-1} \in V$ for all $u \in V$. Hence, if $v \in V$ and $Q(v) \neq 0$, then $v \in \Gamma(V,Q)$. In fact, T_v is *the reflection about the hyperplane orthogonal to* v. We also have the following important subgroup of the Clifford group of $\mathrm{C}\ell(V,Q)$:

$$\mathrm{Spin}\,(V,Q) = \{v_1v_2\cdots v_r\colon r \text{ is even}, v_i \in V, \text{ and } Q(v_i) = \pm 1\}, \tag{69}$$

known as the *spin group of* $\mathrm{C}\ell(V,Q)$.

The following theorem is well known in the literature. Its proof can be found, for instance, in Theorem 2.9 of [19].

Theorem 6.1. *The restriction of the twisted adjoint representation to the spin group of* $\mathrm{C}\ell(V,Q)$ *is a surjective group homomorphism from* $\mathrm{Spin}\,(V,Q)$ *to the special orthogonal group* $\mathrm{SO}(V,Q)$ *of* V. *Its kernel is* $\{1,-1\}$.

Corollary 6.2. *The quotient group* $\mathrm{Spin}\,(V,Q)/\{1,-1\}$ *and the special orthogonal group* $\mathrm{SO}(V,Q)$ *are isomorphic.*

As V is a linear space over $\mathbb{R}$, we can choose an ordered basis for V so that

$$Q(v) = v_1^2 + v_2^2 + \cdots + v_m^2 - v_{m+1}^2 - v_{m+2}^2 - \cdots - v_{m+n}^2$$

for all $v = (v_1, \ldots, v_m, v_{m+1}, \ldots, v_{m+n}) \in \mathbb{R}^{m+n}$, [16, Theorem 4.5]. Hence, $\mathrm{SO}(V,Q) \equiv \mathrm{SO}(m,n)$ and $\mathrm{Spin}\,(V,Q) \equiv \mathrm{Spin}\,(m,n)$. Corollary 6.2 implies that

$$\mathrm{Spin}\,(m,n)/\{1,-1\} \cong \mathrm{SO}(m,n). \tag{70}$$

Hence, we have the following theorem.

Theorem 6.3. *The quotient group*

$$\mathrm{Spin}\,(m,n)/\{1,-1\}$$

has a bi-gyrocommutative bi-gyrodecomposition.

Proof. This theorem follows directly from (70) and Theorems 4.32 and 5.2. □

7. Conclusion

A gyrogroup is a non-associative group-like structure in which the non-associativity is controlled by a special family of automorphisms called gyrations. Gyrations, in turn, result from the extension by abstraction of the relativistic effect known as *Thomas precession*. In this paper we generalize the notion of gyrogroups, which involves a single family of gyrations, to that of bi-gyrogroups, which involves two distinct families of gyrations, collectively called bi-gyrations.

The bi-transversal decomposition $\Gamma = H_L B H_R$, studied in Section 3, naturally leads to a groupoid $(B, \odot)$ that comes with two families of automorphisms, left and right ones. This groupoid is related to the bi-gyrogroupoid $(B, \oplus_b)$, studied earlier in Section 2. Bi-gyrogroupoids $(B, \oplus_b)$ form an intermediate structure that suggestively leads to the desired bi-gyrogroup structure $(B, \oplus)$. The bi-transversal operation $\odot$ arises naturally from the bi-transversal decomposition (8). Under the natural conditions of Definition 4.1, the bi-transversal operation $\odot$ becomes the bi-gyrogroupoid operation $\oplus_b$. The latter operation leads to the desired bi-gyrogroup operation $\oplus$ by means of (38).

As we have shown in Section 4, any bi-gyrodecomposition $\Gamma = H_L B H_R$ of a group Γ induces the bi-gyrogroup structure on B, giving rise to a bi-gyrogroup $(B, \oplus)$ along with left gyrations $\mathrm{lgyr}[a,b]$ and right gyrations $\mathrm{rgyr}[a,b]$, $a, b \in B$. Further, in the case where H_L is the trivial subgroup of Γ, the bi-gyrodecomposition reduces to the decomposition $\Gamma = BH$ studied in [14]. The bi-gyrogroup $(B, \oplus)$ induced by a bi-gyrodecomposition of a group is indeed an abstract version of the bi-gyrogroup $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices studied in [34].

Bi-gyrogroups are group-like structures. For instance, they satisfy the bi-gyroassociative law (Theorem 4.19), which descends to the associative law if their left and right gyrations are the identity automorphism. A concrete realization of a bi-gyrogroup is found in the special pseudo-orthogonal group $\mathrm{SO}(m,n)$ of the pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m,n), as shown in [34] and in Section 5. Moreover, bi-gyrogroups arise in the group counterpart of Clifford algebras as we establish in Section 6 that the quotient group $\mathrm{Spin}\,(m,n)/\{1,-1\}$ of the spin group possesses a bi-gyrodecomposition.

By Theorem 4.28, any bi-gyrogroup is a gyrogroup. Yet, in general, the bi-gyrostructure of a bi-gyrogroup is richer than the gyrostructure of a gyrogroup. To see this clearly, we note that gyrations $\mathrm{gyr}[a,b]$ of a gyrogroup $(B,\oplus)$, $a,b\in B$, are completely determined by the gyrogroup operation according to the *gyrator identity* in Theorem 2.10 (10) of [29]:

$$\mathrm{gyr}[a,b]x = \ominus(a\oplus b)\oplus(a\oplus(b\oplus x)) \tag{71}$$

for all a,b,x in the gyrogroup $(B,\oplus)$. In contrast, the *bi-gyrator identity* analogous to (71) is

$$(\mathrm{lgyr}[a,b]\circ\mathrm{rgyr}[b,a])(x) = \ominus(a\oplus b)\oplus(a\oplus(b\oplus x)) \tag{72}$$

for all a,b,x in a bi-gyrogroup $(B,\oplus)$. Here, the bi-gyrogroup operation completely determines the composite automorphism $\mathrm{lgyr}[a,b]\circ\mathrm{rgyr}[b,a]$. However, it does not determine straightforwardly each of the two automorphisms $\mathrm{lgyr}[a,b]$ and $\mathrm{rgyr}[a,b]$. Thus, the presence of two families of gyrations in a bi-gyrogroup, as opposed to the presence of a single family of gyrations in a gyrogroup, significantly enriches the bi-gyrostructure of bi-gyrogroups.

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Normed Gyrolinear Spaces: A Generalization of Normed Spaces Based on Gyrocommutative Gyrogroups

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Abstract

In this paper, we consider a generalization of the real normed spaces and give some examples.

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1. Introduction

A magma $(S, \circ)$ is a set S with a binary operation $\circ : S \times S \to S$, $(a, b) \mapsto a \circ b$ for any $a, b \in S$. An automorphism ϕ of a magma $(S, \circ)$ is a bijection $\phi : S \to S$ which preserves the magma operation, that is $\phi(a \circ b) = \phi(a) \circ \phi(b)$ for any $a, b \in S$. The set of all automorphisms of $(S, \circ)$ is denoted by $\mathrm{Aut}(S, \circ)$. If there exists an element $e \in (S, \circ)$ such that $e \circ a = a \circ e = a$ for any $a \in S$, then e is called the identity of $(S, \circ)$. For $a \in (S, \circ)$, if there exists an element $a' \in (S, \circ)$ such that $a \circ a' = a' \circ a = e$, then a' is called the inverse of a.

A magma $(G, \oplus)$ is called a gyrogroup if it satisfies the following (G1) to (G5).

(G1) $(G, \oplus)$ has the identity e.

(G2) For any $a \in (G, \oplus)$, a has the inverse $\ominus a$.

(G3) For any $a, b, c \in G$, there exists a unique element $\mathrm{gyr}[a, b]c$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a, b]c.$$

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(G4) For any $a, b \in G$, the map $\mathrm{gyr}[a, b] : G \to G$ defined by $c \mapsto \mathrm{gyr}[a, b]c$ for any c is an automorphism of the magma $(G, \oplus)$, that is $\mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)$. The map $\mathrm{gyr}[a, b]$ is called a gyroautomorphism of $(G, \oplus)$ generated by a and b.

(G5) For any $a, b \in G$, $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a, b]$.

A gyrogroup $(G, \oplus)$ is gyrocommutative if the following (G6) is satisfied.

(G6) For any $a, b \in G$, $a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$.

A concrete example of a gyrocommutative gyrogroup is provided by the addition of relativistically admissible velocities in Einstein's special relativity, and another concrete example is provided by the Poincaré disk model of hyperbolic geometry. Certain gyrocommutative gyrogroups admit scalar multiplication, giving rise to gyrovector spaces. The gyrovector spaces are a generalization of the real inner product spaces, where addition is not necessarily a commutative group but a gyrocommutative gyrogroup. Ungar studied gyrogroups and gyrovector spaces in several books [4, 5, 6, 7, 8, 9, 10]).

The author and O. Hatori define in [1] the generalized gyrovector spaces and give a Mazur-Ulam type theorem for generalized gyrovector spaces. The generalized gyrovector spaces is a common generalization of the gyrovector spaces and of the real normed spaces. A typical example of a generalized gyrovector space is the positive cone of a unital C^*-algebra. The definition of the generalized gyrovector spaces is as follows.

Definition 1.1. [1] Let $(G, \oplus)$ be a gyrocommutative gyrogroup with the map $\otimes : \mathbb{R} \times G \to G$. Let ϕ be an injection from G into a real normed space $(\mathbb{V}, \|\cdot\|)$. We say that $(G, \oplus, \otimes, \phi)$ (or $(G, \oplus, \otimes)$ just for a simple notation) is a generalized gyrovector space or a GGV in short if the following conditions (GGV0) to (GGV8) are fulfilled:

(GGV0) $\|\phi(\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a})\| = \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G$;

(GGV1) $1 \otimes \boldsymbol{a} = \boldsymbol{a}$ for every $\boldsymbol{a} \in G$;

(GGV2) $(r_1 + r_2) \otimes \boldsymbol{a} = (r_1 \otimes \boldsymbol{a}) \oplus (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV3) $(r_1 r_2) \otimes \boldsymbol{a} = r_1 \otimes (r_2 \otimes \boldsymbol{a})$ for any $\boldsymbol{a} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGV4) $(\phi(|r| \otimes \boldsymbol{a}))/\|\phi(r \otimes \boldsymbol{a})\| = \phi(\boldsymbol{a})/\|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G \setminus \{\boldsymbol{e}\}, r \in \mathbb{R} \setminus \{0\}$, where $\boldsymbol{e}$ denotes the identity element of the gyrogroup $(G, \oplus)$;

(GGV5) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{a}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{a}$ for any $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a} \in G,\ r \in \mathbb{R}$;

(GGV6) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = id_G$ for any $\boldsymbol{v} \in G,\ r_1, r_2 \in \mathbb{R}$;

(GGVV) $\|\phi(G)\| = \{\pm\|\phi(\boldsymbol{a})\| \in \mathbb{R} : \boldsymbol{a} \in G\}$ is a real one-dimensional vector space with vector addition $\oplus'$ and scalar multiplication $\otimes'$;

(GGV7) $\|\phi(r \otimes \boldsymbol{a})\| = |r| \otimes' \|\phi(\boldsymbol{a})\|$ for any $\boldsymbol{a} \in G$, $r \in \mathbb{R}$;

(GGV8) $\|\phi(\boldsymbol{a} \oplus \boldsymbol{b})\| \le \|\phi(\boldsymbol{a})\| \oplus' \|\phi(\boldsymbol{b})\|$ for any $\boldsymbol{a}, \boldsymbol{b} \in G$.

One may feel that this definition is complicated. In this paper, we give a definition of a generalization of the real normed spaces, which is simpler and more general than the generalized gyrovector spaces. Also, we give some examples of such a space.

2. Definitions and Examples

In the following definition 2.1 we extract the algebraic structures from a gyrovector space (or a generalized gyrovector space). For consistency, we use the term "gyrolinear space" in this paper.

Definition 2.1. Let $(X, \oplus)$ be a gyrocommutative gyrogroup. Let $\otimes$ be a map $\otimes : \mathbb{R} \times X$, $(r, \boldsymbol{x}) \mapsto r \otimes \boldsymbol{x}$. We say that $(X, \oplus, \otimes)$ is a gyrolinear space if it satisfies the following conditions:

(GL1) $1 \otimes \boldsymbol{x} = \boldsymbol{x}$;

(GL2) $(r_1 + r_2) \otimes \boldsymbol{x} = (r_1 \otimes \boldsymbol{x}) \oplus (r_2 \otimes \boldsymbol{x})$;

(GL3) $(r_1 r_2) \otimes \boldsymbol{x} = r_1 \otimes (r_2 \otimes \boldsymbol{x})$;

(GL4) $\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](r \otimes \boldsymbol{x}) = r \otimes \mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}]\boldsymbol{x}$;

(GL5) $\mathrm{gyr}[r_1 \otimes \boldsymbol{v}, r_2 \otimes \boldsymbol{v}] = \mathrm{id}_X$;

for any $r, r_1, r_2 \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{u}, \boldsymbol{v} \in X$.

We consider a generalization of normed spaces in Definition 2.2. For convenience, we use the term "normed gyrolinear space" in this paper.

Definition 2.2. Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let $\|\cdot\|$ be a map $\|\cdot\| : X \to \mathbb{R}_{\ge 0}$, $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|$. Let f be a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\ge 0}$, where $\|X\| = \{\|\boldsymbol{x}\| \in \mathbb{R}_{\ge 0}; \boldsymbol{x} \in X\}$. We say that $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space if it satisfies the following conditions:

(NG1) $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$;

(NG2) $f(\|\boldsymbol{x} \oplus \boldsymbol{y}\|) \le f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|)$;

(NG3) $f(\|r \otimes \boldsymbol{x}\|) = |r| f(\|\boldsymbol{x}\|)$;

(NG4) $\|\mathrm{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$;

for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

Lemma 2.3. *Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let $\|\cdot\|$ be a map $\|\cdot\| : X \to \mathbb{R}_{\geq 0}$, $\boldsymbol{x} \mapsto \|\boldsymbol{x}\|$. Put $\|X\| = \{\|\boldsymbol{x}\| \in \mathbb{R}_{\geq 0}; \boldsymbol{x} \in X\}$ and $\pm\|X\| = \{\pm\|\boldsymbol{x}\| \in \mathbb{R}; \boldsymbol{x} \in X\}$. Then the following three properties are equivalent.*

(a1) *There is a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\geq 0}$ which satisfies the conditions* (NG1) *to* (NG4) *for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.*

(a2) *There is a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $f(0) = 0$, which satisfies the conditions* (NG1) *to* (NG4) *for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.*

(a3) *There is a one dimensional real linear space $(\pm\|X\|, \oplus', \otimes')$ with addition $\oplus'$ and scalar multiplication $\otimes'$, which satisfies the following conditions:*

(R1) $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$;

(R2) $\|\boldsymbol{x} \oplus \boldsymbol{y}\| \leq \|\boldsymbol{x}\| \oplus' \|\boldsymbol{y}\|$;

(R3) $\|r \otimes \boldsymbol{x}\| = |r| \otimes' \|\boldsymbol{x}\|$;

(R4) $\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$;

for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

Proof. (a1) $\Rightarrow$ (a2) : Let f be a strictly monotone increasing bijection $f : \|X\| \to \mathbb{R}_{\geq 0}$ which satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$. Note that $f(0) = 0$. Define the map $\tilde{f} : \pm\|X\| \to \mathbb{R}$ by

$$\tilde{f}(a) = \begin{cases} f(a) & (a \in \|X\|) \\ -f(-a) & (-a \in \|X\|) \end{cases}$$

then $\tilde{f}$ is a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $\tilde{f}(0) = 0$. It is trivial that $\tilde{f}$ satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$.

(a2) $\Rightarrow$ (a3) : Let $\tilde{f}$ be a strictly monotone increasing bijection $\tilde{f} : \pm\|X\| \to \mathbb{R}$ with $\tilde{f}(0) = 0$, which satisfies the conditions (NG1) to (NG4) for any $r \in \mathbb{R}$ and $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{u}, \boldsymbol{v} \in X$. Define the two operations $\oplus'_f : \pm\|X\| \times \pm\|X\| \to \pm\|X\|$ and $\otimes'_f : \mathbb{R} \times \pm\|X\| \to \pm\|X\|$ by

$$\begin{aligned} a \oplus'_f b &= f^{-1}(f(a) + f(b)), \\ r \otimes'_f a &= f^{-1}(rf(a)) \end{aligned}$$

for any $a, b \in \pm\|X\|$ and $r \in \mathbb{R}$. Then $(\pm\|X\|, \oplus', \otimes')$ is a one dimensional real linear space. It is easy to check $(\pm\|X\|, \oplus', \otimes')$ satisfies the conditions (R1) to (R4).

(a3) $\Rightarrow$ (a1) : Let $(\pm\|X\|, \oplus', \otimes')$ be a one dimensional real linear space which satisfies the conditions (R1) to (R4). Note that 0 is the origin of the linear space $\pm\|X\|$, since $0 \otimes' \|\boldsymbol{x}\| = \|0 \otimes \boldsymbol{x}\| = \|\boldsymbol{e}\| = 0$. Since $(\pm\|X\|, \oplus', \otimes')$ is isomorphic to

$\mathbb{R}$, usual real line, there is an isomorphism $g : \pm\|X\| \to \mathbb{R}$. Since 0 is the origin of $\pm\|X\|$, we have $g(0) = 0$. Note that $-g$ is also an isomorphism from $\pm\|X\|$ to $\mathbb{R}$. Let $\boldsymbol{x}_0 \in X \setminus \{\boldsymbol{e}\}$. We can assume that $a_0 = g(\|\boldsymbol{x}_0\|) > 0$.

First, we prove that $g(\|y\|) > 0$ for any $\boldsymbol{y} \in X \setminus \{\boldsymbol{e}\}$. Assume that there is $\boldsymbol{y} \in X$ such that $g(\|\boldsymbol{y}\|) < 0$. Put $A = \{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}$ and $B = \{\|r \otimes \boldsymbol{y}\|; r \in \mathbb{R}\}$. Clearly, $A \cup B \subset \|X\|$. Since $g(\|r \otimes \boldsymbol{x}_0\|) = g(|r| \otimes' \|\boldsymbol{x}_0\|) = |r|g(\|\boldsymbol{x}_0\|)$, we have $g(A) = \mathbb{R}_{\geq 0}$. Similarly, since $g(\|r \otimes \boldsymbol{y}\|) = g(|r| \otimes' \|\boldsymbol{y}\|) = |r|g(\|\boldsymbol{y}\|)$, we have $g(B) = \mathbb{R}_{\leq 0}$. Thus we have $g(\|X\|) \supset g(A \cup B) = \mathbb{R}$. However, g is a bijection from $\pm\|X\|$ to $\mathbb{R}$, and $\|X\|$ is a proper subset of $\pm\|X\|$. It is a contradiction. So, we have $g(\|\boldsymbol{y}\|) > 0$ for any $\boldsymbol{y} \in X \setminus \{\boldsymbol{e}\}$.

Since g is a bijecton, $g(\{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}) = \mathbb{R}_{\geq 0}$ and $g(\boldsymbol{y}) \in \mathbb{R}_{\geq 0}$ for any $\boldsymbol{y} \in X$, we have $\|X\| = \{\|r \otimes \boldsymbol{x}_0\|; r \in \mathbb{R}\}$. Put $f = g|_{\|X\|}$ then f is a bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$.

Next, we prove that f is a strictly monotone increasing function. For $\boldsymbol{x} \in X$ and $0 \leq \alpha \leq \beta$, we have

$$\begin{aligned}
\alpha \otimes \|\boldsymbol{x}\| &= \|\alpha \otimes \boldsymbol{x}\| \\
&= \left\| \left(\frac{\beta+\alpha}{2} - \frac{\beta-\alpha}{2} \right) \otimes \boldsymbol{x} \right\| \\
&= \left\| \left(\frac{\beta+\alpha}{2} \right) \otimes \boldsymbol{x} \oplus \left(-\frac{\beta-\alpha}{2} \right) \otimes \boldsymbol{x} \right\| \\
&\leq \left(\frac{\beta+\alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \oplus' \left(\frac{\beta-\alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \\
&= \left(\frac{\beta+\alpha}{2} + \frac{\beta-\alpha}{2} \right) \otimes' \|\boldsymbol{x}\| \\
&= \beta \otimes' \|\boldsymbol{x}\|.
\end{aligned}$$

Therefore, we have

$$0 < \alpha < \beta \iff 0 < \alpha \otimes' \|\boldsymbol{x}\| < \beta \otimes' \|\boldsymbol{x}\| \tag{1}$$

for any $\boldsymbol{x} \in X \setminus \{\boldsymbol{e}\}$. Let $a, b \in \|X\|$ and let $\alpha = f(a)/f(\|\boldsymbol{x}_0\|)$, $\beta = f(b)/f(\|\boldsymbol{x}_0\|)$. Then we have

$$\alpha \otimes' \|\boldsymbol{x}_0\| = f^{-1}(\alpha f(\|\boldsymbol{x}_0\|)) = a.$$

Similarly, we have $\beta \otimes' \|\boldsymbol{x}_0\| = b$. Clearly, $\alpha, \beta \geq 0$ and hence

$$0 < \alpha < \beta \iff 0 < a < b$$

as (1). By the definition of α and β, it is trivial that $0 < f(a) < f(b) \iff 0 < \alpha < \beta$. Thus we have,

$$0 < a < b \iff 0 < f(a) < f(b),$$

f is a strictly monotone increasing function.

Recall that

$$f(\|\boldsymbol{x}\| \oplus' \|\boldsymbol{y}\|) = f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|)$$

and

$$f(r \otimes' \|\boldsymbol{x}\|) = rf(\|\boldsymbol{x}\|)$$

for any $\boldsymbol{x}, \boldsymbol{y} \in X$ and $r \in \mathbb{R}$ as f is a restriction of g. Since f is a strictly monotone increasing function, $(\pm\|X\|, \oplus', \otimes')$ satisfies the conditions (R1) to (R4), it is clear that f satisfies the conditions (NG1) to (NG4). $\square$

In the sequel, for a normed gyrolinear space $(X, \oplus, \otimes, \|\cdot\|, f)$, $\tilde{f}$ denotes the function $\tilde{f} : \pm\|X\| \to \mathbb{R}$ which is defined by

$$\tilde{f}(a) = \begin{cases} f(a) & (a \in \|X\|) \\ -f(-a) & (-a \in \|X\|) \end{cases}.$$

Moreover, $(\pm\|X\|, \oplus'_f, \otimes'_f)$ denotes the one dimensional real vector space which is defined by

$$\begin{aligned} a \oplus'_f b &= \tilde{f}^{-1}(\tilde{f}(a) + \tilde{f}(b)), \\ r \otimes'_f a &= \tilde{f}^{-1}(r\tilde{f}(a)) \end{aligned}$$

for any $a, b \in \pm\|X\|$ and $r \in \mathbb{R}$. The following proposition 2.4 is an immediate consequence of Lemma 2.3. The proposition is followed by examples 2.5, 2.6, 2.7 and 2.8.

Proposition 2.4. *Let $(G, \oplus, \otimes, \phi)$ be a GGV with $\phi : G \to (\mathbb{V}, \|\cdot\|)$. Then there is a bijection $f : \|\phi(G)\| \to \mathbb{R}$ which satisfies $\oplus'_f = \oplus'$ and $\otimes'_f = \otimes'$ as Proposition 2.3. We have $(G, \oplus, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = \|\phi(\cdot)\|$. Note that, if $(G, \oplus, \otimes)$ is a gyrovector space, then G is a subset of $\mathbb{V}$ and ϕ is the identity map. Hence $\|\cdot\|' = \|\cdot\|$.*

Example 2.5. A normed vector space $(\mathbb{V}, \|\cdot\|)$ is a normed gyrolinear space $(\mathbb{V}, +, \times, \|\cdot\|, \mathrm{id})$, where $+$ is the vector addition of $\mathbb{V}$, $\times$ is the scalar multiplication of $\mathbb{V}$ and id is the identity map $\mathrm{id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

The admissible velocities in special relativity is a gyrovector space (cf. [6]). Following Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.6. The Einstein gyrovector space is a normed gyrolinear space $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|, \tanh^{-1} \frac{\cdot}{c})$. Note that c is a speed of light in vacuum, $\|\cdot\|$ is the Euclidean norm of $\mathbb{R}^3$ and $\mathbb{R}^3_c = \{\boldsymbol{u} \in \mathbb{R}^3; \|\boldsymbol{u}\| < c\}$. The Einstein gyrogroup addition $\oplus_E$ is given by

$$\boldsymbol{u} \oplus_E \boldsymbol{v} = \frac{1}{1 + \frac{\langle \boldsymbol{u}, \boldsymbol{v} \rangle}{c^2}} \left\{ \boldsymbol{u} + \frac{1}{\gamma_u} \boldsymbol{v} + \frac{1}{c^2} \frac{\gamma_u}{1 + \gamma_u} \langle \boldsymbol{u}, \boldsymbol{v} \rangle \boldsymbol{u} \right\}, \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \mathbb{R}^3_c,$$

where $\langle\cdot,\cdot,\rangle$ is the Euclidean inner product of $\mathbb{R}^3$ and γ_u is a Lorenz factor of $\boldsymbol{u}$,

$$\gamma_u = (1 - \|\boldsymbol{u}\|^2/c^2)^{-\frac{1}{2}}.$$

The Einstein scalar multiplication $\otimes_E$ is given by

$$r \otimes_E \boldsymbol{u} = \begin{cases} c\tanh(r\tanh^{-1}\frac{\|\boldsymbol{u}\|}{c})\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|} & (\boldsymbol{u} \in \mathbb{R}^3_c \setminus \{\boldsymbol{0}\}) \\ 0 & (\boldsymbol{u} = \boldsymbol{0}) \end{cases}$$

for any $r \in \mathbb{R}$.

The Poincaré disk model is an example of a gyrovector space, and it is called the Möbius gyrovector space (cf. [6]). Following Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.7. The Möbius gyrovector space is a normed gyrolinear space $(\mathbb{D}, \oplus_M, \otimes_M, |\cdot|, \tanh^{-1})$. Note that $\mathbb{D}$ is the open unit disc of complex plane $\mathbb{C}$. The Möbius gyrogroup addition is given by

$$a \oplus_M b = \frac{a+b}{1+\bar{a}b}, \qquad \forall a, b \in \mathbb{D}.$$

The Möbius scalar multiplication $\otimes_M$ is given by

$$r \otimes_E \boldsymbol{u} = \begin{cases} \tanh(r\tanh^{-1}|a|)\frac{a}{|a|} & (a \in \mathbb{D} \setminus \{0\}) \\ 0 & (a = 0) \end{cases}$$

for any $r \in \mathbb{R}$.

The positive cone of a unital C^*-algebra is a exapmle of a generalized gyrovector space (cf [1]). As Proposition 2.4, it is an example of a normed gyrolinear space.

Example 2.8. Let $\mathscr{A}$ be a unital C^*-algebra with the norm $\|\cdot\|$ and $\mathscr{A}_+^{-1}$ be the positive cone of $\mathscr{A}$. Define the binary operation $\oplus_A$ on $\mathscr{A}_+^{-1}$ by

$$a \oplus_A b = a^{\frac{1}{2}} b a^{\frac{1}{2}}, \quad a, b \in \mathscr{A}_+^{-1}.$$

Define the scalar multiplication $\otimes_A : \mathbb{R} \times \mathscr{A}_+^{-1} \to \mathscr{A}_+^{-1}$ by

$$r \otimes_A a = a^r, \quad r \in \mathbb{R}, a \in \mathscr{A}_+^{-1}$$

and the norm $\|\cdot\|' = \|\log\cdot\|$. Then $(\mathscr{A}_+^{-1}, \oplus_A, \otimes_A, \|\cdot\|', \mathrm{id})$ is a normed gyrolinear space, where id is the identity map $\mathrm{id} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$.

The density matrices is an example of a gyrolinear space.

Example 2.9. A qubit density matrix is a 2×2 positive semidefinite Hermitian matrix with trace 1. Let $\boldsymbol{D}$ be the set of all invertible qubit density matrices. Define a binary operation on $\boldsymbol{D}$ by

$$A \oplus B = \frac{A^{\frac{1}{2}} B A^{\frac{1}{2}}}{\mathrm{Tr}(A^{\frac{1}{2}} B A^{\frac{1}{2}})},$$

then $(\boldsymbol{D}, \oplus)$ is a gyrocommutative gyrogroup ([2]). The identity of $(\boldsymbol{D}, \oplus)$ is $\dfrac{1}{2}E$, where E is the identity matrix. The inverse of $A \in \boldsymbol{D}$ is $\ominus A = \dfrac{A^{-1}}{\mathrm{Tr}A^{-1}}$. The gyroautomorphism $\mathrm{gyr}[A, B]$ is given by $\mathrm{gyr}[A, B]C = \dfrac{XCX^*}{\mathrm{Tr}(XCX^*)}$ for any $C \in \boldsymbol{D}$, where $X = X(A, B)$ is a unitary matrix given by $X = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{-\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}$. Define the map $\otimes \mathbb{R} \times \boldsymbol{D} \to \boldsymbol{D}$ by

$$r \otimes A = \frac{A^r}{\mathrm{Tr}A^r}$$

then $(\boldsymbol{D}, \oplus, \otimes)$ is a gyrolinear space. Actually, $(\boldsymbol{D}, \oplus, \otimes)$ satisfies the conditions (GL1) to (GL5) as follows.

(GL1): $1 \otimes A = \frac{A^1}{\mathrm{Tr}A^1} = A$, since $\mathrm{Tr}A = 1$.

(GL2): We have

$$\begin{aligned}(r \otimes A) \oplus (s \otimes A) &= \frac{(\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}} (\frac{A^s}{\mathrm{Tr}A^s}) (\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}}}{\mathrm{Tr}((\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}} (\frac{A^s}{\mathrm{Tr}A^s}) (\frac{A^r}{\mathrm{Tr}A^r})^{\frac{1}{2}})} \\ &= \frac{A^{r+s}}{\mathrm{Tr}A^{r+s}} = (r+s) \otimes A.\end{aligned}$$

(GL3): We have

$$r \otimes (s \otimes A) = \frac{(\frac{A^s}{\mathrm{Tr}A^s})^r}{\mathrm{Tr}(\frac{A^s}{\mathrm{Tr}A^s})^r} = \frac{A^{rs}}{\mathrm{Tr}A^{rs}} = (rs) \otimes A.$$

(GL4): Put $X = (A^{\frac{1}{2}} B A^{\frac{1}{2}})^{-\frac{1}{2}} A^{\frac{1}{2}} B^{\frac{1}{2}}$, then

$$\begin{aligned}\mathrm{gyr}[A, B](r \otimes C) &= \frac{X \frac{C^r}{\mathrm{Tr}C^r} X^*}{\mathrm{Tr}(X \frac{C^r}{\mathrm{Tr}C^r} X^*)} \\ &= \frac{XC^rX^*}{\mathrm{Tr}XC^rX^*}\end{aligned}$$

Since X is unitary, we have $XC^rX^* = (XCX^*)^r$ and hence

$$\mathrm{gyr}[A, B](r \otimes C) = \frac{XC^rX^*}{\mathrm{Tr}XC^rX^*} = \frac{(XCX^*)^r}{\mathrm{Tr}(XCX^*)^r} = r \otimes \mathrm{gyr}[A, B]C.$$

(GL5): Put $X = ((r \otimes A)^{\frac{1}{2}} (s \otimes A)(r \otimes A)^{\frac{1}{2}})^{-\frac{1}{2}} (r \otimes A)^{\frac{1}{2}} (s \otimes A)^{\frac{1}{2}}$. Then

$$X = \left\{ \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \left(\frac{A^s}{\mathrm{Tr}A^s} \right) \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \right\}^{-\frac{1}{2}} \left(\frac{A^r}{\mathrm{Tr}A^r} \right)^{\frac{1}{2}} \left(\frac{A^s}{\mathrm{Tr}A^s} \right)^{\frac{1}{2}} = E$$

and hence $\mathrm{gyr}[A,B]C = \dfrac{ECE^*}{\mathrm{Tr}(ECE^*)} = C$ for any $C \in \boldsymbol{D}$.

3. Constructing Normed Gyrolinear Spaces

In this section we construct new normed gyrolinear spaces from given normed gyrolinear spaces.

Proofs of the following Lemma 3.1 and 3.2 are elementary, easy and omitted.

Lemma 3.1. *Let $(G_1, \oplus_1)$ and $(G_2, \oplus_2)$ be (gyrocommutative) gyrogroups. Define the binary operation $\oplus$ on $G = G_1 \times G_2$ by*

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)$$

for any $(x_1, x_2), (y_1, y_2) \in G_1 \times G_2$. Then $(G, \oplus)$ is a (gyrocommutative) gyrogroup. The identity of $(G, \oplus)$ is (e_1, e_2), where e_i is the identity of $(G_i, \oplus_i)$ $(i = 1, 2)$. The inverse of $x = (x_1, x_2) \in G$ is $\ominus x = (\ominus_1 x_1, \ominus_2 x_2)$. The gyroautomorphisms are

$$\mathrm{gyr}[(x_1, x_2), (y_1, y_2)](a_1, a_2) = (\mathrm{gyr}[x_1, y_1]a_1, \mathrm{gyr}[x_2, y_2]a_2)$$

for any $(x_1, x_2), (y_1, y_2), (a_1, a_2) \in G$.

Lemma 3.2. *Let $(X_1, \oplus_1, \otimes_1)$ and $(X_2, \oplus_2, \otimes_2)$ be gyrolinear spaces. Define the binary operation $\oplus$ on $X = X_1 \times X_2$ by*

$$(x_1, x_2) \oplus (y_1, y_2) = (x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)$$

for any $(x_1, x_2), (y_1, y_2) \in G$. Define the scalar multiplication $\otimes$ on G by

$$r \otimes (x_1, x_2) = (r \otimes_1 x_1, r \otimes_2 x_2)$$

for any $r \in \mathbb{R}$ and $(x_1, x_2), (y_1, y_2) \in G$. Then $(G, \oplus, \otimes)$ is a gyrolinear space.

Proposition 3.3. *Let $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ and $(X_2, \oplus_2, \otimes_2, \|\cdot\|_2, f)$ be normed gyrolinear spaces. Then $X = X_1 \times X_2$ is a gyrolinear space with $\oplus$ and $\otimes$ as in Lemma 3.2. Put*

$$\|(x_1, x_2)\| = f^{-1}(f(\|x_1\|_1) + f(\|x_2\|_2))$$

for any $(x_1, x_2) \in X$. Then $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space.

Proof. Since $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ is a normed gyrolinear space, f is a bijection from $\|X_1\|_1$ to $\mathbb{R}$. Similarly, f is also a bijection from $\|X_2\|_2$ to $\mathbb{R}$. It means that $\|X_1\|_1 = \|X_2\|_2$. Since

$$f^{-1}(f(a) + f(b)) \in \|X_1\|_1 = \|X_2\|_2$$

for any $a, b \in \|X_1\|_1 = \|X_2\|_2$, we have $\|X\| = \|X_1\|_1 = \|X_2\|_2$. Thus f is a monotone increasing bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$.

(NG1): Let e_i be the identity of X_i for $i = 1, 2$, then $\boldsymbol{e} = (e_1, e_2)$ is the identity of G. Since $f(0) = 0$,

$$\begin{aligned}
\|(x_1, x_2)\| = 0 \quad &\Longleftrightarrow \quad f^{-1}(f(\|x_1\|_1) + f(\|x_2\|_2)) \\
&\Longleftrightarrow \quad f(\|x_1\|_1) + f(\|x_2\|_2) \\
&\Longleftrightarrow \quad f(\|x_1\|_1) = 0 \text{ and } f(\|x_2\|_2) = 0 \\
&\Longleftrightarrow \quad \|x_1\|_1 = 0 \text{ and } \|x_2\|_2 = 0 \\
&\Longleftrightarrow \quad x_1 = e_1,\ x_2 = e_2.
\end{aligned}$$

It follows that $\|\boldsymbol{x}\| = 0 \iff \boldsymbol{x} = \boldsymbol{e}$.

(NG2): Let $\boldsymbol{x} = (x_1, x_2), \boldsymbol{y} = (y_1, y_2) \in X$. We have

$$\begin{aligned}
f(\|\boldsymbol{x} \oplus \boldsymbol{y}\|) &= f(\|(x_1 \oplus_1 y_1, x_2 \oplus_2 y_2)\|) \\
&= f(\|x_1 \oplus_1 y_1\|_1) + f(\|x_2 \oplus_2 y_2\|_2) \\
&\leq f(\|x_1\|_1) + f(\|y_1\|_1) + f(\|x_2\|_2) + f(\|y_2\|_2) \\
&= f(\|(x_1, x_2)\|) + f(\|(y_1, y_2)\|) \\
&= f(\|\boldsymbol{x}\|) + f(\|\boldsymbol{y}\|).
\end{aligned}$$

(NG3): We have

$$\begin{aligned}
f(\|\alpha \otimes (x_1, x_2)\|) &= f(\|(\alpha \otimes_1 x_1, \alpha \otimes_2 x_2)\|) \\
&= f(\|\alpha \otimes_1 x_1\|) + f(\|\alpha \otimes_2 x_2\|) \\
&= |\alpha| f(\|x_1\|_1) + |\alpha| f(\|x_2\|_2) \\
&= |\alpha| (f(\|x_1\|_1) + f(\|x_2\|_2)) \\
&= |\alpha| f(\|(x_1, x_2)\|)
\end{aligned}$$

for any $(x_1, x_2) \in X$.

(NG4): Let $\boldsymbol{x} = (x_1, x_2), \boldsymbol{u} = (u_1, u_2), \boldsymbol{v} = (v_1, v_2) \in X$. We have

$$\begin{aligned}
f(\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\|) &= f(\|(\operatorname{gyr}[u_1, v_1](x_1), \operatorname{gyr}[u_2, v_2](x_2))\|) \\
&= f(\|(\operatorname{gyr}[u_1, v_1](x_1)\|_1) + f(\|\operatorname{gyr}[u_2, v_2](x_2))\|_2) \\
&= f(\|x_1\|_1) + f(\|x_2\|_2) \\
&= f(\|\boldsymbol{x}\|).
\end{aligned}$$

Thus $\|\operatorname{gyr}[\boldsymbol{u}, \boldsymbol{v}](\boldsymbol{x})\| = \|\boldsymbol{x}\|$. $\square$

Proposition 3.4. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let h be a strictly monotone increasing injection (not necessarily bijection) $h : \|G\| \to \mathbb{R}_{\geq 0}$ with $h(0) = 0$. Put $\|\cdot\|' = h(\|\cdot\|)$. Then $(X, \oplus, \otimes, \|\cdot\|', fh^{-1})$ is a normed gyrolinear space.*

Proof. Note that h is a bijection from $\|X\|$ to $h(\|X\|) = \|X\|'$. Since f is a bijection from $\|X\|$ to $\mathbb{R}_{\geq 0}$, we have fh^{-1} is a bijection from $\|X\|'$ to $\mathbb{R}_{\geq 0}$. Moreover, fh^{-1} is also a strictly monotone increasing function as f and h are strictly monotone increasing.

(NG1): $\|x\|' = 0 \iff h(\|x\|) = 0 \iff \|x\| = 0 \iff x = e$.

(NG2): For any $x, y \in X$, we have

$$\begin{aligned} fh^{-1}(\|x \oplus y\|') &= f(\|x \oplus y\|) \\ &\leq f(\|x\|) + f(\|y\|) = fh^{-1}(\|x\|') + fh^{-1}(\|y\|'). \end{aligned}$$

(NG3): $fh^{-1}(\|\alpha \otimes x\|') = f(\|\alpha \otimes x\|) = |\alpha| f(\|x\|) = |\alpha| fh^{-1}(\|x\|)$.

(NG4): $\|\operatorname{gyr}[u, v]x\|' = h(\|\operatorname{gyr}[u, v]x\|) = h(\|x\|) = \|x\|'$. □

Example 3.5. The Einstein gyrovector space $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|, \tanh^{-1}\frac{\cdot}{c})$ is a normed gyrolinear space. Put $\|\cdot\|' = \tanh^{-1}\frac{\|\cdot\|}{c}$, then $(\mathbb{R}^3_c, \oplus_E, \otimes_E, \|\cdot\|', id)$ is also a normed gyrolinear space as Proposition 3.4.

Example 3.6. The Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M, |\cdot|, \tanh^{-1})$ is a normed gyrolinear space. Put $\|\cdot\| = \tanh^{-1}|\cdot|$, then $(\mathbb{D}, \oplus_M, \otimes_M, \|\cdot\|', id)$ is also a normed gyrolinear space as Proposition 3.4.

Example 3.7. Let $1 \leq p \leq \infty$ and $L_p(X)$ be the L_p space on a measure space X with a measure μ. The L_p-norm $\|\cdot\|_p$ is given by

$$\|f\|_p = \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}.$$

Since $(L_p, \|\cdot\|_p)$ is a normed space, $(L_p, +, \times, \|\cdot\|_p, id)$ is a normed gyrolinear space. By Proposition 3.4, $(L_p, +, \times, \|\cdot\|_p^p, k)$ is also a normed gyrolinear space, where $k(x) = x^{\frac{1}{p}}$.

Proposition 3.8. *Let $(X_1, \oplus_1, \otimes_1, \|\cdot\|_1, f)$ and $(X_2, \oplus_2, \otimes_2, \|\cdot\|_2, g)$ be normed gyrolinear spaces. Then $X = X_1 \times X_2$ is a gyrolinear space with $\oplus$ and $\otimes$ as in Lemma 3.2. Let k be a strictly monotone increasing injection $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $k(0) = 0$. Put*

$$\|(a, b)\|_k = k(f(\|a\|_1) + g(\|b\|_2))$$

for any $(a, b) \in X$. Then $(X, \oplus, \otimes, \|\cdot\|_k, k^{-1})$ is a normed gyrolinear space. In particular, if $k = \mathrm{id}$, then

$$\|(a, b)\| = f(\|a\|_1) + g(\|b\|_2).$$

Proof. Put $\|\cdot\|'_1 = f(\|\cdot\|_1)$ then $(X_1, \oplus_1, \otimes_1, \|\cdot\|'_1, id)$ is a gyrolinear space as Proposition 3.4. Similarly, put $\|\cdot\|'_2 = g(\|\cdot\|_2)$ then $(X_2, \oplus_2, \otimes_2, \|\cdot\|'_2, id)$ is also a gyrolinear space. Put $\|(a, b)\| = \|a\|'_1 + \|b\|'_2$ for any $(a, b) \in X$. Then

$(X, \oplus, \otimes, \|\cdot\|, id)$ is a gyrolinear space as Proposition 3.3. Note that $\|X\| = \mathbb{R}_{\geq 0}$. Let k be a strictly monotone increasing injection $k : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $k(0) = 0$. Since $\|X\| = \mathbb{R}_{\geq 0}$, k is a strictly monotone increasing bijection from $\|X\|$ to $k(\|X\|)$. Put $\|\cdot\|_k = k(\|\cdot\|)$ then $(X, \oplus, \otimes, \|\cdot\|_k, k^{-1})$ is a gyrolinear space as Proposition 3.4. Note that

$$\|(a, b)\| = f(\|a\|_1) + g(\|b\|_2)$$

and

$$\|(a, b)\|_k = k(f(\|a\|_1) + g(\|b\|_2))$$

for any $(a, b) \in X$. $\square$

The following Lemma 3.9 is trivial.

Lemma 3.9. *Let $(X, \oplus, \otimes)$ be a gyrolinear space. Let Y be a set and ϕ be a injection $\phi : X \to Y$. Define the binary operation $\oplus_\phi$ on $\phi(X)$ by*

$$\phi(a) \oplus_\phi \phi(b) = \phi(a \oplus b)$$

and the map $\otimes_\phi : \mathbb{R} \times \phi(X) \to \phi(X)$ by

$$r \otimes_\phi \phi(a) = \phi(r \otimes a)$$

for any $r \in \mathbb{R}$ and $a, b \in X$. Then $(\phi(X), \oplus_\phi, \otimes_\phi)$ is a gyrolinear space. Moreover, if $(X, \oplus, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space, then $(\phi(X), \oplus_\phi, \otimes_\phi, \|\cdot\|', f)$ is a normed gyrolinear space, where

$$\|\phi(a)\|' = \|a\|$$

for any $a \in X$. Note that the identity of $(\phi(X), \oplus_\phi)$ is $\phi(e)$, where e is the identity of $(X, \oplus)$

Proposition 3.10. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number. Define the binary operation $\oplus_\alpha$ on G by*

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in G$. Then $(X, \oplus_\alpha, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = |\alpha| \otimes'_f \|\cdot\|$.

Proof. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number and ϕ be a map $\phi : X \to X$ which is defined by $\phi(x) = \dfrac{1}{\alpha} \otimes x$ for any $x \in X$. Note that ϕ is a bijection. Actually, $\phi^{-1}(x) = \alpha \otimes x$ as $\alpha \otimes (\dfrac{1}{\alpha} \otimes x) = 1 \otimes x = x$

since the conditions (GL3) and (GL1). By Lemma 3.9, $(X, \oplus_\alpha, \otimes_\alpha, \|\cdot\|', f)$ is a normed gyrolinear space, where

$$(\frac{1}{\alpha} \otimes x) \oplus_\alpha (\frac{1}{\alpha} \otimes y) = \frac{1}{\alpha} \otimes (x \oplus y), \tag{2}$$

$$r \otimes_\alpha (\frac{1}{\alpha} \otimes x) = \frac{1}{\alpha} \otimes (r \otimes x) \tag{3}$$

$$\|\frac{1}{\alpha} \otimes x\|' = \|x\| \tag{4}$$

for any $x, y \in X$ and $r \in \mathbb{R}$. Put $a = \dfrac{1}{\alpha} \otimes x$ and $b = \dfrac{1}{\alpha} \otimes y$ then $x = \alpha \otimes a$ and $y = \alpha \otimes b$. Hence we have

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in X$ as (2). Note that $\dfrac{1}{\alpha} \otimes (r \otimes x) = r \otimes (\dfrac{1}{\alpha} \otimes x)$ for any $r \in \mathbb{R}$ and $x \in X$ as the condition (GL3). It follows that $r \otimes_\alpha a = r \otimes_\alpha (\dfrac{1}{\alpha} \otimes x) = r \otimes (\dfrac{1}{\alpha} \otimes x) = r \otimes a$ since (3). So, we have $\oplus_\alpha = \oplus$. The equation (4) follows that $\|a\|' = \|\alpha \otimes a\| = |\alpha| \otimes'_f \|a\|$. □

Proposition 3.11. *Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. Let α be a nonzero real number. Define the binary operation $\oplus_\alpha$ on G by*

$$a \oplus_\alpha b = \frac{1}{\alpha} \otimes (\alpha \otimes a \oplus \alpha \otimes b)$$

for any $a, b \in G$. Then $(X, \oplus_\alpha, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space.

