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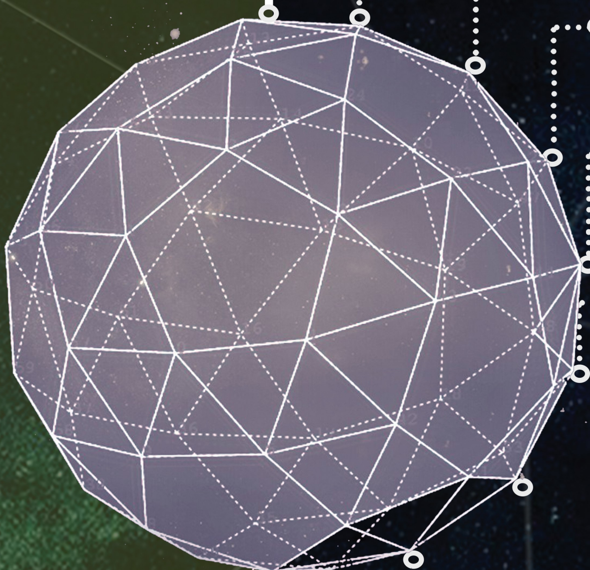

Mathematics Interdisciplinary Research

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Motion of Particles under Pseudo-Deformation

Akhilesh Chandra Yadav⋆

Abstract

In this short article, we observe that the path of particle of mass m moving along $\mathbf{r} = \mathbf{r}(t)$ under pseudo-force $\mathbf{A}(t)$, t denotes the time, is given by $\mathbf{r}_d = \int(\frac{d\mathbf{r}}{dt}\mathbf{A}(t))dt + \mathbf{c}$. We also observe that the effective force $\mathbf{F}_e$ on that particle due to pseudo-force $\mathbf{A}(t)$, is given by $\mathbf{F}_e = \mathbf{F}\mathbf{A}(t) + \mathbf{L}d\mathbf{A}(t)/dt$, where $\mathbf{F} = m\ d^2\mathbf{r}/dt^2$ and $\mathbf{L} = m\ d\mathbf{r}/dt$. We have discussed stream lines under pseudo-force.

Keywords: Right loops, right transversals, gyrotransversals.

2010 Mathematics Subject Classification: 70A05, 74A05, 76A99.

1. Introduction

In [3], we have observed that $\mathbb{R}^n$ is a unique gyrotransversal [1] to the subgroup $O(n)$ in the group $Iso\ \mathbb{R}^n$, the group of motion [3, Corollary 6.8]. If S is a right transversal to the subgroup H of a group G and $g : S \to H$ is a map with $g(e) = e$, e being identity of G. Then, it induces a binary operation o_g [3, 4] on S is given by

$$xo_gy = x\theta g(y)oy.$$

This suggests us that the map g affects the sum xoy of $x, y \in S$ and the effective sum is $x\theta g(y)oy$ instead of xoy. Indeed, this group-theoretic idea makes certain sense.

2. Preliminaries

Let S be a non-empty set. Then, a groupoid (S, o) is called a *right quasigroup* if for each x, y in S, the equation $X\ o\ x = y$ has a unique solution in S, where X is

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unknown in the equation. If there exists $e \in S$ such that $eox = x = xoe$ for every $x \in S$, then the right quasigroup (S, o) is called a *right loop.*

Let H be a subgroup of a group G. Then a set S obtained by selecting one and only one element from each right coset of G modulo H, including identity of G is called a *right transversal* to H in G. The group operation induces a binary operation o on S and an action θ of H on S given by $\{xoy\} = S \cap Hxy$ and $\{x\theta h\} = S \cap Hxh$ respectively, where $x, y \in S$ and $h \in H$. One may easily observe that (S, o) is a right loop with identity e, where e is the identity of group. Indeed, it determines an algebraic structure (S, H, σ, f) known as $c-groupoid$ [2]. Conversely a given c-groupoid (S, H, σ, f) determines a group $G = H \times S$ which contains H as a subgroup and S as a right transversal to H in G so that the corresponding c-groupoid is (S, H, σ, f) [2, Theorem 2.2]. It is also observed that every right loop (S, o) can be embedded as a right transversal to a subgroup $Sym\ S \setminus \{e\}$ in to a group $Sym\ S \setminus \{e\} \times S$ with some universal property [2, Theorem 3.4].

Let S be a fixed right transversal to a subgroup H in a group G. Then every right transversal to H in G determines and is determined uniquely by a map $g : S \to H$ such that $g(e) = e$, the identity of G. The right transversal S_g determined by a map $g : S \to H$ is given by $S_g = \{g(x)x | x \in S\}$. The induced operations o on S and o' on S_g are given by

$$\{xoy\} = Hxy \cap S$$

and

$$\{g(x)xo'g(y)y\} = S_g \cap Hg(x)xg(y)y,$$

respectively. Further, H acts on S from right through an action θ given by $\{x\theta h\} = Hxh \cap S$, $\forall\, x \in S, h \in H$. Indeed, the right loop (S_g, o') is isomorphic to the right loop (S, o_g) where the binary operation o_g on S is given by $xo_gy = x\theta g(y)oy$ [3,4]. This suggests us to say that *the map $g : S \to H$ with $g(e) = e$ affects the operation o on S and the resulting operation on S due to the effect of g is o_g, where $x\ o_gy = x\theta g(y)oy$.*

3. Pseudo-Force, Deformed Path and Effective Force

Consider the group of motion $Iso\ \mathbb{R}^n$. As we have observed in [3] that $\mathbb{R}^n$ is a right transversal (unique gyrotransversal) to the subgroup $O(n)$ in the group $Iso\ \mathbb{R}^n$, the group of motion [3, Corrolary 6.8].

Definition 3.1. A map $g : \mathbb{R}^n \to O(n)$ with $g(\mathbf{0}) = \mathbf{I_n}$ will be called a pseudo-deformation. The image $g(\mathbf{v})$ of $\mathbf{v}$ will be called pseudo-force corresponding to velocity $\mathbf{v}$. The corresponding operation $+_g$ on $\mathbb{R}^n$ given by $\mathbf{v} +_g \mathbf{w} = \mathbf{v}g(\mathbf{w}) + \mathbf{w}$, will be called pseudo-sum on $\mathbb{R}^n$.

Assume that every velocity $\mathbf{v}$ creates a field of force $g(\mathbf{v}) \in O(n)$. Then it determines a pseudo-deformation $g : \mathbb{R}^n \to O(n)$. Let Σ_1, Σ_2 be any two dynamical systems moving with velocities $\mathbf{v}$ and $\mathbf{w}$ respectively. Due to a pseudo-deformation g, the pseudo-sum $+_g$ on $\mathbb{R}^n$ will be $\mathbf{v}+_g\mathbf{w} = \mathbf{v}g(\mathbf{w})+\mathbf{w}$. In other words, we can say that the resultant of $\mathbf{v}$ and $\mathbf{w}$ under pseudo-force $g(\mathbf{w})$ will be $\mathbf{v}+_g\mathbf{w} = \mathbf{v}g(\mathbf{w})+\mathbf{w}$ instead of $\mathbf{v}+\mathbf{w}$. Thus, the relative velocity of Σ_1 with respect to Σ_2 will be $(\mathbf{v}+_g\mathbf{w}) - \mathbf{w} = \mathbf{v}g(\mathbf{w})$ instead of $\mathbf{v}+\mathbf{w}-\mathbf{w} = \mathbf{v}$. This suggests us to define the following:

Definition 3.2. Let $g : \mathbb{R}^n \to O(n)$ be a pseudo-deformation. Assume that Σ_1, Σ_2 be any two dynamical systems moving with velocities $\mathbf{v}$ and $\mathbf{w}$ respectively. Then the difference $(\mathbf{v}+_g\mathbf{w}) - \mathbf{w} = \mathbf{v}g(\mathbf{w})$ will be called effective velocity of Σ_1 under the pseudo-force $g(\mathbf{w})$. The integral $\int \mathbf{v}g(\mathbf{w})dt$ of effective velocity of Σ_1 will be called deformed-path of Σ_1 under the pseudo-force $g(\mathbf{w})$. It is denoted by $\mathbf{r}_d$. Thus, $\mathbf{r}_d = \int \mathbf{v}g(\mathbf{w})dt + \mathbf{c}$ and so the effective velocity will be $\frac{d\mathbf{r}_d}{dt} = \mathbf{v}g(\mathbf{w})$ under pseudo-force $g(\mathbf{w})$. The quantity $\frac{d^2\mathbf{r}_d}{dt^2}$ will be called effective acceleration of Σ_1 under a pseudo-force and the quantity $m\frac{d^2\mathbf{r}_d}{dt^2}$ will be called effective force acting on Σ_1 under a given pseudo-force, where 'm' is the mass of Σ_1.

Proposition 3.3. *Suppose that a particle of mass m is moving with velocity $\mathbf{q}$ in space whose path is $\mathbf{r} = \mathbf{r}(t)$, where t denotes the time. If there is a pseudo-force $\mathbf{A}(t) \in O(3)$ at time t. Then the deformed-path is given by*

$$\mathbf{r}_d = \int \frac{d\mathbf{r}}{dt}\mathbf{A}(t)dt + \mathbf{c} = \int \left(\mathbf{q}\mathbf{A}(t)\right) dt + \mathbf{c}.$$

If $\mathbf{A}(t)$ is constant, say $\mathbf{A}$, throughout the motion, then the deformed path due to pseudo-deformation $\mathbf{A}$ is $\mathbf{r}_d = \mathbf{r}\mathbf{A} + \mathbf{c}$, $\mathbf{c}$ being the constant of integration.

Proof. Since the effective velocity of particle due to presence of pseudo-force $\mathbf{A}(t)$ at time t is $\mathbf{q}\mathbf{A}(t)$, where $\mathbf{q} = \frac{d\mathbf{r}}{dt}$. Thus, the deformed path $\mathbf{r}_d$ is

$$\mathbf{r}_d = \int \mathbf{q}\mathbf{A}(t)dt + \mathbf{c}$$

where $\mathbf{c}$ is the integrating constant. If pseudo-force $\mathbf{A}(t)$ is constant throughout the motion then the deformed path will be $\mathbf{r}_d = \mathbf{r}\mathbf{A} + \mathbf{c}$. □

From this it follows that the orbit of a satellite, planet etc., will be changed due to a pseudo-force. These pseudo-forces may exist due to asteroids, black holes, etc.

Proposition 3.4. *Suppose that a mass particle 'm' is moving in space $\mathbb{R}^3$ along a curve $\mathbf{r} = \mathbf{r}(t) \in \mathbb{R}^3$ under the action of force $\mathbf{F}$. If $\mathbf{A}(t)$ is a pseudo-force acting on the given mass particle. Then, its equation of motion is given by*

$$\mathbf{F}_e = \mathbf{F}\mathbf{A} + \mathbf{L}\frac{d\mathbf{A}}{dt}$$

where $\mathbf{L}$ is the linear momentum of the mass particle and $\mathbf{F} = m\ \frac{d^2\mathbf{r}}{dt^2}$.

Proof. Let P be a particle of mass m moving in space along a path $\mathbf{r} = \mathbf{r}(t)$. Let $\mathbf{F}$ be the force acting at P. Then, its equation of motion is given by

$$\mathbf{F} = m\frac{d^2\mathbf{r}}{dt^2}.$$

Suppose that $\mathbf{A}(t) \in O(3)$ is a pseudo-force acting on the particle at time t. Then, its deformed path $\mathbf{r}_d$ is given by

$$\mathbf{r}_d = \int \frac{d\mathbf{r}}{dt}\mathbf{A}dt + \mathbf{c},$$

and so

$$\begin{aligned}\frac{d^2\mathbf{r}_d}{dt^2} &= \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\mathbf{A}\right)\\ &= \frac{d^2\mathbf{r}}{dt^2}\mathbf{A} + \frac{d\mathbf{r}}{dt}\frac{d\mathbf{A}}{dt}.\end{aligned}$$

Thus, the effective force $\mathbf{F}_e$ which causes the motion is given by

$$\begin{aligned}\mathbf{F}_e &= m\frac{d^2\mathbf{r}_d}{dt^2}\\ &= m\left(\frac{d^2\mathbf{r}}{dt^2}\mathbf{A} + \frac{d\mathbf{r}}{dt}\frac{d\mathbf{A}}{dt}\right)\\ &= \mathbf{FA} + \mathbf{L}\frac{d\mathbf{A}}{dt},\end{aligned}$$

where $\mathbf{L} = m\frac{d\mathbf{r}}{dt}$ is the linear momentum of the mass particle. □

Thus, the motion of a dynamical system will be affected due to the presence of pseudo-force. From the above, it follows that the magnitude F_e of force $\mathbf{F}_e$ at time t will be

$$\sqrt{\|\mathbf{F}\|^2 + 2\mathbf{FA}\left(\frac{d\mathbf{A}}{dt}\right)^T\mathbf{L}^T + \left\|\mathbf{L}\frac{d\mathbf{A}}{dt}\right\|^2}$$

where $\mathbf{A}^T$ denotes the transpose of $\mathbf{A}$.

If $\mathbf{A}(t)$ is independent of time, then $\frac{d\mathbf{A}}{dt} = \mathbf{0}$ and so the equation of motion under the pseudo-force $\mathbf{A}$ is given by $\mathbf{F}_e = \mathbf{FA}$. Thus, we have:

Corollary 3.5. *If* $\mathbf{A}$ *is a constant pseudo-force acting on the mass particle which is moving in space under the action of force* $\mathbf{F}$*. Then, its equation of motion is given by*

$$\mathbf{F}_e = \mathbf{FA}.$$

4. Streamlines under Pseudo-Deformation

Let $\mathbf{q} = (u, v, w)$ be a velocity of a blood particle at point $P(x, y, z)$. Due to electromagnetic field, suppose that the pseudo-force is $\mathbf{A} = [A_1, A_2, A_3] \in O(3)$, where A_1, A_2, A_3 are orthonormal column vectors in $\mathbb{R}^3$. Then, effective velocity of that blood particle at P will be $(\mathbf{q}A_1, \mathbf{q}A_2, \mathbf{q}A_3)$. Thus, the differential equation of streamlines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

are changed into

$$\frac{dx}{\mathbf{q}A_1} = \frac{dy}{\mathbf{q}A_2} = \frac{dz}{\mathbf{q}A_3}$$

This shows that an electromagnetic field affects motion of blood particles. Thus, electromagnetic field affects the motion of blood particles and hence it will exert extra pressure on the heart. Also due to that field, the deformed motion causes tumors in effected area of our body.

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C-Class Functions and Remarks on Fixed Points of Weakly Compatible Mappings in G-Metric Spaces Satisfying Common Limit Range Property

Arslan Hojat Ansari, Diana Dolićanin−Đekić, Feng Gu, Branislav Z. Popović and Stojan Radenović ⋆

Abstract

In this paper, using the contexts of C-class functions and common limit range property, common fixed point result for some operator are obtained. Our results generalize several results in the existing literature. Some examples are given to illustrate the usability of our approach.

Keywords: Generalized metric space, common fixed point, generalized weakly G-contraction, weakly compatible mappings, common (CLR_{ST}) property, C-class functions.

2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

The study of common fixed point theorems satisfying contractive conditions has a wide range of applications in different areas such as, variational and linear inequality problems, optimization and parameterize estimation problems and many others. One of the simplest and most useful results in the fixed point theory is the Banach-Caccioppoli contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical sciences and engineering.

Banach contraction principle has been generalized in different spaces by mathematicians over the years. Mustafa and Sims [22] proposed a new class of generalized metric spaces, which are called as G-metric spaces. In this type of spaces

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a non-negative real number is assigned to every triplet of elements. Many mathematicians studied extensively various results on G-metric spaces by using the concept of weak commutativity, compatibility, non-compatibility and weak compatibility for single valued mappings satisfying different contractive conditions (cf. [1, 3–5, 7, 8, 10–13, 15–27]).

Branciari [9] obtained a fixed point result for a single mapping satisfying an analogue of Banach's contraction principle for an integral type inequality. This influenced many authors, and consequently, a number of new results in this line followed (see, for example [7]). Later on, Aydi [7] proved an integral type fixed point theorem for two self mappings and extended the results of Brianciari [9] to the class of G-metric spaces. The first fixed point theorem without any continuity requirement was proved by Abbas and Rhoades [5] in which they utilized the notion of non-commuting mappings for the existence of fixed points. Shatanawi et al. [27] proved some interesting fixed point results by using φ-contractive condition and generalized the results of Abbas and Rhoades [5]. Most recently, Mustafa et al. [18] defined the notion of the property $(E.A)$ in G-metric space and proved some fixed point results.

In this paper, firstly we prove an integral type fixed point theorem for a pair of weakly compatible mappings in G-metric space satisfying the common limit range property which is initiated by Sintunavarat and Kumam [28]. We extend our main result to two finite families of self mappings by using the notion of pairwise commuting. We also present some fixed point results in G-metric spaces satisfying ϕ-contractions. Some related examples are furnished to support our results.

Now we give preliminaries and basic definitions which are used throughout the paper.