Proof. By Proposition 3.10, $(X, \oplus_\alpha, \otimes, \|\cdot\|', f)$ is a normed gyrolinear space, where $\|\cdot\|' = |\alpha| \otimes'_f \|\cdot\|$. Note that $f(\|\cdot\|) = f(\dfrac{1}{\alpha} \otimes'_f \|\cdot\|') = |\dfrac{1}{\alpha}| f(\|\cdot\|')$. Put $h(a) = f^{-1}(|\dfrac{1}{\alpha}| f(a))$, then h is a strictly monotone increasing bijection from $\|X\|$ to $\|X\|$. Since $\|\cdot\| = h(\|\cdot\|')$, $(X, \oplus_\alpha, \otimes, \|\cdot\|, f)$ is a normed gyrolinear space as Proposition 3.4. □

4. Structures on a Normed Gyrolinear Space and a Mazur-Ulam Theorem

Definition 4.1. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a normed gyrolinear space. The gyrometric ϱ on X is defined by

$$\varrho(a, b) = \|a \ominus b\|$$

for any $a, b \in X$.

Note that the gyrometric ϱ on $(X, \oplus, \otimes, \|\cdot\|, f)$ is not necessarily a metric on X, but $d = f\varrho$ is a metric on X.

Definition 4.2. Let $(X, \oplus, \otimes, \|\cdot\|, f)$ be a gyrolinear space. Put

$$L[a,b](s) = a \oplus s \otimes (\ominus a \oplus b)$$

for any $a, b \in X$ and $s \in \mathbb{R}$. We call $L[a,b](\mathbb{R})$ the unique gyroline that passes through a and b. We call $L[a,b]([0,1])$ the gyrosegment ab. We call $\boldsymbol{p}(a,b) = L[a,b](\dfrac{1}{2})$ the gyromidpoint of a and b.

The gyromidpoint $\boldsymbol{p}(a,b)$ can be rewritten by $\frac{1}{2}\otimes(a \boxplus b)$, where $\boxplus$ is a coaddition of $(X, \oplus)$.

Example 4.3. Let $(\mathbb{V}, +, \times, \|\cdot\|, id)$ be a normed space. The gyrometric $\varrho(a,b) = \|a - b\|$ is the usual metric induced by its norm. $L[a,b](s) = a + s(-a + b)$ and hence the gyroline is the line, the gyrosegment is the segment, the gyromidpoint is the arithmetic mean $\dfrac{a+b}{2}$.

Example 4.4. Let $(\mathscr{A}_+^{-1}, \oplus_A, \otimes_a, \|\cdot\|', id)$ be a normed gyrolinear space of the positive cone. The gyrometric

$$\varrho(a,b) = \|a \ominus b\|' = \|\log a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}}\|$$

is the Thompson metric.

$$L[a,b](s) = a^{\frac{1}{2}}(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-s} a^{\frac{1}{2}}$$

and hence the gyrosegment is the geodesic. The gyromidpoint

$$\boldsymbol{p}(a,b) = a^{\frac{1}{2}}(a^{\frac{1}{2}} b^{-1} a^{\frac{1}{2}})^{-\frac{1}{2}} a^{\frac{1}{2}}$$

is the geometric mean.

The celebrated Mazur-Ulam theorem asserts that surjective isometry between two normed vector spaces is a real linear isomorphism followed by a translation. In [1], author and Hatori give a generalization of the Mazur-Ulam theorem for generalized gyrovector spaces. This theorem holds for normed gyrolinear space as the following Theorem 4.5 and Corollary 4.6. There are no gaps between proofs for normed gyrolinear spaces and for generalized gyrovector spaces in [1]. Refer to [1] for the proofs.

Theorem 4.5. *Let $(X_i, \oplus_i, \otimes_i, \|\cdot\|_i, f_i)$ be a normed gyrolinear space and ϱ_i be the gyrometric for $i = 1, 2$. Let $T : X_1 \to X_2$ be a surjection. If T preserves the gyrometric,*

$$\varrho_2(Ta, Tb) = \varrho_1(a, b)$$

for any $a, b \in X_1$*, then* T *preserves the gyromidpoint,*

$$T\boldsymbol{p}(a,b) = \boldsymbol{p}(Ta, Tb)$$

for any $a, b \in X_1$.

Corollary 4.6. *Let* $(X_i, \oplus_i, \otimes_i, \|\cdot\|_i, f_i)$ *be a normed gyrolinear space and* ϱ_i *be the gyrometric for* $i = 1, 2$. *Let* $T : X_1 \to X_2$ *be a surjection. Suppose that* T *preserves the gyrometric,*

$$\varrho_2(Ta, Tb) = \varrho_1(a, b)$$

for any $a, b \in X_1$. *Then* T *is of the form* $T(\cdot) = T(e_1) \oplus T(\cdot)$, *where* e_1 *is the identity of* X_1 *and* T_0 *is an isometrical isomorphism in the sense that the equalities*

$$T_0(\boldsymbol{a} \oplus_1 \boldsymbol{b}) = T_0(\boldsymbol{a}) \oplus_2 T_0(\boldsymbol{b}); \tag{5}$$

$$T_0(\alpha \otimes_1 \boldsymbol{a}) = \alpha \otimes_2 T_0(\boldsymbol{a}); \tag{6}$$

$$\varrho_2(T_0\boldsymbol{a}, T_0\boldsymbol{b}) = \varrho_1(\boldsymbol{a}, \boldsymbol{b}). \tag{7}$$

for every $\boldsymbol{a}, \boldsymbol{b} \in G_1$ *and* $\alpha \in \mathbb{R}$ *hold.*

5. A Normed Gyrolinear Space Induced by a Metric Space

5.1 A Gyrocommutative Gyrogroup Induced by a Metric Space

A dyadic symset is a magma $(X, \circ)$ satisfying for all $a, b, c \in X$ the following axioms (d1) to (d4):

(d1) $a \circ a = a$;

(d2) $a \circ (a \circ b) = b$;

(d3) $a \circ (b \circ c) = (a \circ b) \circ (a \circ c)$;

(d4) the equation $x \circ a = b$ has a unique solution $x \in X$, called the midpoint of a and b, and denoted $a \sharp b$.

In the paper [3], Lawson and Lim show a strong equivalence between pointed dyadic symsets and uniquely 2-divisible gyrocommutative gyrogroups in the following sense.

Let $(X, \circ)$ be a dyadic symset and $e \in X$. Define a new binary operation $\oplus$ on X by $x \oplus y = (e \sharp x) \circ (e \circ y)$ then $(X, \oplus)$ is a uniquely 2-divisible gyrocommutative gyrogroup with the identity e. Conversely, let $(X, \oplus)$ be a uniquely 2-divisible gyrocommutative gyrogroup. Define a new binary operation $\circ$ on X by $x \circ y = 2 \otimes x \ominus y$, then $(X, \circ)$ is a dyadic symset.

As the consequence of the fact, we have a uniquely 2-divisible gyrocommutative gyrogroup which is induced by a metric space as following Lemma 5.3.

Definition 5.1. Let (X, d) be a metric space. We say that (X, d) satisfies the condition K if the following conditions (K1) to (K3) are hold.

(K1) For any pair $x, y \in X$, there exists a unique element $c \in X$ such that

$$d(x, c) = d(c, y) = \frac{1}{2} d(x, y).$$

We call c the metric midpoint of x and y and write

$$c = \mathrm{mid}(x, y).$$

(K2) For any elements $x, y \in X$, there exists a unique element $z \in X$ such that $x = \mathrm{mid}(y, z)$. We write

$$z = \varphi_x(y)$$

and we call the map $\varphi_x : X \to X$ the metric reflection in the point x.

(K3) The metric reflection $\varphi_x : X \to X$ is an isometry for any $x \in X$.

Note that $\mathrm{mid}(x, y) = \mathrm{mid}(y, x)$. Moreover, $z = \varphi_x(y) \iff z = \mathrm{mid}(y, z) \iff y = \varphi_x(z)$ and hence $\varphi_x^{-1} = \varphi_x$.

Definition 5.2. Let (X, d) be a metric space that satisfies the condition K. For fixed $e \in X$, we define the binary operation $\oplus_e$ on X by

$$x \oplus_e y = \varphi_{\tilde{x}} \varphi_e(y),$$

where $\tilde{x} = \mathrm{mid}(e, x)$ for any $x, y \in X$. We call $\oplus_e$ the binary operation induced by the metric d on X at $e \in X$.

Theorem 5.3. *Let (X, d) be a metric space that satisfies the condition K and let $e \in X$. Let $\oplus_e$ is the binary operation on X induced by the metric d at e. Then $(X, \oplus_e)$ is a uniquely 2-divisible gyrocommutative gyrogroup.*

Proof. Let (X, d) be a metric space that satisfies the condition K and let $e \in X$. Define a binary operation $\circ$ by $x \circ y = \varphi_x(y)$.

First, we prove that $(X, \circ)$ is a dyadic symset.

(d1): $a \circ a = \varphi_a(a) = a$.

(d2): $a \circ (a \circ b) = \varphi_a \varphi_a(b) = b$.

(d3): Since φ_a is an isometry,

$$d(x, b) = d(b, y) = \frac{1}{2} d(x, y)$$

implies

$$d(\varphi_a(x), \varphi_a(y)) = d(\varphi_a(b), \varphi_a(y)) = \frac{1}{2} d((\varphi_a(x), \varphi_a(y))).$$

Therefore, $b = \mathrm{mid}(x, y)$ implies that $\varphi_a(b) = \mathrm{mid}(\varphi_a(x), \varphi_a(y))$. It follows that $\varphi_{\varphi_a(b)}(\varphi_a(x)) = \varphi_a(y)$. Since $y = \varphi_b(x)$, we have

$$(a \circ b) \circ (a \circ x) = \varphi_{\varphi_a(b)}(\varphi_a(x)) = \varphi_a(\varphi_b(x)) = a \circ (b \circ x).$$

(d4): $x \circ a = b \iff \varphi_x(a) = b \iff x = \mathrm{mid}(a, b)$. The midpoint of a and b is $a \sharp b = \mathrm{mid}(a, b)$.

Since $(X, \circ)$ is a dyadic symset, we have $(X, \oplus)$ is a uniquely 2-divisible gyrocommutative gyrogroup, where the binary operation $\oplus$ is defined by $x \oplus y = (e \sharp x) \circ (e \circ y)$. Note that $x \oplus y = (e \sharp x) \circ (e \circ y) = \varphi_{\varphi_e(x)}(\varphi_e(y))$. □

Let $(X, \oplus)$ be a gyrogroup. For $a \in X$, the left translation $\lambda_a : X \to X$ is defined by $\lambda_a(x) = a \oplus x$ for any $x \in X$. It is well known that

$$\mathrm{gyr}[a, b] = \lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b \tag{8}$$

for any $a, b \in X$.

Proposition 5.4. *Let (X, d) be a metric space which satisfies the condition K and let $e \in X$. Let $\oplus_e$ is the binary operation on X induced by the metric d at e. Then*

$$d(a \oplus_e x, a \oplus_e y) = d(x, y) \tag{9}$$

and

$$d(\mathrm{gyr}[a, b]x, \mathrm{gyr}[a, b]y) = d(x, y) \tag{10}$$

for any $a, b, x, y \in X$.

Proof. Let $a \in X$ and put $\tilde{a} = \mathrm{mid}(e, a)$. Since the condition (K3), we have $\varphi_{\tilde{a}}$ and φ_e are isometries. Hence

$$d(a \oplus_e x, a \oplus_e y) = d(\varphi_{\tilde{a}} \varphi_e x, \varphi_{\tilde{a}} \varphi_e y) = d(x, y)$$

for $x, y \in X$. It follows that

$$d(\mathrm{gyr}[a, b]x, \mathrm{gyr}[a, b]y) = d(\lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b(x), \lambda_{\ominus(a \oplus b)} \lambda_a \lambda_b(y)) = d(x, y)$$

for any $a, b, x, y \in X$. □

5.2 Preparations

Let (X, d) be a metric space. A geodesic path joining $x \in X$ and $y \in X$ is a map δ from $[0, l]$ to X such that $\delta(0) = x$, $\delta(l) = y$ and $d(\delta(t_1), \delta(t_2)) = |t_1 - t_2|$ for all $t_1, t_2 \in [0, l]$. In particular, $l = d(x, y)$. (X, d) is called a uniquely geodesic space if any pairs $x, y \in X$ has exactly one geodesic path joining x and y.

In this subsection, (X,d) is a uniquely geodesic space with condition K and $\gamma_{x,y}$ is a map from $[0,1]$ to X defined by

$$\gamma_{x,y}(t) = \delta_{x,y}(td(x,y))$$

for any $t \in [0,1]$, where $\delta_{x,y}$ denotes the geodesic path joining x and y. It is easy to show that $\gamma_{x,y}(0) = x$, $\gamma_{x,y}(1) = y$ and

$$d(\gamma_{x,y}(t_1), \gamma_{x,y}(t_2)) = |t_1 - t_2| d(x,y)$$

for any $t_1, t_2 \in [0,1]$.

Note that $\gamma_{x,y}(s)$ is a unique point c in X which satisfies $d(x,c) = sd(x,y)$ and $d(c,y) = (1-s)d(x,y)$ for any $0 \le s \le 1$.

For $x \in X$, define the map ϕ_x on X by the equation

$$\phi_x(y) = \varphi_y(x)$$

for any $y \in X$. Then we have

$$\phi_x(y) = c \iff \varphi_y(x) = c \iff \mathrm{mid}(x,c) = y.$$

It implies that ϕ_x is a bijection on X for any $x \in X$.

Lemma 5.5. *Let $x, y, z, c \in X$. Then the following holds.*

(y1) For any $s \in \mathbb{R} \setminus \{0\}$,

$$\begin{cases} d(x,z) = |s| d(x,y) \\ d(z,y) = |1-s| d(x,y) \end{cases} \iff \begin{cases} d(x,y) = |\frac{1}{s}| d(x,z) \\ d(y,z) = |1 - \frac{1}{s}| d(x,z). \end{cases}$$

(y2) For any $0 \le s \le 1$,

$$c = \gamma_{x,y}(s) \iff \begin{cases} d(x,c) = sd(x,y) \\ d(c,y) = (1-s)d(x,y). \end{cases}$$

(y3)

$$c = \phi_x(y) \iff \begin{cases} d(x,c) = 2d(x,y) \\ d(c,y) = (2-1)d(x,y). \end{cases}$$

(y4) For any natural number n,

$$c = \phi_x^n(y) \iff \begin{cases} d(x,c) = 2^n d(x,y) \\ d(c,y) = (2^n - 1)d(x,y). \end{cases}$$

(y5) For any $s > 1$,

$$c = \phi_x^n(\gamma_{x,y}(\frac{s}{2^n})) \iff \begin{cases} d(x,c) = sd(x,y) \\ d(c,y) = (s-1)d(x,y), \end{cases}$$

where $n \in \mathbb{N}$ *which satisfies* $2^{n-1} < s \leq 2^n$.

(y6) For any $s > 0$,

$$\begin{cases} d(x,c) = sd(x,y) \\ d(c,y) = |s-1|d(x,y) \end{cases} \iff \begin{cases} d(x,c') = sd(x,y) \\ d(c',y) = (1+s)d(x,y), \end{cases}$$

where $c' = \varphi_x(c)$.

In particular, for any real number s, *there exists a unique point* c *in* X *such that* $d(e,c) = |s|d(e,x)$ *and* $d(c,x) = |1-s|d(e,x)$.

Proof. (y1) and (y2) are obvious.

(y3): We have

$$\begin{aligned} c = \phi_x(y) \quad &\iff \quad \mathrm{mid}(x,c) = y \\ &\iff \quad d(x,y) = d(c,y) = \frac{1}{2}d(x,c) \\ &\iff \quad d(x,c) = 2d(x,y) \ \text{ and } \ d(c,y) = d(x,y). \end{aligned}$$

(y4): We will prove by induction. When $n = 1$, the argument is true by (y2). Let k be a natural number and suppose that $\phi_x^k(y)$ is a unique point $c_k \in X$ which satisfies

$$d(x,c_k) = 2^k d(x,y) \ \text{ and } \ d(c_k,y) = (2^k - 1)d(x,y).$$

Note that

$$\begin{aligned} c = \phi_x^{k+1}(y) \quad &\iff \quad c = \phi_x(c_k) \\ &\iff \quad \mathrm{mid}(x,c) = c_k \\ &\iff \quad d(x,c_k) = d(c,c_k) = \frac{1}{2}d(x,c) \\ &\iff \quad d(x,c) = 2d(x,c_k) \ \text{ and } \ d(c,c_k) = d(x,c_k) \\ &\iff \quad d(x,c) = 2^{k+1}d(x,y) \ \text{ and } \ d(c,c_k) = 2^k d(x,y). \end{aligned}$$

($\Rightarrow$): Let $c = \phi_x^{k+1}(y)$. We have

$$d(x,c) = 2^{k+1}d(x,y)$$

and

$$d(y,c) \leq d(y,c_k) + d(c_k,c) = (2^{k+1} - 1)d(x,y),$$

$$d(y,c) \geq d(x,c) - d(x,y) = (2^{k+1}-1)d(x,y).$$

Therefore

$$d(y,c) = (2^{k+1}-1)d(x,y).$$

($\Leftarrow$): Let c be a point of X which satisfies

$$d(x,c) = 2^{k+1}d(x,y) \quad \text{and} \quad d(c,y) = (2^{k+1}-1)d(x,y).$$

Then

$$d(x,y) = \frac{1}{2^{k+1}}d(x,c) \quad \text{and} \quad d(y,c) = (1-\frac{1}{2^{k+1}})d(x,c)$$

as (y1). It implies that $y = \gamma_{x,c}(\frac{1}{2^{k+1}})$. Put $c' = \mathrm{mid}(x,c)$, then we have

$$d(x,c') = \frac{1}{2}d(x,c) = 2^k d(x,y)$$

and

$$\begin{aligned} d(y,c') &= d(\gamma_{x,c}(\frac{1}{2k+1}), \gamma_{x,c}(\frac{1}{2})) \\ &= (\frac{1}{2} - \frac{1}{2^{k+1}})d(x,c) = (2^k-1)d(x,y). \end{aligned}$$

By the inductive assumption, we have $c' = \phi_x^k(y)$ and hence $c = \phi_x^{k+1}(y)$.

By the principle of induction, the proof of (y4) is complete.

(y5): Let $s > 1$. Then there exist $n \in \mathbb{N}$ such that $2^{n-1} < s \leq 2^n$. Put $s' = \dfrac{s}{2^n}$, then $\dfrac{1}{2} < s' \leq 1$.

($\Rightarrow$): Let c_0 be a point in X that satisfies

$$d(x,c_0) = s'd(x,y) \quad \text{and} \quad d(c_0,y) = (1-s')d(x,y).$$

Following (y1) we have $c_0 = \gamma_{x,y}(s')$. Let c be a point in X that satisfies

$$d(x,c) = 2^n d(x,c_0) \quad \text{and} \quad d(c,c_0) = (2^n-1)d(x,c_0).$$

Since (y3), we have $c = \phi_x^n(c_0)$. Put $b = \mathrm{mid}(x,y)$, then

$$d(c_0,b) = d(\gamma_{x,y}(s'), \gamma_{x,y}(\frac{1}{2})) = (s' - \frac{1}{2})d(x,y) = (1-\frac{1}{2s'})d(x,c_0)$$

and

$$d(x,b) = \frac{1}{2}d(x,y) = \frac{1}{2s'}d(x,c_0) = \frac{1}{2^{n+1}s'}d(x,c) = \frac{1}{2s}d(x,c).$$

Thus,

$$\begin{aligned} d(b,c) \leq d(b,c_0) + d(c_0,c) &= (2^n - \frac{1}{2s'})d(x,c_0) \\ &= (1 - \frac{1}{2^{n+1}s'})d(x,c) = (1-\frac{1}{2s})d(x,c), \end{aligned}$$

$$d(b,c) \geq d(x,c) - d(x,b) = (1 - \frac{1}{2^{n+1}s'})d(x,c) = (1 - \frac{1}{2s})d(x,c)$$

and hence

$$d(b,c) = (1 - \frac{1}{2s})d(x,c).$$

Therefore, we have $b = \gamma_{x,c}(\frac{1}{2s})$. Since $b = \mathrm{mid}(x,y)$, we have $y = \gamma_{x,c}(\frac{1}{s})$, that is,

$$d(x,y) = \frac{1}{s}d(x,c) \ \text{ and } \ d(y,c) = (1 - \frac{1}{s})d(x,c).$$

By (y1) we have

$$d(x,c) = sd(x,y) \ \text{ and } \ d(c,y) = (s-1)d(x,y).$$

($\Leftarrow$): Let c be a point in X which satisfies

$$d(x,c) = sd(x,y) \ \text{ and } \ d(c,y) = (s-1)d(x,y).$$

By (y1) and (y2) we have $y = \gamma_{x,c}(\frac{1}{s})$. Let $c_1 = \mathrm{mid}(x,c)$ and $c_{k+1} = \mathrm{mid}(x, c_k)$ for any $k \in \mathbb{N}$. Then $c_k = \gamma_{x,c}(\frac{1}{2^k})$ and $\phi_x^k(c_k) = c$ for any $k \in \mathbb{N}$. Thus we have

$$d(y, c_n) = (\frac{1}{s} - \frac{1}{2^n})d(x,c) = (1 - \frac{s}{2^n})d(x,y)$$

and

$$d(x, c_n) = \frac{1}{2^n}d(x,c) = \frac{s}{2^n}d(x,y).$$

It implies that $c_n = \gamma_{x,y}(\frac{s}{2^n})$ and hence $c = \phi_x^n(\gamma_{x,y}(\frac{s}{2^n}))$.

(y6): Let $s > 0$.

($\Rightarrow$): Let $c \in X$ be a point in X such that

$$d(x,c) = sd(x,y) \ \text{ and } \ d(c,y) = |1-s|d(x,y).$$

Since φ_x is an isometry and $\varphi_x(x) = x$, we have

$$d(x, \varphi_x(c)) = d(x,c) = sd(x,y)$$

and

$$d(\varphi_x(c), \varphi_x(y)) = d(c,y) = |1-s|d(x,y).$$

By the definition of φ_x, we have $d(a, \varphi_x(a)) = 2d(x,a)$ for any $a \in X$ and hence

$$d(\varphi_x(y), y) = 2d(x,y),$$

$$d(\varphi_x(c), c) = 2d(x,c) = 2sd(x,y).$$

We first assume that $0 < s \leq 1$. Then

$$d(\varphi_x(c), y) \leq d(\varphi_x(c), x) + d(x,y) = (1+s)d(x,y)$$

and
$$d(\varphi_x(c), y) \geq d(\varphi_x(y), y) - d(\varphi_x(y), \varphi_x(c)) = (1+s)d(x,y).$$
It follows that
$$d(\varphi_x(c), y) = (1+s)d(x,y).$$

Next, we assume that $1 < s$. Then
$$d(\varphi_x(c), y) \leq d(\varphi_x(c), x) + d(x,y) = (1+s)d(x,y)$$
and
$$d(\varphi_x(c), y) \geq d(\varphi_x(c), c) - d(c,y) = (1+s)d(x,y).$$
It follows that
$$d(\varphi_x(c), y) = (1+s)d(x,y).$$

($\Leftarrow$): Since (y2) to (y5), there exists a point c in X such that
$$d(x,c) = sd(x,y) \text{ and } d(c,y) = |s-1|d(x,y).$$
We have
$$d(x, \varphi_x(c)) = sd(x,y) \text{ and } d(\varphi_x(c), y) = (1+s)d(x,y)$$
as opposite direction. Let $c' \in X$ be a point such that
$$d(x,c') = sd(x,y) \text{ and } d(c',y) = (1+s)d(x,y).$$
Put $t = 1+s$ then $t > 1$ and
$$d(y,c') = td(y,x) \text{ and } d(c',x) = (t-1)d(y,x).$$
Since (y4), such a point c' is unique in X. Thus $c' = \varphi_x(c)$. □

By Lemma 5.5, for any $x, y \in X$ and any $s \in \mathbb{R}$, there exists a unique point $c \in X$ such that $d(x,c) = |s|d(x,y)$ and $d(c,y) = |1-s|d(x,y)$. We will denote the such point c by $\gamma_{x,y}(s)$.

Lemma 5.6. *For any $x, y \in X$ and $s, t \in \mathbb{R}$, the equation*
$$d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t|d(x,y)$$
holds.

Proof. Put $a_r = \gamma_{x,y}(r)$ for any $r \in \mathbb{R}$. We can assume that $s \leq t$.

(the case: $0 \leq s \leq t \leq 1$): trivial.

(the case: $0 \leq s \leq 1 \leq t$): By the definition of a_r, we have
$$\begin{aligned} d(x,a_s) &= |s|d(x,y) = sd(x,y), \\ d(a_s,y) &= |1-s|d(x,y) = (1-s)d(x,y) \\ d(x,a_t) &= |t|d(x,y) = td(x,y), \\ d(a_t,y) &= |1-t|d(x,y) = (t-1)d(x,y). \end{aligned}$$

Then

$$\begin{aligned} d(a_s, a_t) &\leq d(a_s, y) + d(y, a_t) = (t-s)d(x,y), \\ d(a_s, a_t) &\geq d(x, a_t) - d(x, a_s) = (t-s)d(x,y) \end{aligned}$$

and hence

$$d(a_s, a_t) = (t-s)d(x,y).$$

(the case: $1 \leq s \leq t$): Since (y1), we have $y = \gamma_{x,a_t}(\frac{1}{t})$. Let $c = \gamma_{x,a_t}(\frac{s}{t})$, then, since $0 \leq \frac{1}{t}, \frac{s}{t} \leq 1$, we have

$$\begin{aligned} d(x,c) &= \frac{s}{t} d(x, a_t) = sd(x,y), \\ d(y,c) &= |\frac{1}{t} - \frac{s}{t}| d(x, a_t) = |1-s| d(x,y). \end{aligned}$$

It implies that $c = a_s$. Thus

$$d(a_s, a_t) = |\frac{s}{t} - 1| d(x, a_t) = |s-t| d(x,y).$$

As the above part of the proof, we have

$$0 \leq s, t \Rightarrow d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = |s-t| d(x,y).$$

(the case: $s \leq 0 \leq 1 \leq t$): By the definition of a_t we have

$$\begin{aligned} d(x, a_t) &= td(x,y), \\ d(a_t, y) &= (t-1)d(x,y). \end{aligned}$$

It follows that

$$\begin{aligned} d(a_t, x) &= \frac{t}{t-1} d(a_t, y), \\ d(x,y) &= (1 - \frac{t}{t-1}) d(a_t, y). \end{aligned}$$

It implies that $x = \gamma_{a_t,y}(\frac{t}{t-1})$. Let $c = \gamma_{a_t,y}(\frac{t-s}{t-1})$. Since $0 \leq \frac{t}{t-1}, \frac{t-s}{t-1}$, we have

$$\begin{aligned} d(y,c) &= |1 - \frac{t-s}{t-1}| d(y, a_t) = (1-s)d(x,y), \\ d(c,x) &= |\frac{t-s}{t-1} - \frac{t}{t-1}| d(y, a_t) = |s| d(x,y). \end{aligned}$$

Hence $c = a_s$. Thus

$$d(a_t, a_s) = d(a_t, c) = \frac{t-s}{t-1} d(a_t, y) = (t-s)d(x,y).$$

(the case: $s \leq 0 \leq t \leq 1$): By the definition of a_r and triangle inequality, we have

$$\begin{aligned} d(a_s, a_t) &\leq d(a_s, x) + d(x, a_t) \\ &= -sd(x, y) + td(x, y) = (t-s)d(x, y), \\ d(a_s, a_t) &\geq d(y, a_s) + d(y, a_t) \\ &= (1-s)d(x, y) + (1-t)d(x, y) = (t-s)d(x, y) \end{aligned}$$

and hence

$$d(a_s, a_t) = (t-s)d(x, y).$$

(the case: $s \leq t \leq 0$): For any $r \in \mathbb{R}$, we have

$$\gamma_{x,y}(r) = \gamma_{y,x}(1-r)$$

by the definition. Since $0 \leq 1-s, 1-t$, we have

$$d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = d(\gamma_{y,x}(1-s), \gamma_{y,x}(1-t)) = (t-s)d(y, x).$$

□

Lemma 5.7. *Let $x, y \in X$. The equation*

$$\varphi_{\gamma_{x,y}(s)}(\gamma_{x,y}(t)) = \gamma_{x,y}(2s-t)$$

holds for any $s, t \in \mathbb{R}$.

Proof. Since Lemma 5.6, we have

$$\begin{aligned} d(\gamma_{x,y}(2s-t), \gamma_{x,y}(s)) &= |s-t|d(x, y), \\ d(\gamma_{x,y}(s), \gamma_{x,y}(t)) &= |s-t|d(x, y) \end{aligned}$$

and

$$d(\gamma_{x,y}(2s-t), \gamma_{x,y}(t)) = |2s-2t|d(x, y).$$

Thus

$$d(\gamma_{x,y}(2s-t), \gamma_{x,y}(s)) = d(\gamma_{x,y}(s), \gamma_{x,y}(t)) = \frac{1}{2}d(\gamma_{x,y}(2s-t), \gamma_{x,y}(t))$$

and hence

$$\gamma_{x,y}(s) = \mathrm{mid}(\gamma_{x,y}(t), \gamma_{x,y}(2s-t)).$$

Therefore,

$$\varphi_{\gamma_{x,y}(s)}(\gamma_{x,y}(t)) = \gamma_{x,y}(2s-t).$$

□

5.3 A Normed Gyrolinear Space Induced by a Metric Space

In this subsection, let (X, d) be a uniquely geodesic space which satisifies the condition K. For $x, y \in X$ and $s \in \mathbb{R}$, $\gamma_{e,x}(s)$ denote the unique point $c \in X$ that satisfies $d(x, c) = |s|d(x, y)$ and $d(c, y) = |1 - s|d(x, y)$.

Definition 5.8. Let (X, d) be a uniquely geodesic space that satisfies the condition K. For fixed $e \in X$, we define the scalar multiplication $\otimes_e$ on X by

$$s \otimes_e x = \gamma_{e,x}(s)$$

for any $x \in X$ and $s \in \mathbb{R}$. We call $\otimes_e$ the scalar multiplication induced by the metric d on X at e.

In the following part of this subsection e is a point in X. $\oplus_e$ is the binary operation induced by the metric d on X at e, and $\otimes_e$ is the scalar multiplication induced by the metric d on X at e.

Proposition 5.9. *Let (X, d) be a uniquely geodesic space which satisfies the condition K. Let $e \in X$ and put $\|x\|_e = d(e, x)$ for any $x \in X$. Assume that (X, d) satisfies the following condition:*

(K4) $x \to y$ implies $\varphi_x(a) \to \varphi_y(a)$ for any $x, y, a \in X$.

Then $(X, \oplus_e, \otimes_e, \| \cdot \|_e, id)$ is a normed gyrolinear space with gyrometric d.

Proof. Recall that $(X, \oplus_e)$ is a gyrocommutative gyrogroup by Lemma 5.3.

(GL1): It is an immediate consequence of $\gamma_{e,x}(1) = x$.

(GL2): By Lemma 5.7, we have

$$\begin{aligned}
(r \otimes_e x) \oplus_e (s \otimes_e x) &= \varphi_{\mathrm{mid}(e, r\otimes_e x)}\varphi_e(s \otimes_e x) \\
&= \varphi_{\mathrm{mid}(e, \gamma_{e,x}(r))}((-s) \otimes_e x) \\
&= \varphi_{\gamma_{e,x}(\frac{r}{2})}(\gamma_{e,x}(-s)) \\
&= \gamma_{e,x}(\frac{2r}{2} - (-s)) \\
&= \gamma_{e,x}(r + s) \\
&= (r + s) \otimes_e x
\end{aligned}$$

(GL3): Put $z = r \otimes_e (s \otimes_e x)$, then $z = \gamma_{e,s\otimes_e x}(r)$. Since (y1), $x = \gamma_{e,s\otimes_e x}(\frac{1}{s})$. We have $z = (rs) \otimes_e x$ as

$$d(e, z) = d(e, r \otimes_e (s \otimes_e x)) = |r|d(e, s \otimes_e x) = |rs|d(e, x)$$

and

$$\begin{aligned}
d(x, z) &= d(\gamma_{e,s\otimes_e x}(\frac{1}{s}), \gamma_{e,s\otimes_e x}(r)) \\
&= |r - \frac{1}{s}|d(e, s \otimes_e x) = |rs - 1|d(e, x).
\end{aligned}$$

(GL4): Since $\mathrm{gyr}[x,y]$ is an automorphism, $\mathrm{gyr}[x,y]e = e$. By equation (10) of Proposition 5.4, $\mathrm{gyr}[x,y]$ is a isometry. Hence we have

$$\begin{aligned}
c = r \otimes_e a \quad &\Longleftrightarrow \quad \begin{cases} d(e,c) = |r|d(e,a) \\ d(c,a) = |1-r|d(e,a) \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]c) = |r|d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]a) \\ d(\mathrm{gyr}[x,y]c, \mathrm{gyr}[x,y]a) = |1-r|d(\mathrm{gyr}[x,y]e, \mathrm{gyr}[x,y]a) \end{cases} \\
&\Longleftrightarrow \quad \begin{cases} d(e, \mathrm{gyr}[x,y]c) = |r|d(e, \mathrm{gyr}[x,y]a) \\ d(\mathrm{gyr}[x,y]c, \mathrm{gyr}[x,y]a) = |1-r|d(e, \mathrm{gyr}[x,y]a) \end{cases} \\
&\Longleftrightarrow \quad \mathrm{gyr}[x,y]c = \gamma_{e,\mathrm{gyr}[x,y]a}(r) \\
&\Longleftrightarrow \quad \mathrm{gyr}[x,y]c = r \otimes_e \gamma[x,y]a
\end{aligned}$$

(GL5): For any $x \in X$, since $(X, \oplus_e)$ satisfies the condition (G5), we have

$$\begin{aligned}
\mathrm{gyr}[n \otimes_e x, x] &= \mathrm{gyr}[((n-1) \otimes_e x) \oplus_e x, x] \\
&= \mathrm{gyr}[(n-1) \otimes_e x, x]
\end{aligned}$$

for any integer n. It follows that

$$\mathrm{gyr}[n \otimes_e x, x] = \mathrm{gyr}[0 \otimes_e x, x] = \mathrm{gyr}[e, x] = id_X$$

for any integer n. Also, we have

$$\mathrm{gyr}[x, m \otimes_e x] = \mathrm{gyr}^{-1}[m \otimes_e x, x] = id_X$$

for any integer m. Since gyrocommutative gyrogroup satisfies the equation

$$\mathrm{gyr}[a,b]\,\mathrm{gyr}[b,c]gyr[c,a] = id$$

for any a, b, c ([6]Theorem 3.31), we have

$$\mathrm{gyr}[n \otimes_e x, m \otimes_e x] = \mathrm{gyr}[n \otimes_e x, x]\,\mathrm{gyr}[x, m \otimes_e x]\,\mathrm{gyr}[m \otimes_e x, n \otimes_e x] = id_X$$

for any integers n, m. It follows that

$$\mathrm{gyr}[\frac{n}{m} \otimes_e y, y] = \mathrm{gyr}[n \otimes_e (\frac{1}{m} \otimes_e y), m \otimes_e (\frac{1}{m} \otimes_e y)] = id_X \tag{11}$$

for any $y \in X$ and rational number $\dfrac{n}{m}$.

Let $\{k_n\}$ be a sequence of rational numbers such that $k_n \to \alpha$. Then we have $k_n \otimes_e x = \gamma_{e,x}(k_n) \to \gamma_{e,x}(\alpha) = \alpha \otimes_e x$ by Lemma 5.6. By the condition (K4) we have

$$\lambda_{k_n \otimes_e x}(a) = \varphi_{k_n \otimes_e x}\varphi_e(a) \to \varphi_{\alpha \otimes_e x}\varphi_e(a) = \lambda_{\alpha \otimes_e x}(a)$$

for any $x, a \in X$. Thus we have

$$\begin{aligned}
\operatorname{gyr}[k_n \otimes_e x, x](a) &= \lambda_{\ominus(k_n \otimes_e x \oplus_e x)} \lambda_{k_n \otimes_e x} \lambda_x(a) \\
&= \lambda_{(-k_n - 1)\otimes_e x} \lambda_{k_n \otimes_e x} \lambda_x(a) \\
&\to \lambda_{(-\alpha - 1)\otimes_e x} \lambda_{\alpha \otimes_e x} \lambda_x(a) \\
&= \lambda_{\ominus(\alpha \otimes_e x \oplus_e x)} \lambda_{\alpha \otimes_e x} \lambda_x(a) \\
&= \operatorname{gyr}[\alpha \otimes_e x, x](a)
\end{aligned}$$

for any real number α and $a, x \in X$, where $\{k_n\}$ is a sequence of rational numbers such that $k_n \to \alpha$. Following (11) we have $\operatorname{gyr}[\alpha \otimes_e x, x] = id_X$ for any $x \in X$ and $\alpha \in \mathbb{R}$. Thus we have

$$\operatorname{gyr}[r \otimes_e x, s \otimes_e x] = \operatorname{gyr}[\frac{r}{s} \otimes_e (s \otimes_e x), s \otimes_e x] = id_X$$

for any $x \in X$ and $r, s \in \mathbb{R}$.

(NG1): $\|x\|_e = 0 \iff d(e, x) = 0 \iff x = e$

(NG2): Following Proposition 5.4, we have

$$\begin{aligned}
\|x \oplus_e y\|_e &= d(e, x \oplus_e y) \\
&\leq d(e, x) + d(x, x \oplus_e y) \\
&= d(e, x) + d(e, y) = \|x\|_e + \|y\|_e
\end{aligned}$$

for any $x, y \in X$.

(NG3): For any $x \in X$ and $r \in \mathbb{R}$, we have

$$\begin{aligned}
\|r \otimes_e x\|_e &= d(e, r \otimes_e x) \\
&= d(\gamma_{e,x}(0), \gamma_{e,x}(r)) \\
&= |0 - r| d(e, x) = |r| \|x\|_e
\end{aligned}$$

by Lemma 5.6.

(NG4): Since any gyroautomorphism preserves the identity e and Proposition 5.4, we have

$$\begin{aligned}
\| \operatorname{gyr}[x, y]a\|_e &= d(e, \operatorname{gyr}[x, y]a) \\
&= d(\operatorname{gyr}[x, y]e, \operatorname{gyr}[x, y]a) \\
&= d(e, a) = \|a\|_e
\end{aligned}$$

for any $a, x, y \in X$.

Finally, since Proposition 5.4, we have

$$d(x, y) = d(e, x \ominus_e y) = \|x \ominus_e y\|.$$

□

The following Corollary 5.10 is a immediately consequence of Proposition 5.9 and Proposition 3.4.

Corollary 5.10. *Let X be a set and ϱ be a function $\varrho : X \times X \to X$ that satisfies $\varrho(x, y) = 0$ if and only if $x = y$. Let $e \in X$ and put $\|x\|'_e = \varrho(e, x)$ for any $x \in X$. Let f be a monotone increasing bijection $f : \|X\|'_e \to \mathbb{R}_{\geq 0}$, where $\|X\|'_e = \{\|x\|'_e : x \in X\}$. Put $d = f\varrho$. Suppose that (X, d) is a uniquely geodesic space that satisfies the condition K and the condition (K4). Then $(X, \oplus_e, \otimes_e, \|\cdot\|'_e, f)$ is a normed gyrolinear space with gyrometric ϱ.*

Proof. Put $\|x\|_e = d(e, x)$. By Proposition 5.9, $(X, \oplus_e, \otimes_e, \|\cdot\|_e, id)$ is a normed gyrolinear space. Since f is a monotone increasing bijection $f : \|X\|'_e \to \mathbb{R}_{\geq 0}$, we have

$$\|X\|_e = \{\|x\|_e : x \in X\} = \{f\|x\|'_e : x \in X\} = \mathbb{R}_{\geq 0}$$

and hence f^{-1} is a monotone increasing bijection $f^{-1} : \|X\| \to \|X\|'_e$. Note that $0 \in \|X\|'$ as $\|e\|' = 0$. Since f^{-1} is strictly monotone increasing, we have $f^{-1}(0) = 0$. By Proposition 3.4, we have $(X, \oplus_e, \otimes_e, \|\cdot\|', f)$. Finally, we have

$$\varrho(x, y) = f^{-1}d(x, y) = f^{-1}\|x \ominus_e y\| = \|x \ominus_e y\|'.$$

□

5.4 Examples

Example 5.11. Let $\|\cdot\|$ be the Euclidean norm and d be the Euclidean metric on $\mathbb{R}^n$. Then the Euclidean space $(\mathbb{R}^n, d)$ is a uniquely geodesic metric space that satisfies the condition K with

$$\mathrm{mid}(x, y) = \frac{x + y}{2}$$

and

$$\varphi_x(y) = 2x - y$$

for any $x, y \in \mathbb{R}$. In this case, $(\mathbb{R}^n, \oplus_0) = (\mathbb{R}^n, +)$ as

$$\begin{aligned} x \oplus_0 y &= \varphi_{\mathrm{mid}(0,x)}\varphi_0(y) \\ &= \varphi_{\frac{x}{2}}(-y) \\ &= x + y \end{aligned}$$

for any $x, y \in X$. Moreover, $\otimes_0$ coincides with the usual scalar multiplication on $\mathbb{R}^n$ as

$$d(0, rx) = \|rx\| = |r|\|x\| = rd(0, x)$$

and

$$d(x, rx) = \|x - rx\| = |1 - r|\|x\| = |1 - r|d(0, x)$$

for any $x \in \mathbb{R}^n$ and $r \in \mathbb{R}$.

Example 5.12. Let $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. The Möbius addition $\oplus$ in $\mathbb{D}$ is given by the equation

$$a \oplus b = \frac{a+b}{1+\overline{a}b}$$

for any $a, b \in \mathbb{D}$. $(\mathbb{D}, \oplus)$ is a gyrocommutative gyrogroup (see [6]) and is called the Möbius gyrogroup. The identity of $(\mathbb{D}, \oplus)$ is the origin of $\mathbb{C}$ and $\ominus a = -a$ for every $a \in \mathbb{D}$. Moreover, the Möebius multiplication is given by

$$r \otimes a = \tanh(r \tanh^{-1} |a|) \frac{a}{|a|}$$

for any $a \in \mathbb{D}$ and $r \in \mathbb{R}$. The Möbius gyrometric ϱ is given by the equation

$$\varrho(a, b) = |(\ominus a) \oplus b| \left(= \left| \frac{-a+b}{1-\overline{a}b} \right| \right)$$

for every $a, b \in \mathbb{D}$. $(\mathbb{D}, \varrho)$ is a metiric space in itself, and $(\mathbb{D}, \tanh^{-1} \varrho)$ is a metric space again. $(\mathbb{D}, \varrho)$ doesn't satisfy the condition K. However, $(\mathbb{D}, \tanh^{-1} \varrho)$ satisfies the condition K with

$$\text{mid}(a, b) = \frac{1}{2} \otimes (a \boxplus b)$$

and

$$\varphi_a(b) = (2 \otimes a) \oplus (-b)$$

for any $a, b \in \mathbb{D}$. Let $\oplus_0$ be the binary operation on $\mathbb{D}$ induced by $\tanh^{-1} \varrho$ at 0 then $\oplus_0 = \oplus$. Moreover, $(\mathbb{D}, \tanh^{-1} \varrho)$ is a uniquely geodesic metric space. Let $\otimes_0$ be the scalar multiplication on $\mathbb{D}$ induced by $\tanh^{-1} \varrho$ at 0 then $\otimes_0 = \otimes$.

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Gyrovector Spaces on the Open Convex Cone of Positive Definite Matrices

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Abstract

In this article we review an algebraic definition of the gyrogroup and a simplified version of the gyrovector space with two fundamental examples on the open ball of finite-dimensional Euclidean spaces, which are the Einstein and Möbius gyrovector spaces. We introduce the structure of gyrovector space and the gyroline on the open convex cone of positive definite matrices and explore its interesting applications on the set of invertible density matrices. Finally we give an example of the gyrovector space on the unit ball of Hermitian matrices.

Keywords: Gyrogroup, gyrovector space, gyroline, gyromidpoint, positive definite matrix, density matrix.

2010 Mathematics Subject Classification: Primary 20N05; Secondary 15B48.

1. Introduction

In the theory of special relativity founded by Albert Einstein, the velocities are 3-dimensional vectors with speed bounded by the speed of light $s \approx 3 \times 10^8$ m/s, called the *admissible vectors*. The relativistic sum of two admissible vectors $\mathbf{u}$ and $\mathbf{v}$, called the *Einstein vector addition* is given by

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}^T\mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u} \right\}, \tag{1}$$

where $\gamma_{\mathbf{u}} = \dfrac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}}$ is the well-known *Lorentz factor*. We denote as $\mathbf{u}^T\mathbf{v}$ the usual inner product in matrix form. To study abstractly the Einstein vector addi-

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tion in special relativity, A. Ungar [16] has introduced a group-like structure that he called a *gyrogroup* or *gyrocommutative gyrogroup*. Gyrogroups and gyrocommutative gyrogroups are equivalent to Bol-loops and K-loops (Bruck loops) [6,14], respectively.

As a vector space is used in Euclidean geometry, a *gyrovector space* is a mathematical concept introduced by A. Ungar for studying hyperbolic geometry. We review in Section 2 the definitions of gyrogroup and gyrovector space with two important examples, the Einstein gyrovector space and Möbius gyrovector space. The axioms of gyrovector space in this article are more loose than those proposed by A. Ungar, but they also give a plenty of applications [4,6]. For instance, the Einstein gyrovector space and Möbius gyrovector space provide the algebraic tool to study the Beltrami-Klein ball model and the Poincaré ball model of hyperbolic geometry, respectively. It has been proved that the Einstein and Möbius gyrovector spaces are isomorphic. See [7,16] for more details.

In Section 3 we see examples of gyrovector space on the open convex cone of all positive definite matrices and on the set of all invertible density matrices. Furthermore, we show the isomorphism between the gyrovector space of all qubit invertible density matrices and the Einstein gyrovector space on the Bloch sphere, the open unit ball of $\mathbb{R}^3$. It generalizes the result in Theorem 3.4 of [3].

A *gyroline* uniquely determined by given two points on the gyrovector space plays an important role in the concepts of gyrocentroid and gyroparallelogram law. In Section 4 we discuss a gyroline on the open convex cone of all positive definite matrices and on the set of all invertible density matrices. Finally we give a different example of a gyrovector space on the open unit ball of all Hermitian matrices constructed by the exponential and logarithmic maps.

2. Gyrovector Spaces

We review in this section the algebraic structure of a gyrogroup as a natural extension of a group into the regime of the nonassociative algebra. We then introduce a gyrovector space providing the setting for hyperbolic geometry just as a vector space provides the setting for Euclidean geometry. A. A. Ungar has introduced and intensely studied them in a series of papers and books; see [16] and its bibliography.

The binary operation in a gyrogroup is not associative, in general. The breakdown of associativity for gyrogroup operations is salvaged in a modified form, called gyroassociativity. The axioms for a (gyrocommutative) gyrogroup G are reminiscent of those for a (commutative) group.

Definition 2.1. A binary system $(G, \oplus)$ is a *gyrogroup* if it satisfies the following for all $a, b, c \in G$:

(G1) $e \oplus a = a \oplus e = a$ (existence of identity);

(G2) $a \oplus (\ominus a) = (\ominus a) \oplus a = e$ (existence of inverses);

(G3) There is an automorphism $\mathrm{gyr}[a,b] : G \to G$ for each $a, b \in G$ such that

$$a \oplus (b \oplus c) = (a \oplus b) \oplus \mathrm{gyr}[a,b]c \quad \text{(gyroassociativity);}$$

(G4) $\mathrm{gyr}[a \oplus b, b] = \mathrm{gyr}[a,b]$ (loop property).

A gyrogroup $(G, \oplus)$ is *gyrocommutative* if it satisfies

$$a \oplus b = \mathrm{gyr}[a,b](b \oplus a) \quad \text{(gyrocommutativity).}$$

A gyrogroup is *uniquely* 2*-divisible* if for every $b \in G$, there exists a unique element $a \in G$ such that $a \oplus a = b$.

In (G3) the automorphism $\mathrm{gyr}[a,b]$ for each $a, b \in G$ is called the *Thomas gyration* or the *gyroautomorphism*, or simply, the *gyration* generated by a and b. From (G2) and (G3) we have

$$\mathrm{gyr}[a,b]c = \ominus(a \oplus b) \oplus [a \oplus (b \oplus c)]$$

for all $a, b, c \in G$. In Euclidean space it plays a role of rotation in the plane spanned by $\{a, b\}$ leaving the orthogonal complement fixed.

It has been shown in [14] that gyrocommutative gyrogroups are equivalent to Bruck loops with respect to the same operation. It follows that uniquely 2-divisible gyrocommutative gyrogroups are equivalent to B-loops. The two approaches have remained quite distinctive in the literature, but we primarily use a notion of gyrogroups rather than a notion of loops.

For arbitrary fixed positive constant s, we let

$$\mathbf{B}_s = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < s\}$$

be the open s-ball in the n-dimensional vector space $\mathbb{R}^n$. We consider elements in $\mathbb{R}^n$ naturally as column vectors, so that $\mathbf{u}^T\mathbf{v}$ is the usual inner product written in matrix form. We here see two important examples of gyrogroups [16].

Example 2.2. We define the binary operation $\oplus_E$ in $\mathbf{B}_s$ by

$$\mathbf{u} \oplus_E \mathbf{v} = \frac{1}{1 + \frac{\mathbf{u}^T\mathbf{v}}{s^2}} \left\{ \mathbf{u} + \frac{1}{\gamma_{\mathbf{u}}}\mathbf{v} + \frac{1}{s^2}\frac{\gamma_{\mathbf{u}}}{1+\gamma_{\mathbf{u}}}(\mathbf{u}^T\mathbf{v})\mathbf{u} \right\} \tag{2}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_s$, where $\gamma_{\mathbf{u}}$ is the well-known *Lorentz factor* such that

$$\gamma_{\mathbf{u}} = \frac{1}{\sqrt{1 - \frac{\|\mathbf{u}\|^2}{s^2}}}.$$

The equation (2) is called the *Einstein addition* of relativistically admissible velocities, introduced by Einstein in his 1905 paper. The binary system $(\mathbf{B}_s, \oplus_E)$ forms a gyrocommutative gyrogroup, called the *standard real relativistic gyrogroup* or the *Einstein gyrogroup*.

Example 2.3. In the open s-ball $\mathbf{B}_s$, we define the binary operation $\oplus_M$ by

$$\mathbf{u} \oplus_M \mathbf{v} = \frac{1}{1 + \frac{2}{s^2}\mathbf{u}^T\mathbf{v} + \frac{1}{s^4}\|\mathbf{u}\|^2\|\mathbf{v}\|^2} \left\{ \left(1 + \frac{2\mathbf{u}^T\mathbf{v}}{s^2} + \frac{\|\mathbf{v}\|^2}{s^2}\right)\mathbf{u} + \left(1 - \frac{\|\mathbf{u}\|^2}{s^2}\right)\mathbf{v} \right\} \quad (3)$$

for any $\mathbf{u}, \mathbf{v} \in \mathbf{B}_s$. The equation (3) is called the *Möbius addition*, known as Möbius translation on the open s-ball (see formula (4.5.5) of [13]). The binary system $(\mathbf{B}_s, \oplus_M)$ forms also a gyrocommutative gyrogroup, called the *nonstandard real relativistic gyrogroup* or *Möbius gyrogroup*.

Suksumran and Wiboonton [15] have recently shown by using the Clifford algebra that the open ball $\mathbf{B}_s$ equipped with binary operations $\oplus_E$ and $\oplus_M$, respectively, is a uniquely 2-divisible gyrocommutative gyrogroup.

In the same way that vector spaces are commutative groups of vectors that admit scalar multiplication, gyrovector spaces are gyrocommutative gyrogroups of gyrovectors that admit properly scalar multiplication. We give a definition of gyrovector spaces slightly different from Definition 6.2 in [16].

Definition 2.4. A gyrocommutative gyrogroup $(G, \oplus)$ equipped with a scalar multiplication

$$(t, x) \mapsto t \otimes x : \mathbb{R} \times G \to G$$

is called a *gyrovector space* if it satisfies the following for $s, t \in \mathbb{R}$ and $a, b, c \in G$.

(V1) $1 \otimes a = a$, $0 \otimes a = t \otimes e = e$, and $(-1) \otimes a = \ominus a$.

(V2) $(s + t) \otimes a = s \otimes a \oplus t \otimes a$.

(V3) $(st) \otimes a = s \otimes (t \otimes a)$.

(V4) $\mathrm{gyr}[a, b](t \otimes c) = t \otimes \mathrm{gyr}[a, b]c$.

Definition 2.5. A *topological gyrovector space* is a gyrovector space $(G, \oplus, \otimes)$ equipped with Hausdorff topology such that both $\oplus : G \times G \to G$ and $\otimes : \mathbb{R} \times G \to G$ are continuous.

Remark 2.6. In a topological gyrovector space $(G, \oplus, \otimes)$, it has been proved from [4] that

$$\mathrm{gyr}[s \otimes a, t \otimes a] = \mathrm{id}_G$$

for any $s, t \in \mathbb{R}$ and $a \in G$, where id denotes the identity map on G.

We have seen two distinctive examples of gyrocommutative gyrogroups in the open s-ball $\mathbf{B}_s$ of the n-dimensional vector space $\mathbb{R}^n$. Via defining a scalar multiplication we see two common examples of inner product gyrovector spaces, also corresponding to two models of hyperbolic geometry.

Example 2.7. Let $\mathbf{B}_s$ be the Einstein gyrogroup with Einstein addition $\oplus_E$, or the Möbius gyrogroup with Möbius addition $\oplus_M$, respectively. We define a map $\otimes : \mathbb{R} \times \mathbf{B}_s \to \mathbf{B}_s$ by

$$\begin{aligned} t \otimes \mathbf{v} &= s \cdot \frac{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^t - \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^t}{\left(1 + \frac{\|\mathbf{v}\|}{s}\right)^t + \left(1 - \frac{\|\mathbf{v}\|}{s}\right)^t} \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= s \tanh\left(t \tanh^{-1} \frac{\|\mathbf{v}\|}{s}\right) \frac{\mathbf{v}}{\|\mathbf{v}\|}, \end{aligned} \tag{4}$$

for $t \in \mathbb{R}$ and $\mathbf{v}(\neq \mathbf{0}) \in \mathbf{B}_s$, and define $t \otimes \mathbf{0} := \mathbf{0}$. We call $(\mathbf{B}_s, \oplus_E, \otimes)$ and $(\mathbf{B}_s, \oplus_M, \otimes)$ the *Einstein gyrovector space* and the *Möbius gyrovector space*, respectively.

The Beltrami-Klein ball model of hyperbolic geometry is algebraically regulated by Einstein gyrovector spaces. The geodesics of this model, called gyrolines, are Euclidean straight lines in the open s-ball. On the other hand, the Poincaré ball model of hyperbolic geometry is algebraically regulated by Möbius gyrovector spaces. The geodesics of this model are Euclidean circular arcs in the open s-ball that intersect the boundary of the ball orthogonally.

3. On the Cone of Positive Definite Matrices

We have seen two fundamental examples of a gyrovector space, Einstein gyrovector space and Möbius gyrovector space, on the open s-ball $\mathbf{B}_s$. In this section we give an example of a gyrovector space on the open convex cone $\mathbb{P}$ of all $n \times n$ positive definite matrices.

Example 3.1. [4, Example 2.2, Example 3.2] Let $\mathbb{P}$ be an open convex cone of positive definite Hermitian matrices. Define the binary operation $\oplus$ and a scalar multiplication $\circ$ by

$$\begin{aligned} &\oplus : \mathbb{P} \times \mathbb{P} \to \mathbb{P},\ A \oplus B = A^{1/2} B A^{1/2}, \\ &\circ : \mathbb{R} \times \mathbb{P} \to \mathbb{P},\ t \circ A = A^t \end{aligned}$$

for any $A, B \in \mathbb{P}$ and $t \in \mathbb{R}$. Then the system $(\mathbb{P}, \oplus, \circ)$ forms a gyrovector space, and the gyroautomorphism generated by A and B is given by

$$\mathrm{gyr}[A, B]C = U(A, B) C U(A, B)^{-1}, \tag{5}$$

where $U(A, B) = (A^{1/2} B A^{1/2})^{-1/2} A^{1/2} B^{1/2}$ is a unitary part of the polar decomposition for $A^{1/2} B^{1/2}$ such that

$$A^{1/2} B^{1/2} = (A \oplus B)^{1/2} U(A, B).$$

Indeed, let us check (V4). For any $A, B, X \in \mathbb{P}$

$$\begin{aligned}\mathrm{gyr}[A,B](t \circ X) &= U(A,B)X^t U(A,B)^{-1} \\ &= U(A,B)\exp(t \log X)U(A,B)^{-1} \\ &= \exp[t \log U(A,B)XU(A,B)^{-1}] \\ &= [U(A,B)XU(A,B)^{-1}]^t = t \circ \mathrm{gyr}[A,B]X.\end{aligned}$$

One can easily see that the binary operation $\oplus$ and the scalar multiplication $\circ$ are both continuous. Thus, the system $(\mathbb{P}, \oplus, \circ)$ is a topological gyrovector space.

Remark 3.2. The inner product on $\mathrm{M}_n(\mathbb{C})$, the vector space of all $n \times n$ matrices with complex entries, is naturally defined as $\langle A, B\rangle = \mathrm{tr}(AB^*)$, where X^* is a complex conjugate transpose of a matrix X. The gyroautomorphism on $\mathbb{P}$ preserves the inner product, and so the norm induced by inner product. Indeed, for any $A, B, X, Y \in \mathbb{P}$

$$\begin{aligned}\langle \mathrm{gyr}[A,B]X, \mathrm{gyr}[A,B]Y\rangle &= \mathrm{tr}[U(A,B)XU(A,B)^{-1}(U(A,B)YU(A,B)^{-1})^*] \\ &= \mathrm{tr}[U(A,B)XY^*U(A,B)^*] \\ &= \mathrm{tr}[XY^*] = \langle X, Y\rangle.\end{aligned}$$

A. Ungar has explained a gyrogroup structure for qubit density matrices in Chapter 9, [16]. We now see an example of gyrovector space for arbitrary dimensional density matrices.

Example 3.3. [3] Let $\mathbb{D}_n$ be a set of all $n \times n$ invertible density matrices, which are positive definite Hermitian matrices of trace 1. We define a binary operation $\odot$ and a scalar multiplication $\star$ given by

$$\begin{aligned}\odot :& \mathbb{D}_n \times \mathbb{D}_n \to \mathbb{D}_n, \ \rho \odot \sigma = \frac{\rho^{\frac{1}{2}}\sigma\rho^{\frac{1}{2}}}{\mathrm{tr}(\rho\sigma)} = \frac{\rho \oplus \sigma}{\mathrm{tr}(\rho \oplus \sigma)} \\ \star :& \mathbb{R} \times \mathbb{D}_n \to \mathbb{D}_n, \ t * \rho = \frac{\rho^t}{\mathrm{tr}(\rho^t)} = \frac{t \circ \rho}{\mathrm{tr}(t \circ \rho)}\end{aligned}$$

for any $\rho, \sigma \in \mathbb{D}_n$ and $t \in \mathbb{R}$. Then $(\mathbb{D}_n, \odot, \star)$ is a gyrovector space. Note that the identity element in $(\mathbb{D}_n, \odot, \star)$ is $\frac{1}{n} I_n$ and the inverse of ρ is $(-1) \star \rho = \frac{1}{\mathrm{tr}(\rho^{-1})}\rho^{-1}$, where I_n denotes the $n \times n$ identity matrix.

In [3, Theorem 3.4] it has been shown the relationship between the Einstein gyrogroup $(\mathbf{B}_{s=1}, \oplus_E)$ and the gyrogroup $(\mathbb{D}_2, \odot)$ of 2×2 invertible density matrices. In other words, the map

$$\rho : (\mathbf{B}_{s=1}, \oplus_E) \to (\mathbb{D}_2, \odot), \ \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \mapsto \rho_{\mathbf{v}} = \frac{1}{2}\begin{pmatrix} 1 + v_3 & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 \end{pmatrix}$$

is a gyrogroup isomorphism. We give an extension of the isomorphism between gyrovector spaces.

Theorem 3.4. *The Einstein gyrovector space* $(\mathbf{B}_{s=1}, \oplus_E, \otimes)$ *and the gyrovector space* $(\mathbb{D}_2, \odot, \star)$ *of* 2×2 *invertible density matrices are isomorphic.*

Proof. It remains to show that

$$\rho_{t\otimes\mathbf{v}} = t \star \rho_{\mathbf{v}} = \frac{1}{\operatorname{tr}(\rho_{\mathbf{v}}^t)} \rho_{\mathbf{v}}^t$$

for any $t \in \mathbb{R}$. Set

$$T = \{t \in \mathbb{R} : \rho_{t\otimes\mathbf{v}} = \frac{1}{\operatorname{tr}(\rho_{\mathbf{v}}^t)} \rho_{\mathbf{v}}^t \text{ for any } \mathbf{v} \in \mathbf{B}_{s=1}\}$$

Our goal is to show that the set T contains all dyadic rational numbers, since this implies by the density of dyadic rational numbers that $T = \mathbb{R}$.

Easily $0, 1 \in T$. Moreover, $\dfrac{1}{2} \in T$. Indeed,

$$\frac{1}{2} \otimes \mathbf{v} = \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \mathbf{v}.$$

So we obtain from [3, Lemma 3.3] that

$$\begin{aligned}
\rho_{\frac{1}{2}\otimes\mathbf{v}} &= \frac{1}{2} \begin{pmatrix} 1 + \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1} v_3 & \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}(v_1 - iv_2) \\ \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1}(v_1 + iv_2) & 1 - \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}}+1} v_3 \end{pmatrix} \\
&= \frac{1}{2} \frac{\gamma_{\mathbf{v}}}{\gamma_{\mathbf{v}} + 1} \begin{pmatrix} 1 + v_3 + \frac{1}{\gamma_{\mathbf{v}}} & v_1 - iv_2 \\ v_1 + iv_2 & 1 - v_3 + \frac{1}{\gamma_{\mathbf{v}}} \end{pmatrix} \\
&= \frac{1}{\operatorname{tr}(\rho_{\mathbf{v}}^{1/2})} \rho_{\mathbf{v}}^{1/2}.
\end{aligned}$$

This gives us that $\dfrac{t}{2} \in T$ whenever $t \in T$.

From $\rho_{\mathbf{u}\oplus\mathbf{v}} = \rho_{\mathbf{u}} \odot \rho_{\mathbf{v}}$ we have

$$\rho_{2\otimes\mathbf{v}} = \frac{1}{\operatorname{tr}(\rho_{\mathbf{v}}^2)} \rho_{\mathbf{v}}^2 \text{ and } \rho_{(-1)\otimes\mathbf{v}} = \frac{1}{\operatorname{tr}(\rho_{\mathbf{v}}^{-1})} \rho_{\mathbf{v}}^{-1}.$$

That is, $2t \in T$ and $-t \in T$ whenever $t \in T$. Then for $s, t \in T$

$$\begin{aligned}
\rho_{(2s-t)\otimes\mathbf{v}} &= \rho_{(2s)\otimes\mathbf{v}\oplus(-t)\otimes\mathbf{v}} \\
&= \rho_{(2s)\otimes\mathbf{v}} \odot \rho_{(-t)\otimes\mathbf{v}} \\
&= (2s) \circ \rho_{\mathbf{v}} \odot (-t) \circ \rho_{\mathbf{v}} \\
&= (2s - t) \circ \rho_{\mathbf{v}}.
\end{aligned}$$

In other words, $2s - t \in T$ whenever $s, t \in T$. So the set T contains all dyadic rational numbers in $\mathbb{R}$. $\square$

It is still an open question whether or not the Einstein gyrovector space $(\mathbf{B}_{s=1}, \oplus_E, \otimes)$ and the gyrovector space $(\mathbb{D}_n, \odot, \star)$ are isomorphic for $n > 2$.

4. Gyrolines and Gyromidpoints

The *gyroline* passing through the points a and b in the gyrovector space $(G, \oplus, \otimes)$ is defined in Definition 6.19, [16], by

$$L : \mathbb{R} \times G \times G \to G, \ L(t; a, b) = a \oplus t \otimes (\ominus a \oplus b). \tag{6}$$

The gyroline is uniquely determined by given points, and a left gyrotranslation of a gyroline is again a gyroline by Theorem 6.21 in [16]. In other words,

$$x \oplus L(t; a, b) = L(t; x \oplus a, x \oplus b)$$

for any $x \in G$.

Example 4.1. From Example 3.1 we obtain the gyroline on $(\mathbb{P}, \oplus, \circ)$ passing through A and B such that

$$L(t; A, B) = A \oplus t \circ ((-1) \circ A \oplus B) = A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2} \tag{7}$$

for $t \in [0, 1]$. This is usually called the *weighted geometric mean* of A and B, and denoted by $L(t; A, B) = A\#_t B$. Moreover, it is known in [1, Chapter 6] as a unique geodesic connecting from A to B on $\mathbb{P}$ with respect to the Riemannian trace metric δ:

$$\delta(A, B) = \| \log(A^{-1/2}BA^{-1/2})\|_F,$$

where $\|X\|_F$ denotes the Frobenius norm of X. Note that for any $A, B, C, D \in \mathbb{P}$ and $t \in [0, 1]$

$$\delta(A\#_t B, C\#_t D) \leq (1 - t)\delta(A, C) + t\delta(B, D).$$

It is also satisfied that for any invertible matrix M,

$$M(A\#_t B)M^* = (MAM^*)\#_t(MBM^*).$$

This implies that a left gyrotranslation of a gyroline is again a gyroline.

Since the map $t \in [0, 1] \mapsto A\#_t B$ for any $A, B \in \mathbb{P}$ is introduced by two-variable geometric mean, a variety of approaches to extend it to multivariable geometric means have been recently developed. Among them we introduce a least squares mean as a hot topic of matrix means.

Remark 4.2. It is known in [1, Chapter 6] that $(\mathbb{P}, \delta)$ is a Bruhat-Tits space (a Hadamard space or a non-positive curvature space), which is a complete metric space satisfying the semi-parallelogram law. For an n-dimensional positive probability vector $\omega = (w_1, \dots, w_n)$ and positive definite matrices $A_1, \dots, A_n$, there exists a unique minimizer of the weighted sum of squares of Riemannian distances to each point.

$$\arg\min_{Z \in \mathbb{P}} \sum_{i=1}^{n} w_i \delta^2(Z, A_i). \tag{8}$$

This is called the least squares mean (the Karcher mean or Riemannian barycenter), and denoted by $\Lambda(\omega; A_1, \ldots, A_n)$. Vanishing the gradient of the objective function $f(Z) = \sum_{i=1}^{n} w_i \delta^2(Z, A_i)$, we obtain that the least squares mean coincides with the unique positive solution of the Karcher equation

$$\sum_{i=1}^{n} w_i \log(Z^{1/2} A_i^{-1} Z^{1/2}) = O. \tag{9}$$

Many interesting properties for the least squares mean including the monotonicity have been studied; see [2, 8–11].

A. Ungar has introduced in [16] a *gyrocentroid* as a barycenter of points on the gyrovector space. It would be interesting to find a connection between the least squares mean and the gyrocentroid.

We finally give a formula of the gyroline on the gyrovector space $(\mathbb{D}_n, \odot, \star)$.

Theorem 4.3. *For any $\rho, \sigma \in (\mathbb{D}_n, \odot, \star)$ and $t \in [0, 1]$*

$$L(t; \rho, \sigma) = \frac{1}{\operatorname{tr}(\rho \#_t \sigma)} \rho \#_t \sigma.$$

Proof. Let $\rho, \sigma \in (\mathbb{D}_n, \odot, \star)$ and $t \in [0, 1]$. From Example 3.3 we have

$$(-1) \star \rho \odot \sigma = \frac{(-1) \circ \rho \oplus \sigma}{\operatorname{tr}((-1) \circ \rho \oplus \sigma)},$$

and

$$t \star [(-1) \star \rho \odot \sigma] = \frac{t \circ [(-1) \circ \rho \oplus \sigma]}{\operatorname{tr}(t \circ [(-1) \circ \rho \oplus \sigma])}.$$

Thus, we obtain

$$L(t; \rho, \sigma) = \rho \odot t \star [(-1) \star \rho \odot \sigma] = \frac{\rho \oplus t \circ [(-1) \circ \rho \oplus \sigma]}{\operatorname{tr}(\rho \oplus t \circ [(-1) \circ \rho \oplus \sigma])} = \frac{\rho \#_t \sigma}{\operatorname{tr}(\rho \#_t \sigma)}.$$

$\square$

Remark 4.4. It has been shown in Proposition 3.8 of [5] that the map $L(t; \rho, \sigma)$ is a minimal geodesic on $\mathbb{D}_n$ with respect to the Hilbert projective metric. In Theorem 4.2 and Remark 4.3 of [5], moreover,

$$L\left(\frac{1}{2}; \rho_{\mathbf{u}}, \rho_{\mathbf{v}}\right) = \frac{\rho_{\mathbf{u}} \# \rho_{\mathbf{v}}}{\operatorname{tr}(\rho_{\mathbf{u}} \# \rho_{\mathbf{v}})} = \frac{\gamma_{\mathbf{u}} \mathbf{u} + \gamma_{\mathbf{v}} \mathbf{v}}{\gamma_{\mathbf{u}} + \gamma_{\mathbf{v}}}$$

for any $\mathbf{u}, \mathbf{v} \in \mathbb{B}_{s=1}$. This is known as the *Einstein gyromidpoint* in Theorem 6.92, [16].

5. Applications and Remarks

We have seen in Example 3.1 that $(\mathbb{P}, \oplus, \circ)$ is a gyrovector space. Let us denote $\mathbb{H}$ as the real vector space of all Hermitian matrices. The exponential map from $\mathbb{H}$ to $\mathbb{P}$ given by

$$\exp A = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$$

is a diffeomorphism, and its inverse is the logarithm map denoted by log.

We define a map $f : \mathbb{H} \to \mathbf{B}(\mathbb{H}) := \{A \in \mathbb{H} : \|A\| < 1\}$ by

$$f(X) := \tanh \|X\| \frac{X}{\|X\|}, \ X \neq O,$$

and $f(O) := O$, where $\|\cdot\|$ denotes the Frobenius or Hilbert-Schmidt norm. Since the function $g(x) = \dfrac{\tanh x}{x}$ for $x > 0$ is bijective, so is f. Then the composition defined as

$$g = f \circ \log : \mathbb{P} \to \mathbf{B}(\mathbb{H}), \ g(A) := \tanh \|\log A\| \frac{\log A}{\|\log A\|}, \ A \neq I \tag{10}$$

and $g(I) = O$, is also a bijection. It means that every element in $\mathbf{B}(\mathbb{H})$ can be uniquely written as $g(A)$ for some $A \in \mathbb{P}$.

Furthermore, defining a binary operation $\diamond$ on $\mathbf{B}(\mathbb{H})$ by

$$g(A) \diamond g(B) := g(A \oplus B)$$

gives us an isomorphism g from $(\mathbb{P}, \oplus)$ onto $(\mathbf{B}(\mathbb{H}), \diamond)$. So the binary system $(\mathbf{B}(\mathbb{H}), \diamond)$ is a gyrocommutative gyrogroup. Also, setting

$$t * g(A) := g(t \circ A) = g(A^t)$$

for all $t \in \mathbb{R}$ and $A \in \mathbb{P}$ gives us a gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$. Indeed, the following are satisfied for any $s, t \in \mathbb{R}$ and $A, B, X \in \mathbb{P}$.

(V1) $1 * g(A) = g(A)$ and $0 * g(A) = g(I) = O$.

(V2) Using an isomorphism g we have

$$\begin{aligned} (s+t) * g(A) &= g(A^{s+t}) = g(A^{s/2} A^t A^{s/2}) \\ &= g(A^s \oplus A^t) = g(A^s) \diamond g(A^t) = s * g(A) \diamond t * g(A). \end{aligned}$$

(V3) $(st) * g(A) = g(A^{st}) = g((A^s)^t) = t * (s * g(A))$.

(V4) We note by the gyroassociativity and an isomorphism g that

$$\mathrm{gyr}[g(A), g(B)] g(X) = g(\mathrm{gyr}[A, B] X). \tag{11}$$

Then

$$\begin{aligned}\mathrm{gyr}[g(A), g(B)](t * g(X)) &= \mathrm{gyr}[g(A), g(B)]g(t \star X)\\ &= g(\mathrm{gyr}[A, B](t \star X)) = g(t \star \mathrm{gyr}[A, B]X)\\ &= t * g(\mathrm{gyr}[A, B]X) = t * \mathrm{gyr}[g(A), g(B)]g(X).\end{aligned}$$

Remark 5.1. Since $\mathrm{gyr}[s \circ A, t \circ A] = \mathrm{id}_{\mathbb{P}}$ on the gyrovector space $(\mathbb{P}, \oplus, \circ)$ for any $s, t \in \mathbb{R}$, we also have

$$\begin{aligned}\mathrm{gyr}[s * g(A), t * g(A)]g(X) &= \mathrm{gyr}[g(s \circ A), g(t \circ A)]g(X)\\ &= g(\mathrm{gyr}[s \circ A, t \circ A]X) = g(X).\end{aligned}$$

The second equality follows from the equation (11). This means that

$$\mathrm{gyr}[s * g(A), t * g(A)] = \mathrm{id}_{\mathbf{B}(\mathbb{H})}\,.$$

The following gives us a formula of the gyroline on the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$.

Proposition 5.2. *For given $A, B \in \mathbb{P}$, the gyroline connecting from $g(A)$ to $g(B)$ on the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$ is $g(A \#_t B)$.*

Proof. By the general equation (6) of gyroline on the gyrovector space,

$$\begin{aligned}L(t; g(A), g(B)) &= g(A) \diamond t * ((-1) * g(A) \diamond g(B))\\ &= g(A) \diamond t * (g(A^{-1}) \diamond g(B))\\ &= g(A) \diamond t * g(A^{-1/2}BA^{-1/2})\\ &= g(A) \diamond g((A^{-1/2}BA^{-1/2})^t)\\ &= g(A^{1/2}(A^{-1/2}BA^{-1/2})^t A^{1/2}).\end{aligned}$$

□

Remark 5.3. On the gyrovector space $(\mathbf{B}(\mathbb{H}), \diamond, *)$, it would be interesting to investigate any geometric aspect such as metric relations and gyrocentroids.

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An Extension of Poincaré Model of Hyperbolic Geometry with Gyrovector Space Approach

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Abstract

The aim of this paper is to show the importance of analytic hyperbolic geometry introduced in [9]. In [1], Ungar and Chen showed that the algebra of the group $SL(2,\mathbb{C})$ naturally leads to the notion of gyrogroups and gyrovector spaces for dealing with the Lorentz group and its underlying hyperbolic geometry. They defined the Chen addition and then Chen model of hyperbolic geometry. In this paper, we directly use the isomorphism properties of gyrovector spaces to recover the Chen's addition and then Chen model of hyperbolic geometry. We show that this model is an extension of the Poincaré model of hyperbolic geometry. For our purpose we consider the Poincaré plane model of hyperbolic geometry inside the complex open unit disc $\mathbb{D}$. Also we prove that this model is isomorphic to the Poincaré model and then to other models of hyperbolic geometry. Finally, by gyrovector space approach we verify some properties of this model in details in full analogue with Euclidean geometry.

Keywords: Hyperbolic geometry, gyrogroup, gyrovector space, Poincaré model, analytic hyperbolic geometry.

2010 Mathematics Subject Classification: 51M10, 51N99, 20N99.

1. Introduction

Gyrogroups are noncommutative and nonassociative algebraic structures and this noncommutativity-nonassociativity turns out to be generated by the Thomas precession, well-known in the special theory of relativity. Gyrogroups also revealed themselves to be specially fitting in order to deal with formerly unsolved problems in special relativity (e.g. the problem of determining the Lorentz transformation

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that links given initial and final time-like 4-vectors). Gyrogroups are split up into gyrocommutative gyrogroups and nongyrocommutative. It turns out that introducing (gyrocommutative) gyrogroups, Ungar gave a concrete physical realization to formerly well-known algebraic systems called K-loops discovered by Helmut Karzel(e. g. see [2, 3]) in his study of neardomains. Since his 1988 pioneering paper [7] Ungar has studied gyrogroups and gyrovector spaces in several books [8–14] and many papers.

Some gyrocommutative gyrogroups admit a multiplication which turn them to a gyrovector space. Gyrovector spaces, in turn, provide the setting for hyperbolic geometry in the same way that vector spaces provide the setting for Euclidean geometry, thus enabling the two geometries to be unified. Armed with a gyrovector space structure, hyperbolic geometry is perfect for use in relativity physics. Abraham A. Ungar introduced the analytic hyperbolic geometry in [9]. The nonassociative algebra of gyrovector spaces is the framework for analytic hyperbolic geometry just as the associative algebra of vector spaces is the framework for analytic Euclidean geometry. Moreover, gyrovector spaces include vector spaces as a special, degenerate case corresponding to trivial gyroautomorphisms. Hence, Ungar gyrovector space approach forms the theoretical framework for uniting Euclidean and hyperbolic geometry.

In this paper, our aim is to use the gyrovector space approach of Ungar to investigate the analytical hyperbolic geometry. For our purpose we consider the Poincaré model of hyperbolic geometry defined inside the complex open unit disc $\mathbb{D} = \{a \in \mathbb{C} \mid |a| = \sqrt{a\bar{a}} < 1\}$ where $\bar{a}$ is the conjugate of a. Using the gyrovector spaces isomorphism, we extend the Poincaré model of hyperbolic geometry to the whole plane $\mathbb{C}$ which is called in [1] Chen model of hyperbolic geometry. We recover Chen gyrogroup and Chen gyrovector space of [1]. But our approach is different from [1]. We directly use the isomorphism properties of gyrovector spaces. As an application of gyrovector spaces as the algebraic settings of analytical hyperbolic geometry, we obtain some concepts of the new model by using gyrovector space properties.

2. Preliminaries and Well-Known Results

Definition 2.1. (Gyrogroups). A groupoid $(G, +)$ is a gyrogroup if its binary operation satisfies the following axioms.

G_1. In G there is at least one element, 0, called a left identity, satisfying $0 + a = a$ for all $a \in G$.

G_2. There is an element $0 \in G$ satisfying axiom G_1 such that for each $a \in G$ there is an element $-a \in G$, called a left inverse of a, satisfying $-a + a = 0$.

G_3. For any $a, b, c \in G$ there exists a unique element $\text{gyr}[a, b]c \in G$ such that the binary operation obeys the left gyroassociative law $a + (b + c) = (a + b) + \text{gyr}[a, b]c$.

G_4. The map $\text{gyr}[a, b] : G \to G$ given by $c \mapsto \text{gyr}[a, b]c$ is an automorphism of the groupoid $(G, +)$, i.e. $\text{gyr}[a, b] \in Aut(G, +)$ and the automorphism $\text{gyr}[a, b]$ of G is called the gyroautomorphism of G generated by $a, b \in G$. The operator $\text{gyr} : G \times G \to Aut(G, +)$ is called the gyrator of G.

G_5. Finally, the gyroautomorphism $\text{gyr}[a, b]$ generated by any $a, b \in G$ possesses the left loop property $\text{gyr}[a, b] = \text{gyr}[a, b + a]$.

Definition 2.2. A gyrogroup $(G, +)$ is a gyrocommutative gyrogroup if its binary operation obeys the gyrocommutative law $a + b = \text{gyr}[a, b](b + a)$

Remark 1. Another equivalent definition of gyrocommutative gyrogroup, which also are called K-loops, comes from H. Karzel(cf., [2, 3]) as follows:
A loop $(P, +)$ is said to be a K-loop if the following properties hold:
For all $a, b \in P$,

$\mathbf{K_1}$: $\text{gyr}[a, b] \in Aut(P, +)$

$\mathbf{K_2}$: $\text{gyr}[a, b] = \text{gyr}[a, b + a]$

$\mathbf{K_3}$: $-(a + b) = (-a) + (-b)$

Example 2.3. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the complex open unit disc and $\oplus_E$ be the Einstein's velocity addition in Beltrami-Klein model of hyperbolic geometry, hence for $a, b \in \mathbb{D}$,

$$a \oplus_E b = \frac{a + b}{1+ < a, b >} + \frac{\gamma_a}{1 + \gamma_a}\left(\frac{< a, b > \ a - |a|^2 \ b}{1+ < a, b >} \right)$$

where $\gamma_a = \frac{1}{\sqrt{1-|a|^2}}$. It is proved that $(\mathbb{D}, \oplus_E)$ is a gyrocommutative gyrogroup(e.g see [4] and [1, 9]).

Example 2.4. Let $\mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$ be the complex open unit disc of Poincaré hyperbolic plane. By the Möbius transformation $z \mapsto e^{i\theta} \frac{a+z}{1+a\bar{z}}$ we define $\oplus_M$ on $\mathbb{D}$ as $a \oplus_M b = \frac{a+b}{1+\bar{a}b}$. Then $(\mathbb{D}, \oplus_M)$ is a gyrocommutative gyrogroup, which is called Möbius gyrogroup(e.g. see [1, 9]).

2.1 Gyrovector Space

Gyrovector spaces provide the setting for hyperbolic geometry just as vector spaces provide the setting for Euclidean geometry. The elements of a gyrovector space are called points. Any two points of a gyrovector space give rise to a gyrovector.

Definition 2.5. (Real Inner Product Gyrovector Spaces). A real inner product gyrovector space $(G, \oplus, \otimes)$ (gyrovector space, in short) is a gyrocommutative gyrogroup $(G, \oplus)$ that obeys the following axioms:
(1) G is a subset of a real inner product vector space V called the carrier of $G, G \subset V$, from which it inherits its inner product, $< ., . >$, and norm, $|\cdot|$, which are invariant under gyroautomorphisms, that is, $< \mathrm{gyr}[u, v]a, \mathrm{gyr}[u, v]b >=< a, b >$ for all points $a, b, u, v \in G$.
(2) G admits a scalar multiplication, $\otimes$, possessing the following properties. For all real numbers $r, r_1, r_2 \in \mathbb{R}$ and all points $a \in G$:
V_1 $1 \otimes a = a$.
V_2 Scalar Distributive Law: $(r_1 + r_2) \otimes a = r_1 \otimes a \oplus r_2 \otimes a$.
V_3 Scalar Associative Law: $r_1 \otimes (r_2 \otimes a) = (r_1 r_2) \otimes a$.
V_4 Scaling Property: $\frac{|r| \otimes a}{|r \otimes a|} = \frac{a}{|a|}$.
V_5 Gyroautomorphism Property: $\mathrm{gyr}[\mathrm{u}, \mathrm{v}](r \otimes a) = r \otimes \mathrm{gyr}[u, v]a$.
V_6 Identity Automorphism: $\mathrm{gyr}[r_1 \otimes v, r_2 \otimes v] = I$.
(3) Real vector space structure $(|G|, \oplus, \otimes)$ for the set $|G|$ of one dimensional "vectors" $|G| = \{\pm|a| \mid a \in G\} \subset \mathbb{R}$ with vector addition $\oplus$ and scalar multiplication $\otimes$, such that for all $r \in \mathbb{R}$ and $a, b \in G$,
V_7 Homogeneity Property: $|r \otimes a| = |r| \otimes |a|$.
V_8 Gyrotriangle Inequality: $|a \oplus b| \leq |a| \oplus |b|$.

Definition 2.6. (Gyrovector Space Isomorphisms). Let $(G, \oplus_G, \otimes_G)$ and $(H, \oplus_H, \otimes_H)$ be two gyrovector spaces. A bijective map $\phi : G \to H$ is an isomorphism from G to H if for all $u, v \in G$ and $r \in \mathbb{R}$,
(1) $\phi(u \oplus_G v) = \phi(u) \oplus_H \phi(v)$,
(2) $\phi(r \otimes_G u) = r \otimes_H \phi(u)$ and
(3) $< \frac{u}{|u|}, \frac{v}{|v|} >=< \frac{\phi(u)}{|\phi(u)|}, \frac{\phi(v)}{|\phi(v)|} >$.

Example 2.7. We can form by

$$\otimes : \mathbb{R} \times \mathbb{D} \to \mathbb{D}; (r, a) \mapsto r \otimes a := \tanh(r \cdot \tanh^{-1}(|a|)) \cdot \frac{a}{|a|}, \text{ if } r \neq 0 \text{ and } r \otimes 0 := 0$$

a multiplication of scalars with elements of $\mathbb{D}$. Then $\otimes$ turns gyrogroups $(\mathbb{D}, \oplus_E)$ and $(\mathbb{D}, \oplus_M)$ into gyrovector spaces $(\mathbb{D}, \oplus_E, \otimes_E)$ and $(\mathbb{D}, \oplus_M, \otimes_M)$. The gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ provide algebraic settings for the Beltrami-Klein model of hyperbolic geometry and $(\mathbb{D}, \oplus_M, \otimes_M)$ provide algebraic settings for Poincaré model of hyperbolic geometry. Since $a \oplus_M b = \frac{1}{2} \otimes (2 \otimes a \oplus_E 2 \otimes b)$, Einstein gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ and Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$ are isomorphic. It means Beltrami-Klein model and Poincaré model are isomorphic. The coincidence $\otimes_E = \otimes_M = \otimes$ stems from the fact that for parallel vectors in $\mathbb{D}$, Möbius addition and Einstein addition coincide (cf., [11]).

3. Results

3.1 An Extension of Möbius Gyrovector Space to the Whole Space $\mathbb{C}$

In this section we give a gyrovector space isomorphic to the Möbius gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$. Actually we extend the gyrovector space $(\mathbb{D}, \oplus_M, \otimes_M)$ to the whole gyrovector space $(\mathbb{C}, \oplus, \otimes)$ as follows. Let for $a \in \mathbb{D}$,

$$|a|^2 = a\bar{a}, \ \gamma_a := \frac{1}{\sqrt{1-|a|^2}}, \ \beta_a = \frac{1}{\sqrt{1+|a|^2}}$$

Define $\phi : \mathbb{D} \longrightarrow \mathbb{C}$ by $a \mapsto a\gamma_a$. Therefore $\phi^{-1} : \mathbb{C} \to \mathbb{D}$ is given by $a \mapsto a\beta_a$. Now we extend the Möbius addition $\oplus_M$ to $\oplus$ on $\mathbb{C}$ by the bijection map ϕ as follows:

$$\forall a, b \in \mathbb{C}, \quad a \oplus b := \phi(\phi^{-1}(a) \oplus_M \phi^{-1}(b))$$

Therefore we have

$$a \oplus b = \lambda_{a,b} \frac{a\beta_a + b\beta_b}{1 + \bar{a}b\beta_a\beta_b} = \lambda_{a,b}(a\beta_a \oplus_M b\beta_b)$$

where $\lambda_{a,b} = \sqrt{\frac{1}{\beta_a^2\beta_b^2} + \frac{2<a,b>}{\beta_a\beta_b} + |a|^2|b|^2}$, or equivalently

$$a \oplus b = \frac{1 + a\bar{b}\beta_a\beta_b}{|1 + a\bar{b}\beta_a\beta_b|}(\frac{a}{\beta_b} + \frac{b}{\beta_a})$$

It is not difficult to show that $(\mathbb{C}, \oplus)$ is a gyrocommutative gyrogroup with identity 0 and

$$\mathrm{gyr}[a,b] = \frac{1 + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b}$$

We only prove G_5. Firstly, note that $\beta_{a\oplus b} = \frac{\beta_a\beta_b}{|1+\bar{a}b\beta_a\beta_b|} = \beta_{b\oplus a}$. Therefore

$$\begin{aligned}
\mathrm{gyr}[a, b \oplus a] &= \frac{1 + a(\overline{b \oplus a})\beta_a\beta_{b\oplus a}}{1 + \bar{a}(b \oplus a)\beta_a\beta_{b\oplus a}} \\
&= \frac{1 + a\frac{1+a\bar{b}\beta_a\beta_b}{|1+ab\beta_a\beta_b|}(\frac{\bar{a}}{\beta_b} + \frac{\bar{b}}{\beta_a})\beta_a\beta_{b\oplus a}}{1 + \bar{a}\frac{1+\bar{a}b\beta_a\beta_b}{|1+ab\beta_a\beta_b|}(\frac{a}{\beta_b} + \frac{b}{\beta_a})\beta_a\beta_{b\oplus a}} \\
&= \frac{1 + a\frac{\frac{\bar{a}}{\beta_b}+\frac{\bar{b}}{\beta_a}}{1+\bar{a}b\beta_a\beta_b}\beta_a\beta_a\beta_b}{1 + \bar{a}\frac{\frac{a}{\beta_b}+\frac{b}{\beta_a}}{1+a\bar{b}\beta_a\beta_b}\beta_a\beta_a\beta_b} \\
&= \frac{1 + \bar{a}b\beta_a\beta_b + a\bar{a}\beta^2{}_a + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b + a\bar{a}\beta^2{}_a + a\bar{b}\beta_a\beta_b} \times \frac{1 + a\bar{b}\beta_a\beta_b}{1 + \bar{a}b\beta_a\beta_b} \\
&= \mathrm{gyr}[a, b]
\end{aligned}$$

The addition $\oplus$ for parallel velocities reduces to

$$a \oplus b = \frac{a}{\beta_b} + \frac{b}{\beta_a}$$

Now we define the scalar multiplication as follows:

$$r \otimes v := \phi(r \otimes_M \phi^{-1}(v))$$

So we have

$$r \otimes v = \sinh(r \sinh^{-1}(|v|)) \frac{v}{|v|}$$

or equivalently,

$$r \otimes v = \frac{1}{2}\{(\sqrt{1+|v|^2} + |v|)^r - (\sqrt{1+|v|^2} - |v|)^r\} \frac{v}{|v|}$$

if $v \neq 0$ and $r \otimes 0 := 0$. In particular,

$$a' := \frac{1}{2} \otimes a = \frac{\sqrt{\beta_a}}{\sqrt{1+|a|\beta_a} + \sqrt{1-|a|\beta_a}} \cdot a \quad \text{and} \quad 2 \otimes a = \frac{2a}{\beta_a}.$$

$(\mathbb{C}, \oplus, \otimes)$ inherits its inner product from $\mathbb{C}$ such that its gyroautomorphism preserves the inner product $< \cdot, \cdot >$, hence

$$< \mathrm{gyr}[a,b]u, \mathrm{gyr}[a,b]v >= \frac{1}{2}(\mathrm{gyr}[a,b]u \cdot \overline{\mathrm{gyr}[a,b]v} + \overline{\mathrm{gyr}[a,b]u} \cdot \mathrm{gyr}[a,b]v)$$

Since $\mathrm{gyr}[a,b]\overline{\mathrm{gyr[a,b]}} = \frac{1+a\bar{b}\beta_a\beta_b}{1+\bar{a}b\beta_a\beta_b}\frac{1+\bar{a}b\beta_a\beta_b}{1+a\bar{b}\beta_a\beta_b} = 1$, so we have

$$< \mathrm{gyr}[a,b]u,\ \mathrm{gyr}[a,b]v >= \frac{1}{2}(u\bar{v} + \bar{u}v) =< u, v > .$$

V_1 is trivial.
Let $r_1, r_2 \in \mathbb{R}$ and $a \in \mathbb{C}$. Since $\beta_{r_i \otimes a} = \frac{1}{\cosh(r_i \sinh^{-1}(|a|))}$ so

$$\lambda_{r_1 \otimes a, r_2 \otimes a} = \cosh((r_1 + r_2)\sinh^{-1}(|a|)).$$

Therefore we have

$$\begin{aligned}
r_1 &\otimes a \oplus r_2 \otimes a \\
= &\ \lambda_{r_1 \otimes a, r_2 \otimes a} \frac{\tanh(r_1 \sinh^{-1}(|a|)) + \tanh(r_2 \sinh^{-1}(|a|))}{1 + \tanh(r_1 \sinh^{-1}(|a|))\tanh(r_2 \sinh^{-1}(|a|))} \frac{a}{|a|} \\
= &\ \cosh((r_1 + r_2)\sinh^{-1}(|a|)) \tanh((r_1 + r_2)\sinh^{-1}(|a|)) \frac{a}{|a|} \\
= &\ \sinh((r_1 + r_2)\sinh^{-1}(|a|)) \frac{a}{|a|} \\
= &\ (r_1 + r_2) \otimes a
\end{aligned}$$

Thus V_2 is valid.