Definition 1.1. [22] Let X be a nonempty set, and let $G : X \times X \times X \longrightarrow [0, \infty)$ be a function satisfying the following axioms:
$(G1)$ $G(x, y, z) = 0$ if $x = y = z$;
$(G2)$ $0 < G(x, x, y)$, for all $x, y \in X$ with $x \neq y$;
$(G3)$ $G(x, x, y) \leq G(x, y, z)$,for all $x, y, z \in X$ with $z \neq y$;
$(G4)$ $G(x, y, z) = G(x, z, y) = G(y, z, x) = \ldots$ (symmetry in all three variables);
$(G5)$ $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$, (rectangle inequality)
then the function G is called a generalized metric, or, more specifically a G-metric on X and the pair (X, G) is called a G-metric space.

It is known that the function $G(x, y, z)$ on a G-metric space X is jointly continuous in all three of its variables, and $G(x, y, z) = 0$ if and only if $x = y = z$; see [22] for more details and the reference therein.

Definition 1.2. [22] Let (X, G) be a G-metric space, and let $\{x_n\}$ be a sequence of points in X, a point x in X is said to be the limit of the sequence $\{x_n\}$ if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$, and one says that sequence $\{x_n\}$ is G-convergent to x.

Thus, if $x_n \to x$ in a G-metric space (X, G), then for any $\varepsilon > 0$, there exists $N \in \mathbb{N}$ (throughout this paper we mean by $\mathbb{N}$ the set of all natural numbers) such that $G(x, x_n, x_m) < \varepsilon$, for all n, $m \geq N$.

Proposition 1.3. [22] *Let (X, G) be a G-metric space, then the following are equivalent:*
(1) *$\{x_n\}$ is G-convergent to x.*
(2) *$G(x_n, x_n, x) \to 0$ as $n \to \infty$.*
(3) *$G(x_n, x, x) \to 0$ as $n \to \infty$.*
(4) *$G(x_n, x_m, x) \to 0$ as n, $m \to \infty$.*

Definition 1.4. [22] Let (X, G) be a G-metric space. A sequence $\{x_n\}$ is called G-Cauchy sequence if, for each $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $G(x_n, x_m, x_l) < \varepsilon$ for all $n, m, l \geq N$; i.e., if $G(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$.

Definition 1.5. [22] A G-metric space (X, G) is said to be G-complete (or a complete G-metric space) if every G-Cauchy sequence in (X, G) is G-convergent in X.

Proposition 1.6. [22] *Let (X, G) be a G-metric space. Then the following are equivalent:*
(1) *The sequence $\{x_n\}$ is G-Cauchy.*
(2) *For every $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $G(x_n, x_m, x_m) < \varepsilon$, for all $n, m \geq k$.*

Proposition 1.7. [22] *Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.8. [22] *Let (X, G) be a G-metric space. Then, for all x, y in X it follows that $G(x, y, y) \leq 2G(y, x, x)$.*

Definition 1.9. [2] Let f and g be self maps of a set X. If $w = fx = gx$ for some x in X, then x is called a coincidence point of f and g, and w is called point of coincidence of f and g.

Definition 1.10. [2] Two self mappings f and g on X are said to be weakly compatible if they commute at coincidence points.

Definition 1.11. [8] Let X be a G-metric space. Self mappings f and g on X are said to satisfy the G-(E.A) property if there exists a sequence $\{x_n\}$ in X such that $\{fx_n\}$ and $\{gx_n\}$ are G-convergent to some $t \in X$.

Definition 1.12. [8,28] A pair (f, g) of self mappings of a G-metric space (X, G) is said to satisfy the $(CLRg)$ property if there exists a sequence $\{x_n\}$ such that $\{fx_n\}$ and $\{gx_n\}$ are G-converge to gt for some $t \in X$, that is,

$$\lim_{n\to\infty} G(fx_n, fx_n, gt) = \lim_{n\to\infty} G(gx_n, gx_n, gt) = 0.$$

Definition 1.13. A pair (f, g) of self mappings of a G-metric space (X, G) is said to satisfy the (LRg) property if there exists a sequence $\{x_n\}$ such that $\{fx_n\}$ and $\{gx_n\}$ are G-converge to gt for some $t \in f(X) \cap g(X)$, that is,

$$\lim_{n\to\infty} G(fx_n, fx_n, gt) = \lim_{n\to\infty} G(gx_n, gx_n, gt) = 0.$$

Definition 1.14. Self mappings f and g of a G-metric space (X, G) are said to be compatible if $\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ and $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to} fx_n = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.

Khan et al. [14] introduced the concept of altering distance function that is a control function employed to alter the metric distance between two points enabling one to deal with relatively new classes of fixed point problems. Here, we consider the following notion.

Definition 1.15. [14] The function $\psi : [0, +\infty) \to [0, +\infty)$ is called an altering distance function if the following properties are satisfied:
(1) ψ is continuous and increasing;
(2) $\psi(t) = 0$ if and only if $t = 0$.
We denote Ψ set all of altering distance functions.

In 2014 the concept of C-class functions (see Definition 1.16) was introduced by A. H. Ansari in [6] that is able to notice that can see in numbers (1), (2), (9) and (15) from Example 1.17.

Definition 1.16. A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies following axioms:
(1) $F(s, t) \le s$;
(2) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0, \infty)$.

Note for some F we have that $F(0, 0) = 0$.
We denote C-class functions as $\mathcal{C}$.

Example 1.17. The following functions $F : [0, \infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0, \infty)$:
(1) $F(s, t) = s - t$, $F(s, t) = s \Rightarrow t = 0$;
(2) $F(s, t) = ms$, $0<m<1$, $F(s, t) = s \Rightarrow s = 0$;
(3) $F(s, t) = \frac{s}{(1+t)^r}$; $r \in (0, \infty)$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(4) $F(s, t) = \log(t + a^s)/(1 + t)$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(5) $F(s, t) = \ln(1 + a^s)/2$, $a > e$, $F(s, 1) = s \Rightarrow s = 0$;
(6) $F(s, t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0, \infty)$, $F(s, t) = s \Rightarrow t = 0$;
(7) $F(s, t) = s \log_{t+a} a$, $a > 1$, $F(s, t) = s \Rightarrow s = 0$ or $t = 0$;
(8) $F(s, t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$, $F(s, t) = s \Rightarrow t = 0$;
(9) $F(s, t) = s\beta(s)$, $\beta : [0, \infty) \to [0, 1)$, $F(s, t) = s \Rightarrow s = 0$;

(10) $F(s,t) = s - \frac{t}{k+t}, F(s,t) = s \Rightarrow t = 0$;

(11) $F(s,t) = s - \varphi(s), F(s,t) = s \Rightarrow s = 0$, here $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$;

(12) $F(s,t) = sh(s,t), F(s,t) = s \Rightarrow s = 0$, here $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$;

(13) $F(s,t) = s - (\frac{2+t}{1+t})t$, $F(s,t) = s \Rightarrow t = 0$;

(14) $F(s,t) = \sqrt[n]{\ln(1+s^n)}$, $F(s,t) = s \Rightarrow s = 0$;

(15) $F(s,t) = \phi(s), F(s,t) = s \Rightarrow s = 0$, here $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$, and $\phi(t) < t$ for $t > 0$;

(16) $F(s,t) = \frac{s}{(1+s)^r}$; $r \in (0,\infty)$, $F(s,t) = s \Rightarrow s = 0$.

Problem: Whether can say that for all F we have $F(0,0) = 0$?

Definition 1.18. An ultra altering distance function is a continuous, non$-$decreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ if $t > 0$ and $\varphi(0) \geq 0$.

Remark 1. We denote Φ_u set all of ultra altering distance functions.

In the sequel let Φ be the set of all functions ω such that $\omega : [0,+\infty) \to [0,+\infty)$ is a non$-$decreasing function with $\lim_{n\to+\infty} \omega^n(t) = 0$ for all $t \in (0,+\infty)$. If $\omega \in \Phi$, then ω is called a Φ-mapping. If ω is a Φ-mapping, then it is easy matter to show that:

1. $\omega(t) < t$ for all $t \in (0,+\infty)$,

2. $\omega(0) = 0$.

2. Results

We start with the following theorem.

Theorem 2.1. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{\psi(G(fx,fy,fz))} \varphi(t)dt \leq F\left(\int_0^{\psi(L(x,y,z))} \varphi(t)dt, \int_0^{\phi(L(x,y,z))} \varphi(t)dt\right), \quad (1)$$

for all $x,y,z \in X$, $F : [0,\infty)^2 \to \mathbb{R}$ is a C-class, $\psi \in \Psi$, $\phi \in \Phi_u$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$ where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\}, \quad (2)$$

or

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,gx,fx), G(gy,gy,fy), G(gz,gz,fz)\}. \quad (3)$$

If the pair (f,g) satisfies the $(CLRg)$ property then f and g have a unique common fixed point in X.

Proof. Since the pair (f,g) satisfies the (CLR_g) property, then there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = gu$ for some $u \in X$. We show that $fu = gu$. On using inequality (1), we get

$$\int_0^{\psi(G(fx_n,fx_n,fu))} \varphi(t)dt \le F\left(\int_0^{\psi(L(x_n,x_n,u))} \varphi(t)dt, \int_0^{\phi(L(x_n,x_n,u))} \varphi(t)dt\right), \tag{4}$$

where

$$L(x_n,x_n,u) = \max\{G(gx_n,gx_n,gu), G(gx_n,fx_n,fx_n), G(gx_n,fx_n,fx_n), G(gu,fu,fu)\}.$$

Taking limit as $n \to +\infty$ in (4), we have

$$\int_0^{\psi(G(gu,gu,fu))} \varphi(t)dt \le F\left(\int_0^{\psi(G(gu,fu,fu))} \varphi(t)dt, \int_0^{\phi(G(gu,fu,fu))} \varphi(t)dt\right). \tag{5}$$

Similarly, one can obtain

$$\int_0^{\psi(G(gu,fu,fu))} \varphi(t)dt \le F\left(\int_0^{\psi(G(gu,gu,fu))} \varphi(t)dt, \int_0^{\phi(G(gu,gu,fu))} \varphi(t)dt\right). \tag{6}$$

From (5) and (6), we have

$$\begin{aligned}\int_0^{\psi(G(gu,gu,fu))} \varphi(t)dt &\le \int_0^{\psi(G(gu,fu,fu))} \varphi(t)dt \\ &\le F\left(\int_0^{\psi G(gu,gu,fu))} \varphi(t)dt, \int_0^{\phi(G(gu,gu,fu))} \varphi(t)dt\right).\end{aligned}$$

So,

$$\int_0^{\psi(G(gu,gu,fu))} \varphi(t)dt = 0 \quad \text{or} \quad \int_0^{\phi(G(gu,gu,fu))} \varphi(t)dt = 0,$$

therefore $\psi(G(gu,gu,fu)) = 0$ or $\phi(G(gu,gu,fu)) = 0$. Thus $G(gu,gu,fu) = 0$, that is, $fu = gu$. Suppose that $w = fu = gu$. Since the pair (f,g) is weakly compatible and $w = fu = gu$, therefore $fw = fgu = gfu = gw$. Finally, we prove that $w = fw$. Inequality (1) implies

$$\int_0^{\psi(G(fw,fw,fu))} \varphi(t)dt \le F\left(\int_0^{\psi(L(w,w,u))} \varphi(t)dt, \int_0^{\phi(L(w,w,u))} \varphi(t)dt\right), \tag{7}$$

where

$$\begin{aligned}L(w,w,u) &= \max\{G(gw,gw,gu), G(gw,fw,fw), G(gw,fw,fw), G(gu,fu,fu)\} \\ &= \max\{G(fw,fw,w), G(fw,fw,fw), G(fw,fw,fw), G(w,w,w)\} \\ &= G(fw,fw,w).\end{aligned}$$

Therefore (7) implies

$$\int_0^{\psi(G(fw,fw,w))}\varphi(t)dt \le F\left(\int_0^{\psi(G(fw,fw,w))}\varphi(t)dt, \int_0^{\phi(G(fw,fw,w))}\varphi(t)dt\right),$$

so,

$$\int_0^{\psi(G(fw,fw,w))}\varphi(t)dt = 0 \quad \text{or} \quad \int_0^{\phi(G(fw,fw,w))}\varphi(t)dt = 0.$$

Therefore $\psi(G(fw,fw,w)) = 0$ or $\phi(G(fw,fw,w)) = 0$, thus $G(fw,fw,w) = 0$, that is, $w = fw$. Therefore, w is a common fixed point of the mappings f and g. The proof is similar for condition (3), hence the details are omitted. Uniqueness of the common fixed point is easy consequences of inequalities (1)-(7). □

With choice $F(s,t) = s - t$ in Theorem 2.1 we have the following corollary.

Corollary 2.2. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{\psi(G(fx,fy,fz)}\varphi(t)dt \le \int_0^{\psi(L(x,y,z))}\varphi(t)dt - \int_0^{\phi(L(x,y,z))}\varphi(t)dt,$$

for all $x,y,z \in X$, $\psi \in \Psi$, $\phi \in \Phi_u$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt > 0$, where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\},$$

or

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,gx,fx), G(gy,gy,fy), G(gz,gz,fz)\}.$$

If the pair (f,g) satisfies the (CLRg) property, then f and g have a unique common fixed point in X.

With choice $F(s,t) = ks, 0 \le k < 1$ in Theorem 2.1 we have the following corollary.

Corollary 2.3. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{\psi(G(fx,fy,fz))}\varphi(t)dt \le k\int_0^{\psi(L(x,y,z))}\varphi(t)dt,$$

for all $x,y,z \in X$, $0 \le k < 1$, $\psi \in \Psi$ and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon}\varphi(t)dt > 0$, where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\},$$

or

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,gx,fx), G(gy,gy,fy), G(gz,gz,fz)\}.$$

If the pair (f,g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

With choice $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to [0,1)$, in Theorem 2.1 we have the following corollary.

Corollary 2.4. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \le k \int_0^{G(gx,gy,gz)} \varphi(t)dt,$$

for all $x,y,z \in X$, $0 \le k < 1$, and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$. If the pair (f,g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

With choice $F(s,t) = \omega(s)$, here $\omega : [0,\infty) \to [0,\infty)$ is a continuous function such that $\omega(0) = 0$, and $\omega(t) < t$ for $t > 0$, in Theorem 2.1 we have the following corollary.

Corollary 2.5. *Let (X,G) be a G-metric space and the pair (f,g) of self mappings is weakly compatible such that*

$$\int_0^{\psi(G(fx,fy,fz))} \varphi(t)dt \le \omega\left(\int_0^{\psi(L(x,y,z))} \varphi(t)dt\right)$$

for all $x,y,z \in X$, $\omega \in \Phi$, $\psi \in \Psi$, and $\varphi : [0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, where

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,fx,fx), G(gy,fy,fy), G(gz,fz,fz)\},$$

or

$$L(x,y,z) = \max\{G(gx,gy,gz), G(gx,gx,fx), G(gy,gy,fy), G(gz,gz,fz)\}.$$

If the pair (f,g) satisfies the (CLRg) property then f and g have a unique common fixed point in X.

With choice $\psi(t) = t$, in Corollary 2.5 we have the following corollary.

Corollary 2.6. [8] *Let (X, G) be a G-metric space and the pair (f, g) of self mappings is weakly compatible such that*

$$\int_0^{G(fx,fy,fz)} \varphi(t)dt \le \omega\left(\int_0^{L(x,y,z)} \varphi(t)dt\right)$$

for all $x, y, z \in X$, $\omega \in \Phi$, and $\varphi : [0, +\infty) \to [0, +\infty)$ is a Lebesgue integrable mapping which is summable, non-negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$, where

$$L(x, y, z) = \max\{G(gx, gy, gz), G(gx, fx, fx), G(gy, fy, fy), G(gz, fz, fz)\},$$

or

$$L(x, y, z) = \max\{G(gx, gy, gz), G(gx, gx, fx), G(gy, gy, fy), G(gz, gz, fz)\}.$$

If the pair (f, g) satisfies the (CLRg) property, then f and g have a unique common fixed point in X.

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Unconditionally Stable Difference Scheme for the Numerical Solution of Nonlinear Rosenau-KdV Equation

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Abstract

In this paper we investigate a nonlinear evolution model described by the Rosenau-KdV equation. We propose a three-level average implicit finite difference scheme for its numerical solutions and prove that this scheme is stable and convergent in the order of $O(\tau^2 + h^2)$. Furthermore we show the existence and uniqueness of numerical solutions. Comparing the numerical results with other methods in the literature show the efficiency and high accuracy of the proposed method.

Keywords: Finite difference scheme, solvability, unconditional stability, convergence.