For $r_1, r_2 \in \mathbb{R}$ and $a \in \mathbb{C}$ we have:

$$\begin{aligned} r_1 & \otimes(r_2 \otimes a) = r_1 \otimes (\sinh(r_2 \sinh^{-1}(|a|))\frac{a}{|a|}) \\ = & \sinh(r_1 \sinh^{-1}(|\sinh(r_2 \sinh^{-1}(|a|))\frac{a}{|a|})|)\frac{r_2 \otimes a}{|r_2 \otimes a|} \\ = & \sinh(r_1 \sinh^{-1}(|\sinh(r_2 \sinh^{-1}(|a|)))|)\frac{a}{|a|} \\ = & \sinh((r_1 r_2)\sinh^{-1}(|a|))\frac{a}{|a|} \\ = & (r_1 r_2) \otimes a \end{aligned}$$

Therefore V_3 is also valid.
V_4 comes from definition.
Now since $|\operatorname{gyr}[a, b]| = 1$, then for $r \in \mathbb{R}$ and $a, b, u \in \mathbb{C}$ we have:

$$\begin{aligned} r \otimes \operatorname{gyr}[a, b]u &= \sinh(r \sinh^{-1}(|\operatorname{gyr}[a, b]u|))\frac{\operatorname{gyr}[a, b]u}{|\operatorname{gyr}[a, b]u|} \\ &= \operatorname{gyr}[a, b]\sinh(r \sinh^{-1}(|u|))\frac{u}{|u|} \\ &= \operatorname{gyr}[a, b](r \otimes u) \end{aligned}$$

Hence V_5 holds. Straightforward computations shows that V_6 and V_7 are valid.
Since $\oplus_M$ satisfies in triangle inequality, we can write

$$\begin{aligned} |a \oplus b| &= |\lambda_{a,b}(a\beta_a \oplus_M \beta_b b)| \\ &= \lambda_{a,b}|(a\beta_a \oplus_M b\beta_b)| \leq \lambda_{a,b}(|a\beta_a| \oplus_M |b\beta_b|) = |a| \oplus |b| \end{aligned}$$

So $\oplus$ satisfies in V_8. Thus we have proved that $(\mathbb{C}, \oplus, \otimes)$ is a gyrovector space.

In the following, we show that $(\mathbb{C}, \oplus, \otimes)$ and $(\mathbb{D}, \oplus_M, \otimes_M)$ are gyroisomorphic.
Consider the map $\phi : (\mathbb{D}, \oplus_M, \otimes_M) \longrightarrow (\mathbb{C}, \oplus, \otimes)$ given by $a \mapsto a\gamma_a$.

(i) For $a,\ b \in \mathbb{D}$, $\phi(a \oplus_M b) = \phi(\frac{a+b}{1+\bar{a}b}) = \frac{a+b}{1+\bar{a}b}\gamma_a\gamma_b|1 + \bar{a}b|$.
On the other hand, since $\beta_{a\gamma_a} = \frac{1}{\gamma_a}$ and $\lambda_{a\gamma_a, b\gamma_b} = \gamma_a\gamma_b|1 + \bar{a}b|$,

$$\begin{aligned} \phi(a) \oplus \phi(b) = a\gamma_a \oplus a\gamma_b &= \lambda_{a\gamma_a, b\gamma_b}\frac{a\gamma_a\beta_{a\gamma_a} + b\gamma_b\beta_{b\gamma_b}}{1 + \overline{a\gamma_a\beta_{a\gamma_a}}b\gamma_b\beta_{b\gamma_b}} \\ &= \gamma_a\gamma_b|1 + \bar{a}b|\frac{a + b}{1 + \bar{a}b} \end{aligned}$$

Hence $\phi(a \oplus_M b) = \phi(a) \oplus \phi(b)$.

(ii) $\phi(r \otimes_M a) = r \otimes_M a\gamma_{r\otimes_M a} = \sinh(r \tanh^{-1}(|a|))\frac{a}{|a|}$, on the other hand,

$$\begin{aligned} r \otimes \phi(a) = r \otimes a\gamma_a &= \sinh(r \ln(\frac{|a|}{\sqrt{1-|a|^2}} + \frac{1}{\sqrt{1-|a|^2}}))\frac{a}{|a|} \\ &= \sinh(r \tanh^{-1}(|a|))\frac{a}{|a|} \end{aligned}$$

So $\phi(r \otimes_M a) = r \otimes \phi(a)$.

$$\text{(iii)} \;\; < \frac{\phi(a)}{|\phi(a)|}, \frac{\phi(b)}{|\phi(b)|} >=< \frac{a\gamma_a}{|a|\gamma_a}, \frac{b\gamma_b}{|b|\gamma_b} >=< \frac{a}{|a|}, \frac{b}{|b|} > .$$

From (i),(ii) and (iii) we conclude that ϕ is a gyrovector space isomorphism. Thus we have proved the following theorem:

Theorem 3.1. $(\mathbb{C}, \oplus, \otimes)$ *is a gyrovector space isomorphic to the Möbius gyrovector space* $(\mathbb{D}, \oplus_M, \otimes_M)$.

Note that by example 2.7, $(\mathbb{C}, \oplus, \otimes)$ is isomorphic to the Einstein's gyrovector space $(\mathbb{D}, \oplus_E, \otimes_E)$ and Ungar gyrovector space $(\mathbb{R}^2_c, \oplus_U, \otimes_U)$ described in [1]. Also note that $(\mathbb{C}, \oplus, \otimes)$ is exactly the Chen gyrovector space introduced in [1] by specifying the function $f : \mathbb{R}^+ \to \mathbb{R}^+$ given by $f(r) = \sinh(\frac{r}{2})$ in definition of general addition of the group $SL(2, \mathbb{C})$.

3.2 Extension of Poincaré Model of Hyperbolic Geometry

Since $(\mathbb{D}, \oplus_M, \otimes_M)$ provides the algebraic setting for the Poincaré model of hyperbolic geometry, and the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ is an extension of it to $\mathbb{C}$, so $(\mathbb{C}, \oplus, \otimes)$ provides the algebraic settings for a new model of hyperbolic geometry just as vector spaces provide the algebraic setting for Euclidean geometry. Also our model is an extension of the hyperbolic geometry of the Poincaré model to the whole plane $\mathbb{C}$ in which the unique geodesic through two given points a and b in the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ is given by $a \oplus (\ominus a \oplus b) \otimes t$ with $0 \le t \le 1$. This geodesic (or, gyroline), its segment from a to b, and the midpoint $m_{ab} = a \oplus (\ominus a \oplus b) \otimes \frac{1}{2}$, of the segment are shown in Figure 1. These are Euclidean semi-hyperbolas with asymptotes which intersect at the origin.

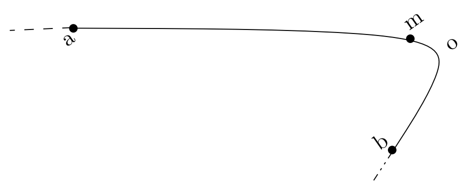

Figure 1. gyroline passing through two points a and b and their midpoint m.

3.3 Trigonometry

One can employ the gyrogroup operation and its gyrovector space to describe the trigonometry of hyperbolic geometry which is called now gyrotrigonometry (e. g. see [6,11]). In the following by using the gyrovector space $(\mathbb{C}, \oplus, \otimes)$ we verify and obtain some trigonometry relations of our model.
Let $a, b \in \mathbb{C}$ and $a \perp b$. Then

$$\beta_{(a\oplus b)\sqrt{2}} = \beta_{a\sqrt{2}}\beta_{b\sqrt{2}} \qquad (*)$$

(i) By using $(*)$, $\beta_{(a\oplus b)\sqrt{2}}|a \oplus b|^2 = \beta_{a\sqrt{2}}\beta_{b\sqrt{2}}(\frac{|a|^2}{\beta_b^2} + \frac{|b|^2}{\beta_a^2})$.

(ii) $\beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2 = \frac{\beta_{a\sqrt{2}}|a|^2}{\beta_{\beta_{b\sqrt{2}}|b|^2}} + \frac{\beta_{b\sqrt{2}}|b|^2}{\beta_{\beta_{b\sqrt{2}}|a|^2}}$. $\qquad (**)$

But $\beta_{\beta_{a\sqrt{2}}|a|^2} = \frac{1}{\sqrt{1+\frac{|a|^4}{1+2|a|^2}}} = \frac{\beta_a^2}{\beta_{a\sqrt{2}}}$, so $(**)$ is equal to $\beta_{a\sqrt{2}}\beta_{b\sqrt{2}}(\frac{|a|^2}{\beta_b^2} + \frac{|b|^2}{\beta_a^2})$.

From (i) and (ii) we get the hyperbolic Pythagorean theorem

$$\beta_{(a\oplus b)\sqrt{2}}|a \oplus b|^2 = \beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2$$

Thus we have proved the following theorem:

Theorem 3.2. *Let $a, b \in \mathbb{C}$ and $a \perp b$. Then the hyperbolic Pythagorean theorem in $(\mathbb{C}, \oplus, \otimes)$ is of the form*

$$\beta_{c\sqrt{2}}|c|^2 = \beta_{a\sqrt{2}}|a|^2 \oplus \beta_{b\sqrt{2}}|b|^2.$$

Note that in general, for any $a, b \in \mathbb{C}$ we have the following relations:

$$\beta_a^2\beta_b^2|a \oplus b|^2 = \beta_a^2|a|^2 + \beta_b^2|b|^2 + 2\beta_a\beta_b < a, b >,$$

$$\beta_{a\oplus b} = \frac{\beta_a\beta_b}{|1 + \bar{a}b\beta_a\beta_b|}.$$

Hyperbolic Distance. Define $d : \mathbb{C} \times \mathbb{C} \to \mathbb{R}^{\geq 0}; (a, b) \mapsto |a \ominus b|$. Equivalently we can write

$$d(a, b) = |\frac{a}{\beta_b} - \frac{b}{\beta_a}|.$$

It is easy to show that d is a metric on $\mathbb{C}$ which is the hyperbolic distance of any two points a and b in our model.

3.3.1 Hyperbolic Angle

For three points a, b and c in gyrovector space $(\mathbb{C}, \oplus, \otimes)$ the cosine of the hyperbolic angle α between two geodesic rays $a \oplus (\ominus a \oplus b) \otimes t$ and $a \oplus (\ominus a \oplus c) \otimes t$ with common point a and respectively containing b and c is given by the equation

$$\cos\alpha = \frac{\ominus a \oplus b}{| \ominus a \oplus b|} \cdot \frac{\ominus a \oplus c}{| \ominus a \oplus c|}$$

This hyperbolic angle α is independent of the choice of the points b and c on the geodesic rays, and it remains invariant under left gyrotranslations and rotations.

Theorem 3.3. *Let $\triangle(a,b,c)$ be any triangle in hyperbolic plane $\mathbb{C}$ with angles α, β and γ in a, b and c respectively and denote the opposite sides of a, b and c respectively with a, b and c. Then*

(i) If $\gamma = \frac{\pi}{2}$ then

$$\cos(\alpha) = \frac{|b|\beta_b(2-\beta_c^2)}{|c|\beta_c(2-\beta_b^2)} = \frac{|b|}{|c|}\frac{\beta^2_{b\sqrt{2}}\beta_c}{\beta^2_{c\sqrt{2}}\beta_b}$$

and

$$\sin(\alpha) = \frac{|a|\beta_c}{|c|\beta_a}$$

(ii) $\cos(\gamma) = \frac{\beta^2_{a\sqrt{2}}\beta^2_{b\sqrt{2}}-\beta^2_{c\sqrt{2}}}{4|a||b|}\beta_a\beta_b$

(iii)

$$\frac{\sin(\alpha)\beta_a}{|a|} = \frac{\sin(\beta)\beta_b}{|b|} = \frac{\sin(\gamma)\beta_c}{|c|}$$

(iv) $\beta^2_{c\sqrt{2}} = \frac{\cos(\alpha)\cos(\beta)+\cos(\gamma)}{\sin(\alpha)\sin(\beta)}$ *or equivalently,*

$$|c|^2 = \frac{\cos(\alpha+\beta)+\cos(\gamma)}{2\sin(\alpha)\sin(\beta)}$$

3.4 Defect and Area

Let $\triangle(a,b,c)$ be a triangle in $(\mathbb{C},\oplus)$, without loss of generality we can assume that $c = o$. It is shown in Proposition 3.3 of [3] that the defect of $\triangle(o,a,b)$, hence δ, is the measure of $\mathrm{gyr}[a,-b]$. Since $\mathrm{gyr}[a,-b] = \frac{1-a\bar{b}\beta_a\beta_b}{1-\bar{a}b\beta_a\beta_b}$ and $\cos(\delta) = \frac{1}{2}(\ \mathrm{gyr}[a,-b] + \overline{\mathrm{gyr}[a,-b]})$, so we obtain

$$\cos(\delta) = \frac{1-2<a,b>\beta_a\beta_b+[(a\bar{b})^2+(\bar{a}b)^2]\beta_a^2\beta_b^2}{1-2<a,b>\beta_a\beta_b+|a|^2|b|^2\beta_a^2\beta_b^2}$$

Thus if we set $b^\perp := ib$ where $i=\sqrt{-1}$, we have

$$\tan(\frac{\delta}{2}) = \frac{<a,b^\perp>\beta_a\beta_b}{1-2<a,b>\beta_a\beta_b}$$

In particular if $a \perp b$, then

$$\tan(\frac{\delta}{2}) = |a||b|\beta_a\beta_b$$

We define area equal to defect, so the area of $\triangle(a,b,c)$ with defect δ is

$$S := 2\tan^{-1}\left(\frac{< a, b^{\perp} > \beta_a\beta_b}{1 - 2 < a, b > \beta_a\beta_b}\right)$$

By similar arguments described in [5] we have the following result about circles in this model:

Theorem 3.4. *Let C_r be any circle of radius r in hyperbolic plane $\mathbb{C}$ with circumference P and area S. Then*

$$P = \frac{4\pi r}{\beta_r}, \qquad S = 4\pi r^2$$

Theorem 3.5. *Let $\triangle(A,B,C)$ be any triangle and C_r be its circumscribed circle with radius r in hyperbolic plane $\mathbb{C}$. If δ be its defect then*

$$\sin\left(\frac{\delta}{2}\right) = \frac{|a||b||c|}{2r\beta_r}\beta_a\beta_b\beta_c(2 - \beta_r^2).$$

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The Principle of Relativity: From Ungar's Gyrolanguage for Physics to Weaving Computation in Mathematics

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Abstract

This paper extends the scope of algebraic computation based on a non standard $\times$ to the more basic case of a non standard $+$, where standard means associative and commutative. Two physically meaningful examples of a non standard $+$ are provided by the observation of motion in Special Relativity, from either outside (3D) or inside (2D or more), We revisit the "gyro"-theory of Ungar to present the multifaceted information processing which is created by a metric cloth W, a relating computational construct framed in a normed vector space V, and based on a non standard addition denoted $\oplus$ whose commutativity and associativity are ruled (woven) by a relator, that is a map which assigns to each pair of admissible vectors in V an automorphism in Aut W. Special attention is given to the case where the relator is directional.

Keywords: Relator, noncommutativity, nonassociativity, induced addition, organ, metric cloth, weaving information processing, cloth geometry, hyperbolic geometry, special relativity, liaison, geodesic, organic line, action at a distance.

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1. About Relating Computation

1.1 Introduction

Hypercomputation, that is nonlinear computation in real multiplicative Dickson algebras $A_k \cong \mathbb{R}^{2^k}$, is developed in (Chatelin 2012 a). For $k \geq 2$ (resp. $k \geq$

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3) multiplication is not commutative (resp. not associative). However addition remains both associative and commutative.

The situation changes in an essential way when computation is merely additive and there exists a *relator* which rules the way any two multidimensional numbers in $\mathbb{R}^n$ (i.e vectors) are to be *added*. This kind of relating computation will be defined in precise terms in Section 2. It includes the special case of an *explicit metric reference* consisting of a positive *finite* number λ, $0 < \lambda < \infty$. The classical structure of an abelian additive group is weakened by considering an addition whose commutativity and associativity are controlled by the relator. A physically meaningful example was provided a century ago by 3D-Special Relativity (Einstein) where the role of λ as a metric reference is played by c, the speed of light in vacuum, and the relator is a plane rotation.

1.2 Special Relativity in the Early Days

It was soon recognised that hyperbolic geometry underlies Einstein's law of addition for admissible velocities (Varičak 1910, Borel 1914) creating the relativistic effect known today as *Thomas precession* (Silberstein 1914, Thomas 1926). But, despite Maxwell's valiant efforts (Maxwell 1871), Hamilton's noncommutative $\times$ of 4-vectors was still unacceptable for most scientists at the dawn of the 20th century. Therefore Einstein's noncommutative $+$ of 3-vectors (representing relativistically admissible velocities) was fully inconceivable: Einstein's geometric vision was far too much ahead of its time! An analytic version of Special Relativity with more appeal to physicists was conceived by Minkowski in 1907, by dressing up as physical concepts the Lorentz transformations which had been introduced by (Poincaré 1905) as the correction of Lorentz preliminary version (1904), see (Walter 1999, Auffray 2005, Damour 2008). This version was quickly grasped by leading physicits (Von Laue, Sommerfeld); it is the version adopted until today in most physics textbooks for students, which carefully avoids any reference to the underlying non commutative quaternionic field $\mathbb{H}$ invented by Hamilton (1843).

1.3 A Mathematical Revival in 1988

Einstein's intuition was left dormant for some 80 years until it was brought back to a new mathematical life in the seminal paper (Ungar 1988). During almost 30 years, Ungar has crafted an algebraic language for hyperbolic geometry lucidly presented in (Ungar, 2008). The book sheds a *natural light* on the physical theories of Special Relativity and Quantum Computation. It dissipates some of the mystery that has shrouded earlier expositions. At the same time, it provides new insight on hyperbolic geometry. Ungar's geometry, which is expressed in "gyrolanguage", is based on the key concepts of gyrator and gyrovector space. They are mathematical concepts abstracted from Thomas precession, a kinematic effect in 3D-special relativity. The *physical* effect was anticipated in (Borel 1913, 1914).

As we shall see, these concepts find an equally natural use beyond physics, in the realm of computation ruled by a relator.

1.4 Geometric Information Processing in Relating Computation

The gyrolanguage is geared towards Hyperbolic Geometry and Physics. In this paper, we export some of Ungar's tools developed for mathematical physics into mathematical computation in a *relating* context (Definition 2.1 below). The reward of the shift of focus from physics to computation is to gain insight about the *geometric* ways by which information can be organically processed in the mind of a computing agent when *relation* prevails. This processing exemplifies the computational thesis posited in (Chatelin 2012 a,b) by revealing geometric aspects of organic intelligence.

The change of focus entails some necessary changes in the vocabulary which are signalled by a reference to the original gyroterm defined in (Ungar 2008). The reader can find all the necessary theoretical background for the presentation to follow in Ungar's work, conveniently put together in his 2008 book which is an algebraic goldmine. Unless otherwise stated, all cited gyroresults are taken from this book.

1.5 Organisation of the Paper

Sections 2 to 6 export parts of Ungar's gyrotheory for physics into relating computation: an organ is a gyrocommutative gyrogroup (Section 2), a metric cloth is a gyrovector space (Section 3). The associated cloth geometry is studied by means of three basic organic lines, the first two corresponding to gyrolines and cogyrolines (Sections 4 to 6)). The rest of the paper (Sections 7 to 9) is original. In Section 7 we restrict our attention to those relators which are directional because they do not depend on the norm of the vectors. This restriction enables us to show that the third organic line enjoys a twofold interpretation in terms of each of two geodesics (Section 7). Section 8 develops the consequences for Weaving Information Processing based on cloth geometry. Finally, epistemological considerations are presented in Section 9.

2. Additive Relating Computation

2.1 Preliminaries

A *groupoid* $(S, \oplus)$ is a set S of elements on which a binary operation called *addition* and denoted $\oplus$ is defined : $(a, b) \in S \times S \mapsto a \oplus b \in S$. An element 0 such that $0 \oplus a = a$ (resp. $a \oplus 0 = a$) is called a left (resp. right) *neutral*. An *automorphism* for $(S, \oplus)$ is a bijective endomorphism φ which preserves $\oplus$: $\varphi(a \oplus b) = \varphi(a) \oplus \varphi(b)$

for all $a, b \in S$. The set of automorphisms forms a group (relative to $\oplus$) denoted Aut $(S, \oplus)$ with the identity map I as unit element. The subtraction is denoted $\ominus$: $a \ominus b = a \oplus (\ominus b)$. In particular $\ominus a$, the left opposite of a, satisfies $\ominus a \oplus a = 0$.

2.2 Relators

We suppose that we are given a map:

$$\begin{aligned} rel \quad : \quad & S \times S \to \text{Aut }(S, \oplus) \\ & (a, b) \mapsto rel(a, b) \end{aligned}$$

such that
$$rel(a \oplus b, b) = rel(a, b). \tag{A1}$$

A map *rel* satisfying the *reduction* axiom (A1) is called a *relator*. We set $\boldsymbol{R} = rel(S, S)$ for the range of the relator in Aut $(S, \oplus)$.

2.3 Organs Underlie Additive Relating Computation

We suppose that $\oplus$ satisfies the additional axioms:

$$a \oplus b = rel(a, b)(b \oplus a), \tag{A2}$$
$$a \oplus (b \oplus c) = (a \oplus b) \oplus rel(a, b)c, \tag{A3}$$

which express by means of $rel(a, b)$ a weak form of commutativity (A2) and associativity (A3). Then $(a \oplus b) \oplus c = a \oplus (b \oplus rel(b, a)c)$ by Theorem 2.35.
The algebraic structure (G, rel) consisting of the additive groupoid $G = (S, \oplus)$ and the relator *rel* is called an *organ*.

Definition 2.1. An additive relating computation refers to any algebraic computation taking place in an organ defined by the data $\{\oplus, rel\}$ satisfying the three axioms (A1), (A2), (A3).

Remark 1. In (Definition 2.7, Ungar 2008), the relator is called gyrator with (A1)$\Longleftrightarrow$ $(G5)$ (=left loop property in multiplicative algebra vocabulary). Next (A3)$\Longleftrightarrow$ $(G3)$ is gyroassociativity and (A2)$\Longleftrightarrow$ $(G6)$ is gyrocommutativity which is optional in a gyrogroup. An organ is a gyrocommutative gyrogroup (Definition 2.8). And $\oplus$ is denoted either $+$ or $\oplus$ therein.

2.4 Some Properties of the Relator

The neutral 0 and the opposite $\ominus a$ are unique: (left=right), and $a \ominus a = \oplus a \ominus a = 0$.
The relator satisfies:

- $\ominus(a \oplus b) \;=\; rel(a, b)(\ominus b \ominus a)$, (Theorem 2.11)
  $\qquad\qquad\;\;\; = \;\ominus a \ominus b$ (Theorem 3.2)
- $rel^{-1}(a, b) \;=\; rel(\ominus b, \ominus a)$ (Theorem 2.32)
- $rel(b, a) \;=\; rel^{-1}(a, b)$ (Theorem 2.34)
  $\qquad\qquad\; = \; rel(a, \ominus rel(a, b)b)$ (Lemma 2.33)

More identities are found in Table 2.2 (Ungar 2008, p.50). In particular:

$$rel(\ominus a, a) = rel(a, \ominus a) = rel(0, a) = rel(a, 0) = rel(0, 0) = I \qquad (2.1)$$

The identities in (2.1) follow from the reduction axiom (A1). Because $\ominus a \oplus a = 0 \oplus 0 = 0$, $rel(\ominus a, a)$ and $rel(0, 0)$ could be arbitrarily chosen in Aut $(S, \oplus)$. In full generality, 0 is a singularity with an indeterminate character for the relator. The indeterminacy disappears under the reduction axiom (A1).
The following additional hypotheses are useful:

- $g \oplus g = 0 \Longrightarrow g = 0$ holds for any $g \in G$ (H$_1$)
- for any $0 \neq g \in G$, there exists at least one half-vector h such that $h \oplus h = g$. (H$_2$)

(H$_1$) is satisfied in the Examples 2.1 to 2.3 that will be given in Section 2.6. It is the additive analogue of the multiplicative notion of 2-torsion free algebra, see Definition 3.32 on p. 72.
Under the 2 assumptions (H$_1$) and (H$_2$) for $\oplus$, the following statements hold:

- the half-vector h is unique (Theorem 3.34),
- $rel(a, b) \neq \ominus I$ (Theorem 3.36), that is anticommutativity is ruled out: $a \oplus b \neq \ominus (b \oplus a)$,
- $rel(a, b)b = \oplus b \Longrightarrow b = 0$ (Theorem 3.37).

2.5 The Two Basic Equations Associated with $\oplus$ and rel

Because $\oplus$ is not commutative we are led to consider $\mathcal{L} = \{L_a = a \oplus \cdot; a \in G\}$ and $\mathcal{R} = \{R_a = \cdot \oplus a; a \in G\}$ Left– (resp. right–) addition $\oplus$ is abbreviated $L\oplus$ (resp. $R\oplus$). We consider the left and right linear equations associated with a, b in G.

$$\begin{aligned} L_a x &= a \oplus x = b, & (2.2)\\ R_a y &= y \oplus a = b, & (2.3) \end{aligned}$$

Each of them has the unique solution

$$\begin{aligned} x &= \ominus a \oplus b, \text{ by Eq.(2.30)}, & (2.4)\\ y &= b \ominus rel(b, a)a. \text{ by Eq.(2.32)}, & (2.5) \end{aligned}$$

The equality (2.5) suggests to consider the composite map $\oplus rel(\cdot, \ominus \cdot)$ as an *induced* addition $\hat{+}$ defined by

$$(a, b) \in G \times G \mapsto a \hat{+} b = a \oplus rel(a, \ominus b)b \text{ (Theorem 2.14)}. \qquad (2.6)$$

The corresponding subtraction, denoted $\hat{-}$, is such that (2.5) can be rewritten as $y = b \hat{-} a$ (Theorem 2.22). Definition (2.6) is equivalent to $a \oplus b = a \hat{+} rel(a, b)b$.

Three properties about $\oplus$ and $\hat{+}$, are noteworthy:

- Aut $(S, \oplus) = $ Aut $(S, \hat{+})$ (Theorem 2.28),
- $\hat{+}$ is classically *commutative* (Theorems 2.38 and 3.4).
- $\ominus a = \hat{-} a$ (Theorem 2.21).

The concept of an *organ* is determined by two data: the addition $\oplus$ and the associated relator (as a map into the automorphisms for $\oplus$). In the relating perspective, the source notion is the pair $(\oplus, rel)$ where the *relator* rules its associated addition $\oplus$. This addition precedes the secondary addition $\hat{+}$, which is induced by $R\oplus$ and the relator combined together. This novel concept reduces to the classical concept of an abelian additive group when the primitive operation is associative and commutative (hence $\oplus = \hat{+}$), that is when the range $\boldsymbol{R}$ reduces to $\{I\}$. By expanding its range to the larger subset $\boldsymbol{R} \subset$ Aut $(S, \oplus)$, the relator controls the weak (or relative) commutativity and associativity of $\oplus$, thus introducing anisotropy in the organic structure. This has the additional benefit to induce the existence of $\hat{+}$, another addition which is classically commutative.

In other words, the expansion $\{I\} \to \boldsymbol{R}$ loosens the rigid structure of an abelian group and provides the more flexible, relating, structure of an organ which lies at the foundation of relating computation.

When the range $\boldsymbol{R}$ is a proper subset of Aut $(S, \oplus)$, its role is to *reduce* the variety of possible automorphisms. The standard group structure appears as a limit case corresponding to the ultimate reduction $\boldsymbol{R} = \{I\}$.

Remark 2. The *group* structure which underlies classical computation guarantees the invariance of its logic. From their logical vantage point, many logicians view the whole mathematical enterprise as a mere giant tautology. It is clear that the reduction of mathematics to a formal axiomatic system does not do justice to the creative power of non linear computation which may lead to a non standard addition ruled by a relator (see Example 2.2 below). We believe that the concept of an *organ* is better suited than that of a group to describe some of the organic logics which are at work in life's computation and are *evolutive* by essence

Organic Information Processing (IP) is a *dynamical* process which reflects the variability of the *relator*. Its operations in G consist of $\oplus$, $\hat{+}$ and their automorphisms. One can view an organ as a new algebraic species, some kind of a "fieldoid", based on the groupoid, in which $\hat{+}$ plays the role attributed to $\times$ in an ordinary field (group-based) structure. The main difference with a field is that the neutral 0 (identical for $\oplus$ and $\hat{+}$) replaces the unit $1 \neq 0$. The analogy is commented next.

Remark 3. The induction $\{R\oplus, rel\} \to \hat{+}$ is analogous to the creation of the product $n \times a$ by n repeated additions of the real number a. In this most familiar case, the multiplication stems from an iterated addition.

2.6 Three Basic Examples

The following Examples are found in Sections 3.4, 3.8 and 3.10 respectively of (Ungar 2008) The explicit formula for $x \oplus y$ entails the determination of $rel(x,y)$.

Example 2.2. The subgroup of all Möbius transformations of the complex open unit disk $D = \{z \in \mathbb{C}; |z| < 1\}$ into itself is defined by: $(a,z) \mapsto e^{i\theta}\frac{a+z}{1+\bar{a}z}$ for $a, z \in D$ and $\theta \in \mathbb{R}$. If we set $a \oplus z = \frac{a+z}{1+\bar{a}z}$, then $z \oplus a = \dfrac{a+z}{1+\bar{z}a}$. The relator is defined by $rel(a,z) = \frac{1+a\bar{z}}{1+\bar{a}z} \in \mathrm{Aut}\ (D, \oplus)$. Hence clearly $a \oplus z = rel(a,z)(z \oplus a)$. Endowed with $\oplus$ the unit disk becomes an organ. Observe that $\oplus$ is expressed by means of the 3 operations $+, \times$, conjugacy defined on $\mathbb{C}$. It is known in mathematics as a *hyperbolic translation* in the plane $\mathbb{R}^2$. The relator is a *rotation* since its modulus is 1. The pseudo-hyperbolic distance in $\bar{D}$ from a to b is $d(a,b) = \left|\dfrac{a-b}{1-\bar{a}b}\right| = |b \ominus a|$. The metric used by Poincaré(1882) in his disc-model for hyperbolic geometry in $\mathbb{R}^2$ is

$$\tanh^{-1} d(a,b) = \frac{1}{2}\ln\frac{1+d(a,b)}{1-d(a,b)},$$

cf. Ungar, Section 6.17, p. 216-217. There exists a *real* version of this addition defined in the open unit disk $B_1 = \{x \in \mathbb{R}^2,\ \|x\| < 1\}$ which reads:

$$x \oplus y = \frac{(1 + 2 < x, y > + \|y\|^2)x + (1 - \|x\|^2)y}{1 + 2 < x, y > + \|x\|^2\|y\|^2},$$

cf.(3.127) in Ungar.

Setting $X = \|x\|\|y\| \geq 0$, $\theta = \measuredangle(x,y)$ the denominator $X^2 + 2X\cos\theta + 1$ has no real roots unless $\cos\theta = -1$, then $X = \|x\|\|y\| = 1$ is a double root. The condition that $x, y \in \bar{B}_1$ entails $\|x\| = \|y\| = 1$, $x = -y$. We observe then that $x \oplus (-x) = \dfrac{x-x}{1-1} = \dfrac{0}{0}$ is an indeterminate form for $x \in \partial B_1$.

When x and y are linearly dependent, $y = rx$, $r \in \mathbb{R}$ (say) then the addition becomes associative and commutative for x, y inside B_1 (so that $1 + r\|x\|^2 = 1 + < x, y > \neq 0$)

$$x \oplus y = \frac{1}{1 + < x, y >}(x + y)$$

Example 2.3. Let c be the vacuum speed of light. We set $B_c = \{x \in \mathbb{R}^3; \|x\| < c\}$ to represent the ball of relativistically admissible velocities.

Einstein's law of addition of velocities $x, y \in B_c$ is

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}}\left[x + y + \frac{1}{c^2}\frac{\gamma_x}{1+\gamma_x} x \wedge (x \wedge y)\right]$$

where $\gamma_x = \left(1 - \frac{1}{c^2}\|x\|^2\right)^{-1/2}$ is the inverse of Lorentz contraction $\sqrt{1 - \left(\frac{\|x\|}{c}\right)^2}$.

Using Grassmann identity in $\mathbb{R}^3$:

$$x \wedge (y \wedge z) = < x, z > y - < x, y > z,$$

(Lamotke 1998, Chapter 7, p. 207), one can also write

$$x \oplus y = \frac{1}{1 + \frac{<x,y>}{c^2}} \left[x + \frac{1}{\gamma_x} y + \frac{1}{c^2} \frac{\gamma_x}{1+\gamma_x} < x, y > x \right]$$

a formula well defined, unless $1 + \frac{\|x\|\|y\|}{c^2} \cos\theta = 0$, where $\theta = \measuredangle(x, y)$. The relation $\cos\theta = -\frac{c^2}{\|x\|\|y\|}$ for $x, y \in \bar{B}_c$ entails that $x, y \in \partial B_c$ and $\cos\theta = -1$. Therefore $x = -y$, and $x \oplus (-x) = \frac{0}{0}$ is an indeterminate form.

The two velocity components, parallel and orthogonal to the relative velocity between inertial systems, were given by Einstein in his 1905-epoch-making paper. The above formula is valid for $n \geq 2$.

Einstein's addition is ruled by a relator which is the *rotation*: $y \oplus x \mapsto x \oplus y$ in the plane spanned by x and y (when independent) with axis parallel to $x \wedge y$ through the angle ε, $0 \leq |\varepsilon| < \pi$ (Borel 1913, Silberstein 1914). The angle ε is related non linearly to $\theta = \measuredangle(x, y)$ and to $\frac{1}{c}\|x\|, \frac{1}{c}\|y\|$ in the following way (Ungar 1988,1991): $\varepsilon = 0$ for $|\theta| \in \{0, \pi\}$ and for $|\theta| \in]0, \pi[$ x and y are independent, yielding:

$$\cos\varepsilon = \frac{(\rho + \cos\theta)^2 - \sin^2\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

$$\sin\varepsilon = -2\frac{(\rho + \cos\theta)\sin\theta}{(\rho + \cos\theta)^2 + \sin^2\theta},$$

with $\rho^2 = \frac{\gamma_x+1}{\gamma_x-1}\frac{\gamma_y+1}{\gamma_y-1}$, $\rho > 1$, and $|\varepsilon| < |\theta|$.

When $\|x\|$ and $\|y\|$ tend to c^-, γ_x and γ_y tend to ∞ and $\rho \to 1^+$. Then $\cos\varepsilon \to \cos\theta$ and $\sin\varepsilon \to -\sin\theta$, that is $\varepsilon \to -\theta$.

We recall that for x, y in $\Im\mathbb{H} \cong \mathbb{R}^3$, $x \times y = - < x, y > + x \wedge y \in \mathbb{H} \cong \mathbb{R}^4$, where $x \wedge y = \frac{1}{2}[x, y] = \frac{1}{2}[x \times y - y \times x] \in \Im\mathbb{H}$. Therefore Einstein's addition wraps up the two distinct operations $+$ and $\times$ in $\Im\mathbb{H}$ into a single *synthetic addition* denoted $\oplus$. The synthesis is realised on independent vectors at the expense of classical commutativity and associativity.

Example 2.4. $V = \mathbb{R}^n$, $n \geq 2$ is the euclidean linear vector space with scalar product $< \cdot, \cdot >$. Let be given λ, $0 < \lambda < \infty$, and define $v_\lambda = \frac{1}{\lambda} v$ for $v \in V$, $\beta_v = \left(1 + \|v_\lambda\|^2\right)^{-1/2}$, $0 < \beta_v \leq 1$. We consider

$$u \oplus v = \left(\frac{1}{\beta_v} + \frac{\beta_u}{1 + \beta_u} < u_\lambda, v_\lambda > \right) u + v$$

defined for $u, v \in V$. For $n = 3$ and $\lambda = c$, this additive law governs the relativistic addition of *proper* velocities expressed in traveller's time. The relator is again a *rotation*. If u and v are dependent, $u \oplus v = \frac{1}{\beta_v} u + \frac{1}{\beta_u} v$ (Eq. (3.214) on p. 96).

The reader can check that in each example above $x \oplus y$ is symmetric in x and y iff x and y are *dependent*.

2.7 Liaison Λ between rel, $\oplus$ and $\hat{+}$

To the linear equations (2.2), (2.3) for $\oplus$, we add the third equation for $\hat{+}$

$$a \,\hat{+}\, \hat{x} = \hat{x} \,\hat{+}\, a = b \tag{2.7}$$

which admits the unique solution

$$\hat{x} = \ominus(\ominus b \oplus a) = b \ominus a. \tag{2.8}$$

Observe that $x = rel(\ominus a, b)\hat{x}$ by (A2)$\Longleftrightarrow \hat{x} = rel(b, \ominus a)x$.

Each of the solutions x, y and $\hat{x}$ is obtained by a respective call to the three following cancellation laws for $\oplus$ and $\hat{+}$:

- left cancellation for $\oplus$: $a \oplus (\ominus a \oplus b) = b$ (2.9)

- right cancellation for $\oplus$: $(b \,\hat{-}\, a) \oplus a = b$ (2.10)

- cancellation for $\hat{+}$: $(b \ominus a) \,\hat{+}\, a = a \,\hat{+}\, (b \ominus a) = b$ (2.11)

Identities (2.10) and (2.11) express a link by means of the relator between $R\oplus$ and $\hat{+}$ which is not present in (2.9) concerning $L\oplus$. If one uses $x = \ominus a \oplus b$,$y = b \ominus rel(a, b)a = b \hat{-} a$ and $\hat{x} = b \ominus a$, the three identities become respectively

$$a \oplus x = b \tag{2.12}$$
$$y \oplus a = b \tag{2.13}$$
$$\hat{x} \,\hat{+}\, a = a \,\hat{+}\, \hat{x} = b \tag{2.14}$$

This rewriting separates $R\oplus$ and $\hat{+}$ in the identities (2.10), (2.11) which appear now as (2.13) = right cancellation for $\oplus$, (2.14)=cancellation for $\hat{+}$.

None of the two writings is a faithful description of the complete computational reality which is, by essence, *connected*. Whichever writing is chosen, the reader should keep in mind that a liaison based on $rel(a, \cdot)$ exists between $\cdot \oplus a$ and

$\cdot\hat{+}a = a\hat{+}\cdot$ for $rel(a,\cdot) \neq I$ when the additive cancellation laws are at work. This liaison reflects the existence of the relator which regulates any relating computation performed in its organ. The liaison concerns $L\oplus$ as well. Indeed, the equality (2.8) $\hat{x} = b\ominus a$ suggests to consider the equation involving L_b:

$$L_b\tilde{x} = b\oplus \tilde{x} = a$$

whose solution is $\tilde{x} = \ominus b\oplus a = \ominus (b\ominus a) = \ominus \hat{x}$.

We call liaison $\Lambda(rel, \oplus, \hat{+})$ the computational consequences of the three fundamental cancellation laws (2.9), (2.10) and (2.11).

Definition 2.5. We call liaison $\Lambda(rel, \oplus, \hat{+})$ the computational consequences of the three fundamental cancellation laws (2.9), (2.10) and (2.11).

The computational dynamics of organic IP results from the shifts $L\oplus$, $R\oplus$ and the automorphisms of G. Given a and b, we shall be concerned in Sections 4 and 7 with the evolution of $\hat{x} = b\ominus a$ (resp. $y = b\,\hat{-}\,a$) when a left (resp. right) shift by an arbitrary $g \in G$ is realised simultaneously on a and b.

Regarding left shift $g\oplus$ and $\ominus$, we have:

$$(g\oplus b)\ominus(g\oplus a) = rel(g,b)(b\ominus a) \quad \text{(Theorem 3.13)}. \tag{2.15}$$

For future reference we mention the following result with right shift:

$$a\,\hat{-}\,b = (a\oplus k)\,\hat{-}\,(b\oplus g) \quad \text{with} \quad k = rel(a,b)g \quad \text{(Theorem 2.23)}. \tag{2.16}$$

3. Metric Cloths

3.1 The Normed Vector Space Frame

Let V be a linear vector space over $\mathbb{R}$ with finite dimension $n \geq 2$, endowed with a scalar product $< a, b >$ for $a, b \in V$ and derived norm $\|a\| = \sqrt{< a, a >}$.

The addition $+$ and scalar multiplication are standard operations in $V \cong \mathbb{R}^n$. Let λ be given, $0 < \lambda < \infty$ and set $B_\lambda = \{x \in V; \|x\| < \lambda\}$. We suppose that the ball B_λ, or V itself, are endowed with the *organic* structure $G = (S, \oplus)$ with relator rel, where S represents B_λ or V as the case may be. The neutral 0 for G is identified with $0 \in V$.

The linear vector space V is the *frame* of the organ G iff the relator preserves the scalar product: $< rel(u,v)x, rel(u,v)y > = < x, y >$ for any quadruple $(u,v,x,y) \in G^4$. It follows that $\|rel(x,y)\| = 1$ for $x, y \in G$, and G inherits from V its scalar product $< \cdot, \cdot >$ and norm $\|\cdot\|$ which are invariant under $\boldsymbol{R} \subset \text{Aut}\, G$.

We assume moreover that, if x and y are linearly dependent in G, then for $x = ry$, $r \in \mathbb{R}$ (say), $(ry)\oplus y = y\oplus (ry)$. Hence $rel(ry,y) = rel(x,y) = I$ ($\Longrightarrow$ $x\oplus y = x\,\hat{+}\,y$). The formula for $\oplus$ becomes symmetric in x and y when x and y are colinear. The property is satisfied for the 3 Examples given in Section 2.6.

3.2 The Scalar Multiplication $\boxtimes$

We suppose that G admits a scalar multiplication $\boxtimes : (\mathbb{R} \times G$ or $G \times \mathbb{R} \to G)$ such that

- $r \boxtimes a = a \boxtimes r$, $r \in \mathbb{R}$, $a \in G$,
- $1 \boxtimes a = a$,
- $(r_1 + r_2) \boxtimes a = (r_1 \boxtimes a) \oplus (r_2 \boxtimes a)$,
- $(r_1 r_2) \boxtimes a = r_1 \boxtimes (r_2 \boxtimes a)$, $a \in G$, $r_1, r_2 \in \mathbb{R}$,
- for r and $a \neq 0$ $\frac{|r| \boxtimes a}{\|r \boxtimes a\|} = \frac{a}{\|a\|}$,
- $rel(u,v)(r \boxtimes a) = r \boxtimes (rel(u,v)a)$ for $u, v, a \in G$, $r \in \mathbb{R}$,
- $rel(r_1 \boxtimes u, r_2 \boxtimes u) = I$, $u \in G$, $r_1, r_2 \in \mathbb{R}$.
- $\|r \boxtimes a\| = |r| \boxtimes \|a\|$, $r \in \mathbb{R}$, $a \in G$.

Even though $\boxtimes$ does not distribute with $\oplus$ in general, the following special identity holds:

$$2 \boxtimes (a \oplus b) = a \oplus (2 \boxtimes b \oplus a) = a \hat{+} (a \oplus 2 \boxtimes b)$$

for any $a, b \in W$ (Theorem 6.7).

3.3 $n = 1$: The Measuring Rod $M = \{\pm\|a\|, a \in G\}$

All elements in M are colinear, hence the relator image reduces to $\{I_1 = 1\}$, and $\oplus = \hat{+}$ on M. M is a 1D-linear vector line equipped with $\oplus$, $\boxtimes$ and $\|\cdot\|$ deriving from G and V. These 3 operations usually *differ* from the standard operations $+, \cdot, |\cdot|$ defined on $\mathbb{R}$.

3.4 $n \geq 2$: The V-Framed Metric Cloth W

We suppose that $\|a \oplus b\| \leq \|a\| \oplus \|b\|$, $a, b \in G$. This ends the list of axioms satisfied by Ungar's gyrovector space in the carrier V (Definition 6.2, Ungar 2008).

To this list, we add that, when a and b are *dependent and nonzero*: $b = ra$, $r \neq 0$, there exists $l \in \mathbb{R}\backslash\{0\}$ such that $b = l \boxtimes a$ with $1 \cdot a = 1 \boxtimes a = a$ for $r = 1$. If $r = 0$, $0 \cdot a = 0 = 0 \boxtimes a = 0$. In other words, $r \cdot (\cdot) = r(\cdot) = l \boxtimes (\cdot)$: the map $l \mapsto r = \mu(l)$, $\mu(0) = 0$, $\mu(1) = 1$, is a change of scale on the axis spanned by $a \neq 0$, induced by the change of context from the linear vector space $V(+, \cdot)$ to the additive cloth $W(\oplus, \boxtimes)$. As a consequence $rel(a, b) = I$ and the vectors $a \oplus b = a \hat{+} b = (1 + l) \boxtimes a$ are colinear with $a + b = (1 + r)a$.

The structure $W = (S, \oplus, \boxtimes) = (G, \boxtimes)$ obeying the assumptions above is a *metric cloth* in the normed vector frame V. The cloth W is organically and metrically woven by $\{\oplus$, relator, $\boxtimes, \|\cdot\|\}$. It satisfies the

Proposition 3.1. *The addition $\oplus$ in the metric cloth W satisfies (H_1) and (H_2).*

Proof. 1) (H_1): observe that $g \oplus g = 2 \otimes g$. Hence if $2 \otimes g = 0$, $\|2 \otimes g\| = 2 \otimes \|g\| = 0$ and $\|g\| = 0$ in $\Longleftrightarrow g = 0$.

2) (H_2): $g = \frac{1}{2} \otimes (2 \otimes g) = \frac{1}{2}(g \oplus g)$. Now Ungar's Theorem 6.4 tells us that $\otimes$ distributes *axially* along the axis spanned by any $g \neq 0$ in G:

$$r \otimes (r_1 \otimes g \oplus r_2 \otimes g) = r \otimes (r_1 \otimes g) \oplus r \otimes (r_2 \otimes g) =$$

$$(rr_1) \otimes g \oplus (rr_2) \otimes g = (r(r_1 + r_2)) \otimes g.$$

Setting $r = 1/2$ and $r_1 = r_2 = 1$ we get $\frac{1}{2} \otimes (g \oplus g) = \frac{1}{2} \otimes g \oplus \frac{1}{2} \otimes g$. Therefore the half-vector h for g exists and is uniquely defined by $\frac{1}{2} \otimes g$. □

Because $\otimes$ distributes axially in W, it follows readily that anticommutativty is ruled out for $\oplus$

Example 3.2. The scalar multiplication for the organ B_c in Example 2.2 is such that $r \otimes 0 = 0$, $r \otimes x = \mu(r)x$ for $0 \neq x \in B_c$. We set $x_c = \frac{1}{c}x$, then Definition 6.86 on p. 218 gives

$$\mu(r) = \frac{1}{\|x_c\|} \tanh(r \tanh^{-1} \|x_c\|), \quad r \in \mathbb{R}, \tag{3.17}$$

with $\mu(0) = 0$, $\mu(1) = 1$. Then B_c becomes the $\mathbb{R}^3$-framed cloth W_E (based on Einstein's addition) which is an alternative framework for Special Relativity in Physics, classically presented by means of Lorentz transformations, hence implicitly on the field of quaternions $\mathbb{H}$.

Let $q = (c\alpha, X)$ be given in $\mathbb{H}$, with real part $c\alpha$, $\alpha \in \mathbb{R}$ and imaginary part X in $\mathbb{R}^3$. Then $q^2 = c^2\alpha^2 - \|X\|^2 + 2c\alpha X$. A Lorentz transformation in $\mathbb{H}$ leaves invariant the quantity

$$\Re q^2 = c^2\alpha^2 - \|X\|^2 = f \quad \text{constant for all} \quad q \in \mathbb{H}$$

(Poincaré 1905). Observe that $\|X\|^2 = c^2\alpha^2 - f$ and $\|\Im q^2\|^2 = 4c^2\alpha^2(c^2\alpha^2 - f)$ are nonnegative iff $c^2\alpha^2 \geq f$ which is always satisfied when $f \leq 0$.

By (11.2) in (Ungar 2008), the Lorentz transformation *without* rotation is a boost $L(u)$ for $u \in B_c$ such that, for $u_c = \frac{1}{c}u$, $q_c = \frac{1}{c}q = (\alpha, X_c)$

$$L(u)q_c = (\gamma_u[\alpha + < u_c, X_c >], \gamma_u u[\alpha + \frac{\gamma_u}{1 + \gamma_u} < u_c, X_c >]).$$

Then by (11.10) for $u, v \in B_c$ we get the composition law:

$$L(u)L(v) = L(u \oplus v)rel(u, v) = rel(u, v)L(v \oplus u).$$

The general case (transformations with rotations in $SO(3)$ is given in (11.15), (11.20).

These formulae shed an interesting light about the connection between hypercomputation in $\mathbb{H}$ based on $\times$ and computation in the cloth W_E based on Einstein addition $\oplus_E$. The connection is developed in the references (Chatelin 2011, 2012b).

Example 3.3. For a given λ, $0 < \lambda < \infty$, we set $x_\lambda = \frac{1}{\lambda}x$ for $x \in \mathbb{R}^n$, and consider the organ $B_\lambda = \{x \in \mathbb{R}^n; \|x\| < \lambda\}$ where the addition is the Poincaré addition

$$x \oplus_P y = \frac{\left(1 + 2 < x_\lambda, y_\lambda > + \|y_\lambda\|^2\right) x + \left(1 - \|x_\lambda\|^2\right) y}{1 + 2 < x_\lambda, y_\lambda > + \|x_\lambda\|^2 \|y_\lambda\|^2}$$

which is not well-defined when $y = -x$ on the sphere ∂B_λ. The scalar multiplication $r \in \mathbb{R} \mapsto r \otimes x = \{0 \; for \; x = 0, \; \mu(r)x \; for \; 0 \neq x \in B_\lambda\}$ is defined by (3.17) where c is replaced by λ (Definition 6.83 where Möbius stands for Poincaré).

Using a common reference λ, $0 < \lambda < \infty$, we obtain two metric cloths W_E and W_P framed in $\mathbb{R}^n$. Remarkably, these two cloths are isomorphic in the following sense. The bijective map $\psi : \; W_E \to W_P$ defined by $x \mapsto x' = \psi(x) = \frac{1}{2} \otimes x$ preserves $\oplus$ and $\otimes$. See Table 6.1 on p. 226 for more, see also (Ungar 2012). The commutator $[x, y] = (x \oplus y) - (y \oplus x)$ is studied for $\oplus_E$ and $\oplus_P$ in (Chatelin, 2012c).

Example 3.4. Let $\oplus_{PV}$ represent the relativistic addition of velocities in the traveller's time defined in Example 2.3. The underscript PV for *proper velocity* used by Ungar indicates that time refers to the traveller, i.e. the moving observer (standing inside the phenomenon) rather than to an outside observer. Definitions 6.87 on p.223 gives

$$\mu_{PV}(r) = \frac{1}{\|x_\lambda\|} \sinh(r \sinh^{-1} \|x_\lambda\|), \; r \in \mathbb{R},$$

which is a modification of (3.1): λ replaces c and sinh replaces tanh. Then $(V \oplus_{PV}, \otimes)$ is the metric cloth W_{PV}.

Because $a - a = 0$ in V, $-a = (-1) \times a = \ominus a$ in W. In general $r \otimes (a \oplus b) \neq (r \otimes a) \oplus (r \otimes b)$, unless a and b are dependent. Scalar multiplication distributes axially (Theorem 6.4). The automorphisms of W form the group Aut (W): they consist of automorphisms of G which preserve also the scalar multiplication $\otimes$ and the scalar product $< \cdot, \cdot >$ (Definition 6.5).

The identification $-a = \ominus a = \hat{-} a$ which holds in W provides more insight on the induced addition $\hat{+}$ by considering the mirror equation for (2.2) where a and b are exchanged:

$$b \oplus \tilde{x} = a. \tag{3.18}$$

Lemma 3.5.

$$\tilde{x} = -\hat{x} \tag{3.19}$$

Proof. (3.1) yields $\tilde{x} = -b \oplus a$ by (2.4) and $\hat{x} = b \ominus a$ by (2.8). Now $\hat{x} = -(-b \oplus a) = -\tilde{x}$. □

In the larger context of a cloth, the liaison Λ includes $+$, as illustrated by the identification $\hat{x} = -\tilde{x}$.

Definition 3.6. An additive weaving computation refers to any algebraic computation taking place in a metric cloth $W = \{S, \oplus, \otimes\}$.

The set of operations that we shall consider in Weaving Information Processing (WIP) is restricted to $\mathrm{Op}(W) = \mathcal{L} \cup \mathcal{R} \cup \mathrm{Aut}\,(W)$.

Definition 3.7. The Weaving Information Processing (WIP) in a metric cloth W is realised in W by means of $\mathrm{Op}(W)$.

We shall study by geometric means the results of WIP. The metric cloth W inherits from its euclidean frame not only a scalar product/norm, but also its *affine* essence with respect to a *real* parameter. Therefore the geometry derived from a cloth is based on *lines* (as affine functions of a real parameter) and in particular on *geodesics* (for which the triangle inequality becomes an equality). In what follows, we build on Ungar's mathematical vision based on physical insight. We develop some aspects of the role of geometry in WIP. The existence of the three additions $\oplus$, $\hat{+}$, $+$ endows cloth geometry with several ways to carry information, shedding a new light on the role of *non euclidean* geometry in Information Processing (IP).

4. The Metrics Associated with $\oplus$ and $\hat{+}$

4.1 Definition

We revisit the three linear equations (2.2), (2.3) (2.7) and their three solutions x (2.4), y (2.5), $\hat{x}$ (2.7). A simplification occurs because $\|rel(a,b)\| = 1$ for $x = rel(-a,b)\hat{x}$, hence $\|x\| = \|\hat{x}\| =\neq \|y\|$. Thus one can associate *two* metrics d in W with the three cancellation laws. They are given by

$$\mathring{d}(a,b) \quad = \quad \|-a \oplus b\| = \|b \ominus a\|, \tag{4.20}$$

$$\hat{d}(a,b) \quad = \quad \|b \,\hat{-}\, a\| = \|b \ominus rel(b,a)a\|. \tag{4.21}$$

where the upperscripts $^\circ$ and $\hat{}$ for d refer to the respective additions $\oplus$ and $\hat{+}$. The values are identical when a and b are dependent.

Ungar's inequality (6.14) (resp. (6.18)) expresses the following triangle (resp relating-triangle) inequality (4.3) (resp. (4.4)):

$$\mathring{d}(a,c) \leq \mathring{d}(a,b) \oplus \mathring{d}(b,c), \tag{4.22}$$

$$\hat{d}(a, rel(a \hat{-} b, b \hat{-} c)c) \leq \hat{d}(a,b) \,\hat{+}\, \hat{d}(b,c). \tag{4.23}$$

It is clear that $\mathring{d}$ defines a distance, whereas $\hat{d}$ does not (pp. 158 and 159). Curves for which (4.22) is an equality are the *geodesics* of $\oplus$ associated with the distance $\mathring{d}$.

4.2 Invariance Properties

The two metrics are invariant under Aut (W). Metric invariance under left shift in $\mathcal{L}$ holds for $\mathring{d}$ by (2.15): $\| -a \oplus b\| = \|b \ominus a\| = \|(g \oplus b) \ominus (g \oplus a)\|$ for any $g \in G$ (Theorem 6.12).

Regarding $\mathcal{R}$-invariance for $\hat{+}$ (based on $\oplus\, g$), if $rel(a,b) = I$, then: $a \mathbin{\hat{-}} b = (a \oplus g) \mathbin{\hat{-}} (b \oplus g)$ implies $\mathcal{R}$-invariance for $\hat{+}$. This is always true when a and b are dependent. In general (2.16) above holds with $k = rel(a,b)g$, $\|k\| = \|g\|$. The topic will be developed further in Section 8.1.

Remark 4. *On the notational dilemma*

It is important to keep in mind that, in the connecting context of weaving computation, the notation itself is, by force, ambiguous. For example the notation $\mathring{d}$ and $\hat{d}$ was suggested by the definitions (4.1), (4.2). But, of course, the notation $\mathring{d}$ measures in an equal fashion both $x = \ominus\, a \oplus b$ associated with $L\oplus$ and $\hat{x} = b \ominus a = rel(a,b)x$ associated with $\hat{+}$. And $\hat{d}$ reflects the *unique* aspect $R\oplus$ converted into $\hat{+}$. In the difficult task to capture as best as possible the subtle relational interplay between $\oplus$ and $\hat{+}$, cloth geometry will prove to be a precious ally.

5. About the Organic Lines Passing through 2 Distinct Points in Cloth Geometry

5.1 Introduction

Let be given $a \neq b$ in $\mathbb{R}^n$. In euclidean geometry there exists a unique straight line passing through a and b, which can be represented by the affine function: $t \in \mathbb{R} \mapsto a + (b-a)t \in \mathbb{R}^n$: the point a (resp. b) corresponds to $t = 0$ (resp. 1). The straight line is the geodesic of the euclidean metric. The segment $[a,b]$ is defined by $0 \leq t \leq 1$. It has a unique midpoint $m_{ab} = a + \frac{1}{2}(b-a) = \frac{1}{2}(b+a) = m_{ba}$. This simple euclidean picture will be modified in cloth geometry since there exist *more than one* affine curve passing through two points due to the existence of more than one cancellation law.

In what follows we restrict our attention to the three fundamental (cancellation) laws (2.9), (2.10), (2.11) that we put at the foundations of our geometric study. The three laws are ordered respectively as first, second and third. They define *three* types of affine functions defining *organic lines* L_i numbered by $i \in \{1,2,3\}$. It is important to distinguish whether a and b are dependent or not.

5.2 Three Fundamental Organic Lines through a and b Independent

To each fundamental law we can associate a unique fundamental (organic) line passing through a for $t = 0$ and b for $t = 1$. The non commutative addition $\oplus$ provides the left-(resp. right-) line $L - L_{ab}$ (resp. $R - L_{ab}$). The commutative addition $\hat{+}$ provides the unique line-$\hat{L}_{ab}$. These lines are given by the table below

symbol	definition	representation, $t \in \mathbb{R}$	
$L_1 = L\text{-}L_{ab}$	left-line for $L\oplus$	$a \oplus ((-a \oplus b) \otimes t)$	(5.24)
$L_2 = R\text{-}L_{ab}$	right-line for $R\oplus$	$((b \,\hat{-}\, a) \otimes t) \oplus a$	(5.25)
$L_3 = \hat{L}_{ab}$	line for $\hat{+}$	$((b \ominus a) \otimes t) \,\hat{+}\, a = a \,\hat{+}\, ((b \ominus a) \otimes t)$	(5.26)

We call a the *origin* of the 3 lines ($t = 0$). The three solutions $x, y, \hat{x}$ are the respective coefficients of t for the lines; they are distinct iff a and b are independent. The 3 representations can be rewritten respectively under the form: $a \oplus x \otimes t$, $y \otimes t \oplus a$, $\hat{x} \otimes t \,\hat{+}\, a = a \,\hat{+}\, \hat{x} \otimes t$.

Lemma 5.1. *If a and b are dependent, non zero and distinct, $y = b \,\hat{-}\, a = b \ominus a = \hat{x} = -a \oplus b = x$ is a real multiple of a.*

Proof. Use $rel(a, b) = I$ to show that $x = y = \hat{x}$. If $b = l \otimes a$, $l \neq 1$, $x = -b \oplus a = (1 - l) \otimes a = \mu(1 - l)a$, where $\mu(1 - l) \neq 0$ for $l \neq 1$. □

5.3 L_1 and L_2 are Geodesics for $\mathring{d}$ and $\hat{d}$

The lines L_1 and L_2 define 2 notions of collinearity between a, b and a third point c which are distinct when a and b are independent.

By Definition 6.22 (resp. 6.55) the 3 points a, b, c are L_1- (resp. L_2-) collinear iff there exists $t \in \mathbb{R}$ such that c satisfies (5.1) (resp. (5.2)). The points c defined for $0 < t < 1$ are between a and b on L_1 (resp. L_2). They define the open organic arc $L_1 - (a, b)$ (resp. $L_2 - (a, b)$).

In view of (4.3), it is not surprising that L_1 is a geodesic for $\mathring{d}$ (Theorem 6.48, Remark 6.49). The less obvious Lemma 6.61 tells us that $rel(a \,\hat{-}\, c, c \,\hat{-}\, b) = I$ when a, b, c are L_2-collinear. It follows that L_2 is a geodesic for $\hat{d}$ (Theorem 6.77, Remark 6.79).

We observe that the noncommutativity of $\oplus$ ($L\oplus \neq R\oplus$) which is controlled by the relator entails the existence of two distinct geodesics related to the metrics (4.1) and (4.2) when a and b are independent, $rel(a, b) \neq I$.

Corollary 5.2. *When $a \neq 0$ and $b = l \otimes a$, $l \neq 1$, the 3 lines L_i, $i = 1, 2, 3$, coalesce into the geodesic for $\mathring{d} = \hat{d}$ which is the euclidean straight line spanned by a.*

Proof. Apply Lemma 5.1. The common geometric image is a euclidean straight line, more precisely the linear axis spanned by $a \neq 0$. □

If $l = 1$, the lines degenerate into the point $a \neq 0$. Observe that it is the *linear independence* of a and b which forces the organic lines to bend, indicating a non linearity in disguise.

6. About Midpoints on Organic Arcs

There are 3 types of fundamental organic arc (a, b) to consider which are denoted $L_i - (a, b)$. We first assume that a and b are independent.

6.1 Midpoints on L_1 and L_2 for $\oplus$

In Chapter 6, Ungar shows that a unique midpoint on $L_1 - (a, b)$ exists for (5.1) by Theorems 6.53, 6.34 and on $L_2 - (a, b)$ for (5.2) by Theorem 6.74:

- $m^L_{ab} = a \oplus (x \otimes \frac{1}{2}) = \frac{1}{2} \otimes (a \hat{+} b) = b \ominus \left(x \otimes \frac{1}{2}\right) = m^L_{ba}$, (6.27)
  $\|a \ominus m^L_{ba}\| = \|b \ominus m^L_{ba}\| = \|x\| \otimes \frac{1}{2}$,
- $m^R_{ab} = \left(y \otimes \frac{1}{2}\right) \oplus a = b \ominus \left(y \otimes \frac{1}{2}\right) = m^R_{ba}$, with $\|y\| \neq \|x\|$, (6.28)
  $\|a \hat{-} m^R_{ab}\| = \|b \hat{-} m^R_{ab}\| = \|y\| \otimes \frac{1}{2}$.

The equality $m^L_{ab} = m^L_{ba} = \frac{1}{2} \otimes (a \hat{+} b)$, suggests that a and b could play a more symmetric role in the definition of the left line L_1 for $\oplus$ under an appropriate change of parameter.

Lemma 6.1. *The line $L_1 = L\text{-}L_{ab}$ can be represented in the four equivalent forms:* $a \oplus x \otimes t = a \otimes (1 - t) \oplus b \otimes t$, $x = -a \oplus b$, *and* $b \oplus \tilde{x} \otimes t' = b \otimes (1 - t') \oplus \otimes t'$, $\tilde{x} = -b \oplus a$, *with* $t + t' = 1$.

Proof. $a \oplus (-a \otimes t \oplus b \otimes t) = a \otimes (1 - t) \oplus b \otimes t$ since $rel(a, a) = I$.
When t' replaces t, a and b are exchanged. □

Letting $t = t' = \frac{1}{2}$ yields m^L_{ab} which admits the fully symmetric representation $\frac{1}{2} \otimes (a \hat{+} b)$. This reflects an essential property of the scalar multiplication $\otimes$ by 2 (Theorem 6.7, Ungar 2008).

$$2 \otimes (a \oplus b) = a \oplus (2 \otimes b \oplus a) = a \hat{+} (a \oplus (2 \otimes b)) \tag{6.29}$$

for any $a, b \in W$. In (6.3), $2 \otimes a$ is split so that a occurs in two places in the rhs of $2 \otimes (a \oplus b)$, yielding three terms.

This yields the remarkable

Theorem 6.2. *For any two independent points $a \neq b$ the three additions $L\oplus$, $R\oplus$ and $\hat{+}$ provide the same arithmetic mean on the geodesic $L_1 = L\text{-}L_{ab}$:*

$$m_{ab}^L = \frac{1}{2} \otimes (a \oplus b) = \frac{1}{2} \otimes (b \oplus a) = \frac{1}{2} \otimes (a \mathbin{\hat{+}} b).$$

Proof. This is Theorems 6.33 and 6.34. Observe that, in addition to the above coincidences, and to (6.1), we also have $m_{ab}^L = b \oplus \left(\tilde{x} \otimes \frac{1}{2}\right) = a \ominus \left(\tilde{x} \otimes \frac{1}{2}\right)$. □

No such remarkable property holds for m_{ab}^R on $L_2 = R\text{-}L_{ab}$. The identities about m_{ab}^R and m_{ba}^R given in (6.2) cannot be further rewritten in general.

6.2 On the Line $\hat{L}$ for $\hat{+}$

The third type of organic arc (a, b) on $\hat{L}$ defined by (5.3) above has *two* pseudo-means: $\hat{m}_{ab} = \left(\hat{x} \otimes \frac{1}{2}\right) \mathbin{\hat{+}} a$ differs from $\hat{m}_{ba} = b \mathbin{\hat{-}} \hat{x} \otimes \frac{1}{2}$ (Section 6.13 in Ungar). However, $\|x\| = \|\hat{x}\|$ guarantees the equality of the respective distances $\|a \mathbin{\hat{-}} \hat{m}_{ab}\| = \|b \mathbin{\hat{-}} \hat{m}_{ba}\| = \|\hat{x}\| \otimes \frac{1}{2}$ and of their counterparts on $L\text{-}L_{ab}$.

Lemma 6.3. *The two pseudo-means $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$ are such that*

$$\|x\| \otimes \frac{1}{2} = \mathring{d}(a, m_{ab}^L) = \mathring{d}(a, \hat{m}_{ab}) = \mathring{d}(b, \hat{m}_{ba}).$$

Proof. Clear by (6.1). □

When a and b are independent, the two midpoints: m^L on $L\text{-}L_{ab}$, m^R on $R\text{-}L_{ab}$, enable dichotomy inside the two arcs $L_1 - (a, b)$, $L_2 - (a, b)$. The existence of *two* pseudo-means $\hat{m}_{ab}$ and $\hat{m}_{ba}$ on $\hat{L}_{ab}$ forbids any appeal to a dichotomy argument on an L_3-arc. For this reason, the general study of $L_3 = \hat{L}_{ab}$ is stopped at this point by Ungar, see. p. 205.

Lemma 6.4. *If $a \neq 0$, $b = ra$, for $r \in \mathbb{R}$, the four means (or midpoints) coalesce into the single point $m = \frac{1}{2} \otimes (a \mathbin{\hat{+}} b)$ on the unique line L_{ab}.*

Proof. When a and b are dependent, $\oplus = \hat{+}$, hence $x = y = \hat{x}$. Then, by Corollary 5.2, the three organic lines L_1, L_2 and L_3 coalesce into a unique one which is the geodesic for $\mathring{d} = \hat{d}$ through a and b. Clearly $m^L = m^R$, and $\hat{m}_{ba} = (b \ominus a) \oplus a \ominus (\hat{x} \otimes \frac{1}{2}) = (\hat{x} \otimes \frac{1}{2}) \oplus a = \hat{m}_{ab} = m_R = m^L$. □

7. Directional Relators

7.1 Definition

In this Section, the admissible relators belong to the subset $\boldsymbol{Q}$ of automorphisms in Aut W which satisfy (H_3):

$$rel(a,b) = rel(\frac{a}{\|a\|}, \frac{b}{\|b\|}) \tag{H_3}$$

for any pair (a,b) of nonzero vectors in V.

In other words, the map *rel* is specified by the unit vectors $1_a = \dfrac{a}{\|a\|}$ and $1_b = \dfrac{b}{\|b\|}$ defining linear directions spanned by a and b. Any relator satisfying (H_3) is called *directional*. It follows that

$$rel(a,b) = rel(a, b \boxtimes t) = rel(a \boxtimes t, b) \quad \text{for any} \quad 0 \neq t \in \mathbb{R}.$$

Example 7.1. Let $x, y \in V = \mathbb{R}^n$ be independent, then $\theta = (x,y) \notin \{0, \pi\}$. Let $R(\theta)$ denote the plane rotation $x \mapsto y$. We define $x \oplus y = x + R(\theta)y$, hence $y \oplus x = y + R(-\theta)x = R(-\theta)(x \oplus y)$ for independent x and y. Otherwise $\oplus = +$. The range $\boldsymbol{Q}$ of the relator is the set of plane rotations $SO(2)$ except the symmetry $-I_2$.

7.2 A Twofold Interpretation of L_3 under (H_3)

We use the generic notation $L_{ab} = L(a,x)$ where x is the coefficient of the parameter t in the equation for the associated line passing through the origin $a(t=0)$ and $b(t=1)$. For example, $\hat{L}_{ab} = L_3 = L(a, \hat{x})$, $\hat{x} = b \ominus a$. The line $\hat{L}_{ab}$ can be interpreted equally as a version of $(i), L_1$ or $(ii), L_2$.

Lemma 7.2. *(i)* $a \hat{+} \hat{x} \boxtimes t = a \oplus x \boxtimes t$ *with* $rel(a,-b)\hat{x} = x$.

(ii) $\hat{x} \boxtimes t \hat{+} a = \hat{x} \boxtimes t \oplus a_2$ *with* $a_2 = rel(b,-a)a$.

(iii) Moreover $\|\hat{x}\| = \|x\|$, $\|a\| = \|a_2\|$*:* $\hat{x}$ *and* a_2 *are rotated about* O *from* x *and* a *through the same angle.*

Proof. (i) $a \hat{+} \hat{x} \boxtimes t = a \oplus (rel(a,-\hat{x})\hat{x}) \boxtimes t$ by (2.6) and (H_3) with $rel(a, -b \oplus a) = rel^{-1}(-b \oplus a, a) = rel^{-1}(-b, a) = rel(a,-b)$ by (A1). And $rel(-a,b)\hat{x} = x$.

(ii) $\hat{x} \boxtimes t \hat{+} a = (b \ominus a) \boxtimes t \oplus rel(\hat{x}, -a)a$ by (2.6) and (H_3) again, and $rel(b \ominus a, -a) = rel(b \oplus (-a), -a) = rel(b, -a)$.

(iii) Clear when we observe that $rel^{-1}(a,-b) = rel(b,-a)$. □

Proposition 7.3. *When the relator is directional, the following two interpretations hold for* $\hat{L}_{ab}$*:*

(i) $\hat{L}_{ab} = L\text{-}L(a,x) = L\text{-}L_{ab}$ *with* $x = rel(a,-b)\hat{x}$, $b = a \oplus x = a \hat{+} \hat{x}$.

(ii) $\hat{L}_{ab} = R\text{-}L(a_2, \hat{x}) = R\text{-}L_{a_2 b_2}$ *with* $a_2 = rel(b,-a)a$, $b_2 = \hat{x} \oplus a_2$.

Proof. Apply Lemma 7.2. For $t = 1$, (i) $a \oplus x = b = a \hat{+} \hat{x}$, (ii) $\hat{x} \oplus a_2 = b_2 \Longleftrightarrow \hat{x} = b_2 \hat{-} a_2 = b \ominus a$. □

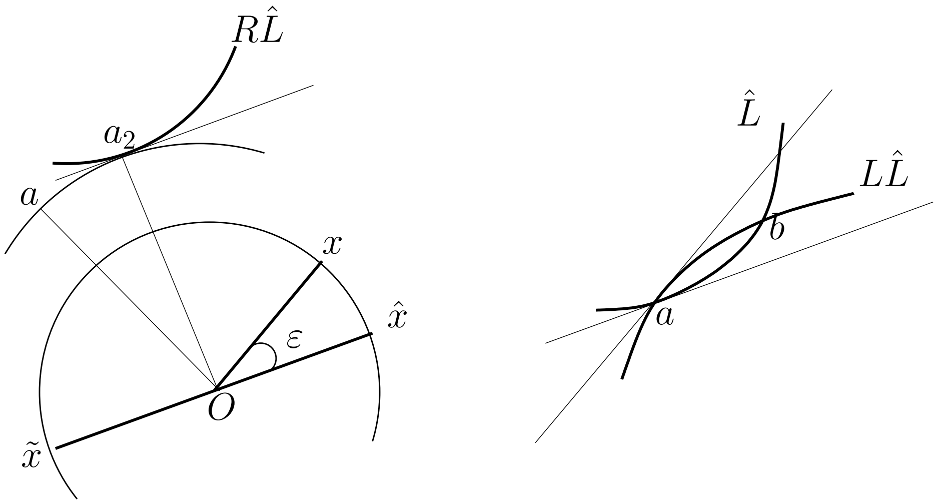

Figure 1: $\hat{L} = \hat{L}_{ab}$ *and its left and right interpretations/images:* $L\hat{L} = L\text{-}L_{ab}$ *and* $R\hat{L} = R\text{-}L(a_2, \hat{x})$.

The line $\hat{L}_{ab} = L(a, \hat{x})$ can be interpreted at the same time as the left or right image of two distinct sources. In the left (resp. right) image, the origin a is preserved (resp. moved to a_2) and the coefficient $\hat{x}$ is moved back to x (resp. preserved). Therefore the line $\hat{L} = \hat{L}_{ab}$ is a *composite* construction resulting from $\oplus$ and $rel(b, -a)$ with a *dual* character: it can be interpreted either leftwise or rightwise. Quite remarkably, the left interpretation $L\hat{L}$ is $L\text{-}L_{ab} = L_1$ itself. The right interpretation $R\hat{L} = R - L_{a_2 b_2}$ can be characterised by the rotation $x \mapsto \hat{x} = rel(b, -a)x$ about O through the angle ε. Then a is rotated into a_2 through ε. See Figure 1.

There are altogether *four* lines of interest associated with a pair (a, b): the three fundamental lines L_i, $i = 1, 2, 3$ through a, b plus the right interpretation or image $R\hat{L}$ through a_2, b_2.

7.3 a and b are Dependent

When a and b are dependent and distinct, nonzero, the 3 points O, a, b are collinear. As we know an *essential simplification* takes place: the *three organic* lines above coalesce geometrically into *one*. When the relator is *directional*, more can be said about the right image $R - L_{a_2 b_2}$ for $\hat{L}$.

Lemma 7.4. *If* $a \neq b$ *are dependent, then* $rel(a, b) = I$, $a_2 = a$, $b_2 = b$ *and* $R\hat{L} \equiv L_i$, $i = 1, 2, 3$. *If* $a = b$, $x = 0$ *and the line reduces to the point* a.

Proof. By assumption $rel(a, b) = I$ then $a \oplus b = a \hat{+} b$, hence $x = y = \hat{x} = b \ominus a$. The 3 lines L_1, L_2, L_3 coalesce into a unique line $a \oplus x \boxtimes t = x \boxtimes t \oplus a = x \boxtimes t \hat{+} a$, if

$x \neq 0 \Longleftrightarrow b \neq a$ (Corollary5.2). If $b = a$, $x = 0$, the lines reduce to the unique point $a = b \neq O$.

For the right image $R\hat{L}$ $a_2 = a$ and $b_2 = b$, yielding the identification $R\hat{L} \equiv L_i$ ☐

When a and b are *dependent* and distinct, a unification takes place. Not only the organic lines coalesce into the geodesic L_1, but also does, when the relator is directional, the right image $R\hat{L}$.

8. Weaving Information Processing (WIP)

8.1 Organic Lines

Among the three fundamental lines passing through a and b independent, the *first two* are geodesics (expressing two different views on the non commutativity of $\oplus$ (Section 5.3)). Remarkably, the third organic line L_3 offers, under (H_3) two geodesic images of itself, either $L\hat{L} = L_1$ as its left image, or $R\hat{L} = R - L_{a_2b_2}$ as its right image.

This Section develops some consequences of these geometric properties on information processing in weaving computation.

Proposition 8.1. *The two additions $\oplus$ and $\hat{+}$ coalesce on the geodesic $L_1 =$ L-L_{ab}.*

Proof. By successive dichotomy arguments based on Theorem 6.2 above: $x \oplus y = x \hat{+} y$ for any x, y between a and b on L-L_{ab}. ☐

Proposition 8.1 indicates that a sort of "differential" commutativity holds for $x \oplus y$ when x and y vary on L_1. Given a and b linearly independent the geodesic for $\mathring{d}$ through a, b describes the unique locus of points for which $\oplus$ is *commutative*, hence $\oplus = \hat{+}$ locally (on L_1). This mechanism underlies the emergence of the axiomatic role of commutativity for addition in classical mathematics.

Let us turn to L_2 which is a geodesic for $\hat{d}$; it plays a very different connecting role in IP that is discovered by revisiting (2.16) above:

Proposition 8.2. *The line $L_2 = R$-L_{ab} is such that for any $w \in W$ and $s \in$ R-L_{ab}, then*

$$b \,\hat{-}\, a = (b \oplus rel(b, s)w) \,\hat{-}\, (a \oplus rel(a, s)w) \tag{8.30}$$

Proof. See Theorem 6.76 in (Ungar 2008) and Figure 2. ☐

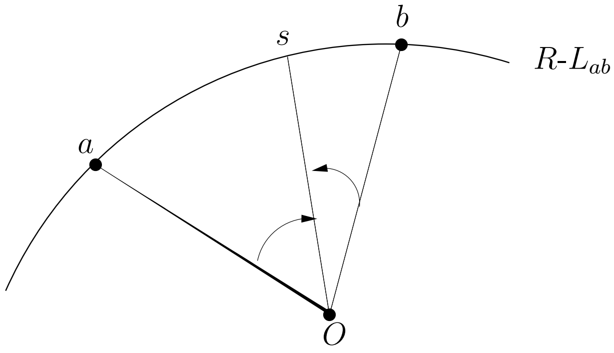

Figure 2: $a, b, s \in R\text{-}L_{ab} = L_2$.

The identity (8.1) is one possible form of the kind of right shift-invariance enjoyed by $\hat{+}$ when a and b are independent, which generalises (2.16). Indeed, if $s = a$, (8.1) yields $b \hat{-} a = (b \oplus rel(a, b)w) \hat{-} (a \oplus w)$ which becomes (2.16) after exchanging a and b, and setting $w = g$. The coefficient $y = b \hat{-} a$ is *invariant* when the same right shift, chosen in $\{\cdot \oplus rel(\cdot, s)w,\ w \in W,\ s \in R\text{-}L_{ab}\}$, is equally applied to a and b, see Figure 2 . This *exact*, albeit special, kind of $\mathcal{R}$-invariance for $\hat{+}$ under right shift should be contrasted with the metric $\mathcal{L}$-invariance for $\oplus$ (which hides the rotation $rel(g, b)$ present in (2.15)).

Definition 8.3. Given $a \neq b$, the property (8.1) for $w \in W$, $s \in L_2 = R\text{-}L_{ab}$ defines the L_2- link between a and b assumed to be independent.

Any s on $R\text{-}L_{ab}$ is uniquely defined by $t \in \mathbb{R}$ through (5.2) which defines the map:

$$t \in \mathbb{R} \mapsto y(t) = \Big(y \otimes t\Big) \oplus a,\ t \in \mathbb{R},\ y = b \hat{-} a.$$

At any $(t, w) \in \mathbb{R} \times W$ we consider in W

$$z_a(t) = rel(a, y(t))w, \quad z_b(t) = rel(b, y(t))w,$$

with $z_a(0) = z_b(1) = w$. By (8.1), $b \hat{-} a = (b \oplus z_b(t)) \hat{-} (a \oplus z_a(t))$ for all $t \in \mathbb{R}$, with $\|z_a(t)\| = \|z_b(t)\| = \|w\|$ for any $w \in W$.

The L_2- link between a and b is ruled by the two values $rel(a, y)$ and $rel(b, y)$ for the relator. Indeed, $rel(a, (b \hat{-} a) \otimes t \oplus a) rel(b \hat{-} a, a) = I$ by (2.16) in Ungar (2008), and $rel^{-1}(b \hat{-} a, a) = rel(-a, a \hat{-} b)$ (Section 2.4 in Ungar).

Proposition 8.4. *When w varies on the sphere $S_r = \{w, \|w\| = r\}$ for $0 < r < \lambda$, the L_2- link between a and b maintains $z_a(t)$ and $z_b(t)$ on S_r for all $t \in \mathbb{R}$. In particular $z_a(0) = z_b(1) = w$.*

Proof. Clear from the above discussion. $\square$

When w is arbitrary in W, the double equality $\|w\| = \|z_a(t)\| = \|z_b(t)\|$ holds for any t, and hides the actual source of the L_2- link (8.1) between a and b which resides in the relator at the pairs $(a, b \,\hat{-}\, a)$ and $(b, b \,\hat{-}\, a)$.

As for the third line, Section 7 has told us that, when the relator is **directional**, the line $\hat{L} = L_3$ is a *shape-shifter*: it can be interpreted as $L\hat{L} \equiv L_1$ and equally as $R\hat{L} = R - L_{a_2b_2}$ which differ when $rel(a, b) \neq I$.

8.2 Weaving Computation and Broadcasting Information

The broadcasting of information from a to b uses the *real* parameter t in $\mathbb{R}$ to channel through the three lines L_i with distinct features.

1) For the geodesic L_1, $\|b \ominus a\|$ is invariant under left shift. We say that L_1 *radiates metric* information. In other words, L_1 is a channel which is blind to rotations performed on the results produced by WIP: it is a *normative* channel. Because the two additions $\oplus$ and $\hat{+}$ yield identical results for any pair of points picked on itself, L_1 draws the *commutative* path from a to b: addition $\oplus$ is *locally* commutative on L_1.

2) By comparison, the geodesic L_2 through a and b (when independent) is a channel which *selects*, from the whole of WIP results, only the ones which enjoy the L_2-link (according to Definition 8.3). We say that L_2 *emanates* selected *exact* information. It is a *discriminative* or *filtering* channel.

The lines L_1 and L_2 are the *two* channels associated with $L\oplus$ and $R\oplus$ respectively: they differ geometrically when a and b are independent.

3) As we have already noticed, the third line $L_3 = \hat{L}$ is a computational construct which can be *interpreted* by means of any of the channels L_1 and L_2 when the relator is directional. The two interpretations differ markedly from $\hat{L}$ and between themselves.

The left image L_1 generally differs from $\hat{L}_{ab}$ for $t \neq 0$ and 1. The right image is $R\hat{L} = R\text{-}L_{a_2b_2}$ which passes through $a_2 \neq a$ $(t = 0)$ and $b_2 \neq b$ $(t = 1)$ in general. This computational property lends weight to the notion of "action at a distance" for information, a possibility which is most often ruled out a priori in empirical science.

By contrast, if a and b are *dependent*, $a \neq b$, there exists a *unique* channel because all L_i coalesce into the axis spanned by $a \neq 0$. When the relator is directional, the right geometric image for $\hat{L}$ is $\hat{L}$ itself.

It appears that there are several *distinct* ways by which information can be broadcast from a to b:

(i) If a and b are independent, there exist two distinct channels of information based on $L\oplus$ and $R\oplus$: the left one is a geodesic for $\mathring{d}$ which is normative and the right one is a filtering geodesic for $\hat{d}$. Provided that the relator be *directional*, these channels enable the computing agent to get a left and right interpretations for the construct $L_3 = \hat{L}$. It is remarkable that the right interpretation sustains the ill-received concept of "action at a distance" for information.

(ii) If a and b are dependent, and if the relator is directional, the two channels L_1 and L_2 coalesce with the organic line L_3 and with its two images.

9. An Epistemological Appraisal

The fact that hyperbolic geometry underlies Special Relativity was quickly realised by a handful of physicists and geometers (Ungar, 2008, Section 3.8); Ungar (2012). But the scope of hyperbolic geometry reaches much further.

9.1 Hyperbolic Geometry in Nature

A number of natural shapes exhibit, at least locally, a hyperbolic character in their geometry. The most famous example is a horse saddle or a mountain pass. Among other natural hyperbolic surfaces, one can cite lettuce leaves, coral reef or some species of marine flatworms with hyperbolic ruffles. According to W. Thurston, if one moves away from a point in hyperbolic plane, the space around the point expands exponentially. The idea was implemented in crochet in 1997 by D. Taimina by ceaselessly increasing the number of stitches in each row of her crochet model (Henderson and Taimina 2001). Experiments have shown that the visual information seen through the eyes and processed by our brain is better explained by hyperbolic geometry (Luneburg 1950). This explains the popularity of hyperbolic browsers among information professionals (Lamping et al. 1995, Allen 2002). Einstein gyrovector spaces are used in (Urribarri et al. 2013) to program an efficient tree layout, with varying levels of detail for data enclosed in a 3D-volume.

9.2 Axiomatic Vs. Cloth Geometries

The classical concept of a *group* underlies the three geometries which can be axiomatically derived from three versions of the parallel postulate: by a point not on a given line in a plane, one can draw a number p of parallels to the line with $p \in \{0, 1, \infty\}$. The best-known case $p = 1$ corresponds to a linear vector space endowed with a scalar product and derived norm. The cases $p = 0$ (elliptic) and $p = \infty$ (hyperbolic) are modifications of the euclidean case, each with many equivalent models.

By comparison, cloth geometry is derived from a metric cloth framed in a linear normed space with dimension $n \geq 2$, and based on an organ $G(\phi, \text{relator})$. It is *not* axiomatically defined, but is a *computational construct* based on ϕ and on the corresponding choice of automorphisms in the relator's range $\boldsymbol{R}$. The computation results in a **trimorphic** geometry in which the relator for $\diamond$, by inducing a secondary addition $\hat{+}$, blurs the clear-cut distinctions created by axiomatisation based on an abelian group. For example, it can be proved that $p = \infty$ and $p = 1$ are co-existing properties (Figure 8.50 on p. 370). Depending on the choice $\boldsymbol{R}$ of isometries, the computed geometry will exhibit, in addition to the euclidean structure of the frame V, *new non-euclidean* features, among which some are considered as characteristic of either hyperbolic or elliptic geometries defined axiomatically. To witness, Chapter 7 in (Ungar 2008) ends on p. 259 with the following statement:

"In modern physics, hyperbolic geometry is the study of manifolds with Riemannian metrics with contant negative curvature. However, we can see from Table 7.1 that in classical hyperbolic geometry, that is, the hyperbolic geometry of Bolyai and Lobacheivsky, constant negative curvatures and variable positive ones are inseparable."

The clear-cut distinction between the three aspects of geometry is relative rather than absolute: it can be by-passed by *weaving computation*.

9.3 Cloth Geometry in the Mind

In (Calude and Chatelin 2010,Chatelin 2012 a,b,d,2015) we have argued that hypercomputation in multiplicative Dickson algebras is part of the algorithmic toolkit for the human mind. Experimental evidence provided by Special Relativity indicates that the mental reconstruction of the observed outside 3D-reality is controlled by cloth geometry based on Einstein addition of 3D-velocities. This may offer a possible clue to what is perceived by some physicists as a pre-established harmony between mathematics and physics (Minkowski 1908,Wigner 1960, Pyenson 1982, Ungar 2003). The paper (Ungar 2003) analyses the twofold harmony which takes place in Special Relativity. Two complementary aspects of *equal importance* are useful to understand SR: either physics and geometry in 3D (Einstein 1905) or analysis in 4D (Poincaré 1905, Minkowski 1908). These complementary aspects are but the two sides of the same coin: mathematical *computations* in the mind.Both aspects have not been equally understandable in the beginnings. Therefore Minkowskian relativity prevailed for a long time, leaving certain theoretical gaps which can be filled elegantly with an appeal to the original idea of Einstein in its geometrically more mature form developed later by Ungar, see (Ungar 2003).

Going back to the human intellectual reconstruction of relativity, we **posit** that, more generally, there exists a commonly shared set of relators for mind computation. This would explain why most people agree on the general appearance of the external landscape, if not on all the details. Two eye-witnesses never agree on the minute details about the scene they both observed at the same place and time. The existence of a common cloth geometry in 3D could be the reason why we, human beings, have the feeling that we share more or less the same external reality, our habitat called Nature.

As for the inner world inside each of us, it differs widely from one individual to the next. Why? Because the number n of dimensions for the frame is not bound to be 3 anymore, but may vary arbitrarily at will, $n \geq 2$.

Cloth geometry provides a plausible mechanism for outer action and inner understanding after observation. In WIP perspective, both processes result from a drive in the mind toward explanation. The observer is *free* to choose to relate a and b by outer or inner observation. However the reader should remember that the physical reference $\lambda = c$ for the speed of light is imposed by physical reality and defines the limit of observable velocities. No such constraint exists for inner observation; in other words the inner reference λ is self-imposed (or chosen).

9.4 On the Poincaré Vs. Einstein Debate about Relativity and Geometry

During the first two decades of the 20th century the intellectual debate about the "true" nature of physical space was structured around Poincaré (and his legacy after 1912) and Einstein, see (Paty 1992). These giants stood at the two endpoints of a continuum of ideas running from Mathematics to Physics. The issues at stakes have been heatedly debated, including a priority dispute which appears rather futile in view of Ungar's isomorphism between W_E and W_P.

On the one hand Poincaré had an axiomatic vision of Geometry based on *groups* which led him to anticipate the "law of relativity" (Poincaré 1902). In special relativity he proved the dynamical invariance of physical laws for Mechanics and Electromagnetism (slightly ahead of Einstein). The relativistic dynamics presented in (Poincaré 1905) bears on group theory and (implicitly) on the field $\mathbb{H}$ of quaternions, two advanced mathematical notions which are now common in theoretical physics. His work wraps up more than 250 years of discoveries about the baffling behaviour of light (Auffray 2005). Poincaré is often criticised because – as Lorentz, Maxwell and Fresnel did before him – he occasionally mentions *ether*, a notion which is considered unnecessary in current physics. We remark that in the cognitive perspective of information processing in the mind, a background reference is required for weaving computation, whatever name is given to it, ether, or cloth geometry, or even riemannian geometry for General Relativity.

On the other hand, it is clear that Einstein did not at first feel the need for a non-euclidean geometry, because he only slowly became aware of the physical consequences of his non symmetric composition law. Together with Ehrenfest, Max Born and others, he realised that an accelerated motion would not permit exact rigidity for the moving body, but would imply elastic deformations and possible explosion. In order to save the relativity principle (by showing that it can apply to all kinds of motions including accelerations) Einstein had to *modify the geometry*, thus uncovering the full breadth of the 1905 paper.