2010 Mathematics Subject Classification: 65N06, 65N12.

1. Introduction

Nonlinear partial differential equations are useful in describing various phenomena. These equations arise in various areas of physics, mathematics and engineering. Analytical solutions of these equations are usually not available. Since only limited classes of equations are solved by analytical means, numerical solution of these nonlinear partial differential equations is of practical importance. KdV equation is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model and is as follows

$$u_t + uu_x + u_{xxx} = 0. \quad (1)$$

In the study of the dynamics of dense discrete systems, the case of wave-wave and wave-wall interactions cannot be described using the well-known KdV equation

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[4], so Rosenau [6, 7] proposed the so-called Rosenau equation

$$u_t + u_{xxxxt} + u_x + uu_x = 0. \tag{2}$$

The existence and the uniqueness of the solution for (2) were proved in [7], but it is difficult to find the analytical solution for (2). So, much works has been done on the numerical methods for (2) [1, 5]. On the other hand, for the further consideration of the nonlinear wave, the viscous term $+u_{xxx}$ needs to be included [4]

$$u_t + u_{xxxxt} + u_x + uu_x + u_{xxx} = 0, \tag{3}$$

which is usually called the Rosenau-KdV equation. Some analytical methods for the solution of this equation are given in [2, 9]. The authors of [4] proposed a conservative three-level linear finite difference scheme for the numerical solution of Rosenau-KdV equation. They proved the stability and convergency of method and existence and uniqueness of numerical solutions. In this paper, we propose a linear three-level average implicit finite difference scheme for the Rosenau-KdV equation (3) with the following boundary conditions

$$\begin{aligned} &u\left(x_L,t\right) = u\left(x_R,t\right) = 0, \qquad && u_x\left(x_L,t\right) = u_x\left(x_R,t\right) = 0, \\ &u_{xx}\left(x_L,t\right) = u_{xx}\left(x_R,t\right) = 0, && t \in [0,T], \end{aligned} \tag{4}$$

and initial condition

$$u\left(x,0\right) = u_0\left(x\right), \qquad x \in \left[x_L, x_R\right]. \tag{5}$$

The solitary wave solution for (3) is [3, 9]

$$\begin{aligned} u\left(x,t\right) = &\left(-\tfrac{35}{24} + \tfrac{35}{312}\sqrt{313}\right) \\ &\times \sec h^4 \left[\tfrac{1}{24}\sqrt{-26+2\sqrt{313}} \times \left(x - \left(\tfrac{1}{2} + \tfrac{1}{26}\sqrt{313}\right)t\right)\right]. \end{aligned} \tag{6}$$

The structure of this paper is as follows. In Section 2, we will describe a three level average implicit finite difference scheme for the Rosenau-KdV equation and discuss the estimate for the difference solution. In Section 3, we will show that the scheme is uniquely solvable. Then, in Section 4, we will prove the convergence and stability for the difference scheme. Finally numerical results are given in Section 5 to verify our theoretical analysis and efficiency of proposed method in comparison with other methods in the literature.

2. Proposed Finite Difference Scheme

Let $h = (x_R - x_L)/J$ and τ be the uniform step size in the spatial and temporal direction, respectively. Denote $x_j = x_L + jh$ $(j = -1, 0, 1, 2 \ldots, J, J+1)$, $t_n =$

$n\tau (n = 0, 1, 2, \ldots, N, N = [T/\tau])$, $u_j^n \approx u(x_j, t_n)$ and $Z_h^0 = \{u = (u_j) \,|\, u_{-1} = u_0 = u_J = u_{J+1} = 0, \quad j = -1, 0, 1, ..., J, J+1\}$. Throughout this paper, we denote C as a generic positive constant independent of h and τ, which may have different values in different occurrences. We introduce the following notations [4]

$$\begin{aligned}
&\left(u_j^n\right)_x = \tfrac{1}{h}\left(u_{j+1}^n - u_j^n\right), \qquad \left(u_j^n\right)_{\bar{x}} = \tfrac{1}{h}\left(u_j^n - u_{j-1}^n\right),\\
&\left(u_j^n\right)_{\hat{x}} = \tfrac{1}{2h}\left(u_{j+1}^n - u_{j-1}^n\right), \qquad \left(u^n, v^n\right) = h\sum_j u_j^n v_j^n,\\
&\left(\bar{u}_j^n\right) = \tfrac{1}{2}\left(u_j^{n+1} + u_j^{n-1}\right), \qquad \|u^n\|^2 = \left(u^n, u^n\right),\\
&\left(u_j^n\right)_{\hat{t}} = \tfrac{1}{2\tau}\left(u_j^{n+1} - u_j^{n-1}\right), \qquad \|u^n\|_\infty = \sup_j \left|u_j^n\right|.
\end{aligned} \tag{7}$$

We note that
$$\left(\frac{u^{p+1}}{p+1}\right)_x = \frac{1}{p+2}\left[u^p u_x + \left(u^{p+1}\right)_x\right], \tag{8}$$
and
$$\left(u_j^n\right)_{\bar{x}x} = \left(u_j^n\right)_{x\bar{x}} = \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{h^2}.$$

We propose the following implicit finite difference scheme for solving Eqs. (3)-(5)

$$\left(u_j^n\right)_{\hat{t}} + \left(u_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(u_j^n\right)_{\hat{x}} + \left(u_j^n\right)_{\hat{x}x\bar{x}} + \tfrac{1}{3}\left[u_j^n\left(\bar{u}_j^n\right)_{\hat{x}} + \left(u_j^n \bar{u}_j^n\right)_{\hat{x}}\right] = 0, \tag{9}$$

$$j = 1, 2, 3, \ldots, J-1, \qquad n = 1, 2, 3, \ldots, N-1, \tag{10}$$

$$u_j^0 = u_0(x_j), \quad j = 0, 1, 2, 3, \ldots, J, \tag{11}$$

$$\begin{aligned}
&u^n \in Z_h^0, \quad (u_0^n)_{\hat{x}} = (u_J^n)_{\hat{x}} = 0,\\
&(u_0^n)_{x\bar{x}} = (u_J^n)_{x\bar{x}} = 0, \quad n = 1, 2, 3, \ldots, N.
\end{aligned} \tag{12}$$

We now state some lemmas which are needed to prove stability and convergence of scheme.

Lemma 2.1. [8] *For any two mesh functions $u, v \in Z_h^0$ we have the following relations*

1. $(u_x, v) = -(u, v_{\bar{x}})$,
2. $(u_{x\bar{x}}, v) = -(u_x, v_x)$,

3. $(u_{x\overline{x}}, u) = -(u_x, u_x) = -\|u_x\|^2$,

4. If $(u_0)_{x\overline{x}} = (u_J)_{x\overline{x}} = 0$, *then* $(u_{xx\overline{x}\overline{x}}, u) = \|u_{xx}\|^2$.

Lemma 2.2. [8] *There exist two constants* C_1 *and* C_2 *such that*

$$\|u^n\|_\infty \le C_1 \|u^n\| + C_2 \|u_x^n\|. \tag{13}$$

Lemma 2.3. [8] *Suppose that* $\omega(k)$ *and* $\rho(k)$ *is nondecreasing. If* $C > 0$, *and*

$$\omega(k) \le \rho(k) + C\tau \sum_{l=0}^{k-1} \omega(l), \qquad \forall k, \tag{14}$$

then

$$\omega(k) \le \rho(k) e^{C\tau k}, \qquad \forall k. \tag{15}$$

Theorem 2.4. *If* u^n *be the solution of (9)-(12),* $u_0 \in H_0^2[x_L, x_R]$ *and* $u(x,t) \in C_{x,t}^{5,3}$ *then we have the following relations*

$$\|u^n\| \le C, \quad \|u_x^n\| \le C, \quad \|u^n\|_\infty \le C, \quad n = 1, 2, \dots N.$$

Proof. Taking an inner product of (9) with $2\overline{u}^n (i.e., u^{n+1} + u^{n-1})$, considering the boundary conditions (12) and Lemma 2.1, we obtain

$$\tfrac{1}{2\tau}\left(\|u^{n+1}\|^2 - \|u^{n-1}\|^2\right) + \tfrac{1}{2\tau}\left(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^{n-1}\|^2\right) + 2(u_{\hat{x}}^n, \bar{u}^n) +$$
$$2(u_{\hat{x}x\bar{x}}^n, \bar{u}^n) + 2(P, \bar{u}^n) = 0, \tag{16}$$

where $P_j = \frac{1}{3}\left[u_j^n (\overline{u}_j^n)_{\hat{x}} + (u_j^n \overline{u}_j^n)_{\hat{x}}\right]$. We can write

$$(P, \bar{u}^n) = 0,$$

so we get

$$\frac{1}{2\tau}\left(\|u^{n+1}\|^2 - \|u^{n-1}\|^2\right) + \frac{1}{2\tau}\left(\|u_{xx}^{n+1}\|^2 - \|u_{xx}^{n-1}\|^2\right) \quad = -2\,(u_{\hat{x}x\bar{x}}^n, \bar{u}^n) - 2\,({u^n}_{\hat{x}}, \bar{u}^n). \tag{17}$$

By Cauchy-Schwarz inequality and Lemma 2.1, we find

$$(u_{\hat{x}x\bar{x}}^n, 2\bar{u}^n) = -(u_{\hat{x}x}^n, 2\bar{u}_x^n),$$

$$\left|(u_{\hat{x}x}^n, u_x^{n+1} + u_x^{n-1})\right| \leqslant \|u_{xx}^n\|^2 + \frac{1}{2}\left(\|u_x^{n+1}\|^2 + \|u_x^{n-1}\|^2\right), \tag{18}$$

$$\|u_x^{n+1}\|^2 \le \tfrac{1}{2}\left(\|u^{n+1}\|^2 + \|u_{xx}^{n+1}\|^2\right),$$
$$\|u_x^{n-1}\|^2 \le \tfrac{1}{2}\left(\|u^{n-1}\|^2 + \|u_{xx}^{n-1}\|^2\right). \tag{19}$$

Substituting (19) into (18), we get

$$|(u^n_{\widehat{x}x\overline{x}}, 2\overline{u}^n)| \leq \|u^n_{xx}\|^2 + \frac{1}{4}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 + \left\|u^{n-1}\right\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right), \quad (20)$$

and

$$\left((u^n)_{\widehat{x}}, u^{n+1} + u^{n-1}\right) \leq \|u^n_x\|^2 + \frac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n-1}\right\|^2\right). \quad (21)$$

It follows from (17)-(21) that

$$\begin{aligned} &\left\|u^{n+1}\right\|^2 - \left\|u^{n-1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 - \left\|u^{n-1}_{xx}\right\|^2 \\ &\leq 2\tau\left(\|u^n_{xx}\|^2 + \tfrac{1}{4}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n+1}_{xx}\right\|^2 + \left\|u^{n-1}\right\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right)\right. \\ &\left. + \|u^n_x\|^2 + \tfrac{1}{2}\left(\left\|u^{n+1}\right\|^2 + \left\|u^{n-1}\right\|^2\right)\right). \end{aligned} \quad (22)$$

Using Lemma 2.1 and Cauchy-Schwarz inequality, we obtain

$$\|u^n_x\|^2 \leq \tfrac{1}{2}\left(\|u^n\|^2 + \|u^n_{xx}\|^2\right), \quad (23)$$

hence, we can write (22) as follows

$$\begin{aligned} &\left(\left\|u^{n+1}\right\|^2 + \|u^n\|^2\right) - \left(\|u^n\|^2 + \left\|u^{n-1}\right\|^2\right) + \left(\left\|u^{n+1}_{xx}\right\|^2 + \|u^n_{xx}\|^2\right) - \\ &\left(\|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2\right) \\ &\leq C\tau\left(\left\|u^{n+1}_{xx}\right\|^2 + \|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2 + \left\|u^{n+1}\right\|^2 + \|u^n\|^2 + \left\|u^{n-1}\right\|^2\right). \end{aligned} \quad (24)$$

Let $B^n = \|u^n\|^2 + \left\|u^{n-1}\right\|^2 + \|u^n_{xx}\|^2 + \left\|u^{n-1}_{xx}\right\|^2$. It follows from (24) that

$$B^{n+1} - B^n \leq C\tau\left(B^{n+1} + B^n\right),$$

so

$$(1 - C\tau)\left(B^{n+1} - B^n\right) \leq 2C\tau B^n.$$

If τ is sufficiently small which satisfies $1 - C\tau = \delta > 0$, then

$$B^{n+1} - B^n \leq C\tau B^n. \quad (25)$$

Summing up (25) from 0 to n-1, we have

$$B^n \leq B^0 + C\tau\sum_{l=0}^{n-1} B^l.$$

It follows from Lemma 2.3 that

$$\|u^n\| \leq C, \qquad \|u^n_{xx}\| \leq C.$$

From (23), we have $\|u^n_x\| \leq C$. Using Lemma 2.2, we get $\|u^n\|_\infty \leq C$. □

3. Solvability

Theorem 3.1. *The difference scheme* (9)-(12) *has a unique solution.*

Proof. We use from mathematical induction to prove. It is obvious that u^0 is uniquely determined by the initial condition (11). We also can get u^1 in order $O(h^2+\tau^2)$ by two-level C-N scheme (that is, u^0 and u^1 are uniquely determined). Now suppose $u^0, u^1, \ldots, u^n$ be solved uniquely. Considering equation (9) for u^{n+1} we can get

$$\tfrac{1}{2\tau}u_j^{n+1} + \tfrac{1}{2\tau}\left(u_j^{n+1}\right)_{xx\bar{x}\bar{x}} + \tfrac{1}{6}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}} + \left(u_j^n u_j^{n+1}\right)_{\hat{x}}\right] = 0. \tag{26}$$

Taking an inner product of (26) with u^{n+1}, we obtain

$$\tfrac{1}{2\tau}\left\|u^{n+1}\right\|^2 + \tfrac{1}{2\tau}\left\|u_{xx}^{n+1}\right\|^2 + \tfrac{h}{6}\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\hat{x}} + u_j^{n-1}\left(u_j^n\right)_{\hat{x}}\right]u_j^{n+1} = 0. \tag{27}$$

We can write

$$\begin{aligned} &\frac{1}{6}h\sum_{j=1}^{J-1}\left[u_j^n\left(u_j^{n+1}\right)_{\widehat{x}} + \left(u_j^n u_j^{n+1}\right)_{\widehat{x}}\right]u_j^{n+1} \\ = \;& \frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^n u_{j+1}^{n+1}u_j^{n+1} + u_{j+1}^n u_{j+1}^{n+1}u_j^{n+1}\right] \\ - \;& \frac{1}{12}\sum_{j=1}^{J-1}\left[u_j^n u_{j-1}^{n+1}u_j^{n+1} + u_{j-1}^n u_{j-1}^{n+1}u_j^{n+1}\right] = 0, \end{aligned} \tag{28}$$

It follows from (27) and (28) that

$$\left\|u^{n+1}\right\|^2 + \left\|u_{xx}^{n+1}\right\|^2 = 0.$$

That is, (26) has only a trivial solution. Therefore, (9)-(12) determines u_j^{n+1} uniquely. □

4. Convergence and Stability

Let $v(x,t)$ be the solution of problem (3)-(5), $v_j^n = v(x_j, t_n)$, then the truncation error of the difference scheme (9)-(12) is as follows:

$$r_j^n = \left(v_j^n\right)_{\widehat{t}} + \left(v_j^n\right)_{xx\overline{x}\overline{x}\widehat{t}} + \left(v_j^n\right)_{\widehat{x}} + \left(v_j^n\right)_{\widehat{x}x\overline{x}} + \tfrac{1}{3}\left[v_j^n\left(\overline{v}_j^n\right)_{\widehat{x}} + \left(v_j^n\overline{v}_j^n\right)_{\widehat{x}}\right]. \tag{29}$$

Using Taylor expansion, we know that $r_j^n = O(h^2+\tau^2)$ holds if $h, \tau \to 0$.