Following (Paty 1992), we may say that: "Poincaré thought Physics with his geometric mind, as much as Einstein viewed Geometry through his physicist's eyes".

The principle of relativity has been observed in light phenomena since the 17th Century. In this intellectual odyssey, history has chosen to emphasise the year 1905 and the sole contribution of the physicist Einstein, This is an ironical twist of fate since the version of Special Relativity which survives today in textbooks rests upon the group structure of Lorentz transformations due to the mathematician Poincaré, while it overlooks the information role played *implicitly* by Einstein's non commutative addition of 3-vectors for the construction of the human image of the world.

In retrospect, one realises that special relativity in physics has two intricate aspects based on *two* algebraic structures: the metric cloth W_E (based on $\oplus_E$) envisioned by Einstein *and* the noncommutative field $\mathbb{H}$ (based on $\times$) implicit in Poincaré. A thorough comparison between the *distinct* computational roles played by these two structures is given in (Chatelin 2011).

9.5 Einstein's Vision of Relativity

In 10 years (1905-1915) Einstein's vision evolved from the commonly shared euclidean view to a highly personal one. By transmuting ideas borrowed from Riemann and Poincaré he was led to General Relativity in 1916. This larger vision he would maintain and refine for the rest of his life (Einstein 1921). Hence his work presents a remarkable continuity of thought since the day he planted the seed of Relativity by positing that admissible velocities do *not* add in a symmetric fashion. The simplicity of this idea – so daring at the time – should strike a chord in any mathematically inclined mind! Simplicity is not triviality ...; it means depth and beauty, conferring a flavour of eternity to Einstein's revolutionary idea. The new idea ran against a couple of centuries of scientific development for physics, which had climaxed in the 19th century with a commutative addition for 2- or 3-vectors in classical Mechanics, symbolised by the parallelogram law. It is fair to say that there exists a world of difference between the two physics papers authored by Einstein and Ungar which are 83 years apart (1905-1988): the difference illustrates the progress of algebraic knowledge in the 20th century. More than a century had to elapse to allow the slow coming of age of the idea of relativity: from its birthplace in experimental physics to its original habitat in the human mind which can add vectors in a *non*commutative way. This evolution would not surprise the perceptive Mach who wrote in *Die Mechanik* (1883): "We should not consider as *foundations* for the real universe the auxiliary intellectual means that we use for the *representation* of the world on the *stage of thought." (italics in original).*

The relativistic formula is routinely put to good use by engineers in telecommunications, geolocalisation and space industries. But is it really understood? A look at textbooks for physics undergraduates casts some doubts. The pristine clarity of Einstein's addition is obscured behind the cloud of Lorentz transformation and its inherent technicalities. The essence is lost in the mist of Minkowski's 4D-spacetime as this is recalled in (Ungar, 2003). It is not uncommon to find only the symmetric formula (valid for parallel velocities) as any quick websurf will confirm. It is no surprise that history has chosen to tout the (physically more difficult) equation $E = mc^2$, which is but one of the many consequences of Einstein's seminal law of noncommutative addition. Why is the analytic Poincaré/Minkowski version still prefered? Because it was the *first* to be accepted in Physics and it offers a satisfactory answer to most questions which have been raised to-date (Ungar 2003). The gaps uncovered in Theoretical Physics have not yet reach the critical mass which would force the physics community to fully endorse the geometric version of Einstein on an equal footing. Hence Ungar is still a lone pioneer.

The result of this unsatisfactory -but all too human- state of affairs is that relativity is not yet fully embraced: it is, at best, interpreted as an exotic law of Nature, with no deeper consequences on everyday life than the use of cellular phones and GPS devices. Relativity is not perceived as giving us a clue about the ways by which the human mind builds its *"imago mundi"*, its image of the world (Chatelin 2012a,b,d). The role of relativity in western science is mostly confined to physics research (nanoscale or high energy) in order to develop ever more sophisticated technologies. More than one century after Einstein's ground breaking invention, relativity has not yet been taken seriously by social scientists. They do not venture beyond the overly simplified version that is called *relativism*, a mental construct which does not do justice to the philosophical depth of relativity.

Information Processing is of paramount importance for human affairs. Information-based activities such as education, medicine, economy and ecology, could benefit greatly from a new relativity-based scientific approach to cognition.

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From the Lorentz Transformation Group in Pseudo-Euclidean Spaces to Bi-Gyrogroups

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Abstract

The Lorentz transformation of order $(m = 1, n)$, $n \in \mathbb{N}$, is the well-known Lorentz transformation of special relativity theory. It is a transformation of time-space coordinates of the pseudo-Euclidean space $\mathbb{R}^{m=1,n}$ of one time dimension and n space dimensions ($n = 3$ in physical applications). A Lorentz transformation without rotations is called a *boost*. Commonly, the special relativistic boost is parametrized by a relativistically admissible velocity parameter $\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^n_c$, whose domain is the c-ball $\mathbb{R}^n_c$ of all relativistically admissible velocities, $\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|\mathbf{v}\| < c\}$, where the ambient space $\mathbb{R}^n$ is the Euclidean n-space, and $c > 0$ is an arbitrarily fixed positive constant that represents the vacuum speed of light. The study of the Lorentz transformation composition law in terms of parameter composition reveals that the group structure of the Lorentz transformation of order $(m = 1, n)$ induces a gyrogroup and a gyrovector space structure that regulate the parameter space $\mathbb{R}^n_c$. The gyrogroup and gyrovector space structure of the ball $\mathbb{R}^n_c$, in turn, form the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry, which underlies the ball $\mathbb{R}^n_c$. The aim of this article is to extend the study of the Lorentz transformation of order (m, n) from $m = 1$ and $n \geq 1$ to all $m, n \in \mathbb{N}$, obtaining algebraic structures called a *bi-gyrogroup* and a *bi-gyrovector space*. A bi-gyrogroup is a gyrogroup each gyration of which is a pair of a left gyration and a right gyration. A bi-gyrovector space is constructed from a bi-gyrocommutative bi-gyrogroup that admits a scalar multiplication.

Keywords: Bi-gyrogroup, bi-gyrovector space, eigenball, gyrogroup, inner product of signature (m,n), Lorentz transformation of order (m,n), Pseudo-Euclidean space, special relativity.

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1. Introduction

Following the parametric realization of the Lorentz transformation group in pseudo-Euclidean spaces [46], the aim of this article is to study Lorentz transformations in pseudo-Euclidean spaces, where each of the resulting generalized Lorentz transformation group is parametrized by a generalized relativistically admissible velocity. A generalized relativistically admissible velocity, in turn, is an element of the *eigenball* $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ of all real $n \times m$ matrices, just as a relativistically admissible velocity in special relativity is an element of the ball $\mathbb{R}_c^n = \{V \in \mathbb{R}^n : \|V\| < c\}$ of the ambient Euclidean n-space $\mathbb{R}^n$. Here $c > 0$ is an arbitrarily fixed positive constant, analogous to the vacuum speed of light in special relativity.

A pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m, n), $m, n \in \mathbb{N}$, is an $(m + n)$-dimensional space with the pseudo-Euclidean inner product of signature (m, n). A Lorentz transformation of order (m, n) is a special linear transformation $\Lambda \in SO(m, n)$ of $\mathbb{R}^{m,n}$ that leaves the pseudo-Euclidean inner product invariant. It is special in the sense that the determinant of the $(m + n) \times (m + n)$ matrix representation of Λ is 1, and the determinant of its first m rows and columns is positive [21, p. 478]. The group $SO(m, n)$ of all Lorentz transformations of order (m, n) is also known as the special pseudo-orthogonal group [21, p. 478], or the group of pseudo-rotations [7]. A Lorentz transformation without rotations is called a boost when $m = 1$ and a bi-boost when $m > 1$. Bi-boosts are studied in [46].

In [46], the bi-boost $B(P)$ in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ is parametrized by $P \in \mathbb{R}^{n\times m}$, $\mathbb{R}^{n\times m}$ being the space of all real rectangular matrices of order $n \times m$. In the special case when $m = 1$, the parameter P is a column vector in $\mathbb{R}^n$ that represents a proper velocity. In physical applications $n = 3$, and a proper velocity in $\mathbb{R}^3$ is a velocity measured by proper (or, traveler's) time rather than observer's time, as explained in [37, 40].

In Sections 2-5 we review part of the study in [46] of the bi-boost $B(P)$ in order to reach the position enabling us to change the parameter $P \in \mathbb{R}^{n\times m}$ to a new parameter $V \in \mathbb{R}_c^{n\times m}$ in Section 6. Here, the space $\mathbb{R}_c^{n\times m}$ of the new parameter V is the *c-eigenball* of the ambient space $\mathbb{R}^{n\times m}$, given by

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m} :\ \text{Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\}\\ &= \{V \in \mathbb{R}^{n\times m} :\ \text{Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0 \le \lambda < c^2\}\end{aligned}$$

where $c > 0$ is an arbitrarily fixed positive constant, said to be the *eigenradius* of the eigenball.

In the special case when $m = 1$, the space $\mathbb{R}^{n\times m}$ specializes to the Euclidean n-space $\mathbb{R}^{n\times 1} = \mathbb{R}^n$ of n-dimensional column vectors. Accordingly, as shown in Example 8.2 (for c normalized to $c = 1$), the eigenball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ specializes to the c-ball of the ambient space $\mathbb{R}^n$, given by $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n = \{V \in \mathbb{R}^n : \|V\| < c\}$. Thus, when $m = 1$, the concepts of the c-eigenball and the c-ball coincide, and the Lorentz transformation of order (m, n) specializes to the Lorentz transformation of special relativity theory in one time dimension and n space dimensions ($n = 3$ in physical applications).

Eigenballs $\mathbb{R}_c^{n\times m}$ are studied in Section 7, and in Section 8 any eigenball $\mathbb{R}_c^{n\times m}$ forms the parameter space of a Lorentz transformation Λ of order (m,n). It is then shown in Sections 9-10 that the resulting bi-boosts $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, and $B_c(P)$, $P \in \mathbb{R}^{n\times m}$, leave invariant the inner product of signature (m,n), as expected.

The crucial step of this article is performed in Sections 11-12, where the composition law of two successive Lorentz transformations of order (m,n) is expressed in terms of a resulting parameter composition law, $V_1 \oplus V_2$, in the parameter space $\mathbb{R}_c^{n\times m}$. The parameter composition law, in turn, gives rise in Sections 13-16 to the novel bi-gyrogroup and bi-gyrovector space structure of the eigenball $\mathbb{R}_c^{n\times m}$. These novel algebraic structures, finally, pave in Section 17 the road leading to the novel non-Euclidean geometry of the eigenball $\mathbb{R}_c^{n\times m}$, $m,n \in \mathbb{N}$.

The algebraic and geometric structure of the parameter space $\mathbb{R}_c^{n\times m}$ is of interest in nonassociative algebra, non-Euclidean geometry, and relativity physics. In the special case when $m = 1$, it gives rise to

1. the group-like structure called a *gyrogroup*; to
2. the vector space-like structure called a *gyrovector space*; to
3. improved understanding of the hyperbolic geometry of Lobachevsky and Bolyai in terms of novel analogies with Euclidean geometry; and to
4. improved understanding of the way hyperbolic geometry regulates Einstein's special theory of relativity.

These structures and their use in hyperbolic geometry and in special relativity, along with other applications, are studied in many papers as, for instance, [2–5, 29], [9–13], [31–34], [8, 22–24, 26–28, 47], [25, 38, 44, 46], and in seven books [36, 37, 40–43, 45]. Hence, the extension of these structures from $m = 1$ and all $n \in \mathbb{N}$ to all $m, n \in \mathbb{N}$ is a most promising step towards revealing the non-Euclidean geometry that underlies the eigenball $\mathbb{R}_c^{n\times m}$, $m,n \in \mathbb{N}$. Accordingly, along with [46], this article initiates the extension of the exploration of the algebraic and geometric structure of the eigenball $\mathbb{R}_c^{n\times m}$ from $m = 1$ to $m \geq 1$, for all $n \geq 1$, and the related extension from gyrogroups and gyrovector spaces to bi-gyrogroups and bi-gyrovector spaces.

2. On the Generalized Lorentz Transformation

The (generalized) Lorentz transformation group $SO(m,n)$, $m,n \in \mathbb{N}$, is a group of special linear transformations in a pseudo-Euclidean space $\mathbb{R}^{m,n}$ of signature (m,n) that leave the pseudo-Euclidean inner product invariant. A Lorentz transformation Λ of order (m,n), $\Lambda = SO(m,n)$, is special in the sense that the determinant of the $(m+n)\times(m+n)$ matrix representation of Λ is 1, and the determinant of its first m rows and columns is positive [21, p. 478]. In the first part of this paper we present results from [46], where the set $SO(m,n)$ is described in detail.

Let $\mathbb{R}^{n\times m}$ be the set of all $n\times m$ real matrices, let $SO(n)$ be the special orthogonal group of order n, let I_n be the $n\times n$ identity matrix, and let $0_{m,n}$ be the $m\times n$ zero matrix.

Theorem 2.1 below realizes the Lorentz transformations $\Lambda\in SO(m,n)$ parametrically, with the three matrix parameters $P\in\mathbb{R}^{n\times m}$, $O_m\in SO(m)$ and $O_n\in SO(n)$.

Embedding each matrix parameter in an $(m+n)\times(m+n)$ matrix, we define (i) bi-boosts; (ii) right rotations; and (iii) left rotations as follows:

Bi-Boosts: A bi-boost is an $(m+n)\times(m+n)$ matrix $B(P)$ parametrized by $P\in\mathbb{R}^{n\times m}$,

$$B(P) := \begin{pmatrix} \sqrt{I_m+P^tP} & P^t \\ P & \sqrt{I_n+PP^t} \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,, \tag{1}$$

where P^t is the transpose of P.

Right Rotations: A right rotation is an $(m+n)\times(m+n)$ block orthogonal matrix $\rho(O_m)$ parametrized by $O_m\in SO(m)$,

$$\rho(O_m) := \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,. \tag{2}$$

Left Rotations: A left rotation is an $(m+n)\times(m+n)$ block orthogonal matrix $\lambda(O_n)$ parametrized by $O_n\in SO(n)$,

$$\lambda(O_n) := \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}\,. \tag{3}$$

Theorem 2.1. (Lorentz Transformation Bi-Gyration Decomposition, P) ([46, Theorem 8]). *A matrix $\Lambda\in\mathbb{R}^{(m+n)\times(m+n)}$ is a Lorentz transformation of order (m,n), $\Lambda\in SO(m,n)$, $m,n\in\mathbb{N}$, if and only if it is given uniquely by the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \sqrt{I_m+P^tP} & P^t \\ P & \sqrt{I_n+PP^t} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{4}$$

or, prarmetrically in short,

$$\Lambda = \Lambda(O_m,P,O_n) = \rho(O_m)B(P)\lambda(O_n) = \begin{pmatrix} P \\ O_n \\ O_m \end{pmatrix}\,. \tag{5}$$

Results (4) – (5) of Theorem 2.1 indicate the notations we use with the generic Lorentz transformation Λ of order (m,n).

We now take the results in ([46, Theorem 13]) as definitions in Def. 2.2 below, giving rise to a binary operation, $\oplus$, in $\mathbb{R}^{n\times m}$ along with two families of automorphisms of $\mathbb{R}^{n\times m}$, called *bi-gyrations*, which are the left gyrations $\mathrm{lgyr}[\cdot,\cdot]$ and the right gyrations $\mathrm{lgyr}[\cdot,\cdot]$.

Definition 2.2. (Bi-Gyroaddition and Bi-Gyration). The bi-gyroaddition $\oplus$ and bi-gyration (lgyr, rgyr) in the parameter bi-gyrogroupoid $(\mathbb{R}^{n\times m},\oplus)$ are given by the equations

$$\begin{aligned}
&P_1\oplus P_2 = P_1\sqrt{I_m+P_2^tP_2}+\sqrt{I_n+P_1P_1^t}P_2\in\mathbb{R}^{n\times m}\\
&\mathrm{lgyr}[P_1,P_2]=\\
&\sqrt{I_n+(P_1\oplus P_2)(P_1\oplus P_2)^t}^{\,-1}\left\{P_1P_2^t+\sqrt{I_n+P_1P_1^t}\sqrt{I_n+P_2P_2^t}\right\}\in SO(n)\\
&\mathrm{rgyr}[P_1,P_2]=\\
&\left\{P_1^tP_2+\sqrt{I_m+P_1^tP_1}\sqrt{I_m+P_2^tP_2}\right\}\sqrt{I_m+(P_1\oplus P_2)^t(P_1\oplus P_2)}^{\,-1}\in SO(m)
\end{aligned}\tag{6}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$.

Def. 2.2 proves useful in Theorem 2.3 below, which presents the Lorentz transformation composition law in terms of parameter composition.

Theorem 2.3. (Lorentz Transformation Product Law) ([46, Theorem 21]) *The product of two generic Lorentz transformations*

$$\begin{aligned}
\Lambda_1 &= (P_1,O_{n,1},O_{m,1})^t\\
\Lambda_2 &= (P_2,O_{n,2},O_{m,2})^t
\end{aligned}\tag{7}$$

of order (m,n), $m,n\in\mathbb{N}$, *is given by*

$$\Lambda_1\Lambda_2=\begin{pmatrix}P_1\\O_{n,1}\\O_{m,1}\end{pmatrix}\begin{pmatrix}P_2\\O_{n,2}\\O_{m,2}\end{pmatrix}=\begin{pmatrix}P_1O_{m,2}\oplus O_{n,1}P_2\\ \mathrm{lgyr}[P_1O_{m,2},O_{n,1}P_2]O_{n,1}O_{n,2}\\ O_{m,1}O_{m,2}\mathrm{rgyr}[P_1O_{m,2},O_{n,1}P_2]\end{pmatrix}.\tag{8}$$

where $\oplus$, lgyr *and* rgyr *are given by* (6) *in terms of the parameters* $P_1,P_2\in\mathbb{R}^{n\times m}$.

Illustrative examples follow.

Example 2.4. *In the special case when* $P_1=P_2=0_{n,m}$ *and* $O_{m,1}=O_{m,2}=I_m$, *the parameter composition law* (8) *yields the equation*

$$\lambda(O_{n,1})\lambda(O_{n,2})=\begin{pmatrix}0_{n,m}\\O_{n,1}\\I_m\end{pmatrix}\begin{pmatrix}0_{n,m}\\O_{n,2}\\I_m\end{pmatrix}=\begin{pmatrix}0_{n,m}\\O_{n,1}O_{n,2}\\I_m\end{pmatrix}=\lambda(O_{n,1}O_{n,2})\tag{9}$$

demonstrating that under the parameter composition law (8) *the parameter* O_n *forms the special orthogonal group* $SO(n)$.

Example 2.5. *In the special case when* $P_1 = P_2 = 0_{n,m}$ *and* $O_{n,1} = O_{n,2} = I_n$, *the parameter composition law* (8) *yields the equation*

$$\rho(O_{m,1})\rho(O_{m,2}) = \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1} \end{pmatrix} \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,2} \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ O_{m,1}O_{m,2} \end{pmatrix} = \rho(O_{m,1}O_{m,2}) \tag{10}$$

demonstrating that under the parameter composition law (8) *the parameter* O_m *forms the special orthogonal group* $SO(m)$.

Example 2.6. *In the special case when* $O_{n,1} = O_{n,2} = I_n$ *and* $O_{m,1} = O_{m,2} = I_m$ *the parameter composition law* (8) *yields the equation*

$$B(P_1)B(P_2) = \begin{pmatrix} P_1 \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} P_2 \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P_1 \oplus P_2 \\ \mathrm{lgyr}[P_1, P_2] \\ \mathrm{rgyr}[P_1, P_2] \end{pmatrix} . \tag{11}$$

Clearly, under the parameter composition law (8) *the parameter* $P \in \mathbb{R}^{n\times m}$ *does not form a group, owing to the presence of bi-gyrations. Indeed,* (11) *demonstrates that, in general, the composition of two bi-boosts is not a bi-boost but, rather, a bi-boost associated with a bi-gyration.*

In the special case when $P_1 = P$ *and* $P_2 = \ominus P$, (11) *gives*

$$B(P)B(\ominus P) = \begin{pmatrix} P \\ I_n \\ I_m \end{pmatrix} \begin{pmatrix} \ominus P \\ I_n \\ I_m \end{pmatrix} = \begin{pmatrix} P \ominus P \\ \mathrm{lgyr}[P, \ominus P] \\ \mathrm{rgyr}[P, \ominus P] \end{pmatrix} = \begin{pmatrix} 0_{n,m} \\ I_n \\ I_m \end{pmatrix} , \tag{12}$$

so that the inverse of $B(P)$ *is* $B(\ominus P) = B(-P)$. *In* (12) *we use the results* $\mathrm{lgyr}[P, \ominus P] = I_n$ *and* $\mathrm{rgyr}[P, \ominus P] = I_m$, *which are verified in* ([46, Eq. (114)]).

The product rule (8) is neither commutative nor associative. However, it possesses a rich algebraic structure. Thus, in particular, it obeys a commutative-like and an associative-like laws, called the bi-gyrocommutative and the bi-gyroassociative law of the bi-gyrogroupoid $(\mathbb{R}^{n\times m}, \oplus)$.

Theorem 2.7. (Bi-Gyrocommutative Law in $(\mathbb{R}^{n\times m}, \oplus)$) ([46, Theorem 25]). *The binary operation* $\oplus$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyrocommutative law*

$$P_1 \oplus P_2 = \mathrm{lgyr}[P_1, P_2](P_2 \oplus P_1)\mathrm{rgyr}[P_1, P_2] \tag{13}$$

for all $P_1, P_2 \in \mathbb{R}^{n\times m}$.

In Theorem 2.7 the bi-gyration $(\mathrm{lgyr}[P_1,P_2],\mathrm{rgyr}[P_1,P_2])$ takes $P_2\oplus P_1$ into $P_1\oplus P_2$. It rotates the $n\times m$ matrix $P_2\oplus P_1\in\mathbb{R}^{n\times m}$ from the left by the orthogonal matrix $\mathrm{lgyr}[P_1,P_2]\in SO(n)$, and from the right by the orthogonal matrix $\mathrm{rgyr}[P_1,P_2]\in SO(m)$.

Theorem 2.8. (Bi-Gyroassociative Law in $(\mathbb{R}^{n\times m},\oplus)$**)** ([46, Theorem 27]). *The binary operation* $\oplus$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyroassociative law*

$$(P_1\oplus P_2)\oplus\mathrm{lgyr}[P_1,P_2]P_3=P_1\mathrm{rgyr}[P_2,P_3]\oplus(P_2\oplus P_3) \tag{14}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$.

Note that P_1 and P_2 are grouped together on the left side of (14), while P_2 and P_3 are grouped together on the right side of (14).

3. Bi-Gyrogroups

It proves useful in [46] to replace the binary operation $\oplus$ in $\mathbb{R}^{n\times m}$ by a new binary operation, $\oplus'$, according to the following definition.

Definition 3.1. (Bi-Gyrogroup Operation, Bi-Gyrogroups) ([46, Definition 35]). Let $(\mathbb{R}^{n\times m},\oplus)$ be a bi-gyrogroupoid. A new bi-gyrogroup binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ is given by

$$P_1\oplus' P_2=(P_1\oplus P_2)\mathrm{rgyr}[P_2,P_1] \tag{15}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$. The resulting groupoid $(\mathbb{R}^{n\times m},\oplus')$ is called a bi-gyrogroup.

The *bi-gyrogroup* $(\mathbb{R}^{n\times m},\oplus')$ is defined in Def. 3.1 in terms of the *bi-gyrogroupoid* $(\mathbb{R}^{n\times m},\oplus)$.

It is shown in [46] that (15) implies the following four identities that exhibit an interesting symmetry between the binary operations $\oplus$ and $\oplus'$ in $\mathbb{R}^{n\times m}$.

$$\begin{aligned}P_1\oplus' P_2&=(P_1\oplus P_2)\mathrm{rgyr}[P_2,P_1]\\ P_1\oplus P_2&=(P_1\oplus' P_2)\mathrm{rgyr}[P_1,P_2]\\ P_1\oplus' P_2&=\mathrm{lgyr}[P_1,P_2](P_2\oplus P_1)\\ P_1\oplus P_2&=\mathrm{lgyr}[P_1,P_2](P_2\oplus' P_1)\end{aligned} \tag{16}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$.

Theorem 3.2. (Bi-Gyrocommutative Law in $(\mathbb{R}^{n\times m},\oplus')$**)** ([46, Theorem 42]). *The binary operation* $\oplus'$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyrocommutative law*

$$P_1\oplus' P_2=\mathrm{lgyr}[P_1,P_2](P_2\oplus' P_1)\mathrm{rgyr}[P_2,P_1] \tag{17}$$

for all $P_1,P_2\in\mathbb{R}^{n\times m}$.

It follows from (13) and (17) that the binary operations $\oplus$ and $\oplus'$ possess the same bi-gyrocommutative law. This is, however, not the case with the bi-gyroassociative law, as shown in Theorem 3.3.

Theorem 3.3. (Bi-Gyrogroup Left and Right Bi-Gyroassociative Law) ([46, Theorem 41]). *The binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ possesses the left bi-gyroassociative law*

$$P_1\oplus'(P_2\oplus'X) = (P_1\oplus'P_2)\oplus'\mathrm{lgyr}[P_1,P_2]X\mathrm{rgyr}[P_2,P_1] \tag{18}$$

and the right bi-gyroassociative law

$$(P_1\oplus'P_2)\oplus'X = P_1\oplus'(P_2\oplus'\mathrm{lgyr}[P_2,P_1]X\mathrm{rgyr}[P_1,P_2]) \tag{19}$$

for all $P_1, P_2, X \in \mathbb{R}^{n\times m}$.

4. Gyrogroup Gyrations

The bi-gyroassociative laws (18) – (19) and the bi-gyrocommutative law (17) suggest the following definition of gyrations in terms of left and right gyrations.

Definition 4.1. (Gyrogroup Gyrations) ([46, Definition 43]). The gyrator

$$\mathrm{gyr} : \mathbb{R}^{n\times m} \times \mathbb{R}^{n\times m} \to \mathrm{Aut}(\mathbb{R}^{n\times m}, \oplus')$$

generates automorphisms called gyrations, $\mathrm{gyr}[P_1,P_2] \in \mathrm{Aut}(\mathbb{R}^{n\times m}, \oplus')$, given by the equation

$$\mathrm{gyr}[P_1,P_2]X = \mathrm{lgyr}[P_1,P_2]X\mathrm{rgyr}[P_2,P_1] \tag{20}$$

for all $P_1, P_2, X \in \mathbb{R}^{n\times m}$, where left gyrations, $\mathrm{lgyr}[P_1,P_2]$, and right gyrations, $\mathrm{rgyr}[P_2,P_1]$, are given in (6), p. 233. The gyration $\mathrm{gyr}[P_1,P_2]$ is said to be the gyration generated by $P_1, P_2 \in \mathbb{R}^{n\times m}$. Being automorphisms of $(\mathbb{R}^{n\times m}, \oplus')$, gyrations are also called gyroautomorphisms.

Def. 4.1 will turn out rewarding, leading to the elegant result that any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ is a gyrocommutative gyrogroup.

Theorem 4.2. (Gyrogroup Gyroassociative and Gyrocommutative Laws) ([46, Theorem 44]). *The binary operation $\oplus'$ in $\mathbb{R}^{n\times m}$ obeys the left and the right gyroassociative law*

$$P_1\oplus'(P_2\oplus'X) = (P_1\oplus'P_2)\oplus'\mathrm{gyr}[P_1,P_2]X \tag{21}$$

and

$$(P_1\oplus'P_2)\oplus'X = P_1\oplus'(P_2\oplus'\mathrm{gyr}[P_2,P_1]X) \tag{22}$$

and the gyrocommutative law

$$P_1\oplus'P_2 = \mathrm{gyr}[P_1,P_2](P_2\oplus'P_1)\,. \tag{23}$$

Proof. Identities (21) – (22) follow immediately from Def. 4.1 and the left and right bi-gyroassociative law (18) – (19). Similarly, (23) follow immediately from Def. 4.1 and the bi-gyrocommutative law (17). □

Lemma 4.3. ([46, Lemma 45]). *The relation* (20) *between gyrations* $\mathrm{gyr}[P_1, P_2]$ *and corresponding bi-gyrations* $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_2, P_1])$*,* $P_1, P_2 \in (\mathbb{R}^{n\times m}, \oplus')$*, is bijective.*

It is obvious from (20) that a gyration $\mathrm{gyr}[P_1, P_2]$ is determined uniquely by the bi-gyration $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$. It follows from Lemma 4.3 that also the converse is true, that is, a bi-gyration $(\mathrm{lgyr}[P_1, P_2], \mathrm{rgyr}[P_1, P_2])$ is determined uniquely by the gyration $\mathrm{gyr}[P_1, P_2]$.

It is anticipated in Def. 4.1 that gyrations are automorphisms. The following theorem asserts that this is indeed the case.

Theorem 4.4. (Gyroautomorphism) ([46, Theorem 46]). *Gyrations* $\mathrm{gyr}[P_1, P_2]$ *of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *are automorphisms of the bi-gyrogroup.*

Theorem 4.5. (Left Gyration Reduction Properties) ([46, Theorem 47]). *Left gyrations of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *possess the left gyration left reduction property*

$$\mathrm{lgyr}[P_1, P_2] = \mathrm{lgyr}[P_1 \oplus' P_2, P_2] \tag{24}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[P_1, P_2] = \mathrm{lgyr}[P_1, P_2 \oplus' P_1]\,. \tag{25}$$

Theorem 4.6. (Right Gyration Reduction Properties) ([46, Theorem 48]). *Right gyrations of a bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$ *possess the right gyration left reduction property*

$$\mathrm{rgyr}[P_1, P_2] = \mathrm{rgyr}[P_1 \oplus' P_2, P_2] \tag{26}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[P_1, P_2] = \mathrm{rgyr}[P_1, P_2 \oplus' P_1]\,. \tag{27}$$

Theorem 4.7. (Gyration Reduction Properties) ([46, Theorem 49]). *The gyrations of any bi-gyrogroup* $(\mathbb{R}^{n\times m}, \oplus')$*,* $m, n \in \mathbb{N}$*, possess the left and right reduction property*

$$\mathrm{gyr}[P_1, P_2] = \mathrm{gyr}[P_1 \oplus' P_2, P_2] \tag{28}$$

and

$$\mathrm{gyr}[P_1, P_2] = \mathrm{gyr}[P_1, P_2 \oplus' P_1]\,. \tag{29}$$

Proof. Identities (28) and (29) follow from Def. 4.1 of gyr in terms of lgyr and rgyr, and from Theorems 4.5 and 4.6. □

5. Gyrogroups and Bi-Gyrogroups

We are now in a position to present the definition of the abstract (gyrocommutative) gyrogroup, and note the proof in [46, Theorem 52] that any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, is a gyrocommutative gyrogroup.

Forming a natural generalization of groups, gyrogroups emerged in the 1988 study of the parametrization of the Lorentz group of Einstein's special relativity theory [35, 36]. Einstein velocity addition, thus, provides a concrete example of a gyrocommutative gyrogroup operation in the ball of all relativistically admissible velocities.

Definition 5.1. (Gyrogroups) ([46, Definition 50]). *A groupoid* $(G, \oplus)$ *is a gyrogroup if its binary operation satisfies the following axioms (G1) – (G5). In G there is at least one element,* 0*, called a left identity, satisfying*

(G1) $\quad 0\oplus a = a$

for all $a \in G$*. There is an element* $0 \in G$ *satisfying axiom* $(G1)$ *such that for each* $a \in G$ *there is an element* $\ominus a \in G$*, called a left inverse of* a*, satisfying*

(G2) $\quad \ominus a\oplus a = 0\,.$

Moreover, for any $a, b, c \in G$ *there exists a unique element* $\mathrm{gyr}[a, b]c \in G$ *such that the binary operation obeys the left gyroassociative law*

(G3) $\quad a\oplus(b\oplus c) = (a\oplus b)\oplus\mathrm{gyr}[a, b]c\,.$

The map $\mathrm{gyr}[a, b] : G \to G$ *given by* $c \mapsto \mathrm{gyr}[a, b]c$ *is an automorphism of the groupoid* $(G, \oplus)$*, that is,*

(G4) $\quad \mathrm{gyr}[a, b] \in \mathrm{Aut}(G, \oplus)\,,$

and the automorphism $\mathrm{gyr}[a, b]$ *of* G *is called the gyroautomorphism, or the gyration, of* G *generated by* $a, b \in G$*. The operator* $\mathrm{gyr} : G \times G \to \mathrm{Aut}(G, \oplus)$ *is called the gyrator of* G*. Finally, the gyroautomorphism* $\mathrm{gyr}[a, b]$ *generated by any* $a, b \in G$ *possesses the left reduction property*

(G5) $\quad \mathrm{gyr}[a, b] = \mathrm{gyr}[a\oplus b, b]\,,$

called the reduction axiom.

The gyrogroup axioms $(G1) - (G5)$ in Definition 5.1 are classified into three classes:

1. The first pair of axioms, $(G1)$ and $(G2)$, is a reminiscent of the group axioms.

2. The last pair of axioms, $(G4)$ and $(G5)$, presents the gyrator axioms.

3. The middle axiom, $(G3)$, is a hybrid axiom linking the two pairs of axioms in (1) and (2).

As in group theory, we use the notation $a\ominus b = a\oplus(\ominus b)$ in gyrogroup theory as well.

In full analogy with groups, gyrogroups are classified into gyrocommutative and non-gyrocommutative gyrogroups.

Definition 5.2. (Gyrocommutative Gyrogroups) ([46, Definition 51]). *A gyrogroup $(G, \oplus)$ is gyrocommutative if its binary operation obeys the gyrocommutative law*

(G6) $\qquad a \oplus b = \mathrm{gyr}[a, b](b \oplus a)$

for all $a, b \in G$.

Theorem 5.3. (Gyrocommutative Gyrogroup) ([46, Theorem 52]).
Any bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$, $n, m \in \mathbb{N}$, is a gyrocommutative gyrogroup.

Following the definition of the abstract (gyrocommutative) gyrogroup, we are now in the position to present the definition of the abstract (bi-gyrocommutative) bi-gyrogroup.

Definition 5.4. (Bi-Gyrogroups) ([46, Definition 53]). *A (gyrocommutative) gyrogroup whose gyrations are bi-gyrations is said to be a (bi-gyrocommutative) bi-gyrogroup.*

A detailed study of the abstract bi-gyrogroup is presented in [32].

A concrete example of a nontrivial bi-gyrogroup is provided by the Einstein bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ that stems in Section 2 from the (generalized) Lorentz transformation of order (m, n), $m, n \in \mathbb{N}$. In the special case when $m = 1$ we have $\mathbb{R}^{n\times m} = \mathbb{R}^n$, and the Einstein bi-gyrogroup $(\mathbb{R}^{n\times m}, \oplus')$ specializes to the Einstein gyrogroup $(\mathbb{R}^n, \oplus')$. It turns out that $(\mathbb{R}^n, \oplus')$=$(\mathbb{R}^n, \oplus_{\mathrm{U}})$, $(\mathbb{R}^n, \oplus_{\mathrm{U}})$ being the Einstein *proper velocity* gyrogroup associated with the Einstein addition law of proper (traveler's) velocities rather than the common observer's velocities. Einstein PV (proper velocity) gyrogroups, in turn, stem from the *proper velocity Lorentz group* studied, for instance, in [38, 40] and [1]. We, therefore, call $(\mathbb{R}^{n\times m}, \oplus')$ *a PV-bi-gyrogroup.*

As a goal of this paper, we now face the task of changing the parameter $P \in \mathbb{R}^{n\times m}$, which represents generalized proper (traveler's) relativistic velocities, to a new parameter, V, which represents generalized relativistically admissible (observer's) velocities. Achieving the goal, we will obtain Einstein bi-gyrogroups associated with generalized observer's, rather than traveler's, velocities.

6. Bi-Boost Parameter Change, $P \to V$

It is now useful to introduce a positive parameter $c > 0$ into the bi-boost $B(P)$ in (1), obtaining the bi-boost $B_c(P)$,

$$B_c(P) = \begin{pmatrix} \sqrt{I_m + c^{-2}P^tP} & \frac{1}{c^2}P^t \\ P & \sqrt{I_n + c^{-2}PP^t} \end{pmatrix}, \tag{30}$$

so that $B(P) = B_{c=1}(P)$ is a normalized form of $B_c(P)$.

Definition 6.1. For any $m, n \in \mathbb{N}$ let $\phi : \mathbb{R}^{n\times m} \to \mathbb{R}^{n\times m}$ be the map given by

$$\phi : \ P \ \mapsto \ V = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P \tag{31}$$

where $c > 0$ is an arbitrarily fixed positive constant.

The image

$$\mathbb{B}^{n\times m} := \phi(\mathbb{R}^{n\times m}) \subset \mathbb{R}^{n\times m} \tag{32}$$

of ϕ in $\mathbb{R}^{n\times m}$ is called the (open) c-eigenball of $\mathbb{R}^{n\times m}$.

The term "eigenball" will be justified in Theorem 7.2 following the observation that $\mathbb{B}^{n\times m} \subset \mathbb{R}^{n\times m}$ is the set $\mathbb{R}_c^{n\times m}$ of all $V \in \mathbb{R}^{n\times m}$ such that each eigenvalue λ of the Gramian matrices VV^t and V^tV is nonnegative and smaller than c, $0 \le \lambda < c$. We will see in Example 8.2 that in the special case when $m = 1$, the c-eigenball $\mathbb{R}_c^{n\times m}$ of $\mathbb{R}^{n\times 1}$ specializes to the c-ball $\mathbb{R}_c^n$ of $\mathbb{R}^n$, that is, $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$.

Lemma 6.2. *The map ϕ in Def. 6.1 can be written equivalently as*

$$\phi : \ P \ \mapsto \ V = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} . \tag{33}$$

Proof. The proof follows immediately from (31) and from the commuting relation

$$P\sqrt{I_m + c^{-2}P^tP} = \sqrt{I_n + c^{-2}PP^t}P \tag{34}$$

for all $P \in \mathbb{R}^{n\times m}$, proved in [46, Eq. (53)] for $c = 1$. The passage from $c = 1$ to $c > 0$ is immediate in this case. □

Theorem 6.3. *For any $P \in \mathbb{R}^{n\times m}$,*

$$V = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} \tag{35}$$

if and only if

$$P = \sqrt{I_n - c^{-2}VV^t}^{\,-1} V = V\sqrt{I_m - c^{-2}V^tV}^{\,-1} . \tag{36}$$

Proof. The proof is divided into four parts. In Parts I_A and I_B we prove that (35) implies (36), and in Parts II_A and II_B we prove that (36) implies (35).

Part $\boldsymbol{I_A}$: Assuming (35), we have

$$P = \sqrt{I_n + c^{-2}PP^t}V \tag{37}$$

and the commuting relation

$$P^t\sqrt{I_n + c^{-2}PP^t}^{\,-1} = \sqrt{I_m + c^{-2}P^tP}^{\,-1} P^t \tag{38}$$

so that by (38) and (35)

$$PP^t\sqrt{I_n + c^{-2}PP^t}^{\,-1} = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} P^t = \sqrt{I_n + c^{-2}PP^t}^{\,-1} PP^t . \tag{39}$$

Then, by (35) and (39),

$$\begin{aligned} VV^t &= \sqrt{I_n + c^{-2}PP^t}^{\,-1} PP^t \sqrt{I_n + c^{-2}PP^t}^{\,-1} \\ &= PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)^{-1}PP^t\,. \end{aligned} \tag{40}$$

Hence,

$$PP^t = (I_n + c^{-2}PP^t)VV^t \tag{41}$$

and

$$PP^t = VV^t(I_n + c^{-2}PP^t)\,. \tag{42}$$

A rearrangement of (41) yields

$$VV^t = PP^t(I_n - c^{-2}VV^t) \tag{43}$$

implying

$$PP^t = VV^t(I_n - c^{-2}VV^t)^{-1}\,. \tag{44}$$

Similarly, a rearrangement of (42) yields

$$VV^t = (I_n - c^{-2}VV^t)PP^t \tag{45}$$

implying

$$PP^t = (I_n - c^{-2}VV^t)^{-1}VV^t\,. \tag{46}$$

Following (44) we have

$$\begin{aligned} I_n + c^{-2}PP^t &= I_n + c^{-2}VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)(I_n - c^{-2}VV^t)^{-1} + c^{-2}VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t + c^{-2}VV^t)(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)^{-1} \end{aligned} \tag{47}$$

so that

$$\sqrt{I_n + c^{-2}PP^t} = \sqrt{I_n - c^{-2}VV^t}^{\,-1}\,. \tag{48}$$

Hence, by (37) and (48),

$$P = \sqrt{I_n + c^{-2}PP^t}V = \sqrt{I_n - c^{-2}VV^t}^{\,-1}V \tag{49}$$

thus validating the first equation in (36), In Part I_B of the proof we validate the second equation in (36),

Part I_B: Assuming (35), we have

$$P = V\sqrt{I_m + c^{-2}P^tP} \tag{50}$$

and the commuting relation, as in (38),

$$P^t \sqrt{I_n + c^{-2}PP^t}^{\,-1} = \sqrt{I_m + c^{-2}P^tP}^{\,-1} P^t \tag{51}$$

so that by (35) and (51),

$$P^tP\sqrt{I_m + c^{-2}P^tP}^{\,-1} = P^t\sqrt{I_n + c^{-2}PP^t}^{\,-1}P = \sqrt{I_m + c^{-2}P^tP}^{\,-1}P^tP\,. \tag{52}$$

Then, by (35) and (52),

$$\begin{aligned} V^tV &= \sqrt{I_m + c^{-2}P^tP}^{\,-1}P^tP\sqrt{I_m + c^{-2}P^tP}^{\,-1} \\ &= P^tP(I_m + c^{-2}P^tP)^{-1} \\ &= (I_m + c^{-2}P^tP)^{-1}P^tP\,. \end{aligned} \tag{53}$$

Hence,

$$P^tP = (I_m + c^{-2}P^tP)V^tV \tag{54}$$

and

$$P^tP = V^tV(I_m + c^{-2}P^tP) \tag{55}$$

A rearrangement of (54) yields

$$V^tV = P^tP(I_m - c^{-2}V^tV) \tag{56}$$

implying

$$P^tP = V^tV(I_m - c^{-2}V^tV)^{-1}\,. \tag{57}$$

Similarly, a rearrangement of (55) yields

$$V^tV = (I_m - c^{-2}V^tV)P^tP \tag{58}$$

implying

$$P^tP = (I_m - c^{-2}V^tV)^{-1}V^tV\,. \tag{59}$$

Following (57) we have

$$\begin{aligned} I_m + c^{-2}P^tP &= I_m + c^{-2}V^tV(I_m - c^{-2}V^tV)^{-1} \\ &= (I_m - c^{-2}V^tV)(I_m - c^{-2}V^tV)^{-1} + c^{-2}V^tV(I_m - c^{-2}V^tV)^{-1} \\ &= (I_m - c^{-2}V^tV + c^{-2}V^tV)(I_m - c^{-2}V^tV)^{-1} \\ &= (I_m - c^{-2}V^tV)^{-1} \end{aligned} \tag{60}$$

so that

$$\sqrt{I_m + c^{-2}P^tP} = \sqrt{I_m - c^{-2}V^tV}^{\,-1}\,. \tag{61}$$

Hence, by (50) and (61),

$$P = V\sqrt{I_m + c^{-2}P^tP} = V\sqrt{I_m - c^{-2}V^tV}^{\;-1} . \tag{62}$$

Equations (49) and (62) validate the two equations in (36).

Conversely, in Parts II_A and II_B we show that (36) implies (35).

Part $\boldsymbol{II_A}$: Assuming (36), we have

$$V = \sqrt{I_n - c^{-2}VV^t}P \tag{63}$$

and the commuting relation

$$V^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} = \sqrt{I_m - c^{-2}V^tV}^{\;-1}V^t \tag{64}$$

so that, by (64) and (36)

$$\begin{aligned} VV^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} &= V\sqrt{I_m - c^{-2}V^tV}^{\;-1}V^t \\ &= \sqrt{I_n - c^{-2}VV^t}^{\;-1}VV^t . \end{aligned} \tag{65}$$

Then, by (36) and (65),

$$\begin{aligned} PP^t &= \sqrt{I_n - c^{-2}VV^t}^{\;-1}VV^t\sqrt{I_n - c^{-2}VV^t}^{\;-1} \\ &= VV^t(I_n - c^{-2}VV^t)^{-1} \\ &= (I_n - c^{-2}VV^t)^{-1}VV^t . \end{aligned} \tag{66}$$

Hence,

$$VV^t = (I_n - c^{-2}VV^t)PP^t \tag{67}$$

and

$$VV^t = PP^t(I_n - c^{-2}VV^t) . \tag{68}$$

A rearrangement of (67) yields

$$PP^t = VV^t(I_n + c^{-2}PP^t) \tag{69}$$

implying

$$VV^t = PP^t(I_n + c^{-2}PP^t)^{-1} \tag{70}$$

Similarly, a rearrangement of (68) yields

$$PP^t = (I_n + c^{-2}PP^t)VV^t \tag{71}$$

implying

$$VV^t = (I_n + c^{-2}PP^t)^{-1}PP^t . \tag{72}$$

Following (70) we have

$$\begin{aligned} I_n - c^{-2}VV^t &= I_n - c^{-2}PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)(I_n + c^{-2}PP^t)^{-1} - c^{-2}PP^t(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t - c^{-2}PP^t)(I_n + c^{-2}PP^t)^{-1} \\ &= (I_n + c^{-2}PP^t)^{-1} \end{aligned} \tag{73}$$

so that

$$\sqrt{I_n - c^{-2}VV^t} = \sqrt{I_n + c^{-2}PP^t}^{\,-1}. \tag{74}$$

Hence, by (63) and (74),

$$V = \sqrt{I_n - c^{-2}VV^t}P = \sqrt{I_n + c^{-2}PP^t}^{\,-1}P \tag{75}$$

thus validating the first equation in (35). In Part II_B of the proof we validate the second equation in (35).

Part $\boldsymbol{II_B}$: Assuming (36), we have

$$V = P\sqrt{I_m - c^{-2}V^tV} \tag{76}$$

and the commuting relation, as in (64),

$$V^t\sqrt{I_n - c^{-2}VV^t}^{\,-1} = \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^t \tag{77}$$

so that by (36) and (77),

$$V^tV\sqrt{I_m - c^{-2}V^tV}^{\,-1} = V^t\sqrt{I_n - c^{-2}VV^t}^{\,-1}V = \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^tV\,. \tag{78}$$

Then, by (36) and (78),

$$\begin{aligned} P^tP &= \sqrt{I_m - c^{-2}V^tV}^{\,-1}V^tV\sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ &= V^tV(I_m - c^{-2}V^tV)^{-1} \\ &= (I_m - c^{-2}V^tV)^{-1}V^tV\,. \end{aligned} \tag{79}$$

Hence,

$$V^tV = (I_m - c^{-2}V^tV)P^tP \tag{80}$$

and

$$V^tV = P^tP(I_m - c^{-2}V^tV)\,. \tag{81}$$

A rearrangement of (80) yields

$$P^tP = V^tV(I_m + c^{-2}P^tP) \tag{82}$$

implying

$$V^tV = P^tP(I_m + c^{-2}P^tP)^{-1}\,. \tag{83}$$

Similarly, a rearrangement of (81) yields

$$P^tP = (I_m + c^{-2}P^tP)V^tV \tag{84}$$

implying

$$V^tV = (I_m + c^{-2}P^tP)^{-1}P^tP\,. \tag{85}$$

Following (83) we have

$$\begin{aligned}
I_m - c^{-2}V^tV &= I_m - c^{-2}P^tP(I_m + c^{-2}P^tP)^{-1} \\
&= (I_m + c^{-2}P^tP)(I_m + c^{-2}P^tP)^{-1} - c^{-2}P^tP(I_m + c^{-2}P^tP)^{-1} \\
&= (I_m + c^{-2}P^tP - c^{-2}P^tP)(I_m + c^{-2}P^tP)^{-1} \\
&= (I_m + c^{-2}P^tP)^{-1}
\end{aligned} \tag{86}$$

so that

$$\sqrt{I_m - c^{-2}V^tV} = \sqrt{I_m + c^{-2}P^tP}^{\;-1}\,. \tag{87}$$

Hence, by (76) and (87),

$$V = P\sqrt{I_m - c^{-2}V^tV} = P\sqrt{I_m + c^{-2}P^tP}^{\;-1}\,. \tag{88}$$

Equations (75) and (88) validate the two equations in (35), and the proof is complete. □

Theorem 6.4. *Let $\phi:\ \mathbb{R}^{n\times m} \to \mathbb{B}^{n\times m}$, $m, n \in \mathbb{N}$, be the map given by each of the two mutually equivalent equations*

$$\begin{aligned}
\phi:\ P &\mapsto V = \sqrt{I_n + c^{-2}PP^t}^{\;-1}P \\
\phi:\ P &\mapsto V = P\sqrt{I_m + c^{-2}P^tP}^{\;-1}
\end{aligned} \tag{89}$$

where $\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$ is the image of $\mathbb{R}^{n\times m}$ under ϕ.

Then, ϕ is bijective, and the inverse $\phi^{-1}: \mathbb{B}^{n\times m} \to \mathbb{R}^{n\times m}$ of ϕ is given by each of the two mutually equivalent equations

$$\begin{aligned}
\phi^{-1}:\ V &\mapsto P = \sqrt{I_n - c^{-2}VV^t}^{\;-1}V \\
\phi^{-1}:\ V &\mapsto P = V\sqrt{I_m - c^{-2}V^tV}^{\;-1}\,.
\end{aligned} \tag{90}$$

Proof. The proof follows immediately from Theorem 6.3. □

7. Eigenballs

In order to characterize the image $\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$ of $\mathbb{R}^{n\times m}$ under ϕ in terms of eigenvalues, we present the following well-known theorem.

Theorem 7.1. ([6, p. 56]). *If a square matrix A has the eigenvalue λ and the corresponding eigenvector $\mathbf{x}$, then any rational function $R(A)$ of A has the eigenvalue $R(\lambda)$ and the eigenvector $\mathbf{x}$.*

Theorem 7.1 enables us to prove the following theorem, which characterizes $\mathbb{B}^{n\times m}$ in terms of eigenvalues.

Theorem 7.2. *Let*

$$\mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m}) \tag{91}$$

and

$$\mathbb{R}_c^{n\times m} = \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\}. \tag{92}$$

Then,

$$\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m}\,. \tag{93}$$

Proof. Let $V \in \mathbb{B}^{n\times m} = \phi(\mathbb{R}^{n\times m})$. Then there exists $P \in \mathbb{R}^{n\times m}$ such that

$$V = \phi(P) = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P \tag{94}$$

and, hence, by (72),

$$VV^t = (I_n + c^{-2}PP^t)^{-1}PP^t\,. \tag{95}$$

Let λ_i, $i = 1, \dots, n$, be the eigenvalues of PP^t. Then $\lambda_i \ge 0$ and, by (95) and Theorem 7.1, the eigenvalues μ_i of VV^t are

$$\mu_i = \frac{\lambda_i}{1 + \lambda_i/c^2} \tag{96}$$

so that $0 \le \mu_i < c^2$. Hence $V \in \mathbb{R}_c^{n\times m}$, implying the inclusion $\mathbb{B}^{n\times m} \subseteq \mathbb{R}_c^{n\times m}$.

To prove the reverse inclusion, let $V \in \mathbb{R}_c^{n\times m}$ and let μ_i, $i = 1, \dots, n$ be the eigenvalues of VV^t. Then $0 \le \mu_i < c^2$, so that we can define $P \in \mathbb{R}^{n\times m}$ by the equation

$$P = \sqrt{I_n - c^{-2}VV^t}^{\,-1} V\,. \tag{97}$$

By means of Theorem 6.3, (97) implies

$$V = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P \tag{98}$$

so that $V = \phi(P) \in \mathbb{B}^{n\times m}$, implying the reverse inclusion $\mathbb{R}_c^{n\times m} \subseteq \mathbb{B}^{n\times m}$. Hence, $\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m}$, as desired. □

For any $V \in \mathbb{R}^{n\times m}$ the set of nonzero eigenvalues of VV^t equals the set of nonzero eigenvalues of V^tV. Hence, following (92) we have

$$\begin{aligned}\mathbb{R}_c^{n\times m} &= \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\} \\ &= \{V \in \mathbb{R}^{n\times m} : \text{ Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0 \le \lambda < c^2\}\end{aligned} \tag{99}$$

Result (93) of Theorem 7.2 suggests calling $\mathbb{B}^{n\times m} = \mathbb{R}_c^{n\times m}$ the *eigenball of* $\mathbb{R}^{n\times m}$ *of eigenradius* c, or the c-eigenball in short.

8. Reparametrizing the Bi-Boost

We now wish to change the bi-boost parameter $P \in \mathbb{R}^{n\times m}$, the domain of which is the set $\mathbb{R}^{n\times m}$ of all $n \times m$ real matrices, to the new parameter $V \in \mathbb{R}_c^{n\times m}$, the domain of which is the eigenball $\mathbb{R}_c^{n\times m}$ of $\mathbb{R}^{n\times m}$. We, therefore, recall the following equations, which are taken from (36), (61) and (48).

$$\begin{aligned}P = \sqrt{I_n - c^{-2}VV^t}^{\,-1} V &= V\sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ \sqrt{I_m + c^{-2}P^tP} &= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \\ \sqrt{I_n + c^{-2}PP^t} &= \sqrt{I_n - c^{-2}VV^t}^{\,-1} .\end{aligned} \tag{100}$$

A generic parameter $V \in \mathbb{R}_c^{n\times m}$ in the eigenball $\mathbb{R}_c^{n\times m}$ is constructed by constructing a generic parameter $P \in \mathbb{R}^{n\times m}$ and employing (35).

The equations in (100) along with analogies with the gamma factor of special relativity theory suggest the definition of a *left gamma factor* $\Gamma^L_{n,V}$ and a *right gamma factor* $\Gamma^R_{m,V}$ by the following equations.

$$\begin{aligned}\Gamma^L_{n,V} &:= \sqrt{I_n - c^{-2}VV^t}^{\,-1} \in \mathbb{R}^{n\times n} \\ \Gamma^R_{m,V} &:= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \in \mathbb{R}^{m\times m} .\end{aligned} \tag{101}$$

Naturally, the pair $(\Gamma^L_{n,V},\ \Gamma^R_{m,V})$ of a left and a right gamma factor is called a *bi-gamma factor*. Practically, it is sometimes convenient to use the short notation

$$\gamma_{m,V} := \Gamma^R_{m,V} , \qquad \gamma_{n,V} := \Gamma^L_{n,V} \tag{102}$$

in which a left (right) gamma factor is implicitly indicated by the subscript n (m). It proves useful to use *interchangeably* the short notation with γ and the full notation with Γ in (102). We will use the short notation mainly in lengthy intermediate results as, for instance, in (146), p. 255.

Following (36), the left and right gamma factors are related by the first commuting relation in (103) below. The remaining commuting relations in (103) follow immediately from the first one, noting that left and right gamma factors are symmetric matrices.

$$\begin{aligned}\Gamma^L_{n,V} V &= V\Gamma^R_{m,V}\\ \Gamma^R_{m,V} V^t &= V^t\Gamma^L_{n,V}\\ \Gamma^L_{n,V} VV^t &= VV^t\Gamma^L_{n,V}\\ \Gamma^R_{m,V} V^tV &= V^tV\Gamma^R_{m,V}\,.\end{aligned} \tag{103}$$

Moreover, by Theorem 6.3 with P replaced by E,

$$\begin{aligned}E = \Gamma^L_{n,V} V = V\Gamma^R_{m,V} \quad\Longleftrightarrow\quad V &= \sqrt{I_n + c^{-2}EE^t}^{\;-1} E\\ &= E\sqrt{I_m + c^{-2}E^tE}^{\;-1}\,.\end{aligned} \tag{104}$$

The result in (104) will prove useful in (163), p. 259.

In the *bi-gamma notation* (101), the equations in (100) take the form

$$\begin{aligned}P = \Gamma^L_{n,V} V = V\Gamma^R_{m,V} &\in \mathbb{R}^{n\times m}\\ \sqrt{I_n + c^{-2}PP^t} = \Gamma^L_{n,V} &\in \mathbb{R}^{n\times n}\\ \sqrt{I_m + c^{-2}P^tP} = \Gamma^R_{m,V} &\in \mathbb{R}^{m\times m}\,.\end{aligned} \tag{105}$$

Introducing the arbitrarily fixed positive constant $c > 0$ into the bi-boost $B(P)$ in (1) we obtain the bi-boost $B_c(P)$, shown in (106) below, parametrized by $P \in \mathbb{R}^{n\times m}$. The bi-boost $B_c(P)$ leaves invariant the inner product of signature (m, n), $m, n \in \mathbb{N}$, shown in (138), p. 254, as we will prove straightforwardly in Theorem 10.1, p. 256.

The bi-boost $B_c(P)$ can be written as a bi-boost $\bar{B}_c(V)$ parametrized by the new parameter $V \in \mathbb{R}_c^{n\times m}$. Abusing notation, instead of $\bar{B}_c(V)$ we write $B_c(V)$ since no confusion may arise. Thus, following the change of parameter from $B_c(P)$ with parameter $P \in \mathbb{R}^{n\times m}$ to $B_c(V)$ with parameter $V \in \mathbb{R}_c^{n\times m}$ we have by means of (30) and (105),

$$\begin{aligned}B_c(P) &= \begin{pmatrix}\sqrt{I_m + c^{-2}P^tP} & \frac{1}{c^2}P^t\\ P & \sqrt{I_n + c^{-2}PP^t}\end{pmatrix}\\ &= \begin{pmatrix}\Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V}V^t = \frac{1}{c^2}V^t\Gamma^L_{n,V}\\ \Gamma^L_{n,V}V = V\Gamma^R_{m,V} & \Gamma^L_{n,V}\end{pmatrix} =: B_c(V)\,.\end{aligned} \tag{106}$$

It can be shown that when $m = 1$ the bi-boost $B_c(V)$ specializes to the standard Lorentz boost in one time dimension and n space dimensions, studied in [35]. It, therefore, proves useful to replace the bi-boost $B_c(P)$ parametrized by $P \in \mathbb{R}^{n\times m}$

by the equivalent bi-boost $B_c(V)$ parametrized by $V \in \mathbb{R}_c^{n\times m}$, obtaining

$$B_c(V) = \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \tag{107}$$

as we see from (106).

Accordingly, the generic Lorentz transformation $\Lambda(P, O_n, O_m)$ of order (m, n), $m, n \in \mathbb{N}$, in (4) becomes $\Lambda = \Lambda(V, O_n, O_m)$ given by the unique bi-gyration decomposition in theorem 8.1 below.

Owing to the bijective correspondence between the old parameter $P \in \mathbb{R}^{n\times m}$ and the new parameter $V \in \mathbb{R}_c^{n\times m}$, Theorem 2.1 can be translated into the following theorem.

Theorem 8.1. (Lorentz Transformation Bi-Gyration Decomposition, V), *A matrix $\Lambda \in \mathbb{R}^{(m+n)\times(m+n)}$ is the matrix representation of a Lorentz transformation of order (m, n), $\Lambda \in SO(m, n)$, if and only if it is given uniquely by the bi-gyration decomposition*

$$\Lambda = \begin{pmatrix} O_m & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V} & \frac{1}{c^2}\Gamma^R_{m,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & O_n \end{pmatrix} \tag{108}$$

or, prarmetrically in short,

$$\Lambda = \Lambda(O_m, V, O_n) = \rho(O_m)B(V)\lambda(O_n) = \begin{pmatrix} V \\ O_n \\ O_m \end{pmatrix} \tag{109}$$

for any $V \in \mathbb{R}_c^{n\times m}$, $O_m \in SO(m)$ and $O_n \in SO(n)$.

Example 8.2. In this example we show that in the special case when $m = 1$ the eigenball $\mathbb{R}_c^{n\times 1}$ specializes to the open c-ball $\mathbb{R}_c^n$ of $\mathbb{R}^{n\times 1} = \mathbb{R}^n$.

For $m = 1$, $V \in \mathbb{R}^{n\times m} = \mathbb{R}^n$ is a column vector in the Euclidean n-space $\mathbb{R}^n$, and $V^tV = \|V\|^2$ is a 1×1 matrix the eigenvalue of which is $\lambda = \|V\|^2$. Hence, following (99) and Theorem 7.2 we have $V \in \mathbb{R}_c^{n\times m}$ and

$$\begin{aligned} \mathbb{R}_c^{n\times 1} &= \{V \in \mathbb{R}^n : \text{ The eigenvalue } \|\mathrm{V}\|^2 \text{ of } \mathrm{V^tV} \text{ satisfies } 0 \le \|V\|^2 < c^2\} \\ &= \{V \in \mathbb{R}^n : \ 0 \le \|V\| < c\} \\ &=: \mathbb{R}_c^n \,. \end{aligned} \tag{110}$$

Indeed, in special relativity, the relativistically admissible velocities are elements of the c-ball $\mathbb{R}_c^3$, where c represents the vacuum speed of light.

Example 8.3. In this example we show that when $m = 1$ the right gamma factor equals the gamma factor of special relativity theory.

When $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector so that $P^tP = \|P\|^2$. Then, by (35),

$$V = \phi(P) = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} = \frac{P}{\sqrt{1 + c^{-2}\|P\|^2}} \tag{111}$$

so that $V \in \mathbb{R}^n$ is a column vector and

$$\|V\|^2 = V^tV = \frac{\|P\|^2}{1 + c^{-2}\|P\|^2}\,. \tag{112}$$

Hence, $0 \le \|V\| < c$ and, by (101),

$$\Gamma^R_{m=1,V} = \frac{1}{\sqrt{1 - c^{-2}\|V\|^2}} =: \gamma_V \tag{113}$$

for all $V \in \mathbb{B}^{n\times 1} = \phi(\mathbb{R}^{n\times 1})$. Here γ_V is the gamma factor that plays an important role in special relativity and in its underlying hyperbolic geometry [36, 37, 40, 42, 43, 45].

Example 8.4. Extending (113) to $m \ge 1$, it can be shown that the left and right gamma factors,

$$\Gamma^L_{n,V} := \sqrt{I_n - c^{-2}VV^t}^{\,-1} = \sqrt{I_n + c^{-2}PP^t} \tag{114}$$

and

$$\Gamma^R_{m,V} := \sqrt{I_m - c^{-2}V^tV}^{\,-1} = \sqrt{I_m + c^{-2}P^tP}\,, \tag{115}$$

are related by the equation

$$-I_n + \Gamma^L_{n,V} = \frac{1}{c^2}P(I_m + \Gamma^R_{m,V})^{-1}P^t \tag{116}$$

where P and V are related by Theorem 6.3. Note that by means of (114)−(115), (116) is equivalent to the elegant matrix identity (117), which we prove in the following lemma.

Lemma 8.5. *The matrix identities*

$$-I_n + \sqrt{I_n + c^{-2}PP^t} = \frac{1}{c^2}P\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^{-1}P^t \tag{117}$$

and

$$-I_m + \sqrt{I_m + c^{-2}P^tP} = \frac{1}{c^2}P^t\left(I_n + \sqrt{I_n + c^{-2}PP^t}\,\right)^{-1}P \tag{118}$$

hold for all $P \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$.

Proof. Clearly,

$$\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^2 = 2\left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right) + c^{-2}P^tP\,. \tag{119}$$

Let

$$R := \left(I_m + \sqrt{I_m + c^{-2}P^tP}\,\right)^{-1} \tag{120}$$

so that (119) can be written as

$$2R^{-1} + c^{-2}P^tP - (R^{-1})^2 = 0_{m,m}\,. \tag{121}$$

Left multiplying and right multiplying (121) by R yields

$$2R + c^{-2}RP^tPR - I_m = 0_{m,m}\,. \tag{122}$$

Left multiplying by P and right multiplying by P^t, (122) yields

$$P(2R + c^{-2}RP^tPR - I_m)P^t = 0_{m,m} \tag{123}$$

so that

$$2PRP^t + c^{-2}PRP^tPRP^t = PP^t \tag{124}$$

and hence,

$$I_n + c^{-2}(2PRP^t + c^{-2}PRP^tPRP^t) = I_n + c^{-2}PP^t\,. \tag{125}$$

Identity (125) can be written as

$$(I_n + c^{-2}PRP^t)^2 = I_n + c^{-2}PP^t \tag{126}$$

implying

$$I_n + c^{-2}PRP^t = \sqrt{I_n + c^{-2}PP^t}\,. \tag{127}$$

Finally, by means of (120), (127) yields (117), as desired.

The proof of (118) is similar to that of (117). □

Example 8.6. In the special case when $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector, $P^tP = \|P\|^2$, and PP^t is an $n\times n$ matrix, so that (117) specializes to

$$\sqrt{I_n + c^{-2}PP^t} = I_n + \frac{1}{c^2}\frac{1}{1 + \sqrt{1 + c^{-2}\|P\|^2}}PP^t\,. \tag{128}$$

We now manipulate (128) in the following chain of equations, which are numbered for subsequent explanation. For all $V \in \mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$,

$$\begin{aligned}
\sqrt{I_n - c^{-2}VV^t}^{\;-1} &\overbrace{=\!=}^{(1)} I_n + \frac{1}{c^2}\frac{1}{1 + \frac{1}{\sqrt{1 - c^{-2}\|V\|^2}}}\left(\Gamma^L_{n,V}\right)^2 VV^t \\
&\overbrace{=\!=}^{(2)} I_n + \frac{1}{c^2}\frac{1}{1 + \gamma_V}V\left(\Gamma^R_{m,V}\right)^2 V^t \\
&\overbrace{=\!=}^{(3)} I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1 + \gamma_V}VV^t\,.
\end{aligned} \tag{129}$$

Derivation of the numbered equalities in (129) follows:

1. This equation is equivalent to (128) since (i) the left sides of the two equations are equal by (48); and (ii) their right sides are equal by (61) with $m = 1$, and by (66) along with (101).

2. Follows from Item (1) by (113) and by the first commuting relation in (103).

3. Follows from Item (2) by (113), noting that $m = 1$.

Noting (101), the chain of equation (129) yields the important equation

$$\Gamma^L_{n,V} = I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V} VV^t \,, \qquad (\mathrm{m}=1)\,, \tag{130}$$

which holds for $m = 1$ and all $n \in \mathbb{N}$.

The importance of (130) is revealed in Example 8.7 below, enabling us to show straightforwardly that the bi-boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, $m, n \in \mathbb{N}$, specializes to the Lorentz boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$, of special relativity in the special case when $m = 1$.

Example 8.7. When $m = 1$ the bi-boost $B_c(V)$ in (107) can be manipulated by means of (103) and by means of (113) and (130), obtaining the following chain of equations.

$$\begin{aligned}
B_c(V) &= \begin{pmatrix} \Gamma^R_{m=1,V} & \frac{1}{c^2}\Gamma^R_{m=1,V}V^t \\ \Gamma^L_{n,V}V & \Gamma^L_{n,V} \end{pmatrix} \\
&= \begin{pmatrix} \Gamma^R_{m=1,V} & \frac{1}{c^2}\Gamma^R_{m=1,V}V^t \\ V\Gamma^R_{m=1,V} & \Gamma^L_{n,V} \end{pmatrix} \\
&= \begin{pmatrix} \gamma_V & \frac{1}{c^2}\gamma_V V^t \\ \gamma_V V & I_n + \frac{1}{c^2}\frac{\gamma_V^2}{1+\gamma_V}VV^t \end{pmatrix}, \qquad (m=1),
\end{aligned} \tag{131}$$

where $V \in \mathbb{R}_c^{n\times 1} \subset \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector in the ball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ of $\mathbb{R}^n$,

$$\mathbb{R}_c^n = \{V \in \mathbb{R}^n :\ \|V\| < c\}\,. \tag{132}$$

The extreme right side of (131) turns out to be the standard special relativistic $(n+1)\times(n+1)$ matrix representation of the Lorentz group in one time dimension and n space dimensions [35] [36, p. 254] [40, p. 447]. Accordingly, it follows from (131) that in the special case when $m = 1$ the Lorentz group of order (m, n) specializes to the Lorentz group of special relativity theory.

Example 8.8. In the special case when $m = 1$, $P \in \mathbb{R}^{n\times 1} = \mathbb{R}^n$ is a column vector so that $P^tP = \|P\|^2$. Accordingly, when $m = 1$ Identity (117) specializes to Identity (128),

$$\sqrt{I_n + c^{-2}PP^t} = I_n + \frac{1}{c^2}\frac{PP^t}{1+\sqrt{1+c^{-2}\|P\|^2}}\,. \tag{133}$$

Hence, when $m = 1$, the boost $B_c(P)$ in (106) specializes to the *proper velocity (PV) bi-boost*

$$B_c(P) = \begin{pmatrix} \sqrt{1+c^{-2}\|P\|^2} & \frac{1}{c^2}P^t \\ P & I_n + \frac{1}{c^2}\frac{PP^t}{1+\sqrt{1+c^{-2}\|P\|^2}} \end{pmatrix} \tag{134}$$

in one proper-time dimension and n space dimensions, where $P \in \mathbb{R}^n$ is the *proper velocity* of special relativity (in physical applications $n = 3$).

The PV-bi-boost (134) leaves invariant the relativistic inner product in (138) below.

The PV-bi-boost $B_c(P)$ involves the proper-velocity parameter $P \in \mathbb{R}^n$, which is measured by means of proper-time. The need for a search for a proper-time boost, like the one in (134), arises in several papers as, for instance, [14–20] and [37–39, 46].

The application $B_c(P)(t, \mathbf{x})^t$ of the PV-bi-boost $B_c(P)$ to time space coordinates $(t, \mathbf{x})^t$ is linear, and it keeps the relativistic norm

$$\tau = \sqrt{t^2 - \mathbf{x}^2/c^2} \tag{135}$$

invariant.

Similarly, the application $B_c(P)(\sqrt{\tau^2 + \mathbf{x}^2/c^2}, \mathbf{x})^t$ of the PV-bi-boost $B_c(P)$ to proper-time space coordinates $(\tau, \mathbf{x})^t$ is nonlinear, and it keeps the proper-time τ invariant.

9. The Bi-Boost $B_c(V)$

We know by construction that the bi-boost $B_c(V)$, $V \in \mathbb{R}_c^{n\times m}$, of order (m, n), $m, n \in \mathbb{N}$, preserves the inner product of signature (m, n) in the pseudo-Euclidean space $\mathbb{R}^{m,n}$. However, solely owing to the commuting relations in (103), a direct proof is straightforward, simple and, hence, instructive. Accordingly, the aim of this section is to prove directly that the bi-boost $B_c(V)$ in (139) below preserves the pseudo-Euclidean inner product of signature (m, n), $m, n \in \mathbb{N}$, in (138) below.

Let

$$\mathbf{t} = \begin{pmatrix} t_1 \\ \vdots \\ t_m \end{pmatrix} \in \mathbb{R}^m, \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n, \tag{136}$$

so that

$$\begin{pmatrix} \mathbf{t} \\ \mathbf{x} \end{pmatrix} = (t_1, \ldots, t_m, x_1, \ldots, x_n)^t \in \mathbb{R}^{m,n} \tag{137}$$

is a generic point of the pseudo-Euclidean space $\mathbb{R}^{m,n}$. The inner product of signature (m, n) in $\mathbb{R}^{m,n}$ involves the constant $c > 0$ according to the equation

$$\begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} := \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix}^t \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & -c^{-2}I_n \end{pmatrix} \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \mathbf{t}_1 \cdot \mathbf{t}_2 - c^{-2}\mathbf{x}_1 \cdot \mathbf{x}_2 \tag{138}$$

for all $(\mathbf{t}_1, \mathbf{x}_1)^t, (\mathbf{t}_2, \mathbf{x}_2)^t \in \mathbb{R}^{m,n}$, where $\mathbf{t}_1 \cdot \mathbf{t}_2 = \mathbf{t}_1^t \mathbf{t}_2$ and $\mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^t \mathbf{x}_2$ are the standard inner product in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively.

The bi-boost $B_c(V)$ is given by its $(m + n) \times)(m + n)$ matrix representation (107),

$$B_c(V) = \begin{pmatrix} \Gamma_{m,V}^R & c^{-2}\Gamma_{m,V}^R V^t \\ \Gamma_{n,V}^L V & \Gamma_{n,V}^L \end{pmatrix} \tag{139}$$

$m, n \in \mathbb{N}$, where the left and right gamma factors are given by (101),

$$\begin{aligned} \Gamma_{n,V}^L &= \sqrt{I_n - c^{-2}VV^t}^{\,-1} \in \mathbb{R}^{n \times n} \\ \Gamma_{m,V}^R &= \sqrt{I_m - c^{-2}V^tV}^{\,-1} \in \mathbb{R}^{m \times m} . \end{aligned} \tag{140}$$

The space of the parameter V in (139) – (140) is the c-eigenball $\mathbb{R}_c^{n \times m} \subset \mathbb{R}^{n \times m}$ which is given by

$$\begin{aligned} \mathbb{R}_c^{n \times m} &= \{V \in \mathbb{R}^{n \times m} : \text{ Each eigenvalue } \lambda \text{ of } VV^t \text{ satisfies } 0 \le \lambda < c^2\} \\ &= \{V \in \mathbb{R}^{n \times m} : \text{ Each eigenvalue } \lambda \text{ of } V^tV \text{ satisfies } 0 \le \lambda < c^2\} . \end{aligned} \tag{141}$$

The generic parameter $V \in \mathbb{R}_c^{n \times m}$ in the c-eigenball $\mathbb{R}_c^{n \times m}$ of $\mathbb{R}^{n \times m}$ is constructed by constructing a generic parameter $P \in \mathbb{R}^{n \times m}$ and employing (35),

$$V = \sqrt{I_n + c^{-2}PP^t}^{\,-1} P = P\sqrt{I_m + c^{-2}P^tP}^{\,-1} . \tag{142}$$

Theorem 9.1. *The bi-boost*

$$B_c(V) = \begin{pmatrix} \Gamma_{m,V}^R & c^{-2}\Gamma_{m,V}^R V^t \\ \Gamma_{n,V}^L V & \Gamma_{n,V}^L \end{pmatrix} \tag{143}$$

$V \in \mathbb{R}_c^{n \times m}$, $m, n \in \mathbb{N}$, *leaves the pseudo-Euclidean inner product* (138) *invariant, that is*

$$B_c(V) \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot B_c(V) \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{t}_1 \\ \mathbf{x}_1 \end{pmatrix} \cdot \begin{pmatrix} \mathbf{t}_2 \\ \mathbf{x}_2 \end{pmatrix} \tag{144}$$

for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^m$ *and* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

Proof. For convenient, we use in the proof the short notation in (102).

$$B_c(V)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}\gamma_{m,V} & c^{-2}\gamma_{m,V}V^t\\ \gamma_{n,V}V & \gamma_{n,V}\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix}=\begin{pmatrix}\gamma_{m,V}\mathbf{t}+c^{-2}\gamma_{m,V}V^t\mathbf{x}\\ \gamma_{n,V}V\mathbf{t}+\gamma_{n,V}\mathbf{x}\end{pmatrix}. \tag{145}$$

Hence, by (136) – (138), (103) and (105), we have the following chain of equations.

$$\begin{aligned}
&B_c(V)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(V)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}\\
&=\begin{pmatrix}\gamma_{m,V}\mathbf{t}_1+c^{-2}\gamma_{m,V}V^t\mathbf{x}_1\\ \gamma_{n,V}V\mathbf{t}_1+c^{-2}\gamma_{n,V}\mathbf{x}_1\end{pmatrix}^t\begin{pmatrix}I_m & 0_{m,n}\\ 0_{n,m} & -c^{-2}I_n\end{pmatrix}\begin{pmatrix}\gamma_{m,V}\mathbf{t}_2+c^{-2}\gamma_{m,V}V^t\mathbf{x}_2\\ \gamma_{n,V}V\mathbf{t}_2+\gamma_{n,V}\mathbf{x}_2\end{pmatrix}\\
&=(\mathbf{t}_1^t\gamma_{m,V}+c^{-2}\mathbf{x}_1^tV\gamma_{m,V},\ -c^{-2}\mathbf{t}_1^tV^t\gamma_{n,V}-c^{-2}\mathbf{x}_1^t\gamma_{n,V})\\
&\qquad\qquad\times\begin{pmatrix}\gamma_{m,V}\mathbf{t}_2+c^{-2}\gamma_{m,V}V^t\mathbf{x}_2\\ \gamma_{n,V}V\mathbf{t}_2+\gamma_{n,V}\mathbf{x}_2\end{pmatrix}\\
&=\mathbf{t}_1^t\gamma_{m,V}^2\mathbf{t}_2+c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2+c^{-4}\mathbf{x}_1^tV\gamma_{m,V}^2V^t\mathbf{x}_2\\
&\quad-c^{-2}(\mathbf{t}_1^tV^t\gamma_{n,V}^2V\mathbf{t}_2+\mathbf{t}_1^tV^t\gamma_{n,V}^2\mathbf{x}_2+\mathbf{x}_1^t\gamma_{n,V}^2V\mathbf{t}_2+\mathbf{x}_1^t\gamma_{n,V}^2\mathbf{x}_2)\\
&=\mathbf{t}_1^t\gamma_{m,V}^2\mathbf{t}_2+c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2+c^{-4}\mathbf{x}_1^t\gamma_{n,V}^2VV^t\mathbf{x}_2\\
&\quad-(c^{-2}\mathbf{t}_1^tV^tV\gamma_{m,V}^2\mathbf{t}_2+c^{-2}\mathbf{t}_1^t\gamma_{m,V}^2V^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^tV\gamma_{m,V}^2\mathbf{t}_2+c^{-2}\mathbf{x}_1^t\gamma_{n,V}^2\mathbf{x}_2)\\
&=\mathbf{t}_1^t(I_m-c^{-2}V^tV)\gamma_{m,V}^2\mathbf{t}_2-c^{-2}\mathbf{x}_1^t\gamma_{n,V}^2(I_n-c^{-2}VV^t)\mathbf{x}_2\\
&=\mathbf{t}_1^t\mathbf{t}_2-c^{-2}\mathbf{x}_1^t\mathbf{x}_2\\
&=\mathbf{t}_1\cdot\mathbf{t}_2-c^{-2}\mathbf{x}_1\cdot\mathbf{x}_2\\
&=\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix},
\end{aligned}\tag{146}$$

as desired. □

Example 9.2. Following (140) we have the obvious limits of large c,

$$\begin{aligned}\lim_{c\to\infty}\Gamma_{m,V}^R&=I_m\\ \lim_{c\to\infty}\Gamma_{n,V}^L&=I_n\,.\end{aligned}\tag{147}$$

Hence, in that limit we have

$$B_\infty(V):=\lim_{c\to\infty}B_c(V)=\begin{pmatrix}I_m & 0_{m,n}\\ V & I_n\end{pmatrix}\tag{148}$$

so that the limit of (145) as c approaches infinity yields an obvious generalization of the familiar Galilei transformation in a pseudo-Euclidean space of signature (m, n),

$$B_\infty(V)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}I_m & 0_{m,n}\\ V & I_n\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}\mathbf{t}\\ \mathbf{x}+V\mathbf{t}\end{pmatrix}. \tag{149}$$

10. The Bi-Boost $B_c(P)$

We know by its construction in [46] that the bi-boost $B_{c=1}(P)$, $P \in \mathbb{R}^{n\times m}$, of order (m, n), $m, n \in \mathbb{N}$, preserves the inner product (138) of signature (m, n) in the pseudo-Euclidean space $\mathbb{R}^{m,n}$. However, solely owing to the commuting relations in (154) below, a direct proof is straightforward, simple and, hence, instructive. Accordingly, the aim of this section is to prove directly that the bi-boost $B_c(P)$ preserves the inner product (138) for an arbitrarily fixed positive constant c.

Theorem 10.1. *The bi-boost*

$$B_c(P) = \begin{pmatrix}\sqrt{I_m + c^{-2}P^tP} & \frac{1}{c^2}P^t\\ P & \sqrt{I_n + c^{-2}PP^t}\end{pmatrix} \tag{150}$$

$P \in \mathbb{R}^{n\times m}$, $m, n \in \mathbb{N}$, *leaves the pseudo-Euclidean inner product* (138) *invariant, that is*

$$B_c(P)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(P)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix} = \begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix} \tag{151}$$

for any $\mathbf{t}_1, \mathbf{t}_2 \in \mathbb{R}^m$ *and* $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^n$.

Proof. It is convenient to use in the proof the short notation

$$\begin{aligned} b_m &:= \sqrt{I_m + c^{-2}P^tP}\\ b_n &:= \sqrt{I_n + c^{-2}PP^t}\end{aligned} \tag{152}$$

so that, by (150),

$$B_c(P) = \begin{pmatrix}b_m & \frac{1}{c^2}P^t\\ P & b_n\end{pmatrix} \tag{153}$$

and, by (34), we have the commuting relations

$$\begin{aligned} Pb_m &= b_nP\\ P^tb_n &= b_mP^t\,. \end{aligned} \tag{154}$$

The application of the bi-boost $B_c(P)$ to $(\mathbf{t}, \mathbf{x})^t \in \mathbb{R}^{m,n}$ is given by

$$B_c(P)\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}b_m & \frac{1}{c^2}P^t\\ P & b_n\end{pmatrix}\begin{pmatrix}\mathbf{t}\\ \mathbf{x}\end{pmatrix} = \begin{pmatrix}b_m\mathbf{t} + c^{-2}P^t\mathbf{x}\\ P\mathbf{t} + b_n\mathbf{x}\end{pmatrix}. \tag{155}$$

Hence, by (136) – (138), (154) and (155) we have the following chain of equations.

$$
\begin{aligned}
&B_c(P)\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot B_c(P)\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}\\
&\quad=\begin{pmatrix}b_m\mathbf{t}_1+c^{-2}P^t\mathbf{x}_1\\ P\mathbf{t}_1+b_n\mathbf{x}_1\end{pmatrix}^t\begin{pmatrix}I_m & 0_{m,n}\\ 0_{n,m} & -c^{-2}I_n\end{pmatrix}\begin{pmatrix}b_m\mathbf{t}_2+c^{-2}P^t\mathbf{x}_2\\ P\mathbf{t}_2+b_n\mathbf{x}_2\end{pmatrix}\\
&\quad=(\mathbf{t}_1^t b_m+c^{-2}\mathbf{x}_1^t P,\quad -c^{-2}(\mathbf{t}_1^t P^t+\mathbf{x}_1^t b_n))\begin{pmatrix}b_m\mathbf{t}_2+c^{-2}P^t\mathbf{x}_2\\ P\mathbf{t}_2+b_n\mathbf{x}_2\end{pmatrix}\\
&\quad=\mathbf{t}_1^t b_m^2\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2+c^{-4}\mathbf{x}_1^t PP^t\mathbf{x}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t P^t b_n\mathbf{x}_2+\mathbf{x}_1^t b_n P\mathbf{t}_2+\mathbf{x}_1^t b_n^2\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t(I_m+c^{-2}P^tP)\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2+c^{-4}\mathbf{x}_1^t PP^t\mathbf{x}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t P^t b_n\mathbf{x}_2+\mathbf{x}_1^t b_n P\mathbf{t}_2+\mathbf{x}_1^t(I_n+c^{-2}PP^t)\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t(I_m+c^{-2}P^tP)\mathbf{t}_2+c^{-2}\mathbf{t}_1^t b_m P^t\mathbf{x}_2+c^{-2}\mathbf{x}_1^t P b_m\mathbf{t}_2\\
&\qquad-c^{-2}(\mathbf{t}_1^t P^t P\mathbf{t}_2+\mathbf{t}_1^t b_m P^t\mathbf{x}_2+\mathbf{x}_1^t P b_m\mathbf{t}_2+\mathbf{x}_1^t\mathbf{x}_2)\\
&\quad=\mathbf{t}_1^t\mathbf{t}_2-c^{-2}\mathbf{x}_1^t\mathbf{x}_2\\
&\quad=\begin{pmatrix}\mathbf{t}_1\\ \mathbf{x}_1\end{pmatrix}\cdot\begin{pmatrix}\mathbf{t}_2\\ \mathbf{x}_2\end{pmatrix}.
\end{aligned}
\tag{156}
$$

as desired. □

11. Bi-Boost Product with the Parameter V

The Lorentz transformation product law, expressed in terms of the old parameter $P \in \mathbb{R}^{n\times m}$ in Theorem 2.3, was derived in [46, Theorem 21]. Accordingly, an important objective of the present article is to derive the Lorentz transformation product law expressed in terms of the new parameter $V \in \mathbb{R}_c^{n\times m}$.

Let $B_c(V_k)$, $k = 1, 2$, be two bi-boosts parametrized by $V_k \in \mathbb{R}_c^{n\times m}$,

$$
B_c(V_k) = \begin{pmatrix}\gamma_{m,V_k} & \frac{1}{c^2}\gamma_{m,V_k}V_k^t\\ \gamma_{n,V_k}V_k & \gamma_{n,V_k}\end{pmatrix} \tag{157}
$$

where we use the short notation in (102), p. 247.

By matrix multiplication and the commuting relations (103),

$$B_c(V_1)B_c(V_2) = \begin{pmatrix} \gamma_{m,V_1} & \frac{1}{c^2}\gamma_{m,V_1}V_1^t \\ \gamma_{n,V_1}V_1 & \gamma_{n,V_1} \end{pmatrix} \begin{pmatrix} \gamma_{m,V_2} & \frac{1}{c^2}\gamma_{m,V_2}V_2^t \\ \gamma_{n,V_2}V_2 & \gamma_{n,V_2} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{m,V_1}\gamma_{m,V_2} + \frac{1}{c^2}\gamma_{m,V_1}V_1^t\gamma_{n,V_2}V_2 & \frac{1}{c^2}(\gamma_{m,V_1}\gamma_{m,V_2}V_2^t + \gamma_{m,V_1}V_1^t\gamma_{n,V_2}) \\ \gamma_{n,V_1}V_1\gamma_{m,V_2} + \gamma_{n,V_1}\gamma_{n,V_2}V_2 & \frac{1}{c^2}\gamma_{n,V_1}V_1\gamma_{m,V_2}V_2^t + \gamma_{n,V_1}\gamma_{n,V_2} \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_{m,V_1}(I_m + \frac{1}{c^2}V_1^tV_2)\gamma_{m,V_2} & \frac{1}{c^2}\gamma_{m,V_1}(V_1+V_2)^t\gamma_{n,V_2} \\ \gamma_{n,V_1}(V_1+V_2)\gamma_{m,V_2} & \gamma_{n,V_1}(I_n + \frac{1}{c^2}V_1V_2^t)\gamma_{n,V_2} \end{pmatrix} =: \begin{pmatrix} E_{11} & \frac{1}{c^2}E_{12} \\ E_{21} & E_{22} \end{pmatrix}. \tag{158}$$

As we see from (158), the product of two bi-boosts need not be a bi-boost. However, it is a Lorentz transformation and, as such, it uniquely possesses the bi-gyration decomposition (108). Hence, by (108), we can express the bi-boost product $B_c(V_1)B_c(V_2)$ as follows,

$$\begin{aligned} B_c(V_1)B_c(V_2) &= \\ &= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2] & 0_{m,n} \\ 0_{n,m} & I_n \end{pmatrix} \begin{pmatrix} \Gamma^R_{m,V_{12}} & \frac{1}{c^2}\Gamma^R_{m,V_{12}}V_{12}^t \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}} \end{pmatrix} \begin{pmatrix} I_m & 0_{m,n} \\ 0_{n,m} & \mathrm{lgyr}[V_1,V_2] \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}} & \frac{1}{c^2}\mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_{12}}V_{12}^t\mathrm{lgyr}[V_1,V_2] \\ \Gamma^L_{n,V_{12}}V_{12} & \Gamma^L_{n,V_{12}}\mathrm{lgyr}[V_1,V_2] \end{pmatrix} \\ &= \begin{pmatrix} E_{11} & \frac{1}{c^2}E_{12} \\ E_{21} & E_{22} \end{pmatrix} \end{aligned} \tag{159}$$

where the composite parameter $V_{12} \in \mathbb{R}_c^{n\times m}$,

$$V_{12} =: V_1 \oplus V_2 \tag{160}$$

and the bi-gyration $(\mathrm{lgyr}[V_1,V_2], \mathrm{rgyr}[V_1,V_2]) \in SO(n) \times SO(m)$ are to be determined in terms of V_1 and V_2.

The uniqueness of the Lorentz transformation bi-gyration decomposition, insured by the Bi-gyration Decomposition Theorem 8.1, implies that the matrix entries E_{ij}, $i,j = 1,2$, defined in (158), and the matrix entries E_{ij} defined in (159) are identically equal.

Hence, the expressions

$$\begin{aligned} &V_{12} =: V_1 \oplus V_2 \in \mathbb{B}^{n\times m} \\ &\mathrm{lgyr}[V_1,V_2] \in SO(n) \\ &\mathrm{rgyr}[V_1,V_2] \in SO(m) \end{aligned} \tag{161}$$

that appear in (159) are uniquely determined by the Bi-gyration Decomposition Theorem 8.1. Employing (158) – (159), in the following Subsections – we determine each of the expressions in (161) in terms of V_1 and V_2.

11.1. E_{21}

In this subsection we study the consequences of the equality between E_{21} in (158) and E_{21} in (159).

With $V_{12} = V_1 \oplus V_2$, we see from (159) that

$$E_{21} = \Gamma^L_{n,V_1\oplus V_2}(V_1\oplus V_2)\,. \tag{162}$$

Hence, by (104), the binary operation $\oplus$ in $\mathbb{R}^{n\times m}_c$ is given by

$$V_1\oplus V_2 = \sqrt{I_n + c^{-2}E_{21}E_{21}^t}^{\,-1} E_{21} = E_{21}\sqrt{I_m + c^{-2}E_{21}^tE_{21}}^{\,-1}\,, \tag{163}$$

where, by (158),

$$E_{21} = \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2}\,, \tag{164}$$

$V_1, V_2 \in \mathbb{R}^{n\times m}_c$.

Thus, the *bi-gyrosum* $V_1\oplus V_2$ is expressed in (163) – (164) in terms of V_1 and V_2.

It is interesting to note that following (141), (163) – (164) and (147), we have the limits

$$\begin{aligned}\lim_{c\to\infty} \mathbb{R}^{n\times m}_c &= \mathbb{R}^{n\times m} \\ \lim_{c\to\infty}(V_1\oplus V_2) &= V_1 + V_2\,.\end{aligned} \tag{165}$$

Thus, as expected, in the limit of large c, the binary operation $\oplus$ in the eigenball $\mathbb{R}^{n\times m}_c$ tends to the common matrix addition, $+$, in the ambient space $\mathbb{R}^{n\times m}$.

In the special case when $m = 1$, the binary operation $\oplus$ in the eigenball $\mathbb{R}^{n\times m}_c$ specializes to Einstein velocity addition of special relativity in the ball $\mathbb{R}^n_c$, as indicated in Example 8.7. Einstein velocity addition in the ball $\mathbb{R}^n_c$ is studied, for instance, in [36, 40].

Additionally, the equality between E_{21} in (159) and in (158), along with the first commuting relation in (103), yields the elegant equations

$$\begin{aligned}\Gamma^L_{n,V_1\oplus V_2}(V_1\oplus V_2) &= \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2} \\ (V_1\oplus V_2)\Gamma^R_{m,V_1\oplus V_2} &= \Gamma^L_{n,V_1}(V_1 + V_2)\Gamma^R_{m,V_2}\end{aligned} \tag{166}$$

which show how closely the binary operations $\oplus$ and $+$ are related to each other.