Theorem 4.1. *Suppose that $u_0 \in H_0^2[x_L, x_R]$, then the solution u^n of (9)-(12) converges to the solution $v(x,t)$ of problem (3)-(5) in norm $\|.\|_\infty$ and the rate of convergence is $O(\tau^2 + h^2)$.*

Proof. Subtracting (9) from (29) and letting $e_j^n = v_j^n - u_j^n$, we have

$$r_j^n = \left(e_j^n\right)_{\hat{t}} + \left(\bar{e}_j^n\right)_{xx\bar{x}\bar{x}\hat{t}} + \left(e_j^n\right)_{\hat{x}} + \left(e_j^n\right)_{\hat{x}x\overline{x}} + R_{1,j} + R_{2,j}, \tag{30}$$

where

$$\begin{aligned} R_{1,j} &= \tfrac{1}{3}\left[v_j^n \left(\bar{v}_j^n\right)_{\hat{x}} - u_j^n \left(\overline{u}_j^n\right)_{\hat{x}}\right], \\ R_{2,j} &= \tfrac{1}{3}\left[\left(v_j^n \bar{v}_j^n\right)_{\hat{x}} - \left(u_j^n \overline{u}_j^n\right)_{\hat{x}}\right]. \end{aligned}$$

Computing the inner product of (30) with $2\bar{e}^n$, we obtain

$$\begin{aligned} (r^n, 2\bar{e}^n) = \tfrac{1}{2\tau}\left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \tfrac{1}{2\tau}\left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) + \\ (e_{\hat{x}}^n, 2\bar{e}^n) + (e_{\hat{x}x\bar{x}}^n, 2\bar{e}^n) + (R_1, 2\bar{e}^n) + (R_2, 2\bar{e}^n)\,. \end{aligned} \tag{31}$$

We can write (31) as follows

$$\begin{aligned} \left(\left\|e^{n+1}\right\|^2 - \left\|e^{n-1}\right\|^2\right) + \left(\left\|e_{xx}^{n+1}\right\|^2 - \left\|e_{xx}^{n-1}\right\|^2\right) = \\ 2\tau\left[(r^n, 2\bar{e}^n) - (e_{\hat{x}x\bar{x}}^n, 2\bar{e}^n) - ((e^n)_{\hat{x}}, 2\bar{e}^n) - (R_1, 2\bar{e}^n) - (R_2, 2\bar{e}^n)\right]. \end{aligned} \tag{32}$$

By Lemma 2.1, Theorem 2.1, and Cauchy-Schwarz inequality, we have

$$\begin{aligned} (R_1, 2\bar{e}^n) &= \tfrac{2}{3}h\sum_j \left(v_j^n \left(\overline{v}_j^n\right)_{\hat{x}} - u_j^n \left(\overline{u}_j^n\right)_{\hat{x}}\right)\left(\bar{e}_j^n\right) \\ &= \tfrac{1}{3}h\sum_j \left[v_j^n \left(v_j^{n+1} + v_j^{n-1}\right)_{\hat{x}} - v_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\hat{x}}\right. \\ &+ \left. v_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\hat{x}} - u_j^n \left(u_j^{n+1} + u_j^{n-1}\right)_{\hat{x}}\right]\left(\overline{e}_j^n\right) \\ &= \tfrac{2}{3}h\sum_j \left(v_j^n \left(\overline{e}_j^n\right)_{\hat{x}} - e_j^n \left(\overline{u}_j^n\right)_{\hat{x}}\right)\left(\overline{e}_j^n\right) \\ &= \tfrac{2}{3}h\sum_j v_j^n \left(\overline{e}_j^n\right)_{\hat{x}}\left(\overline{e}_j^n\right) - \tfrac{2}{3}h\sum_j e_j^n \left(\overline{u}_j^n\right)_{\hat{x}}\left(\overline{e}_j^n\right) \\ &\le \tfrac{2}{3}Ch\sum_j \left(\left|\left(\overline{e}_j^n\right)_{\hat{x}}\right| + \left|e_j^n\right|\right)\left|\overline{e}_j^n\right| \\ &\le C\left[\|\overline{e}_x^n\|^2 + \|e^n\|^2 + 2\|\overline{e}^n\|^2\right] \\ &\le C\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + 2\left\|e^{n+1}\right\|^2 + \|e^n\|^2 + 2\left\|e^{n-1}\right\|^2\right), \end{aligned} \tag{33}$$

and

$$\begin{aligned}
(R_2, 2\bar{e}^n) &= \tfrac{2}{3}h\sum_j \left(\left(v_j^n\bar{v}_j^n\right)_{\hat{x}} - \left(u_j^n\bar{u}_j^n\right)_{\hat{x}}\right)\left(\bar{e}_j^n\right) \\
&= \tfrac{2}{3}h\sum_j \left\{\left(v_j^n\overline{v}_j^n\right)_{\widehat{x}} - \left(v_j^n\overline{u}_j^n\right)_{\widehat{x}} + \left(v_j^n\overline{u}_j^n\right)_{\widehat{x}} - \left(u_j^n\overline{u}_j^n\right)_{\widehat{x}}\right\}\left(\overline{e}_j^n\right) \\
&= -\tfrac{2}{3}h\sum_j \left\{\left(v_j^n\right)\overline{e}_j^n\left(\overline{e}_j^n\right)_{\widehat{x}} + \left[v_j^n - u_j^n\right]\overline{u}_j^n\left(\overline{e}_j^n\right)_{\widehat{x}}\right\} \\
&\leq \tfrac{2}{3}Ch\sum_j \left(\left|\left(\bar{e}_j^n\right)\right| + \left|e_j^n\right|\right)\left|\left(\bar{e}_j^n\right)_{\hat{x}}\right| \\
&\leq C\left[\|\bar{e}_x^n\|^2 + \|e^n\|^2 + \|\bar{e}^n\|^2\right] \\
&\leq C\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2 + \left\|e^{n+1}\right\|^2 + \|e^n\|^2 + \left\|e^{n-1}\right\|^2\right).
\end{aligned} \tag{34}$$

Noting that similar to (18)-(21) we have

$$(r^n, 2\bar{e}^n) = \left(r^n, e^{n+1} + e^{n-1}\right) \leq \|r^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right), \tag{35}$$

$$\left((e^n)_{\hat{x}x\bar{x}}, 2\bar{e}^n\right) = -\left((e^n)_{\hat{x}x}, e_x^{n+1} + e_x^{n-1}\right) \leqslant \|e_{xx}^n\|^2 + \frac{1}{2}\left(\left\|e_x^{n+1}\right\|^2 + \left\|e_x^{n-1}\right\|^2\right), \tag{36}$$

$$\left((e^n)_{\hat{x}}, 2\bar{e}^n\right) = \left((e^n)_{\hat{x}}, e^{n+1} + e^{n-1}\right) \leq \|e_x^n\|^2 + \frac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2\right). \tag{37}$$

From (32)-(37), we have

$$\begin{aligned}
&\left(\left\|e^{n+1}\right\|^2 + \|e^n\|^2\right) - \left(\|e^n\|^2 + \left\|e^{n-1}\right\|^2\right) + \\
&\left(\left\|e_{xx}^{n+1}\right\|^2 + \|e_{xx}^n\|^2\right) - \left(\|e_{xx}^n\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right) \\
&\leq C\tau\left(\left\|e^{n+1}\right\|^2 + \left\|e^{n-1}\right\|^2 + \|e^n\|^2 + \left\|e_x^{n+1}\right\|^2 + \|e_x^n\|^2 + \left\|e_x^{n-1}\right\|^2 + \|e_{xx}^n\|^2\right) + \\
&2\tau\|r^n\|^2.
\end{aligned} \tag{38}$$

Similar to the proof of (23), we obtain

$$\begin{aligned}
\left\|e_x^{n+1}\right\|^2 &\leq \tfrac{1}{2}\left(\left\|e^{n+1}\right\|^2 + \left\|e_{xx}^{n+1}\right\|^2\right), \\
\|e_x^n\|^2 &\leq \tfrac{1}{2}\left(\|e^n\|^2 + \|e_{xx}^n\|^2\right), \\
\left\|e_x^{n-1}\right\|^2 &\leq \tfrac{1}{2}\left(\left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2\right).
\end{aligned} \tag{39}$$

Let $D^n = \|e^n\|^2 + \|e_{xx}^n\|^2 + \left\|e^{n-1}\right\|^2 + \left\|e_{xx}^{n-1}\right\|^2$, then (38) can be rewritten as follows

$$D^{n+1} - D^n \leq 2\tau\|r^n\|^2 + C\tau\left(D^{n+1} + D^n\right), \tag{40}$$

which yields

$$(1 - C\tau)\left(D^{n+1} - D^n\right) \leq 2C\tau D^n + 2\tau\|r^n\|^2. \tag{41}$$

If τ is sufficiently small, which satisfies $1 - C\tau > 0$, then

$$D^{n+1} - D^n \leq C\tau D^n + C\tau\|r^n\|^2. \tag{42}$$

Summing up (42) from 1 to n, we have

$$D^n \leq D^0 + C\tau \sum_{l=0}^{n-1} \left\|r^l\right\|^2 + C\tau \sum_{l=0}^{n-1} D^l. \tag{43}$$

First, we can get u^1 in order $O(h^2+\tau^2)$ that satisfies $D^0 \leq C(h^2+\tau^2)^2$ by two-level C-N scheme. Since

$$\tau \sum_{l=0}^{n-1} \left\|r^l\right\|^2 \leq n\tau \max_{0\leq l\leq n-1} \left\|r^l\right\|^2 \leq T.O\left(\tau^2 + h^2\right)^2, \tag{44}$$

we obtain

$$D^n \leq O\left(\tau^2 + h^2\right)^2 + C\tau \sum_{l=0}^{n-1} D^l. \tag{45}$$

From Lemma 2.3 we get

$$D^n \leq O\left(\tau^2 + h^2\right)^2, \tag{46}$$

that is

$$\|e^n\| \leq O\left(\tau^2 + h^2\right), \qquad \|e^n_{xx}\| \leq O\left(\tau^2 + h^2\right). \tag{47}$$

From (39) we have

$$\|e^n_x\| \leq O\left(\tau^2 + h^2\right). \tag{48}$$

By Lemma 2.2, we obtain

$$\|e^n\|_\infty \leq O\left(\tau^2 + h^2\right). \tag{49}$$

This completes the proof. □

Finally, we can state similarly the following theorem.

Theorem 4.2. *Under the conditions of Theorem 4.1, the solution u^n of* (9)-(12) *is stable in norm $\|.\|_\infty$.*

5. Numerical Results

In this section we present the numerical results of the proposed method on a test problem. We performed our computations using **Matlab** 7 software on a PC with Intel Core 2 Duo, 2.8 GHz CPU and 2 GB RAM. We tested the accuracy and stability of the method presented in this paper by performing the mentioned method for different values of Δt and h.

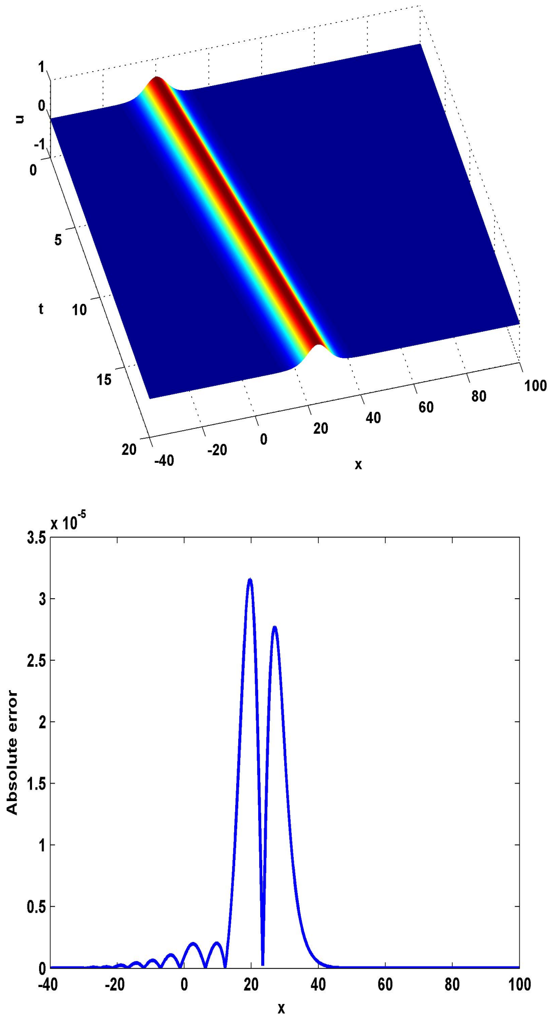

Figure 1: Surface plot of approximate solution (left panel) and plot of absolute error (right panel) with $h = 0.05$, $\tau = 0.05$ at $T = 20$.

Table 1: Errors and computational orders obtained at different final times.

	$h=\tau=0.2$	$h=\tau=0.1$	$h=\tau=0.05$
$T=10$			
$\|e^n\|_\infty$	2.718820×10^{-4}	6.853283×10^{-5}	1.718933×10^{-5}
$C-Order$	$-$	1.988	1.995
$T=20$			
$\|e^n\|_\infty$	5.026183×10^{-4}	1.261183×10^{-4}	3.157146×10^{-5}
$C-Order$	$-$	1.995	1.998
$T=30$			
$\|e^n\|_\infty$	7.217771×10^{-4}	1.810695×10^{-4}	4.532327×10^{-5}
$C-Order$	$-$	1.995	1.998
$T=40$			
$\|e^n\|_\infty$	9.396398×10^{-4}	2.356919×10^{-4}	5.899417×10^{-5}
$C-Order$	$-$	1.995	1.998

Also we calculated the computational orders of the method presented in this article (denoted by C-Order) with the following formula

$$\frac{\log(\frac{E_1}{E_2})}{\log(\frac{h_1}{h_2})},$$

in which E_1 and E_2 are errors correspond to grids with mesh size h_1 and h_2 respectively. Also we put $x_L=-40$ and $x_R=100$.

5.1 Propagation of a Single Solitary Wave

We consider the equation (3) with the following exact solution

$$\begin{aligned} u(x,t) = &\left(-\tfrac{35}{24}+\tfrac{35}{312}\sqrt{313}\right) \\ &\times \sec h^4\left[\tfrac{1}{24}\sqrt{-26+2\sqrt{313}}\right. \\ &\left.\times\left(x-\left(\tfrac{1}{2}+\tfrac{1}{26}\sqrt{313}\right)t\right)\right]. \end{aligned}$$

The initial condition can be obtained from exact solution. Table 1 shows the computational orders and errors of proposed method with different values of $h=\tau$ at different final times. Numerical results of this table confirm the second order of accuracy of method. In Tables 2, 3 we compare the errors of proposed method with the results of [4]. As we see the new method has better accuracy. Figure 1 shows the surface plot of approximate solution (left panel) and plot of absolute error (right panel) with $h=0.05$, $\tau=0.05$ at $T=20$.

Table 2: Comparison of $\|e^n\|_\infty$ error at various time steps.

$\|e^n\|_\infty$	$h=\tau=0.1$		$h=\tau=0.05$	
Method	Present	Scheme [4]	Present	Scheme [4]
$t=10$	2.718×10^{-4}	2.507×10^{-3}	1.719×10^{-5}	1.585×10^{-4}
$t=20$	5.026×10^{-4}	4.489×10^{-3}	3.157×10^{-5}	2.836×10^{-4}
$t=30$	7.218×10^{-4}	6.081×10^{-3}	4.532×10^{-5}	3.834×10^{-4}
$t=40$	9.396×10^{-4}	7.444×10^{-3}	5.899×10^{-5}	4.709×10^{-4}

Table 3: Comparison of $\|e^n\|$ error at various time steps.

$\|e^n\|$	$h=\tau=0.2$		$h=\tau=0.05$	
Method	Present	Scheme [4]	Present	Scheme [4]
$t=10$	7.389×10^{-4}	6.525×10^{-3}	4.663×10^{-5}	4.113×10^{-4}
$t=20$	1.443×10^{-3}	1.209×10^{-2}	9.070×10^{-5}	7.631×10^{-4}
$t=30$	2.132×10^{-3}	1.683×10^{-2}	1.339×10^{-5}	1.063×10^{-3}
$t=40$	2.818×10^{-3}	2.101×10^{-2}	1.769×10^{-5}	1.328×10^{-3}

5.2 Interaction of Two Solitary Waves

We investigate the interaction of two solitary waves for equation (3) using the following initial condition

$$u(x,0)=\sum_{j=1}^{2}3d_j\sec h^2\left(k_j\left(x-x_j\right)\right),$$

in which $k_1=0.4$, $k_2=0.3$, $x_1=15$, $x_2=35$, and

$$d_j=\frac{4k_j^2}{1-4k_j^2},\qquad j=1,2.$$

From the above initial conditions, the solitary waves are propagated rightwards. Shapes of both waves at times $t=10,15,20,25$ and with $h=\tau=0.1$ are shown in Figure 2. We see that as the time progresses the collision occurs and after collision two waves leave each other without changing their shape.

6. Conclusion

In this article, we constructed an implicit finite difference scheme for the solution of Rosenau-KdV equation. We proved that this scheme is stable and convergent in the order of $O(\tau^2+h^2)$. Furthermore we showed the existence and uniqueness of numerical solutions. We compared the numerical results of this paper with other methods in the literature and concluded that the proposed method has better results.

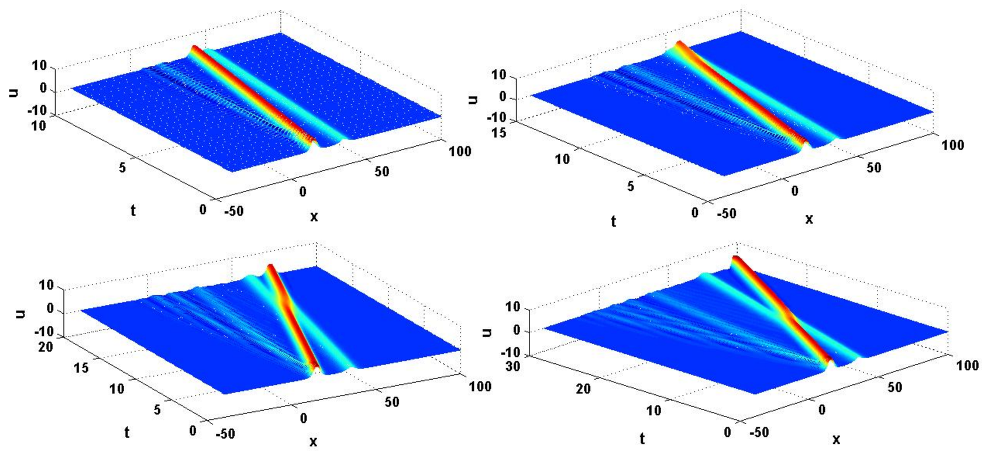

Figure 2: The numerical solutions of two solitary waves with $h = \tau = 0.1$ obtained at final times $t = 10$ (left-top), $t = 15$ (right-top), $t = 20$ (left-bottom) and $t = 30$ (right-bottom).

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Wiener Polarity Index of Tensor Product of Graphs

Mojgan Mogharrab**, Reza Sharafdini and Somayeh Musavi*

Abstract

Mathematical chemistry is a branch of theoretical chemistry for discussion and prediction of the molecular structure using mathematical methods without necessarily referring to quantum mechanics. In theoretical chemistry, distance-based molecular structure descriptors are used for modeling physical, pharmacologic, biological and other properties of chemical compounds. The Wiener Polarity index of a graph G, denoted by $W_P(G)$, is the number of unordered pairs of vertices of distance 3. The Wiener polarity index is used to demonstrate quantitative structure-property relationships in a series of acyclic and cycle-containing hydrocarbons. Let G and H be two simple connected graphs, then the tensor product of them is denoted by $G \otimes H$ whose vertex set is $V(G \otimes H) = V(G) \times V(H)$ and edge set is $E(G \otimes H) = \{(a,\ b)(c,\ d) \mid ac \in E(G)\ ,\ bd \in E(H)\}$. In this paper, we aim to compute the Wiener polarity index of $G \otimes H$ which was computed wrongly in [J. Ma, Y. Shi and J. Yue, The Wiener polarity index of graph products, *Ars Combin.* 116 (2014) 235-244].

Keywords: Topological index, Wiener polarity index, tensor product, graph, distance.

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1. Introduction

In this section, we first describe some notations which will be kept throughout. A ***graph*** is a structure composed of points (***vertices*** or nodes), connected by lines (***edges*** or links).

A graph is called ***finite*** if both its vertex set and edge setare finite. If $e = uv$ is an edge of a graph, then we say that e joins the pair vertices u and v. Also

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the vertices u and v are named the end vertices of the edge e. An edge with identical end vertices is called *a* ***loop***. We say that a graph is ***simple*** whenever it has no loop and no two of its edges join the same pair of vertices. The set of finite simple graphs is shown by Γ and the set of finite graphs in which loops are admitted is denoted as Γ_0, so $\Gamma \subset \Gamma_0$ [13]. From now on, when we say that **"*G is a graph*"** it means $G \in \Gamma$, otherwise $G \in \Gamma_0$. For a given graph G, we show the ***vertex*** and ***edge set*** of G by $V(G)$ and $E(G)$, respectively. If x is a vertex of the graph G, the degree of x in G is denoted by $deg_G(x)$. In the other words, if for any vertex $x \in G$, $N_G(x)$ denotes the set of ***neighbors*** that $x \in G$, i.e. $N_G(x) = \{y \in V(G) | \, xy \in E(G)\}$, then $deg_G(x) = |N_G(x)|$. The minimum and maximum degree of all vertices x of a graph G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. A ***walk*** in G is a sequence of (not necessarily distinct) vertices $v_1 v_2 \, ...v_n$, such that $v_i v_{i+1} \in E(G)$ for $i = 1, 2, \ldots, n-1$. We call such a walk a $(v_1, v_n)-$walk. A ***path*** in the graph is a walk without traversing any vertex twice. So, a ***path*** in the graph is a sequence of adjacent edges without traversing any vertex twice. The graph is called ***connected*** when there is a path between any pair of vertices in it, otherwise the graph is disconnected. For the vertices $u, v \in V(G)$, the ***distance*** between u and v in G is denoted by $d_G(u, v)$ and it is the length of a shortest (u, v)-path in G. If G is a disconnected graph, then we assume that the distance between any two vertices belonging to different components of G, is infinity. For a given vertex $x \in V(G)$, its ***eccentricity*** $ecc(x)$ is the largest distance between x and any other vertex $y \in V(G)$, that is $ecc(x) = Max\{d_G(x, y) \, | y \in V(G)\}$. The maximum eccentricity over all vertices of G is called the ***diameter*** of G and denoted by $D(G)$. Also, the minimum eccentricity among the vertices of G is called the ***radius*** of G and denoted by $R(G)$. Let G and H be two graphs such that $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Then we say that H is a ***subgraph*** of G and write $H \leq G$. Let us denote a cycle and a path on n vertices by C_n and P_n, respectively. For a graph H, a graph G is called H-free if it has no subgraph isomorphic to H. So, a graph is called triangle free if it has no subgraph isomorphic to C_3. The ***adjacency matrix*** of a graph G, denoted by $A(G)$, is a $(0, 1)-$matrix whose rows and columns are indexed by $V(G)$ and the element $A(G)_{u,v} = 1$ if and only if $uv \in E(G)$ for each $u, v \in V(G)$, otherwise $A(G)_{u,v} = 0$.

Mathematical Chemistry is a branch of theoretical chemistry for studying the molecular structure using mathematical methods. Molecular Graphs or Chemical Graphs are models of molecules in which atoms are represented by vertices and chemical bonds by edges of a graph. The ***Chemical Graph Theory*** is a branch of mathematical chemistry concerned with the study of chemical graphs. In theoretical chemistry correlation of chemical structure with various physical properties, chemical reactivity or biological activity are often modeled by means of molecular-graph-based ***structure-descriptors***, which are also referred to as topological indices. A ***topological index*** is a function TOP from Γ to the set of real numbers $\mathbb{R}$ with this property that $TOP(G) = TOP(H)$, whenever G and H are isomorphic.

There exist several types of such indices, especially those based on vertex and edge distances. The most well-known and successful topological indices with several applications in QSAR/QSPR studies in chemistry, was introduced by *H.* Wiener [27] for acyclic molecules. It is defined as the sum of distances between all pairs of vertices of the molecular graph. Let G be a simple connected graph. Then the ***Wiener index*** of G is defined as $W(G) = \frac{1}{2}\sum_{x,y\in V(G)} d(x,y)$. Let $\gamma(G,k)$ be the number of unordered vertex pairs of G for which the distance of them is equal to k and therefore one can write $W(G) = \sum_{k\geq 1} k\gamma(G,k)$. In the case $k = 3$, the number $\gamma(G,3)$ is called the ***Wiener polarity index*** of G and denoted by $W_P(G)$.

It is believed that the Wiener index was the first reported distance-based topological index. This topological index was used for modeling the shape of organic molecules and for calculating several of their physico-chemical properties [11]. For example, Wiener used a linear formula to calculate the boiling points of the paraffin (alkanes). More precisely, let A bean alkane with the corresponding (Hydrogen suppressed) molecular graph G. Then the boiling point $t_B(A)$ of A is estimated as follows

$$t_B(A) = aW(G) + bW_P(G) + c,$$

where a, b and c are constants for a given isomeric group.

Using the Wiener polarity index, Lukovits and Linert demonstrated quantitative structure–property relationships in a series of acyclic and cycle-containing hydrocarbons [20]. Hosoya [14] found a physical-chemical interpretation of $W_P(G)$. Du, Li and Shi [9] described a linear time algorithm APT for computing the index of trees and characterized the trees maximizing the index among all trees of a given order. Deng, Xiao and Tang [7] characterized the extremal trees with respect to this index among all trees of order n and diameter d. Deng and Xiao [6] studied the Wiener polarity index of molecular graphs of alkanes with a given number of methyl group. They also found the maximum Wiener polarity index of chemical trees with k pendants and characterized the extremal graphs [9]. Deng [8] also gave the extremal Wiener polarity index of all chemical trees with order n. Liu, Hou and Huang [18] studied Wiener polarity index of trees with maximum degree for given number of leaves. Hou, Liu and Huang [15] characterized the maximum Wiener polarity index of unicyclic graphs. Also Liu and Liu [19] established a relation between Wiener polarity index and other indices like Zagreb indices and Wiener index. They also obtained some extremal unicyclic graphs on n vertices with respect to Wiener polarity index. Behmaram, Yousefi-Azari and Ashrafi [3] determined the Wiener polarity of fullerenes and hexagonal systems. Chen, Du and Fan [5] computed the Wiener polarity index of cactus graphs. A. Ilić and M. Ilić [16] introduced a generalized Wiener polarity index and described a linear time algorithm for computing these indices for trees and partial cubes, and characterized extremal trees maximizing the generalized Wiener polarity index among trees of given order n. Ou, Feng and Liu [23] characterized minimum Wiener polarity index of unicyclic graphs with prescribed maximum degree. Ashrafi, Dehghanzade and Sharafdini [1] computed maximum Wiener polarity index of bicyclic graphs.

Many graphs can be constructed from simpler graphs via certain operations called graph products [13, 17]. It is believed that the most difficult one in many aspects among standard products is the tensor product of graphs. The tensor product of graphs has been extensively studied in relation to the areas such as graph colorings, graph recognition, decompositions of graphs, graph embeddings, matching theory and design theory. For any two graphs $G, H \in \Gamma_0$, their ***tensor product*** (also known as direct product, Kronecker product, categorical product, cardinal product, relational product, weak direct product conjunction, ...) is denoted by $G \otimes H$ whose vertex set and edge set are as follows:

$$\begin{aligned} V(G \otimes H) &= V(G) \times V(H) \\ E(G \otimes H) &= \{(a,b)(c,d) \mid ac \in E(G), bd \in E(H)\}. \end{aligned}$$

The vertices $(a,\ b)$ and (c,d) are adjacent in $G \otimes H$, whenever ac is an edge in G and bd is an edge in H. From the definition, one can get immediately that

$$|V(G \otimes H)| = |V(G)||V(H)|$$

and if $(a,\ b)(c,\ d) \in E(G \otimes H)$, then also $(a,\ d)(c,\ b) \in E(G \otimes H)$ and hence

$$|E(G \otimes H)| = 2|E(G)||E(H)|.$$

Furthermore, we can see $deg_{G\otimes H}((a,b)) = deg_G(a)deg_H(b)$.

Note. Since a connected graph G is Eulerian if and only if it has no vertices of odd degree. Therefore, if $G \otimes H$ is a connected graph and one of the graphs G or H is Eulerian graph, then $G \otimes H$ is also an Eulerian graph.

We can consider the tensor product as a binary operation on the set Γ_0 [26]. It is known that, up to isomorphism, this product is commutative and associative in a natural way [24]. Also if the graph $I \in \Gamma_0$ denotes a vertex on which there is a loop, then $G \otimes I \cong G$ for any $G \in \Gamma_0$. Therefore I is the identity element for tensor product as a binary operation.

Note. If we consider the tensor product of graphs as a binary operation on the set Γ, then this binary operation has no identity element.

By an appropriate ordering of $V(G) \times V(H)$, it follows that $A(G \otimes H) = A(G) \otimes A(H)$, where $A(G) \otimes A(H)$ is the Kronecker product of matrices $A(G)$ and $A(H)$ [25].

Lemma 1. ([4, 13]) Let G and H be two connected graphs. Then the graph $G \otimes H$ is connected if and only if any G or H contains an odd cycle if and only if at least G or H is non-bipartite. For example, Figures 1, 2 illustrate two examples of tensor products. Note that, in all Figures the vertex (x, y) in the tensor product $G \otimes H$ is shown by xy.

As is depicted in Figure 2, $P_3 \otimes P_5$ is disconnected.

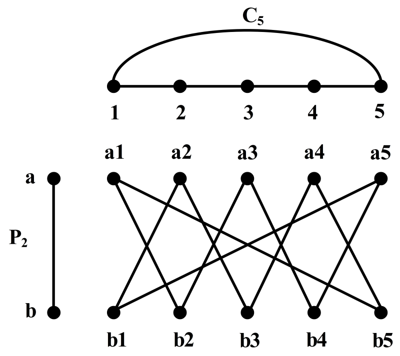

Figure 1: $\boldsymbol{P_2 \otimes C_5 \cong C_{10}}$.

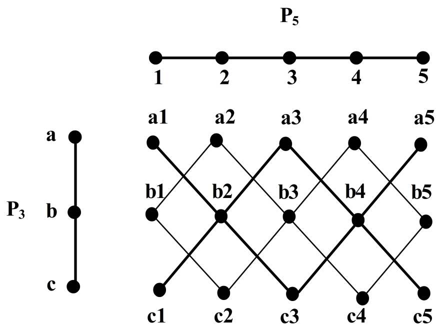

Figure 2: The disconnected graph $\boldsymbol{P_3 \otimes P_5}$.

Moradi [22] computed Wiener type indices of the tensor product of graphs. In this article, we concerned about the Weiner polarity index of the tensor product of graphs. The Wiener polarity index of tensor product of graphs was wrongly computed in [21]. In order to show a counter example, we need to express the following lemma.

Lemma 2. For the positive integer $n \geq 3$,

$$W_P(C_n) = \begin{cases} 0, & n = 3, 4, 5; \\ 3, & n = 6; \\ n, & n \geq 7. \end{cases}$$

Remark. In [21] the authors (wrongly) stated in Theorem 4.1 that

$$W_P\,(G \otimes H) = 2W_P\,(G)\,W_P\,(H) + 2W_P\,(H)\,m\,(G) + 2W_P\,(G)\,m\,(H)$$

where G and H are two non-trivial connected graphs and at least one of them is non-bipartite, $m\,(G)$ and $m\,(H)$ are the number of edges of the graphs G and H, respectively. Let us show that this formula is wrong as it is seen in Figure 1, for which $P_2 \otimes C_5 \cong C_{10}$ and it follows from Lemma 3 that the left hand side of the equation in the above formula is $W_P(P_2 \otimes C_5) = W_P(C_{10}) = 10$, while the right hand side is zero.

In this article we aim to obtain Wiener polarity index of tensor product of graphs. Note that our technique is that of used in [22].

2. Main Results

Let G and H be two graphs. In this section, we consider the Wiener Polarity index of $G \otimes H$. Since this topological index is defined on the connected graphs, it follows from Lemma 2 that we need to assume that at least one of the graph G or H is non-bipartite and thus in this case $G \otimes H$ is connected.

Now, let us study some distance properties of the tensor product of graphs.

Definition 3. [22] Let G be a graph and $x, y \in V(G)$. Define $d'_G(x, y)$ as follows

1. If $d_G(x, y)$ is an odd number, then $d'_G(x, y)$ is defined as the length of a shortest even walk joining x and y in G, and if there is no shortest even walk, then $d'_G(x, y) = +\infty$.
2. If $d_G(x, y)$ is an even number, then $d'_G(x, y)$ is defined as the length of a shortest odd walk joining x and y in G, and if there is no shortest odd walk, then $d'_G(x, y) = +\infty$.
3. If $d_G(x, y) = +\infty$, then $d'_G(x, y) = +\infty$.

Note. We take $d_G(x, y) = +\infty$, if there is no shortest odd walk and no shortest even walk between x and y in G. Also, if $d'_G(x, y) < +\infty$, then $d'_G(x, y) \geq 2$ and so $d'_G(x, y) \neq 1$.

Definition 4. [21, 22] Let G and H be two graphs and $(a, b), (c, d) \in V(G \otimes H)$. The relation $\sim$ on the vertex set $V(G \otimes H)$ is defined as $(a, b) \sim (c, d)$ whenever $d_G(a, c), d_H(b, d) < +\infty$ and $d_G(a, c) + d_H(b, d)$ is an even number, hence the parity of $d_G(a, c)$ and $d_H(b, d)$ are the same. Therefore $(a, b) \nsim (c, d)$ whenever $d_G(a, c) = +\infty$ or $d_H(b, d) = +\infty$ or $d_G(a, c) + d_H(b, d)$ is an odd number.

Lemma 5. [21, 22] Let G and H be graphs and $(a, b), (c, d) \in V(G \otimes H)$. Then

$$d_{G\otimes H}((a,b),(c,d)) = \begin{cases} d_1((a,b),(c,d)) & if \ (a,b) \sim (c,d) \\ d_2((a,b),(c,d)) & if (a,b) \nsim (c,d) \end{cases}$$

where

$$\begin{aligned} d_1((a,b),(c,d)) &= Max\{d_G(a,c), d_H(b,d)\} \\ d_2((a,b),(c,d)) &= min\{Max\{d_G(a,c), d'_H(b,d)\},\ Max\{d'_G(a,c), d_H(b,d)\}\} \end{aligned}$$

Now let $(a,b),(c,d) \in V(G\otimes H)$ be arbitrary. Then one can see that the distance between them is depended on $(a,b) \sim (c,d)$ or $(a,b) \nsim (c,d)$. Hence, if $d_{G\otimes H}(a,b),(c,d) = 3$ then one may conclude that $d_1(a,b)(c,d) = 3$ or $d_2(a,b)(c,d) = 3$. So, we compute the distance between any two vertices in the graph $G\otimes H$, in two cases. To proceed, let us discus the first case as follows.

Proposition 6. Suppose that

$$\mathfrak{R}_1(G\otimes H) = \{\{(a,b),(c,d)\} \subseteq V(G\otimes H)|\,(a,\ b)\sim(c,\ d)\ \&\ d_1((a,b),(c,d)) = 3\}.$$

Then $|\mathfrak{R}_1(G\otimes H)| = 2\left[m_2W_P(G) + m_1W_P(H) + W_P(G)W_P(H)\right]$, where $m_1 = |E(G)|$ and $m_2 = |E(H)|$.

Proof. If $\{(a,\ b),(c,\ d)\} \in \mathfrak{R}_1(G\otimes H)$, then can conclude that $d_G(a,c) + d_H(b,d)$ is an even positive integer. Then only three cases occur as follows:

1. $d_G(a,c) = 1$ and $d_H(b,d) = 3$. In this case, the number of all unordered pairs that are satisfied in this case is equal to $2m_1W_P(H)$.
2. $d_G(a,c) = 3$ and $d_H(b,d) = 1$. In this case, the number of all unordered pairs that are satisfied in this case is equal to $2m_2W_P(G)$.
3. $d_G(a,c) = 3$ and $d_H(b,d) = 3$. In this case, the number of all unordered pairs that are satisfied in this case is equal to $2W_P(G)W_P(H)$,

and our proof is complete. □

Note. One can easily see that if $\{(a,\ b),(c,\ d)\}$ is satisfied in the condition of one part, then also $\{(a,\ d),(c,\ b)\}$ is satisfied.