Lemma 11.1. *The expression* E_{21} *in* (162) *possesses the commuting relations*

$$\begin{aligned} E_{21}E_{21}^t\sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1} &= \sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1}E_{21}E_{21}^t \\ E_{21}^tE_{21}\sqrt{I_m+c^{-2}E_{21}^tE_{21}}^{\,-1} &= \sqrt{I_m+c^{-2}E_{21}^tE_{21}}^{\,-1}E_{21}^tE_{21} \end{aligned} \tag{167}$$

and the identities

$$\begin{aligned} \Gamma^L_{n,V_1\oplus V_2} &:= \sqrt{I_n-c^{-2}(V_1\oplus V_2)(V_1\oplus V_2)^t}^{\,-1} = \sqrt{I_n+c^{-2}E_{21}E_{21}^t} \\ \Gamma^R_{m,V_1\oplus V_2} &:= \sqrt{I_m-c^{-2}(V_1\oplus V_2)^t(V_1\oplus V_2)}^{\,-1} = \sqrt{I_m+c^{-2}E_{21}^tE_{21}}\,. \end{aligned} \tag{168}$$

Proof. The commuting relations in (167) follow immediately from the commuting relation in (163).

By (163) and (167) we have

$$\begin{aligned} (V_1\oplus V_2)(V_1\oplus V_2)^t &= \sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1}E_{21}E_{21}^t\sqrt{I_n+c^{-2}E_{21}E_{21}^t}^{\,-1} \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t\,. \end{aligned} \tag{169}$$

Hence,

$$\begin{aligned} I_n-c^{-2}(V_1\oplus V_2)(V_1\oplus V_2)^t &= I_n-c^{-2}(I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}(I_n+c^{-2}E_{21}E_{21}^t)-c^{-2}(I_n+c^{-2}E_{21}E_{21}^t)^{-1}E_{21}E_{21}^t \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1}(I_n+c^{-2}E_{21}E_{21}^t-c^{-2}E_{21}E_{21}^t) \\ &= (I_n+c^{-2}E_{21}E_{21}^t)^{-1} \end{aligned} \tag{170}$$

thus proving the first identity in (168). The proof of the second identity in (168) is similar. □

11.2. E_{11} and E_{22}

In this subsection we study the consequences of the equality between E_{11} (E_{22}) in (158) and E_{11} (E_{22}) in (159).

With $V_{12}=V_1\oplus V_2$, we see from (159) that

$$\begin{aligned} E_{11} &= \mathrm{rgyr}[V_1,V_2]\Gamma^R_{m,V_1\oplus V_2} \\ E_{22} &= \Gamma^L_{n,V_1\oplus V_2}\mathrm{lgyr}[V_1,V_2]\,, \end{aligned} \tag{171}$$

so that for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$,

$$\begin{aligned} \mathrm{lgyr}[V_1, V_2] &= (\Gamma^L_{n,V_1\oplus V_2})^{-1} E_{22} \\ \mathrm{rgyr}[V_1, V_2] &= E_{11} (\Gamma^R_{m,V_1\oplus V_2})^{-1}\,, \end{aligned} \tag{172}$$

where, by (158),

$$\begin{aligned} E_{11} &= \Gamma^R_{m,V_1}(I_m + \frac{1}{c^2}V_1^t V_2)\Gamma^R_{m,V_2} \\ E_{22} &= \Gamma^L_{n,V_1}(I_n + \frac{1}{c^2}V_1 V_2^t)\Gamma^L_{n,V_2} \end{aligned} \tag{173}$$

and where, by (101) and Lemma 11.1,

$$\begin{aligned} \Gamma^L_{n,V_1\oplus V_2} &= \sqrt{I_n - \frac{1}{c^2}(V_1\oplus V_2)(V_1\oplus V_2)^t}^{\,-1} = \sqrt{I_n + \frac{1}{c^2}E_{21}E_{21}^t} \\ \Gamma^R_{m,V_1\oplus V_2} &= \sqrt{I_m - \frac{1}{c^2}(V_1\oplus V_2)^t(V_1\oplus V_2)}^{\,-1} = \sqrt{I_m + \frac{1}{c^2}E_{21}^t E_{21}}\,. \end{aligned} \tag{174}$$

Following (172) - (174) we have

$$\begin{aligned} &\mathrm{lgyr}[V_1, V_2] = \\ &\sqrt{I_n + \frac{1}{c^2}E_{21}E_{21}^t}^{\,-1} \sqrt{I_n - \frac{1}{c^2}V_1V_1^t}^{\,-1} (I_n + \frac{1}{c^2}V_1V_2^t)\sqrt{I_n - \frac{1}{c^2}V_2V_2^t}^{\,-1} \\ &\mathrm{rgyr}[V_1, V_2] = \\ &\sqrt{I_m - \frac{1}{c^2}V_1^tV_1}^{\,-1} (I_m + \frac{1}{c^2}V_1^tV_2)\sqrt{I_m - \frac{1}{c^2}V_2^tV_2}^{\,-1} \sqrt{I_m + \frac{1}{c^2}E_{21}^tE_{21}}^{\,-1}\,. \end{aligned} \tag{175}$$

Equations (171) and (173) yield the *bi-gamma identities*

$$\begin{aligned} \mathrm{rgyr}[V_1, V_2]\Gamma^R_{m,V_1\oplus V_2} &= \Gamma^R_{m,V_1}(I_m + \frac{1}{c^2}V_1^tV_2)\Gamma^R_{m,V_2} \\ \Gamma^L_{n,V_1\oplus V_2}\mathrm{lgyr}[V_1, V_2] &= \Gamma^L_{n,V_1}(I_n + \frac{1}{c^2}V_1V_2^t)\Gamma^L_{n,V_2}\,. \end{aligned} \tag{176}$$

For $V \in \mathbb{R}^{n\times m}$, the left (right) gamma factor $\Gamma^L_{n,V}$ ($\Gamma^R_{m,V}$) is real if and only if $V \in \mathbb{R}_c^{n\times m}$, as we see from (99) and (101), p. 247. Hence, each of the two equations in (176) yields the following implication: $V_1, V_2 \in \mathbb{R}_c^{n\times m} \Rightarrow V_1\oplus V_2 \in \mathbb{R}_c^{n\times m}$, so that $\oplus$ is a binary operation in $\mathbb{R}_c^{n\times m}$ as expected.

Example 11.2. In the special case when $m = 1$, $\mathrm{rgyr}[V_1, V_2] \in SO(1) = \{1\}$, so that $\mathrm{rgyr}[V_1, V_2] = 1$. Hence, the first identity in (176) specializes to the gamma identity,

$$\gamma_{V_1\oplus V_2} = \gamma_{V_1}\gamma_{V_2}(1 + \frac{1}{c^2}V_1{\cdot}V_2), \qquad (m = 1)\,, \tag{177}$$

which plays an important role in special relativity and its underlying hyperbolic geometry [36, 40, 43].

In fact, The gamma identity (177) signaled the emergence of hyperbolic geometry in special relativity when it was first studied by Sommerfeld [30] and Varičak [48, 49] in terms of *rapidities* [40, p. 90].

11.3. E_{12}

In this subsection we study the consequences of the equality between E_{12} in (158) and E_{12} in (159).

The equality between E_{12} in (159) and in (158) yields the equation

$$\mathrm{rgyr}[V_1, V_2]\Gamma^R_{m,V_1 \oplus V_2}(V_1 \oplus V_2)^t \mathrm{lgyr}[V_1, V_2] = \Gamma^R_{m,V_1}(V_1 + V_2)^t \Gamma^L_{n,V_2} \tag{178}$$

for all $V_1, V_2 \in \mathbb{R}^{n\times m}_c$, $m, n \in \mathbb{N}$. Transposing (178), noting that

$$\begin{aligned}(\mathrm{lgyr}[V_1, V_2])^t &= (\mathrm{lgyr}[V_1, V_2])^{-1} = \mathrm{lgyr}[V_2, V_1] \\ (\mathrm{rgyr}[V_1, V_2])^t &= (\mathrm{rgyr}[V_1, V_2])^{-1} = \mathrm{rgyr}[V_2, V_1]\end{aligned} \tag{179}$$

we obtain the equation

$$\mathrm{lgyr}[V_2, V_1](V_1 \oplus V_2)\Gamma^R_{m,V_1 \oplus V_2}\mathrm{rgyr}[V_2, V_1] = \Gamma^L_{n,V_2}(V_1 + V_2)\Gamma^R_{m,V_1}\,. \tag{180}$$

Manipulating the left side of (180) by means of the first commuting relation in (103), and manipulating the right side of (180) by means of (166) we obtain the equation

$$\mathrm{lgyr}[V_2, V_1]\Gamma^L_{n,V_1 \oplus V_2}(V_1 \oplus V_2)\mathrm{rgyr}[V_2, V_1] = \Gamma^L_{n,V_2 \oplus V_1}(V_2 \oplus V_1) \tag{181}$$

for all $V_1, V_2 \in \mathbb{R}^{n\times m}_c$. The resulting elegant equation demonstrates that the application of the bi-gyration $(\mathrm{lgyr}[V_2, V_1], \mathrm{rgyr}[V_2, V_1])$ takes $\Gamma^L_{n,V_1 \oplus V_2}(V_1 \oplus V_2)$ into $\Gamma^L_{n,V_2 \oplus V_1}(V_2 \oplus V_1)$. Equation (181) thus gives rise to a nice bi-gyrocommutative-like law.

12. Product of Lorentz Transformations, V

Techniques have been developed in [46] enabling the product of Lorentz transformations in the parameter P to be determined by Theorem 2.3, p. 233. By similar techniques one can determine the product of Lorentz transformations in the parameter V as well, obtaining the following theorem.

Theorem 12.1. (Lorentz Transformation Product Law, V) *The product of two generic Lorentz transformations*

$$\begin{aligned}\Lambda_1 &= (V_1, O_{n,1}, O_{m,1})^t \\ \Lambda_2 &= (V_2, O_{n,2}, O_{m,2})^t\end{aligned} \tag{182}$$

of order (m,n)*,* $m,n \in \mathbb{N}$*, in terms of parameter composition is given by*

$$\Lambda_1\Lambda_2 = \begin{pmatrix} V_1 \\ O_{n,1} \\ O_{m,1} \end{pmatrix} \begin{pmatrix} V_2 \\ O_{n,2} \\ O_{m,2} \end{pmatrix} = \begin{pmatrix} V_1 O_{m,2} \oplus O_{n,1} V_2 \\ \mathrm{lgyr}[V_1 O_{m,2}, O_{n,1} V_2] O_{n,1} O_{n,2} \\ O_{m,1} O_{m,2} \mathrm{rgyr}[V_1 O_{m,2}, O_{n,1} V_2] \end{pmatrix}, \tag{183}$$

where $\oplus$*,* lgyr *and* rgyr *are given by* (163) – (164) *and* (175) *in terms of the parameters* $V_1, V_2 \in \mathbb{R}_c^{n\times m}$*.*

Interestingly, the Lorentz transformation product laws in (183) and (8) of Theorem 12.1 and of Theorem 2.3, p. 233, respectively, have the same form when we interchange V_i and P_i, $i = 1,2$. Note, however, that the definitions of $\oplus$, lgyr and rgyr in Theorems 12.1 and 2.3 do not share the same form.

Similarly, as one can check, the gyrogroupoid $(\mathbb{R}_c^{n\times m}, \oplus)$ possesses the same bi-gyrocommutative law as that of the gyrogroupoid $(\mathbb{R}^{n\times m}, \oplus)$, with the parameter $P \in \mathbb{R}^{n\times m}$ replaced by the parameter $V \in \mathbb{R}_c^{n\times m}$. We thus obtain the following Theorem 12.2 from its P-counterpart Theorem 2.7, p. 234, by replacing (P_1, P_2) by (V_1, V_2).

Theorem 12.2. (Bi-Gyrocommutative Law in $(\mathbb{R}_c^{n\times m}, \oplus)$**).** *The binary operation* $\oplus$ *in* $\mathbb{R}^{n\times m}$ *possesses the bi-gyrocommutative law*

$$V_1 \oplus V_2 = \mathrm{lgyr}[V_1, V_2](V_2 \oplus V_1)\mathrm{rgyr}[V_1, V_2] \tag{184}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$*.*

Similarly, as one can check, the gyrogroupoid $(\mathbb{R}_c^{n\times m}, \oplus)$ possesses the same bi-gyroassociative law as that of the gyrogroupoid $(\mathbb{R}^{n\times m}, \oplus)$, with the parameter $P \in \mathbb{R}^{n\times m}$ replaced by the parameter $V \in \mathbb{R}_c^{n\times m}$. We thus obtain the following Theorem 12.3 from its P-counterpart Theorem 2.8, p. 235, by replacing (P_1, P_2) by (V_1, V_2).

Theorem 12.3. (Bi-Gyroassociative Law in $(\mathbb{R}_c^{n\times m}, \oplus)$**).** *The binary operation* $\oplus$ *in* $\mathbb{R}_c^{n\times m}$ *possesses the bi-gyroassociative law*

$$(V_1 \oplus V_2) \oplus \mathrm{lgyr}[V_1, V_2] V_3 = V_1 \mathrm{rgyr}[V_2, V_3] \oplus (V_2 \oplus V_3) \tag{185}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$*.*

13. Bi-Gyrogroups

As in Section 3 with the parameter $P \in \mathbb{R}^{n\times m}$, it proves useful with the parameter $V \in \mathbb{R}_c^{n\times m}$, as well, to replace the binary operation $\oplus$ in $\mathbb{R}_c^{n\times m}$ by a new binary operation, $\oplus'$, according to the following definition.

Definition 13.1. (Bi-Gyrogroup Operation, Bi-Gyrogroups). Let $(\mathbb{R}_c^{n\times m}, \oplus)$ be a bi-gyrogroupoid. A new bi-gyrogroup binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ is given by

$$V_1 \oplus' V_2 = (V_1 \oplus V_2)\mathrm{rgyr}[V_2, V_1] \tag{186}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$. The resulting groupoid $(\mathbb{R}_c^{n\times m}, \oplus')$ is called a bi-gyrogroup.

Having the form of Def. 3.1, Def. 13.1 defines the *bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ in terms of the *bi-gyrogroupoid* $(\mathbb{R}_c^{n\times m}, \oplus)$.

Remark 1. In the special case when $m = 1$, the binary operations $\oplus'$ and $\oplus$ coincide since $\mathrm{rgyr}[V_2, V_2] = 1$, as noted in Example 11.2. Accordingly, when $m = 1$, the two binary operations $\oplus'$ and $\oplus$ in $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ coincide with Einstein velocity addition of special relativity.

It is shown in [46] that (186) implies the following four identities that exhibit an interesting symmetry between the binary operations $\oplus$ and $\oplus'$ in $\mathbb{R}_c^{n\times m}$.

$$\begin{aligned} V_1 \oplus' V_2 &= (V_1 \oplus V_2)\mathrm{rgyr}[V_2, V_1] \\ V_1 \oplus V_2 &= (V_1 \oplus' V_2)\mathrm{rgyr}[V_1, V_2] \\ V_1 \oplus' V_2 &= \mathrm{lgyr}[V_1, V_2](V_2 \oplus V_1) \\ V_1 \oplus V_2 &= \mathrm{lgyr}[V_1, V_2](V_2 \oplus' V_1) \end{aligned} \tag{187}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$.

Bi-gyrogroups $(\mathbb{R}_c^{n\times m}, \oplus')$ possess a commutative-like and an associative-like law. Indeed, by [46, Theorems 42, 41] with P replaced by V we have the following two theorems.

Theorem 13.2. (Bi-Gyrocommutative Law in $(\mathbb{R}_c^{n\times m}, \oplus')$). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ possesses the bi-gyrocommutative law*

$$V_1 \oplus' V_2 = \mathrm{lgyr}[V_1, V_2](V_2 \oplus' V_1)\mathrm{rgyr}[V_2, V_1] \tag{188}$$

for all $V_1, V_2 \in \mathbb{R}_c^{n\times m}$.

Theorem 13.3. (Bi-Gyrogroup Left and Right Bi-Gyroassociative Law of $\oplus'$). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ possesses the left bi-gyroassociative law*

$$V_1 \oplus' (V_2 \oplus' X) = (V_1 \oplus' V_2) \oplus' \mathrm{lgyr}[V_1, V_2] X \mathrm{rgyr}[V_2, V_1] \tag{189}$$

and the right bi-gyroassociative law

$$(V_1 \oplus' V_2) \oplus' X = V_1 \oplus' (V_2 \oplus' \mathrm{lgyr}[V_2, V_1] X \mathrm{rgyr}[V_1, V_2]) \tag{190}$$

for all $V_1, V_2, X \in \mathbb{R}_c^{n\times m}$.

14. Gyrogroup Gyrations

The bi-gyroassociative laws (189) – (190) and the bi-gyrocommutative law (188) suggest the following definition of gyrations in terms of left and right gyrations.

Definition 14.1. (Gyrogroup Gyrations) ([46, Definition 43]). The gyrator gyr,

$$\mathrm{gyr} : \mathbb{R}_c^{n\times m} \times \mathbb{R}_c^{n\times m} \to \mathrm{Aut}(\mathbb{R}_c^{n\times m}, \oplus') \tag{191}$$

generates automorphisms called gyrations, $\mathrm{gyr}[V_1, V_2] \in \mathrm{Aut}(\mathbb{R}_c^{n\times m}, \oplus')$, given by the equation

$$\mathrm{gyr}[V_1, V_2]X = \mathrm{lgyr}[V_1, V_2]X\mathrm{rgyr}[V_2, V_1] \tag{192}$$

for all $V_1, V_2, X \in \mathbb{R}_c^{n\times m}$, where left gyrations, $\mathrm{lgyr}[V_1, V_2]$, and right gyrations, $\mathrm{rgyr}[V_2, V_1]$, are given in (175). The gyration $\mathrm{gyr}[V_1, V_2]$ is said to be the gyration generated by $V_1, V_2 \in \mathbb{R}_c^{n\times m}$. Being automorphisms of $(\mathbb{R}_c^{n\times m}, \oplus')$, gyrations are also called gyroautomorphisms.

Def. 14.1 will turn out rewarding, leading to the elegant result that any bi-gyrogroup $(\mathbb{R}_c^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, is a gyrocommutative gyrogroup.

Theorem 14.2. (Gyrogroup Gyroassociative and Gyrocommutative Laws). *The binary operation $\oplus'$ in $\mathbb{R}_c^{n\times m}$ obeys the left and the right gyroassociative law*

$$V_1\oplus'(V_2\oplus'X) = (V_1\oplus'V_2)\oplus'\mathrm{gyr}[V_1, V_2]X \tag{193}$$

and

$$(V_1\oplus'V_2)\oplus'X = V_1\oplus'(V_2\oplus'\mathrm{gyr}[V_2, V_1]X) \tag{194}$$

and the gyrocommutative law

$$V_1\oplus'V_2 = \mathrm{gyr}[V_1, V_2](V_2\oplus'V_1)\,. \tag{195}$$

Proof. Identities (193) – (194) follow immediately from Def. 14.1 and the left and right bi-gyroassociative law (189) – (190). Similarly, (195) follow immediately from Def. 14.1 and the bi-gyrocommutative law (188). □

Lemma 14.3. ([46, Lemma 45]). *For any $V_1, V_2 \in (\mathbb{R}_c^{n\times m}, \oplus')$, the relation* (192) *between bi-gyrations* $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_2, V_1])$ *and gyrations* $\mathrm{gyr}[V_1, V_2]$ *is bijective.*

It is obvious from (192) that a gyration $\mathrm{gyr}[V_1, V_2]$ is determined uniquely by the bi-gyration $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_1, V_2])$. It follows from Lemma 14.3 that also the converse is true, that is, a bi-gyration $(\mathrm{lgyr}[V_1, V_2], \mathrm{rgyr}[V_1, V_2])$ is determined uniquely by the gyration $\mathrm{gyr}[V_1, V_2]$.

It is anticipated in Def. 14.1 that gyrations are automorphisms. The following theorem asserts that this is indeed the case.

Theorem 14.4. (Gyroautomorphism) ([46, Like Theorem 46]). *For all* $V_1, V_2 \in \mathbb{R}_c^{n\times m}$*, gyrations* $\mathrm{gyr}[V_1, V_2]$ *of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *are automorphisms of the bi-gyrogroup.*

Theorem 14.5. (Left Gyration Reduction Properties) ([46, Like Theorem 47]). *Left gyrations of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *possess the left gyration left reduction property*

$$\mathrm{lgyr}[V_1, V_2] = \mathrm{lgyr}[V_1 \oplus' V_2, V_2] \tag{196}$$

and the left gyration right reduction property

$$\mathrm{lgyr}[V_1, V_2] = \mathrm{lgyr}[V_1, V_2 \oplus' V_1]\,. \tag{197}$$

Theorem 14.6. (Right Gyration Reduction Properties) ([46, Like Theorem 48]). *Right gyrations of a bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$ *possess the right gyration left reduction property*

$$\mathrm{rgyr}[V_1, V_2] = \mathrm{rgyr}[V_1 \oplus' V_2, V_2] \tag{198}$$

and the right gyration right reduction property

$$\mathrm{rgyr}[V_1, V_2] = \mathrm{rgyr}[V_1, V_2 \oplus' V_1]\,. \tag{199}$$

Theorem 14.7. (Gyration Reduction Properties) ([46, Like Theorem 49]). *The gyrations of any bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$*,* $m, n \in \mathbb{N}$*, possess the left and right reduction property*

$$\mathrm{gyr}[V_1, V_2] = \mathrm{gyr}[V_1 \oplus' V_2, V_2] \tag{200}$$

and

$$\mathrm{gyr}[V_1, V_2] = \mathrm{gyr}[V_1, V_2 \oplus' V_1]\,. \tag{201}$$

Proof. Identities (200) and (201) follow from Def. 14.1 of gyr in terms of lgyr and rgyr, and from Theorems 14.5 and 14.6. □

Finally, we have the most important theorem, which is the V-counterpart of Theorem 5.3..

Theorem 14.8. (Gyrocommutative Gyrogroup) ([46, Like Theorem 52]). *Any bi-gyrogroup* $(\mathbb{R}_c^{n\times m}, \oplus')$*,* $n, m \in \mathbb{N}$*, is a gyrocommutative gyrogroup.*

15. Scalar Multiplication for the Parameter V

Let M_1 and M_2 be two square matrices such that the inverse, M_2^{-1}, of M_2 exists. If the two matrices satisfy the commuting relation

$$M_1 M_2^{-1} = M_2^{-1} M_1\,, \tag{202}$$

then we may use the convenient notation

$$\frac{M_1}{M_2} := M_1 M_2^{-1} = M_2^{-1} M_1\,. \tag{203}$$

We are motivated by the scalar multiplication in $(\mathbb{R}_c^{n\times m}, \oplus')$, with $m=1$, which is the scalar multiplication in $(\mathbb{R}_c^n, \oplus_{\text{E}})$ studied, for instance, in [37, Eq. (6.267), p. 195]. We wish to extend it from $m=1$ to all $m \geq 1$. Accordingly, we define scalar multiplication in $(\mathbb{R}_c^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, by each of the following two equations, which are mutually equivalent.

$$
\begin{aligned}
r\otimes V &:= \frac{I_n - \left(\Gamma_{n,V}^L - \sqrt{(\Gamma_{n,V}^L)^2 - I_n}\,\right)^{2r}}{I_n + \left(\Gamma_{n,V}^L - \sqrt{(\Gamma_{n,V}^L)^2 - I_n}\,\right)^{2r}} \frac{\Gamma_{n,V}^L}{\sqrt{(\Gamma_{n,V}^L)^2 - I_n}} V \\
&= V \frac{I_m - \left(\Gamma_{m,V}^R - \sqrt{(\Gamma_{m,V}^R)^2 - I_m}\,\right)^{2r}}{I_m + \left(\Gamma_{m,V}^R - \sqrt{(\Gamma_{m,V}^R)^2 - I_m}\,\right)^{2r}} \frac{\Gamma_{m,V}^R}{\sqrt{(\Gamma_{m,V}^R)^2 - I_n}}
\end{aligned}
\tag{204}
$$

for all $r \in \mathbb{R}$ and $V \in \mathbb{R}_c^{n\times m}$. In the special case when $m = 1$, the scalar multiplication in (204) specializes to the one in [37, Eq. (6.267), p. 195].

As expected, the scalar multiplication in (204) satisfies the equation

$$
B_c(r\otimes V) = B_c(V)^r \tag{205}
$$

so that $V \in \mathbb{R}_c^{n\times m} \Rightarrow r\otimes V \in \mathbb{R}_c^{n\times m}$. In fact, (204) is derived from (205) by calculating the matrix $B_c(V)^r$ for $r \in \mathbb{N}$ and then analytically continuing r off the positive integers.

Furthermore, (205) implies the *scalar distributive law* and the *scalar associative law*

$$
\begin{aligned}
(r_1 + r_2)\otimes V &= r_1\otimes V \oplus' r_2\otimes V \\
(r_1 r_2)\otimes V &= r_1\otimes(r_2\otimes V)
\end{aligned}
\tag{206}
$$

and, hence, the *monodistributive law*

$$
r\otimes(r_1\otimes V \oplus' r_2\otimes V) = r\otimes(r_1\otimes V) \oplus' r\otimes(r_2\otimes V) \tag{207}
$$

for all $r, r_1, r_2 \in \mathbb{R}$ and all $V \in \mathbb{R}_c^{n\times m}$.

Naturally in gyrolanguage, the triple $(\mathbb{R}_c^{n\times m}, \oplus', \otimes)$ is said to be a *bi-gyrovector space*. Here $\oplus'$ is the binary operation in $\mathbb{R}_c^{n\times m}$ given by (186).

16. Scalar Multiplication for the Parameter P

In this section we continue using the notation in (202) – (203).

We introduce the following β-notation,

$$
\begin{aligned}
\beta_{n,P}^L &:= \sqrt{I_n + c^{-2} P P^t}^{\;-1} \in \mathbb{R}^{n\times n} \\
\beta_{m,P}^R &:= \sqrt{I_m + c^{-2} P^t P}^{\;-1} \in \mathbb{R}^{m\times m}\,,
\end{aligned}
\tag{208}
$$

in analogy with the Γ-notation in (140).

We are motivated by the scalar multiplication in $(\mathbb{R}^{n\times m}, \oplus')$ with $m = 1$, which is the scalar multiplication in $(\mathbb{R}^n, \oplus_{\mathrm{U}})$ studied, for instance, in [37, Eq. (6.285), p. 200]. We wish to extend it from $m = 1$ to all $m \geq 1$. Accordingly, we define scalar multiplication in $(\mathbb{R}^{n\times m}, \oplus')$, $m, n \in \mathbb{N}$, by each of the following two equations, which are mutually equivalent.

$$
\begin{aligned}
r \otimes P &:= \frac{1}{2} \frac{\left(I_n + \sqrt{I_n - (\beta^L_{n,P})^2}\,\right)^r - \left(I_n - \sqrt{I_n - (\beta^L_{n,P})^2}\,\right)^r}{(\beta^L_{n,P})^{r-1}\sqrt{I_n - (\beta^L_{n,P})^2}} P \\
&= \frac{1}{2} P \frac{\left(I_m + \sqrt{I_m - (\beta^R_{m,P})^2}\,\right)^r - \left(I_m - \sqrt{I_m - (\beta^R_{m,P})^2}\,\right)^r}{(\beta^R_{m,P})^{r-1}\sqrt{I_m - (\beta^R_{m,P})^2}}
\end{aligned}
\tag{209}
$$

for all $r \in \mathbb{R}$ and $P \in \mathbb{R}^{n\times m}$. In the special case when $m = 1$, the scalar multiplication in (209) specializes to the one in [37, Eq. (6.285), p. 200].

As expected, the scalar multiplication in (209) satisfies the equation

$$
B_c(r \otimes P) = B_c(P)^r \tag{210}
$$

where $B_c(P)$ is the bi-boost in (134). In fact, (209) is derived from (210) by calculating the matrix $B_c(P)^r$ for $r \in \mathbb{N}$ and then analytically continuing r off the positive integers.

Identity (210) implies the *scalar distributive law* and the *scalar associative law*

$$
\begin{aligned}
(r_1 + r_2) \otimes P &= r_1 \otimes P \oplus' r_2 \otimes P \\
(r_1 r_2) \otimes P &= r_1 \otimes (r_2 \otimes P)
\end{aligned}
\tag{211}
$$

and, hence, the *monodistributive law*

$$
r \otimes (r_1 \otimes P \oplus' r_2 \otimes P) = r \otimes (r_1 \otimes P) \oplus' r \otimes (r_2 \otimes P) \tag{212}
$$

for all $r, r_1, r_2 \in \mathbb{R}$ and all $P \in \mathbb{R}^{n\times m}$.

Hence, the triple $(\mathbb{R}^{n\times m}, \oplus', \otimes)$ is a bi-gyrovector space. Here $\oplus'$ is the binary operation in $\mathbb{R}^{n\times m}$ given by (15).

17. Paving the Road to the Eigenball Geometry

We have exposed the structure of the bi-gyrovector space $(\mathbb{R}^{n\times m}_c, \oplus', \otimes)$ of the eigenball $\mathbb{R}^{n\times m}_c$ of the ambient space $\mathbb{R}^{n\times m}$ of all rectangular real matrices of order $n \times m$, $m, n \in \mathbb{N}$. The bi-gyrovector space structure forms the algebraic setting for the non-Euclidean geometry that underlies the eigenball, just as the vector space structure forms the algebraic setting for the standard model of Euclidean geometry [41]. Indeed, in the special case when $m = 1$ the situation is well-known:

In this special case, when $m = 1$,

1. the eigenball $\mathbb{R}_c^{n\times m}$ specializes to the ball $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ of the Euclidean n-space $\mathbb{R}^n$, as shown in Example 8.2, p. 249; and

2. the binary operation $\oplus'$ in $\mathbb{R}_c^{n\times 1} = \mathbb{R}_c^n$ specializes to the binary operation given by Einstein's velocity addition law of relativistically admissible velocities in special relativity, as indicated in Example 8.7, p. 252.

Thus, when $m = 1$ the bi-gyrovector space $(\mathbb{R}_c^{n\times m}, \oplus', \otimes)$ specializes to the gyrovector space $(\mathbb{R}_c^n, \oplus', \otimes)$. The latter, in turn, forms the algebraic setting for the Beltrami-Klein ball model of hyperbolic geometry that underlies the ball $\mathbb{R}_c^n$, where $\oplus'$ in $\mathbb{R}_c^{n\times m}$ specializes to Einstein addition in $\mathbb{R}_c^n$. The resulting analytic hyperbolic geometry has been studied since 2001 in the seven books [36,37,40–43, 45] and in many articles.

It is, therefore, expected that the bi-gyrovector space structure, studied in [46] and in the present article, paves the road to to the discovery of the extended analytic hyperbolic geometry that regulates the eigenball $\mathbb{R}_c^{n\times m}$ of the ambient space $\mathbb{R}^{n\times m}$ for any $m, n \in \mathbb{N}$.

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ABSTRACTS

IN

PERSIAN

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خودزندگی نامه آبراهام اونگار

چکیده

این خودزندگی نامه به بیان زندگی علمی آبراهام اونگار و نقش او در گروه‌ها و فضاهای برداری چرخنده اختصاص دارد.

کلمات کلیدی: گروه چرخنده، فضای برداری چرخنده.

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ادیتور رابط: علی‌رضا اشرفی

The Intrinsic Beauty, Harmony and Interdisciplinarity in Einstein Velocity Addition Law: Gyrogroups and Gyrovector Spaces

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زیبایی ذاتی، هماهنگی و رویکرد بین‌رشته‌ای در قانون جمع سرعت اینشتین: گروه‌های چرخنده و فضاهای برداری چرخنده

چکیده

به نظر می‌رسد تنها توجیه قانون جمع سرعت اینشتین، مشاهدات تجربی باشد، چنانکه زیبایی و هماهنگی ذاتی جمع اینشتین برای مدت‌های طولانی به عنوان یک رمز و راز برای گشوده‌شدن باقی ماند. بر این اساس، هدف از این مقاله‌ی توصیفی ارائه (الف) جمع برداری نسبی اینشتین (ب) نتیجه‌گیری ضرب اسکالر اینشتین (ج) جرم نسبی اینشتین و (د) انرژی جنبشی نسبی اینشتین، به همراه شباهت‌های قابل توجه با نتایج کلاسیک در گروه‌ها و فضاهای برداری است و نشان می‌دهیم که این مفاهیم اینشتینی در گروه‌های چرخنده و فضاهای برداری چرخنده وارد می‌شوند. این شباهت‌ها با معرفی آشناهایی ناآشنا، پژوهش‌های بین رشته‌ایِ آن و زیبایی و هماهنگی ذاتی در قانون جمع برداری سرعت‌های قابل‌قبول نسبی اینشتین را آشکار می‌کند.

کلمات کلیدی: جمع اینشتین، گروه چرخنده، فضای برداری چرخنده، هندسه هذلولوی، نسبیت خاص.

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Special Subgroups of Gyrogroups: Commutators, Nuclei and Radical

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زیرگروه‌های خاص از گروه‌های چرخنده: جابه‌جایی‌پذیرها، هسته و رادیکال

چکیده

گروه چرخنده یک ساختار شبه‌گروه شرکت‌ناپذیری است که روی فضای سرعت‌های مجاز نسبیتی با یک جفت عملگر داده شده توسط قانون جمع اینشتین مدل سازی شده است. در این مقاله، تعدادی از گروه‌هایی که درون یک گروه چرخنده G نشسته اند مانند زیرگروه چرخنده جابه‌جایی‌پذیر، هسته‌ی چپ و رادیکال G را ارائه می‌کنیم. بستار نرمال زیرگروه چرخنده جابه‌جایی‌پذیر، هسته‌ی چپ و رادیکال G جزء زیرگروه‌های نرمال خاصی از G هستند. سپس معیارهایی ارائه می‌دهیم که مشخص می‌کند چه زمانی یک زیرگروه چرخنده H از گروه چرخنده متناهی G که اندیس $[G : H]$ کوچکترین عدد اولی است که مرتبه‌ی G را می‌شمارد، در G نرمال است.

کلمات کلیدی: گروه چرخنده، زیرگروه جابه‌جایی‌پذیر، هسته‌ی گروه چرخنده، زیرگروه چرخنده شاخص اول، رادیکال گروه چرخنده.

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Gyroharmonic Analysis on Relativistic Gyrogroups

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چکیده

گروه‌های چرخنده اینشتینی، موبیوس و سرعت مناسب، جزء گروه‌های چرخنده نسبیتی هستند که در جامه‌ی سه گروه لورنتس خاص در فضا-زمان حقیقی منیکوفسکی $\mathbb{R}^{n,1}$ دیده می‌شوند. با استفاده از زبان چرخنده، به مطالعه‌ی آنالیز هارمونیک چرخنده آن‌ها می‌پردازیم. اگرچه که یک یکریختی چرخنده جبری بین این سه مدل وجود دارد، نشان می‌دهیم که تفاوت‌هایی نیز بین آن‌ها وجود دارد. مطالعات ما روی ترجمه و پیچیدگی‌های عملگرها، تابع ویژه عملگر لاپلاس-بلترامی، تبدیل پواسون، تبدیل فوریه-هلگاسون، معکوس آن‌ها و قضیه‌ی پلانچرل معطوف می‌شود. ما نشان می‌دهیم که وقتی $t \to +\infty$، نتیجه‌ی آنالیز هارمونیک چرخنده به آنالیز هارمونیک اقلیدسی استاندارد $\mathbb{R}^n$ میل می‌کند. بنابراین کاری که انجام می‌دهیم، متحد کردن آنالیز هارمونیک هذلولوی و اقلیدسی است.

کلمات کلیدی: گروه‌های چرخنده، آنالیز هارمونیک چرخنده، عملگر لاپلاس-بلترامی، تابع ویژه، تبدیل هلگاسون-فوریه‌ی تعمیم‌یافته، قضیه‌ی پلانچرل.

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Bi-Gyrogroup: The Group-Like Structure Induced by Bi-Decomposition of Groups

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دو-گروه چرخنده: ساختار شبه‌گروه القا شده توسط دو-تجزیه‌ی گروه‌ها

چکیده

تجزیه $\Gamma = BH$ از گروه Γ به زیرمجموعه‌ی B و زیرگروه H از Γ تحت شرایط کلی ساختار یک شبه‌گروه را برای B القا می‌کند که تحت عنوان گروه چرخنده شناخته می‌شود. ادراک ملموس و البته معروف از یک گروه چرخنده، که انگیزه‌ای برای قرارگیری به عنوان یک هدف اصلی است، فضای تمام سرعت‌های مجاز نسبیتی به همراه یک جفت عملگر داده شده توسط قانون جمع سرعت اینشتین از نظریه‌ی نسبیت خاص است. بعداً در فضای شبه اقلیدسی نشانه $(۱, n)$، خود را به گروه تبدیلات لورنتس $S(n, ۱)$ برای $n \in \mathbb{N}$ محدود می‌کنیم. مطالعه‌ی این مقاله روی گروه لورنتس تعمیم یافته‌ی $SO(m, n)$ برای $n, m \in \mathbb{N}$ در فضای شبه اقلیدسی نشانه (m, n) است. بر این اساس، این مقاله به جستجوی دو-تجزیه $\Gamma = H_L B H_R$ از گروه Γ به زیرمجموعه‌ی B و زیرگروه‌های H_L و H_R به همراه یک ساختار دو-گروه چرخنده جدید از B که توسط دو-تجزیه Γ تولید می‌شود، می‌پردازد. برای مثال، با استفاده از روش جبرهای کلیفورد نشان می‌دهیم که گروه خارج قسمتی از گروه دوران $\mathrm{Spin}(m, n)$ دارای ساختار دو-تجزیه است.

کلمات کلیدی: دو-تجزیه‌ی گروه، دو-گروه چرخنده، گروه چرخنده، گروه دوران، گروه شبه عمود.

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Normed Gyrolinear Spaces: A Generalization of Normed Spaces Based on Gyrocommutative Gyrogroups

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فضاهای خطی چرخنده نرم‌دار: تعمیمی از فضاهای نرم‌دار براساس گروه‌های چرخنده جابه‌جایی‌پذیر چرخنده

چکیده

در این مقاله تعمیمی از فضاهای نرم‌دار حقیقی را بررسی می‌کنیم و چند مثال از آن‌ها ارائه می‌دهیم.

کلمات کلیدی: گروه چرخنده، فضاهای برداری چرخنده.

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ادیتور رابط: آبراهام اونگار

Gyrovector Spaces on the Open Convex Cone of Positive Definite Matrices

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فضاهای برداری چرخنده روی مخروط محدب بازِ ماتریس‌های مثبت معین

چکیده

در این مقاله مروری روی مفهوم جبری گروه چرخنده و نسخه‌ی ساده شده‌ی فضای برداری چرخنده به همراه دو مثال اساسی روی گوی باز فضای اقلیدسی با بعد متناهی که فضاهای برداری چرخنده اینشتین و موبیوس هستند، خواهیم داشت. ساختار فضای برداری چرخنده و خط چرخنده روی مخروط محدب باز از ماتریس‌های مثبت معین را معرفی کرده و کاربردهای جالبی از آن را در رابطه با مجموعه ماتریس‌های چگال وارون‌پذیر کشف می‌کنیم. در پایان، مثالی از فضای برداری چرخنده روی گوی یکه از ماتریس‌های هرمیتی ارائه می‌دهیم.

کلمات کلیدی: گروه چرخنده، فضای برداری چرخنده، نقطه چرخنده میانی، خط چرخنده، ماتریس هرمیتی، ماتریس مثبت معین، ماتریس چگالی.

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ادیتور رابط: آبراهام اونگار

An Extension of Poincaré Model of Hyperbolic Geometry with Gyrovector Space Approach

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توسیعی از مدل پوانکاره‌ی هندسه‌ی هذلولی با رویکرد فضای برداری چرخنده

چکیده

هدف این مقاله نشان دادن اهمیت هندسه هذلولوی تحلیلیِ معرفی شده در مرجع [۹] است. در مرجع [۱] نشان داده شده است که جبر گروه $(2,\mathbb{C})$ به طورِ طبیعی مفهوم گروه‌های چرخنده و فضاهای برداری چرخنده را در مواجهه با گروه لورنتس و هندسه‌ی هذلولوی زمینه‌ای آن نتیجه می‌دهد. همچنین در آن جمع چنِ و مدل هندسه‌ی هذلولوی چن تعریف شده است. در این مقاله، با استفاده‌ی مستقیم از خواص یکریختی فضاهای برداری چرخنده، جمع چن و مدل هندسه‌ی هذلولوی چن را بهبود می‌بخشیم. همچنین نشان می‌دهیم که این مدل تعمیمی از مدل هندسه هذلولی پوانکاره است. برای این منظور، مدل هندسه‌ی هذلولوی سطح پوانکاره را داخل دیسک باز واحد مختلط $\mathbb{D}$ در نظر می‌گیریم. همچنین ثابت می‌کنیم که این مدل با مدل پوانکاره و سپس با دیگر مدل‌های هذلولوی، یکریخت است. سرانجام با رویکرد فضای برداری چرخنده، چند ویژگی از این مدل را با جزئیاتی کاملاً شبیه به هندسه‌ی اقلیدسی اثبات می‌کنیم.

کلمات کلیدی: هندسه هذلولوی، گروه چرخنده، فضای برداری چرخنده، مدل پوانکاره، هندسه هذلولوی تحلیلی.

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The Principle of Relativity: From Ungar's Gyrolanguage for Physics to Weaving Computation in Mathematics

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اصل نسبیت: از زبان چرخنده اونگار برای فیزیک تا بافندگی محاسبات در ریاضیات

چکیده

این مقاله محدوده‌ی محاسبات جبری که بر پایه‌ی $\times$ غیر استاندارد استوار است را به حالت اساسی‌تر $+$ غیر استاندارد توسعه می‌دهد که منظور از استاندارد، شرکت‌پذیری و جابه‌جایی است. دو مثال ملموس فیزیکی از $+$ غیر استاندارد، در حرکت نسبیتی خاص در بیرون فضای سه‌بعدی یا درون فضای دوبعدی (یا بیشتر) قابل مشاهده است. با بازنگری نظریه‌ی چرخنده اونگار، تحلیل اطلاعات چندجانبه را ارائه می‌دهیم که توسط متر پارچه‌ای W ساخته شده است و یک ساختار محاسباتی مرتبط می‌سازد که در چارچوب یک فضای نرم دار مانند V قرار می‌گیرد و برپایه‌ی جمع غیر استاندارد که با نماد $\oplus$ نمایش داده می‌شود، استوار است. به‌علاوه جابه‌جایی و شرکت‌پذیری بودن در آن به کمک یک رابطه‌گر برقرار است و رابطه‌گر نگاشتی است که به هر جفت بردار مجاز در V یک خودریختی به روی W نسبت می‌دهد. ما به حالتی که یک رابطه‌گر جهت‌پذیر باشد، توجه خاصی خواهیم داشت.

کلمات کلیدی: رابطه‌گر، ناجابه‌جایی، شرکت‌ناپذیر، جمع القا شده، ارگان، لباس متریک، پردازش اطلاعات بافندگی، هندسه‌ی لباس، هندسه‌ی هذلولوی، نسبیت خاص، رابط، کوتاه ترین خط ترسیم شده بین دو نقطه روی سطح، خط آلی، کنش در یک فاصله.

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From the Lorentz Transformation Group in Pseudo-Euclidean Spaces to Bi-Gyrogroups

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از گروه تبدیل لورنتس در فضاهای شبه-اقلیدسی تا دو-گروه‌های چرخنده

چکیده

تبدیل لورنتس از مرتبه‌ی $(m = 1, n)$ که با $n \in \mathbb{N}$ به تبدیل لورنتس نظریه‌ی نسبیت خاص معروف است، یک تبدیل مختصات فضا-زمان از فضای شبه اقلیدسی $\mathbb{R}^{m=1,n}$ است که بعد زمان آن یک و بعد فضا، n است و درکاربردهای فیزیکی، $n = 3$ است. تبدیل لورنتس بدون دوران‌ها، یک خیز نامیده می‌شود. عموماً خیز نسبیتیِ خاص توسط یک متغیر سرعت مجاز نسبیتی $\mathbf{v}$ $(v \in \mathbb{R}^n_c)$ پارامتری می‌شود که دامنه‌ی آن c-گوی $\mathbb{R}^n_c$ از تمام سرعت‌های مجاز نسبیتی است و $\mathbb{R}^n_c = \{\mathbf{v} \in \mathbb{R}^n : \|v\| < c\}$ که در آن، فضای محیط $\mathbb{R}^n$ فضای اقلیدسی n-بعدی است و c یک عدد دلخواه مثبت ثابت است که سرعت نور در خلأ را نشان می‌دهد. مطالعه‌ی قانون ترکیب تبدیل لورنتس بر حسب ترکیب پارامتر آشکار می‌کند که ساختار گروه تبدیل لورنتس از مرتبه‌ی $(m = 1, n)$، یک گروه چرخنده و یک فضای برداری چرخنده القا می‌کند که فضای پارامتری $\mathbb{R}^n_c$ را تنظیم می‌کند. ساختار گروه چرخنده و فضای برداری چرخنده از گوی $\mathbb{R}^n_c$ به نوبه‌ی خود یک محیط جبری برای مدل گوی بلترامی-کلاین در هندسه‌ی هذلولوی تشکیل می‌دهد، که زیربنای گوی $\mathbb{R}^n_c$ است. هدف این مقاله، توسعه مطالعه‌ی تبدیل لورنتس از مرتبه‌ی (m, n) از $m = 1$ و $n \geqslant 1$ به همه‌ی $m, n \in \mathbb{N}$، و به دست آوردن ساختارهای جبری است که دو-گروه چرخنده و فضای برداری دو-چرخنده نامیده می‌شود. یک دو-گروه چرخنده، گروهی چرخنده است که هر گردش به صورت یک جفت گردش چپ و گردش راست است و یک فضای برداری دو-چرخنده از دو-جابه‌جاییِ چرخنده دو-گروه چرخنده تشکیل می‌شود که یک ضرب اسکالر داشته باشد.

کلمات کلیدی: دو-گروه چرخنده، فضای برداری دو-چرخنده، گوی ویژه، گروه چرخنده، ضرب داخلی نشان (m, n)، تبدیل لورنتس از مرتبه‌ی (m, n)، فضای شبه اقلیدسی، نسبیت خاص.

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ادیتور رابط: علی‌رضا اشرفی

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