Definition 7. For a graph G, the following notation is useful for the main results of this paper. Suppose that

$$\begin{aligned}
\mathbb{A}(G) &= \{uv\in E(G)\mid N_G(u)\cap N_G(v)=\emptyset\} = \{uv\in E(G)\mid \forall C_3\le G; uv\notin E(C_3)\},\\
\mathbb{B}(G) &= \{x\in V(G)\mid \exists u,v\in N_G(x); uv\in E(G)\} = \{x\in V(G)\mid \exists C_3\le G; x\in V(C_3)\},\\
\mathbb{C}(G) &= \{\{u,v\}\subseteq V(G)|d_G(u,v)=2 \textit{ and } \exists\,(u,v)-\textit{walk of length } 3\},
\end{aligned}$$

also $\mathcal{A}_G = |\mathbb{A}(G)|$, $\mathcal{B}_G = |\mathbb{B}(G)|$ and $\mathcal{C}_G = |\mathbb{C}(G)|$.

Proposition 8. Suppose that

$$\mathfrak{R}_2(G\otimes H) = \{\{(a,b)(c,d)\} \subseteq V(G\otimes H)|\,(a,b)\nsim(c,d)\ \&\ d_2((a,b),(c,d)) = 3\}.$$

Then $|\mathfrak{R}_2(G\otimes H)| = 2\left[\mathcal{C}_H\varphi(G) + \mathcal{C}_G\varphi(H)\right] + \mathcal{B}_H\varphi(G) + \mathcal{B}_G\varphi(H)$, where $\varphi(G) = \mathcal{A}_G + W_P(G)$ and $\varphi(H) = \mathcal{A}_H + W_P(H)$.

Proof. Let $\{(a,b)(c,d)\} \in \mathfrak{R}_2(G\otimes H)$, then we can conclude that $d_G(a,c) + d_H(b,d)$ is an odd natural number and

$$Min\{Max\{d_G(a,c), d'_H(b,d)\}, Max\{d'_G(a,c), d_H(b,d)\}\} = 3.$$

Then two cases occur as follows:

Case (1): $Max\{d_G(a,c), d'_H(b,d)\} = 3$. In this case, we can easily see that one of the following holds,

(i) $d_G(a,c) = 1$, $d'_H(b,d) = 3$;

(ii) $d_G(a,c) = 3$, $d'_H(b,d) = 3$.

Let us investigate each case as follows:

(i) In this case we have $d_G(a,c) = 1$ and $d'_H(b,d) = 3$. If $d_G(a,c) = 1$, then $Max\{d_G(a,c), d'_H(b,d)\} \leq Max\{d'_G(a,c), d_H(b,d)\}$ implies that the even number $d'_G(a,c) \neq 2$ and so $d'_G(a,c) \geq 4$. Also, $d'_H(b,d) = 3$ follows that $d_H(b,d) = 0$ or 2. Therefore we have two cases as follows:

If $d'_H(b,d) = 3$ and $d_H(b,d) = 0$, imply that $b = d$ and $b \in V(C_3)$ for some $C_3 \leq H$. If $d'_H(b,d) = 3$ and $d_H(b,d) = 2$, imply that $\{b,d\} \in \mathbb{C}(H)$. It follows that the number of all unordered vertex pairs $\{(a,\ b), (c,\ d)\} \subseteq V(G \otimes H)$ satisfying **(i)** is equal to $\mathcal{A}_G\mathcal{B}_H + 2\mathcal{A}_G\mathcal{C}_H$.

(ii) Let $d_G(a,c) = 3$, furthermore the hypothesize $d'_H(b,d) = 3$ implies that $d_H(b,d) = 0$ or 2. If $d_H(b,d) = 0$, then we can conclude that $b = d$ and $b \in V(C_3)$ for some $C_3 \leq H$. If $d_H(b,d) = 2$, then $\{b,d\} \in \mathbb{C}(H)$. It follows that the number of all unordered vertex pairs$\{(a,\ b), (c,\ d)\} \subseteq V(G \otimes H)$ satisfying **(ii)** is equal to $\mathcal{B}_H W_P(G) + 2\mathcal{C}_H W_P(G)$.

Consequently, the number of all unordered vertex pairs $\{(a,\ b), (c,\ d)\} \subseteq V(G \otimes H)$ satisfying in Case (1) is equal to $\mathcal{A}_G\mathcal{B}_H + 2\mathcal{A}_G\mathcal{C}_H + \mathcal{B}_H W_P(G) + 2\mathcal{C}_H W_P(G)$.

Case (2): $Max\{d'_G(a,c), d_H(b,d)\} = 3$. By a similar argument as Case (1), we can conclude that $|\mathfrak{R}_2(G \otimes H)| = 2[\mathcal{C}_H\varphi(G) + \mathcal{C}_G\varphi(H)] + \mathcal{B}_H\varphi(G) + \mathcal{B}_G\varphi(H)$. □

The following theorem is our main result which is a direct consequence of Propositions 6 and 8.

Theorem 9. Let G and H be two graphs at least one of them is non-bipartite. Then

$$\begin{aligned} W_P(G \otimes H) &= 2[(m_G + \mathcal{C}_H) W_P(G) + (m_H + \mathcal{C}_G) W_P(H) + \mathcal{A}_G\mathcal{C}_H + \mathcal{A}_H\mathcal{C}_G] \\ &+ \mathcal{B}_H(\mathcal{A}_G + W_P(G)) + \mathcal{B}_G(\mathcal{A}_H + W_P(H)), \end{aligned}$$

where m_G and m_H denote the number of edges of G and H, respectively.

Note. Let G be a simple connected graph. For any unordered vertex pair $\{u,v\} \in \mathbb{C}(G)$, either of the following holds:

i. The vertices u and v are two non-adjacent vertices of a cycle of length 5 of G.

ii. The vertices u and v are the vertices as depicted in Figure 3.

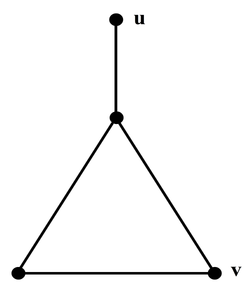

Figure 3: Example of part ii of Note up to isomorphism.

Therefore, if G is a triangle free graph, then one can see that $\mathcal{A}_G = |E(G)|$, $\mathcal{B}_G = 0$. Also, $\{u, v\} \in \mathbb{C}(G)$ if and only if u, v are vertices of a cycle of length 5 of the graph G. Therefore, $\mathcal{C}_G = 0$ whenever the graph G is C_5-free.

Corollary 10. Let G and H be two simple connected C_k-free graphs for $k \in \{3, 5\}$. Then

$$W_P(G \otimes H) = 2[m_H W_P(G) + m_G W_P(H)],$$

where m_G and m_H denote the number of edges of G and H, respectively.

A graph is called ***strongly triangular*** if every pair of its vertices has a common neighbor.

Corollary 11. Let G be a connected graph and H be a strongly triangular graph. Then

$$W_P(G \otimes H) = 2\left[\binom{n_H}{2} W_P(G) + \mathcal{A}_G(\binom{n_H}{2} - m_H)\right] + n_H(\mathcal{A}_G + W_P(G)),$$

where m_H and n_H denote the number of edges and the number of vertices of H, respectively.

Proof. Let H be strongly triangular graph. Then each edge of H belongs to a cycle of length 3, i.e. C_3. It follows that $\mathcal{A}_H = 0$, $\mathcal{B}_H = |V(G)|$. Let u and v be two arbitrary non-adjacent vertices of H. Since H is strongly triangular, u and v have a common neighbor say w. Let zv be common neighbor of vertices u and w. In this case uwv and $uzwv$ are paths of length 2 and 3, respectively. Therefore $C_G = \binom{n}{2} - m$. In the other hand, $W_P(H) = 0$ since every two arbitrary vertices of H are adjacent or of distance 2. Hence the proof is done by Theorem 9. $\square$

Let P denote the Petersen graph. Now we apply our main theorem to $P \otimes G$, where G is one of the following well-known graphs,

- K_n= The complete graph on n vertices;
- W_n = The wheel on n vertices;
- S_n = The star graph on n vertices;
- $K_{r,s}$ = The complete bipartite graph whose parts are of size r and s;
- Q_3 = The hyper cube graph on 8 vertices.

Our computations are summarized in the Table 1.

Table 1: $W_p(P \otimes -)$.

G	n	m	$D(G)$	$\mathcal{A}_G$	$\mathcal{B}_G$	$\mathcal{C}_G$	$W_P(G)$	$W_P(P \otimes G)$
P	10	15	2	15	0	30	0	1800
C_3	3	3	1	0	3	0	0	45
C_4	4	4	2	4	0	0	0	240
C_5	5	5	2	5	0	5	0	450
C_6	6	6	3	6	0	0	3	630
C_n , $n \geq 7$	n	n	$[n/2]$	n	0	0	n	$150n$
S_n	n	$n-1$	2	$n-1$	0	0	0	$60(n-1)$
K_n	n	$n(n-1)/2$	1	0	n	0	0	$15n$
$K_{r,s}$	$r+s$	rs	2	rs	0	0	0	$60rs$
W_n , $n \geq 5$	n	$2n-2$	2	0	n	$n-1$	0	$15(3n-2)$
Q_3	8	12	3	12	0	0	4	1080

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Diameter Two Graphs of Minimum Order with Given Degree Set

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Abstract

The *degree set* of a graph is the set of its degrees. Kapoor et al. [Degree sets for graphs, Fund. Math. 95 (1977) 189-194] proved that for every set of positive integers, there exists a graph of diameter at most two and radius one with that degree set. Furthermore, the minimum order of such a graph is determined. A graph is *2-self-centered* if its radius and diameter are two. In this paper for a given set of natural numbers greater than one, we determine the minimum order of a 2-self-centered graph with that degree set.

Keywords: Degree set, self-centered graph, radius, diameter.

2010 Mathematics Subject Classification: 05C07, 05C12.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For a vertex v of G, the *degree* of v in G, denoted by $deg_G(v)$. We denote the minimum and maximum degrees of the vertices of G by $\delta(G)$ and $\Delta(G)$, respectively. The *distance* between two vertices u and v of a connected graph G is denoted by $d_G(u,v)$ and it is the number of edges in a shortest path connecting u and v. The *eccentricity* $e_G(u)$ of a vertex u, of a connected graph G, is $max\{d_G(u,v)|v \in V(G)\}$. The radius of a connected graph G, *rad(G)*, is the minimum eccentricity among the vertices of G, while the diameter of G, *diam(G)*, is the maximum eccentricity. If *rad(G)=diam(G)=r*, then G is an *r-self-centered* graph. We use *r-sc* as a notation for r-self-centered graph. F. Buckley [2] worked on *r-sc* graphs, but the concept of r-sc graphs was developed independently by Akiyama, Ando, and Avis [1], who called them *r-equi* graphs. They proved that if G is an r-sc graph, then G is a block and $\Delta(G) \leq |V(G)| - 2(r-1)$.

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Hence, for $r = 2$ we have the following corollary.

Corollary 1.1. *If G is a 2-sc graph and v is a vertex of G, then* $2 \leq deg_G(v) \leq |V(G)| - 2$.

In this paper we study 2-sc graphs in terms of the degree sets, where for a given graph G the *degree set* of G, denoted by $D(G)$, is the set of degrees of the vertices of G. It is a simple observation that any set of positive integers forms the degree set of a graph. So it is natural to investigate the minimum order of such graphs. This question is completely answered by a result of Kapoor, Polimeni and Wall [4]. Their result can be stated as follows.

Theorem 1.2 (S. F. Kapoor et al. [4])**.** *For every set $S = \{a_1, ..., a_n\}$ of positive integers, with $a_1 < \cdots < a_n$, there exists a graph G such that $D(G) = S$ and furthermore,*

$$\mu(S) = a_n + 1,$$

where $\mu(S)$ represents the minimum order of such a graph G.

The graph G in Theorem 1.2 has order $a_n + 1$. Therefore G has diameter at most two and radius one. Hence G is not a 2-sc graph. Corollary 1.1 implies that every 2-sc graph has no vertex of degree less than or equal to 1. In this paper, we show that for a finite, nonempty set S of positive integers greater than 1, there exists a 2-sc graph G such that $D(G) = S$. Furthermore, the minimum order of such a graph G is determined.

2. Results

We write K_n and C_n for the *Complete* graph and the *Cycle* of order n, respectively. Also for a graph G, the graph $\bar{G}$ is the *Complement* of G. The *union* of graphs G and H is the graph $G \cup H$ which consists of copies of graphs G and H. Two graphs are *disjoint* if they have no vertex in common. If a graph G consists of $k(\geq 2)$ disjoint copies of a graph H, then we write $G = kH$.

Let S be a set of positive integers, where $S = \{a_1, ..., a_n\}$ and $1 < a_1 < \cdots < a_n$. We define $\mu_r(S)$ to be the minimum order of an r-sc graph G for which $D(G) = S$. In the case when $S = \{a_1\}$, the following theorem implies that there exists an a_1-regular 2-sc graph of minimum order.

Theorem 2.2. *Let a_1 be a positive integer greater than 1 and $S = \{a_1\}$. There exists a 2-sc graph G such that $D(G) = S$ and furthermore,*

$$\mu_2(S) = \begin{cases} a_1 + 2 & \text{if } a_1 \text{ is even} \\ a_1 + 3 & \text{if } a_1 \text{ is odd} \end{cases}.$$

Proof. Let a_1 be a positive integer greater than 1. If a_1 is even, then the graph

$$G = \overline{(\frac{a_1}{2} + 1)K_2},$$

is clearly an a_1-regular graph with a_1+2 vertices. The graph G is also a 2-sc graph [3]. Additionally, Corollary 1.1 implies that, every 2-sc graph of order a_1+1 has no vertex of degree a_1. Therefore, we need at least a_1+2 vertices to construct an a_1-regular 2-sc graph. Hence $\mu_2(S) = a_1 + 2$.

If a_1 is odd, then the graph

$$H = \overline{C}_{a_1+3},$$

is an a_1-regular graph of order $a_1 + 3$. The graph H is also a 2-sc graph. Since the graph H has order at least 6 and for each pair of nonadjacent vertices u and v of H there exists at least one common neighbour, it follows that $d_H(u, v) = 2$. Since, in any graph, the number of vertices of odd degree is even. Thus there is no a_1-regular graph of order $a_1 + 2$. Therefore, the graph H has the minimum order among all such 2-sc a_1-regular graphs. Hence $\mu_2(S) = a_1 + 3$. □

The following lemma which is obtained by Z. Stanic [5] has an interesting applications for constructing 2-sc graphs from other not necessarily 2-sc graphs and also it will be needed in the proof of our results for non-regular graphs. Recall that the *join* $G+H$ of two disjoint graphs G and H is the graph consisting of the union $G \cup H$, together with edges xy, where $x \in V(G)$ and $y \in V(H)$.

Lemma 2.3. (*Z. Stanic* [5]) *Let G and H be simple nontrivial graphs with $\Delta(G) \leq |V(G)| - 2$ and $\Delta(H) \leq |V(H)| - 2$, then $G + H$ is a 2-sc graph.*

Now, we extend Theorem 2.2 for non-regular graphs in following theorems.

Theorem 2.4. *Let a_1 be even and S be a set of positive integers, where $S = \{a_1, \ldots, a_n\}$, $2 \leq a_1 < \cdots < a_n$ and $n > 1$. Then there exists a 2-sc graph G such that $D(G) = S$ and furthermore,*

$$\mu_2(S) = a_n + 2.$$

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = S_1$. Consider the graph

$$G = (H \cup K_1) + F,$$

where $F = \overline{\frac{a_1}{2} K_2}$. The graph G has order a_n+2. We observe that for each vertices v of G, one of the following cases occurs:

1) If $v \in V(K_1)$, then $deg_G(v) = |V(F)| = a_1$.

2) If $v \in V(F)$, then $deg_G(v) = deg_F(v) + |V(K_1)| + |V(H)| = (a_1 - 2) + 1 + (a_n - a_1 + 1) = a_n$.

3) If $v \in V(H)$, then $deg_G(v) = deg_H(v) + |V(F)| = deg_H(v) + a_1$.

Thus $D(G) = S$. Moreover, by considering Lemma 2.3, G is a 2-sc graph and since there is no 2-sc graph of order $a_n + 1$, hence $\mu_2(S) = a_n + 2$. □

Theorem 2.5. *Let a_1 be odd and S be a set of positive integers, where $S = \{a_1, ..., a_n\}$, $3 \leq a_1 < \cdots < a_n$ and $n > 1$. Then there exists a 2-sc graph G of order $a_n + 3$ such that $D(G) = S$.*

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$, where for $1 \leq i \leq n$, $a_i \in S$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = \{a_2 - a_1, ..., a_n - a_1\}$. Consider the graph

$$G = (H \cup 2K_1) + F_1,$$

where $F_1 = \overline{C}_{a_1}$. The graph G has order $a_n + 3$. We observe that for each vertices v of G, one of the following cases occurs.
1) If $v \in V(2K_1)$, then $deg_G(v) = |V(F_1)| = a_1$.
2) If $v \in V(F_1)$, then $deg_G(v) = deg_{F_1}(v) + |V(2K_1)| + |V(H)| = a_n$.
3) If $v \in V(H)$, then $deg_G(v) = deg_H(v) + |V(F_1)| = deg_H(v) + a_1$.
Thus $D(G) = S$. Moreover, by Lemma 2.3, G is a 2-sc graph. □

Note that we considered $S = \{a_1, ..., a_n\}$ and presented a construction method in Theorem 2.4 to ascertain the value of $\mu_2(S)$, where a_1 is even, whereas if a_1 is odd, the graph G described in the proof of Theorem 2.5 has not necessarily the minimum order. As an example, for $S = \{3, 4\}$, the graph G_1 of Figure 1 which is obtained by the method of Theorem 2.5 has order 7, whereas the 2-sc graph G_2 where $G_2 = \overline{P}_6$ with 6 vertices has also the same degree set (see Figure 1).

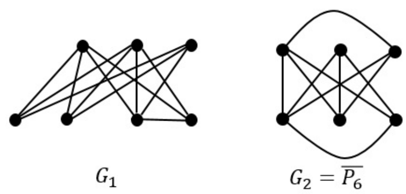

Figure 1: The 2-sc graphs G_1 and G_2 with different orders and the same degree sets.

In this section, we prove that if at least one element of S is even then $\mu_2(S) = a_n + 2$. We begin with a simple case.

Theorem 2.6. *Let S be a set of positive integers, where $= \{a_1, ..., a_n\}$, $n > 1$, $1 < a_1 < a_2 < \cdots < a_n$, a_1 is odd and $a_n = a_{n-1} + 1$ then $\mu_2(S) = a_n + 2$.*

Proof. Let $S_1 = \{a_2 - a_1, a_3 - a_1, ..., a_n - a_1\}$. By Theorem 1.2, there exists a graph H of order $a_n - a_1 + 1$ such that $D(H) = S_1$. Consider the graph

$$F = (H \cup K_1) + \overline{C}_{a_1}.$$

Lemma 2.3 implies that the graph F is a 2-sc graph. The graph F has order $a_n + 2$, and $D(F) = S$. Since there is no 2-sc graph of order $a_n + 1$, therefore, $\mu_2(S) = a_n + 2$. □

Now we consider the set $S = \{a_1, ..., a_n\}$ of positive integers. We prove that if all the elements of S are odd, then $\mu_2(S) = a_n + 3$, otherwise $\mu_2(S) = a_n + 2$. Before proving the main result, we need to have the following theorem.

Theorem 2.7. (*I. Zverovich* [6]) *Let S be a set of positive integers, where $S = \{a_1, ..., a_n\}$ and $3 \leq a_1 < \cdots < a_n$. Then there exists a Hamiltonian graph G such that $D(G) = S$ and $|V(G)| = a_n + 1$.*

Lemma 2.8. *For a graph G, if $\Delta(G) = |V(G)| - 2$ and G contains at least two non−adjacent vertices of degree $\Delta(G)$, then G is a 2-sc graph.*

Proof. Let x and y be two non−adjacent vertices of G with $deg_G(x) = deg_G(y) = \Delta(G) = |V(G)| - 2$. Obviously, x and y are adjacent to all other vertices of G. Therefore, $e_G(x) = e_G(y) = d_G(x, y) = 2$. Moreover, since $\Delta(G) = |V(G)| - 2$, it follows that for all other vertices v of G there is at least one non−adjacent vertex. Hence $e_G(v) = 2$. Therefore G is a 2-sc graph. □

Lemma 2.9. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$ and $2 \leq a_1 < \cdots < a_n$. Then there exists a graph G of order $a_n + 1$ such that $D(G) = S$ and G has a Hamilton path.*

Proof. Let $S' = \{a_1 + 1, ..., a_n + 1\}$. Since $a_1 + 1 \geq 3$, Theorem 2.7 implies that there exists a Hamiltonian graph G' of order $a_n + 2$ such that $D(G') = S'$. Let C' be a Hamilton cycle in G' where $C' = (v_1, v_2, ..., v_{a_n+2}, v_1)$. Without loss of generality, let v_1 be a vertex of degree $a_n + 1$ which is connected to all other vertices of G'. Let $G = G' - v_1$. Thus $D(G) = S$, $|V(G)| = a_n + 1$. Furthermore, by removing the vertex v_1 of C' we obtain the Hamilton path P where $P = (v_2, v_3, ..., v_{a_n+1}, v_{a_n+2})$. □

Lemma 2.10. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$, a_n be odd and $3 \leq a_1 < \cdots < a_n$. Then there exists a graph G of order $a_n + 1$ such that $D(G) = S$ and G has at least two adjacent vertices x and y of degree a_n. Moreover, G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy.*

Proof. Let $S' = \{a_1 - 1, ..., a_n - 1\}$. By Lemma 2.9, there is a graph G' of order a_n with $D(G') = S'$ and a Hamilton path P such that $P = (v_1, v_2, ..., v_{a_n})$ where $v_i \in V(G')$ for $1 \leq i \leq a_n$. Let x be a vertex of degree $a_n - 1$ of G'. We construct G by adding a new vertex y to G' adjacent to all vertices of G'. For $1 \leq i \leq a_n$ we have

$$deg_G(v_i) = deg_{G'}(v_i) + 1.$$

Clearly, we have two adjacent vertices x and y of degree a_n and also G is a graph of order $a_n + 1$ such that $D(G) = S$. We claim that G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy. Obviously, P is a path in G. Let M' be a maximal matching of P such that the vertex x is unsaturated. The size of matching M' is at least $\frac{a_n-3}{2}$. Let $M = M' \cup \{xy\}$. Clearly M is a matching of G such that $|M| = \frac{a_n-1}{2}$, which completes the proof. □

Now we prove our main theorem.

Theorem 2.11. *Let S be a set of positive integers where $S = \{a_1, ..., a_n\}$ and $1 < a_1 < \cdots < a_n$. If all elements of S are odd, then $\mu_2(S) = a_n + 3$, otherwise $\mu_2(S) = a_n + 2$.*

Proof. Consider the case when all elements of S are odd. Theorem 2.5 implies that there exists a 2-sc graph G of order a_n+3 such that $D(G) = S$. Moreover, as noted earlier, in any graph, there is an even number of odd vertices. Hence there is no graph of order $a_n + 2$ with S as its degree set. Therefore G is a 2-sc graph of minimum order such that $D(G) = S$, Hence $\mu_2(S) = a_n + 3$.

Now assume that at least one even element a_i exists in S where $1 \leq i \leq n$. If a_1 is even, then by Theorem 2.4, $\mu_2(S) = a_n+2$. Now let a_1 be odd. There exists at least one i where $2 \leq i \leq n$ such that a_i is even. Now we have two cases as follows:

First we consider the case in which a_n is even. Hence $|V(G)|$ is odd. Since $a_1 \geq 3$, Theorem 2.7 implies that there exists a Hamiltonian graph G of order a_n+1 such that $D(G) = S$. Let C be a Hamilton cycle in G such that $C = (v_1, v_2, ..., v_{a_n+1}, v_1)$ where $v_i \in V(G)$ for $1 \leq i \leq a_n + 1$. Without loss of generality, let v_1 be a vertex of degree a_n.

Let M be a matching of G where $M = \{v_2v_3, v_4v_5, ..., v_{a_n}v_{a_n+1}\}$ and the edge v_iv_{i+1} for $2 \leq i \leq a_n$ is an edge of the Hamilton cycle C (Notice that exactly one vertex v_1 of G is not saturated by M, hence $|M| = \frac{|V(G)|-1}{2}$). Let $G^* = G - M$. Clearly, for $2 \leq i \leq a_n+1$, we have $deg_{G^*}(v_i) = deg_G(v_i)-1$ and also $deg_{G^*}(v_1) = deg_G(v_1)$. Now, we construct a new graph H by adding a new vertex v adjacent to each vertex of G^* except v_1. Since $deg_H(v) = deg_H(v_1) = deg_G(v_1) = a_n$ and for $2 \leq i \leq a_n$, we have $deg_H(v_i) = deg_G(v_i)$, it follows immediately that $D(H) = D(G) = S$. Furthermore, since $|V(H)| = a_n + 2$ and H has at least two non$-$adjacent vertices v and v_1 of degree a_n, by Lemma 2.8, H is a 2-sc graph. Therefore, $\mu_2(S) = a_n + 2$.

Now we consider the case in which a_n is odd. Lemma 2.10 implies that there exists a graph G of order $a_n + 1$ such that $D(G) = S$. Furthermore, the graph G has at least two adjacent vertices x and y of degree a_n and also G has a matching of size $\frac{a_n-1}{2}$ which contains the edge xy . Let v_i be a vertex of degree a_i where a_i is even and $2 \leq i \leq n-1$. Consider the matching M of size $\frac{a_i}{2}$ of G which contains the edge xy. Let

$$G^* = G - M.$$

We construct H by adding a new vertex v to G^* such that

$$E(H) = E(G^*) \cup \{vu_i \mid u_i \textit{ is the vertex of } G \textit{ which is saturated by } M, \textit{ where } 1 \leq i \leq n\}.$$

Clearly, $D(H) = S$ and H has an order $a_n + 2$. Since H has at least two non$-$adjacent vertex x and y such that $deg_H(x) = deg_H(y) = a_n$, Lemma 2.8 implies that the graph H is a 2-sc graph and $\mu_2(S) = a_n + 2$. □

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Eigenfunction Expansions for Second-Order Boundary Value Problems with Separated Boundary Conditions

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Abstract

In this paper, we investigate some properties of eigenvalues and eigenfunctions of boundary value problems with separated boundary conditions. Also, we obtain formal series solutions for some partial differential equations associated with the second order differential equation, and study necessary and sufficient conditions for the negative and positive eigenvalues of the boundary value problem. Finally, by the sequence of orthogonal eigenfunctions, we provide the eigenfunction expansions for twice continuously differentiable functions.

Keywords: Boundary value problem, eigenvalue, eigenfunction, completeness, eigenfunction expansion.

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1. Introduction

In the present paper, we consider the boundary value problem L, defined by the following second-order differential equation of the Sturm-Liouville type

$$(k(x)y')' + (\lambda w(x) - g(x))y = 0, \qquad x \in [a, b], \tag{1}$$

with the separated boundary conditions

$$\cos\alpha \; y'(a, \lambda) - \sin\alpha \; y(a, \lambda) = 0, \tag{2}$$

$$\cos\beta \; y'(b, \lambda) - \sin\beta \; y(b, \lambda) = 0, \tag{3}$$

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which have many interesting applications in basic sciences, some branches of natural sciences and engineering. For example, in mathematical physics, the *one dimentional diffusion equation* is the form

$$\frac{\partial v}{\partial t} = \frac{\partial}{\partial x}(k(x)\frac{\partial v}{\partial x}) - g(x)v, \tag{4}$$

where $a \le x \le b$, $0 \le t < \infty$, and $k(x), g(x)$ are real-valued functions. By the separation of variables technique, we can write the solution $v(x,t)$ in the form

$$v(x,t) = e^{-\lambda t} y(x). \tag{5}$$

In this case, the function y is seen to be a solution of the differential equation (1) with $w(x) = 1$. For other examples, we refer to [2, 4, 17].

Studying the properties of spectrum of the boundary value problem L is an interesting subject for many authors. For example, in [6, 11], the authors considered regular Sturm-Liouville problems with $k = w \equiv 1$ and an integrable potential g, or with locally integrable functions k^{-1}, g, w on finite intervals, and obtained asymptotic approximations of eigenvalues and eigenfunctions. In [8, 18, 19], inverse spectral problems with non-separated boundary conditions with an integrable function $g(x)$ were investigated. Also, inverse problems for singular differential operators on finite intervals with $k = w \equiv 1$ were studied in [7, 13, 15]. Boundary value problems consisting of (1) with $g \equiv 0$, together with Dirichlet boundary condition or irregular decomposing boundary conditions were investigated in [9, 10], and eigenfunction expansions and their uniformly convergent were studied. In [1], the authors considered (1) in the case when the equation has a singularity and discontinuity inside the interval $[0, T]$, and investigated the properties of the spectrum and its associated inverse problem. Also, in [3, 16], the authors considered the problems consisting of (1)-(2) with $k = w \equiv 1$ or with discontinuous weight function $w(x)$ on symmetric intervals under discontinuity conditions in y and y', together with the boundary condition (3) or an spectral parameter dependent boundary condition at the right endpoint, respectively. They investigated the properties of eigenvalues and eigenfunctions, and obtained asymptotic approximation formulas for fundamental solutions. Finally, in [14], a new type of boundary value problem consisting of a second order differential equation with an abstract operator in a Hilbert space on two disjoint intervals together with eigenparameter dependent boundary conditions and with transmission conditions was investigated.

In the present article, in section 2, we study some properties of eigenvalues and eigenfunctions of the boundary value problem L consisting of the equation (1) together with the boundary conditions (2)-(3), in special cases. Here, λ is the eigenvalue parameter, $k(x)$, $w(x)$ and $g(x)$ are real continuous functions on $[a, b]$, and $k(x), w(x)$ are positive. We will provide the formal series solutions for the function $v(x,t)$ of the diffusion equation (4). Then, we study necessary and sufficient conditions for negative and positive eigenvalues of L, and prove that each twice continuously differentiable function can be expanded in terms of the eigenfunctions of the problem, under a sufficient condition (see section 3).

2. The Eigenfunctions and Formal Series Solutions

In this section, first, we consider some special forms of (1)-(3), and find the eigenfunctions and the formal series solutions in several examples.

Example 2.1. Let $a = 0$, $b = \pi$, $k(x) = w(x) = 1$, $g(x) = 0$, $\alpha = \frac{\pi}{2}$ and $\beta = -\frac{\pi}{4}$. Then, we can rewrite the problem L as follows:

$$y'' + \lambda y = 0, \qquad 0 \le x \le \pi, \tag{6}$$
$$y(0) = 0, \qquad y'(\pi) + y(\pi) = 0. \tag{7}$$

The differential equation (6) has two linearly independent solutions

$$y_1(x) = \cos(\sqrt{\lambda}x), \qquad y_2(x) = \sin(\sqrt{\lambda}x).$$

Thus, it follows from the first condition of (7) that the solution $y(x, \lambda)$ of (6)-(7) is the form

$$y(x, \lambda) = c\sin(\sqrt{\lambda}x),$$

where c is constant. Therefore, according to the secondary condition of (7), we get

$$\sqrt{\lambda} + \tan(\sqrt{\lambda}\pi) = 0. \tag{8}$$

Example 2.2. Let $a = 0$, $k(x) = w(x) = 1$, $g(x) = 0$, $\alpha = 0$ and $\beta = \frac{\pi}{2}$. In this case, the problem L be transformed to

$$y'' + \lambda y = 0, \qquad 0 \le x \le b,$$
$$y'(0) = 0, \qquad y(b) = 0.$$

Therefore,

$$y(x, \lambda) = c\cos(\sqrt{\lambda}x).$$

Moreover, L has a countable set of the eigenvalues

$$\lambda_n = (\frac{(2n+1)\pi}{2b})^2, \qquad n = 1, 2, 3, ...,$$

and so, their corresponding eigenfunctions are

$$y_n(x) = \cos(\frac{(2n+1)\pi x}{2b}), \qquad n = 1, 2, 3, ...$$

Definition 2.3. We define the function $< ., . >_w : C[a, b] \to C[a, b]$ by

$$< y, z >_w = \int_a^b y(t)\overline{z(t)}dt$$

for all complex functions $y, z \in C[a, b]$. Here, $\overline{z(t)}$ is the conjugate of $z(t)$.

The function $< .,. >_w$ defines an *inner product* on $C[a,b]$.

Theorem 2.4. Suppose that λ_1 and λ_2 are distinct eigenvalues of the boundary value problem L with corresponding eigenfunctions y_1 and y_2, respectively. Then, y_1 and y_2 are orthogonal with respect to the weight function $w(x)$.

Proof. Since y_i is the corresponding eigenfunction of λ_i, we have

$$(k(x)y_i')' - g(x)y_i = -\lambda_i w(x)y_i, \qquad i = 1,2. \tag{9}$$

For $i = 1$, multiply (9) by y_2 and integrate from a to b with respect to x, then

$$\int_a^b y_2(k(x)y_1')'dx - \int_a^b y_1y_2g(x)dx = -\lambda_1 \int_a^b y_1y_2w(x)dx.$$

Integrate by parts yields

$$y_2y_1'k(x)|_a^b - \int_a^b y_1'y_2'k(x)dx - \int_a^b y_1y_2g(x)dx = -\lambda_1 \int_a^b y_1y_2w(x)dx. \tag{10}$$

Similarly, for $i = 2$, multiply (9) by y_1 and integrate from a to b, we get

$$y_1y_2'k(x)|_a^b - \int_a^b y_1'y_2'k(x)dx - \int_a^b y_1y_2g(x)dx = -\lambda_2 \int_a^b y_1y_2w(x)dx. \tag{11}$$

Subtracting the two equations (10) and (11), we obtain

$$y_1y_2'k(x)|_a^b - y_2y_1'k(x)|_a^b = (\lambda_1 - \lambda_2)\int_a^b y_1y_2w(x)dx. \tag{12}$$

Now, let $\alpha, \beta = k\pi + \pi/2$, $k \in Z$, then it follows from (2)-(3) that

$$y_i(a) = 0 = y_i(b), \qquad i = 1,2. \tag{13}$$

Applying (12)-(13) we conclude that

$$(\lambda_1 - \lambda_2)\int_a^b y_1y_2w(x)dx = 0. \tag{14}$$

Similarly, let $\alpha, \beta \neq k\pi + \pi/2$, $k \in Z$, then we have

$$y_i'(a) = \tan\alpha\ y_i(a), \qquad y_i'(b) = \tan\beta\ y_i(b), \qquad i = 1,2. \tag{15}$$

Substituting (15) into (12), we arrive at (14). Also, in the case when $\alpha = k\pi + \pi/2$ and $\beta \neq k\pi + \pi/2$, (14) is valid. Hence, in general, (14) together with $\lambda_1 \neq \lambda_2$ yields

$$\int_a^b y_1y_2w(x)dx = 0,$$

and our proof is complete. □

Example 2.5. Consider the equation (1) with $k(x) = w(x) = 1$, $g(x) = 0$, $a = 0$, $\alpha, \beta = k\pi + \pi/2$, $k \in Z$. In this case, applying the method which used for finding the eigenvalues in Example 2.1, we obtain the eigenvalues and the corresponding eigenfunctions as follows

$$\lambda_n = \frac{n^2\pi^2}{b^2}, \qquad y_n(x) = \sin(\frac{n\pi x}{b}), \qquad n = 1, 2, 3, ...$$

Therefore, we deduce that

$$< y_m, y_n >_w = \int_a^b \sin(\frac{m\pi x}{b}) \sin(\frac{n\pi x}{b}) dx = 0, \qquad m \neq n.$$

Hence, according to (5), we may construct a formal series solution for $v(x,t)$ by superposition

$$v(x,t) = \sum_{n=1}^{\infty} k_n e^{-\lambda_n t} \sin(\frac{n\pi x}{b}). \tag{16}$$

For finding the Fourier coefficients k_n, substituting $t = 0$ and multiply both sides of (16) by $\sin(\frac{m\pi x}{b})$, and integrate from 0 to b give us

$$\int_0^b v(x,0) \sin(\frac{m\pi x}{b}) dx = \int_0^b \sum_{n=1}^{\infty} k_n \sin(\frac{m\pi x}{b}) \sin(\frac{n\pi x}{b}) dx.$$

Assuming uniform convergence and by orthogonality we obtain (with $m = n$)

$$\int_0^b v(x,0) \sin(\frac{n\pi x}{b}) dx = \int_0^b \sum_{n=1}^{\infty} k_n \sin^2(\frac{n\pi x}{b}) dx,$$

therefore, we get

$$k_n = \frac{2}{b} \int_0^b v(x,0) \sin(\frac{n\pi x}{b}) dx.$$

Hence, for $M = \frac{2}{b} \int_0^b |v(x,0)| dx$,

$$|k_n| \leq M, \qquad n = 1, 2, 3, ...$$

Moreover,

$$|k_n e^{-\lambda_n t} \sin(\frac{n\pi x}{b})| \leq M e^{-(\frac{n\pi x}{b})^2 t} \to 0$$

as $t \to \infty$. This together with (16) yields

$$v(x,t) \to 0 \quad as\ t \to \infty,$$

by uniform convergence.

Example 2.6. Let $k(x) = w(x) = x$, $g(x) = \frac{m^2}{x}$, $m \geq 0$, $a = 0$, $\alpha, \beta = k\pi + \pi/2$, $k \in Z$. Then,

$$\lambda_n = \frac{j^2_{m,n}}{b^2}, \qquad y_n(x) = J_m(\frac{j_{m,n}}{b}x), \qquad n = 1, 2, 3, ...,$$

where $j_{m,n}$ is the n^{th} positive zero of the Bessel function J_m. Moreover,

$$< y_m, y_n >_w = \int_0^b J_m(\frac{j_{m,n}}{b}x) J_m(\frac{j_{m,s}}{b}x) dx = 0, \qquad n \neq s.$$

By (5) and the method applied in Example 2.5, we can write

$$v(x,t) = \sum_{n=1}^{\infty} r_n e^{-\lambda_n t} J_m(\frac{j_{m,n}}{b}x) dx,$$

where the coefficients r_n are obtain as follows

$$r_n = \frac{\int_0^b v(x,0) J_m(\frac{j_{m,n}}{b}x) dx}{\int_0^b \{J_m(\frac{j_{m,n}}{b}x)\}^2 dx}.$$

Theorem 2.7. The eigenvalues of the boundary value problem L, defined by (1)-(3), are real.

Proof. It follows from the relation (14) that for each eigenvalues λ_1, λ_2 of L,

$$(\lambda_1 - \overline{\lambda_2}) \int_a^b y_1 \overline{y_2} w(x) dx = 0. \tag{17}$$

Now, we choose $y_2 = y_1$. This together with (17) yields

$$(\lambda_1 - \overline{\lambda_1}) \int_a^b |y_1|^2 w(x) dx = 0.$$

Since $|y_1|^2 w(x) > 0$, we obtain $\lambda_1 = \overline{\lambda_1}$. Consequently, the eigenvalue λ_1 is real. Since λ_1 was an arbitrary eigenvalue of L, the proof is complete. □

3. The Eigenfunction Expansion

In this section, necessary and sufficient conditions for the negative and positive eigenvalues of the boundary value problem L are obtained. Also, we discuss completeness of the eigenfunctions of L, and prove that each function can be expanded in terms of the eigenfunctions of L, under a sufficient condition.

Lemma 3.1. Suppose that λ is an eigenvalue of L with corresponding eigenfunction y. Then,

$$\lambda = \frac{\int_a^b \{k(x)(y')^2 + g(x)y^2\}dx - (k(x)yy')|_a^b}{\int_a^b y^2(x)w(x)dx}. \tag{18}$$

Proof. First, multiply (1) by y and integrate the result with respect to x on the interval $[a, b]$, we obtain

$$\int_a^b (k(x)y')'y - \int_a^b g(x)y^2)dx = -\lambda \int_a^b y^2 w(x)dx. \tag{19}$$

Now, after an integration by parts from (19), we arrive at (18). □

According to Lemma 3.1, we have the following corollary.

Corollary 3.2. *Let* $\alpha, \beta = k\pi + \pi/2$, $k \in Z$. *If* $\tan\alpha \geq 0$, $\tan\beta \leq 0$, *and* $g(x) \geq 0$ *for every* $x \in [a, b]$, *then the eigenvalue* λ *of* L *is always positive.*

Remark 1. We note that all eigenvalues of L are simple, because otherwise, let y_1 and y_2 be the linearly independent eigenfunctions correspond to the eigenvalue λ. Thus, y_1 and y_2 satisfy the boundary conditions (2)-(3). Moreover, we can write every solution $y(x, \lambda)$ of (1) corresponding to λ in the form

$$y(x, \lambda) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1, c_2 are arbitrary constants, and $y(x, \lambda)$ must be satisfied in the boundary conditions (2)-(3). On the other hand, we know that the problem consisting of (1) together with arbitrary initial conditions that be incompatible with (2)-(3), has a unique solution. Hence, these give us a contradiction.

The following assertion can be proved like Theorem 4.8, p. 157 of [12].

Lemma 3.3. If $H(x, t)$ is a continuous, real-valued and symmetric function, $f : [a, b] \to R$ is a continuous function defined by

$$f(x) = \int_a^b H(x, t)r(t)dt, \qquad x \in [a, b]$$

for some continuous $r : [a, b] \to R$, then f may be expanded in the uniformly convergent series $f = \sum_{n=1}^{\infty} \alpha_n y_n$ on $[a, b]$, where $\{y_n\}$ is the sequence of the the eigenfunctions of L, and

$$\alpha_n = \frac{\int_a^b f(x)y_n(x)dx}{\int_a^b y_n^2(x)dx}, \qquad n \geq 1.$$

The Lemma 3.3 plays an important role for proving the following theorem which is the main result of this section.

Theorem 3.4. If $h : [a,b] \to R$ is an arbitrary twice continuously differentiable function satisfy (2)-(3), then h can be expanded in the uniformly convergent series $h = \sum_{n=1}^{\infty} \beta_n y_n$ on $[a,b]$, where

$$\beta_n = \frac{\int_a^b h(x) y_n(x) w(x) dx}{\int_a^b y_n^2(x) w(x) dx}, \qquad n \geq 1. \tag{20}$$

Proof. First, since h is twice continuously differentiable, there exists a continuous function $p : [a,b] \to R$ such that

$$\ell(h(x)) = -p(x)\sqrt{k(x)},$$

where

$$\ell h := \frac{d}{dx}(k(x)\frac{dh}{dx}) - g(x)h.$$

Second, h satisfies (2)-(3), thus by the method of variation of parameters, $h(x)$ can be written as

$$h(x) = -\int_a^b G(x,t) p(t) \sqrt{w(t)} dt, \qquad x \in [a,b],$$

where

$$G(x,t) = \begin{cases} \frac{u(x)v(t)}{k(t)W(u,v)(t)}, & a \leq x \leq t \leq b, \\ \frac{u(t)v(x)}{k(t)W(u,v)(t)}, & a \leq t \leq x \leq b, \end{cases}$$

where u and v are two arbitrary linearly independent solutions of (1), and $W(u,v)(x)$ is the Wronskian of u and v. Therefore,

$$h(x)\sqrt{w(x)} = -\int_a^b G(x,t) p(t) \sqrt{w(x)w(t)} dt, \qquad x \in [a,b]. \tag{21}$$

Since,

$$\begin{aligned} \frac{d}{dx}\{k(x)W(u,v)(x)\} &= u(x)(k(x)v'(x))' - v(x)(k(x)u'(x))' \\ &= 0, \end{aligned}$$

thus, $k(x)W(u,v)(x)$ is constant. Hence, $G(x,t)\sqrt{w(x)w(t)}$ is continuous and symmetric. Therefore, by the relation (21) and Lemma 3.3 we derive the following uniformly convergent expansion

$$h\sqrt{w} = \sum_{n=1}^{\infty} \beta_n y_n \sqrt{w},$$

where β_n is of the form (20). This completes the proof of Theorem 3.4. □

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ABSTRACTS

IN

PERSIAN

Motion of Particles under Pseudo-Deformation

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حرکت ذرات تحت شبه-تغییرشکل

چکیده

در این مقاله‌ی کوتاه مشاهده می‌کنیم که مسیر ذرات با جرم m که در راستای مسیر $r = r(t)$ در حالت حرکت هستند تحت شبه نیروی $\mathbf{A}(t)$ که t بیانگر زمان است، به‌صورت $\mathbf{r}_d = \int(\frac{d\mathbf{r}}{dt}\mathbf{A}(t))dt + \mathbf{c}$ می‌باشد. همچنین خواهیم دید که نیروی مؤثر $\mathbf{F}_e$ روی ذره‌ای که از شبه‌نیروی $\mathbf{A}(t)$ ناشی می‌شود به‌صورت $\mathbf{F}_e = \mathbf{F}\mathbf{A}(t) + \mathbf{L}d\mathbf{A}(t)/dt$ است که در آن $\mathbf{F} = m\ d^۲\mathbf{r}/dt$. ما در مورد خطوط جریان تحت شبه‌نیرو نیز بحث کرده‌ایم.

کلمات کلیدی: حلقه‌های راست، مقاطع عرضی راست، مقاطع عرضی چرخنده.

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C-Class Functions and Remarks on Fixed Points of Weakly Compatible Mappings in G-Metric Spaces Satisfying Common Limit Range Property

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توابع C-کلاس و ملاحظاتی روی نقاط ثابتِ نگاشت‌های به‌طورضعیف سازگار در فضاهای G-متریک که در ویژگیِ برد محدود مشترک صدق می‌کنند

چکیده

در این مقاله با استفاده از مبانی توابع C-کلاس و ویژگی برد حدی مشترک، قضیه‌ی نقطه ثابت مشترک را برای بعضی از عملگرها به دست می‌آوریم. نتایج ما چندین نتیجه‌ی موجود در ادبیات کنونی را تعمیم می‌دهند. در پایان، چند مثال برای نشان‌دادن قابلیت استفاده از رویکردمان ارائه می‌شود.

کلمات کلیدی: فضای متریک تعمیم یافته، نقطه ثابت مشترک، به‌طورضعیف G-انقباضی تعمیم یافته، توابع به‌طورضعیف سازگار، ویژگی (CLR_{ST}) مشترک، توابع C-کلاس.

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Unconditionally Stable Difference Scheme for the Numerical Solution of Nonlinear Rosenau-KdV Equation

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طرح تفاضل پایدار بی‌قید و شرط برای حل عددی معادله رسناو-KdV غیرخطی

چکیده

در این مقاله یک مدل تکامل غیرخطی که به وسیله معادله رسناو-KdV توصیف شده است را بررسی می‌کنیم. ما برای حل عددی آن، طرح تفاضل متناهی ضمنی میانگین سه-مرحله‌ای را پیشنهاد می‌کنیم و ثابت می‌کنیم که این طرح پایدار و همگرا از مرتبه‌ی $O(\tau^۲ + h^۲)$ است. به علاوه وجود و یکتایی جواب‌های عددی را اثبات می‌کنیم. سپس با مقایسه‌ی نتایج عددی با روش‌های دیگر نشان می‌دهیم که روش پیشنهادی ما، کارآمد و دارای دقت بالایی است.

کلمات کلیدی: طرح تفاضل متناهی، حل‌پذیری، پایداری بی‌قید و شرط، همگرایی.

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ادیتور رابط: عباس سعادتمندی

Wiener Polarity Index of Tensor Product of Graphs

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اندیس قطبی وینر ضرب تانسوری گراف‌ها

چکیده

ریاضی شیمی یکی از شاخه‌های شیمی نظری است که به بحث و پیش‌بینی ساختار مولکولی به کمک روش‌های ریاضی بدون التزام به استفاده از مکانیک کوانتوم می‌پردازد. در شیمی نظری، از توصیف‌گرهای ساختار مولکولی فاصله-محور برای مدل‌سازی خواص فیزیکی، دارویی، زیستی و دیگر ویژگی‌های ترکیبات شیمیایی استفاده می‌شود. اندیس قطبی وینر گراف G، که با نماد $W_P(G)$ نمایش داده می‌شود، عدد جفت‌های نامرتب رئوس با فاصله‌ی ۳ است. اندیس قطبی وینر برای نشان دادن رابطه‌های ویژگی ساختاری کمی در یک سری از هیدروکربن‌های فاقد دور و شامل دور، مورد استفاده قرار می‌گیرد. فرض کنید G و H دو گراف ساده‌ی همبند باشند، در این صورت ضرب تانسوری آن‌ها با نماد $G \otimes H$ نمایش داده می‌شود و مجموعه رئوس آن برابر با $V(G \otimes H) = V(G) \times V(H)$، و مجموعه یال‌های آن برابر

$$E(G \otimes H) = \{(a,b)(c,d) | ac \in E(G), bd \in E(H)\}.$$

است در این مقاله تلاش می‌کنیم تا اندیس قطبی وینر $G \otimes H$ را به‌دست آوریم که در مرجع

J. Ma, Y. Shi and J. Yue, The Wiener polarity index of graph products, Ars Combin. 116 (2014) 235-244

به اشتباه محاسبه شده است.

کلمات کلیدی: اندیس توپولوژیکی، اندیس قطبی وینر، ضرب تانسوری، گراف، فاصله.

رده بندی موضوعی انجمن ریاضی امریکا: 05C20.

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Diameter Two Graphs of Minimum Order with Given Degree Set

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گراف‌هایی قطر دو از کمترین مرتبه با مجموعه‌ی درجات داده شده

چکیده

مجموعه درجات یک گراف برابر مجموعه درجات رئوس آن می باشد. کاپور و همکارانش در مرجع

[Degree sets for graphs, Fund. Math. 95 (1977) 189-194]

ثابت کردند که برای هر مجموعه از اعداد صحیح مثبت، گرافی از قطر حداکثر دو و شعاع یک وجود دارد که مجموعه درجات آن برابر با مجموعه اعداد صحیح داده شده است. بعلاوه آن‌ها کمترین مرتبه چنین گراف‌هایی را تعیین کردند. یک گراف ۲- خودمرکز است هر گاه شعاع و قطرش برابر دو باشد. در این مقاله برای یک مجموعه داده شده از اعداد طبیعی بزرگتر از یک، کمترین مرتبه یک گراف ۲-خودمرکز را تعیین می‌کنیم که مجموعه درجات این گراف برابر مجموعه اعداد طبیعی داده شده باشد.

کلمات کلیدی: مجموعه درجات، گراف خودمرکز، شعاع، قطر.

رده بندی موضوعی انجمن ریاضی امریکا: 05C07، 05C12.

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Eigenfunction Expansions for Second-Order Boundary Value Problems with Separated Boundary Conditions

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بسط‌های تابع ویژه مسائل مقدار مرزی مرتبه دوم دارای شرایط مرزی تفکیک‌پذیر

چکیده

در این مقاله، ما برخی ویژگی‌های مقادیر ویژه و توابع ویژه مسائل مقدار مرزی دارای شرایط مرزی تفکیک‌پذیر را بررسی می‌کنیم. همچنین، جواب‌های به شکل سری متعارفی را برای معادلات دیفرانسیل جزئی مربوط به معادله دیفرانسیل مرتبه دوم به دست می‌آوریم، و شرایط لازم و کافی برای وجود مقادیر ویژه مثبت و منفی مساله مقدار مرزی را مطالعه می‌کنیم. در پایان، با استفاده از دنباله توابع ویژه متعامد، بسط‌های تابع ویژه مربوط به توابع دوبار به‌طور پیوسته مشتق‌پذیر را ارائه می‌کنیم.

کلمات کلیدی: مساله مقدار مرزی، مقدار ویژه، تابع ویژه، کامل بودن، بسط تابع ویژه.

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