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Note

On the Finite Groups that all Their Semi-Cayley Graphs are Quasi-Abelian

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Abstract

In this paper, we prove that every semi-Cayley graph over a group G is quasi-abelian if and only if G is abelian.

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1. Introduction

Let G be a finite group and S be an inverse-closed subset of G not containing the identity element of G. Then $\Gamma = \mathrm{Cay}(G,S)$, the Cayley graph of G with respect to the Cayley set S, is a graph with vertex set G and edge set $\{\{g,sg\} \mid g \in G, s \in S\}$. It is easy to see that $R(G) = \{r_g : G \to G \mid g \in G\}$, where $x^{r_g} = xg$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$. Let $L(G) = \{l_g : G \to G \mid g \in G\}$, where $x^{l_g} = gx$. If G is abelian then $L(G) = R(G)$ is a subgroup of $\mathrm{Aut}(\Gamma)$. For an arbitrary group G, $\mathrm{Cay}(G,S)$ is called *quasi-abelian*, according to [8], whenever $L(G) \leq \mathrm{Aut}(\Gamma)$. Quasi-abelian Cayley graphs have been considered in various contexts since 1979. For a survey up to 2002, see [10]. It is easy to see that the function $g \mapsto g^{-1}$, from

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G to G, is an automorphism of $\mathrm{Cay}(G,S)$ if and only if $\mathrm{Cay}(G,S)$ is quasi-abelian [9, Lemma 2]. In 2010, Goldstone and Weld defined the concept of *graphically abelian groups*; a group G is graphically abelian if the function $g \mapsto g^{-1}$ induces an automorphism of every Cayley graph of G. They proved that a finite group G is a graphically abelian group if and only if G is abelian or a direct product of an elementary abelian 2-group and the quaternion group Q_8 of order 8.

By a theorem of Sabidussi, a graph Γ is a Cayley graph of G if and only if $\mathrm{Aut}(\Gamma)$ contains a regular subgroup isomorphic to G [7]. Since every regular group is a semi-regular group with only one orbit, Resmini and Jungnickel [6], in analogous to Sabidussi's theorem, introduced graphs that their automorphism group contains a semi-regular subgroup with two orbits (of equal size) and called them *semi-Cayley graphs*. Recently, semi-Cayley graphs have been considered in various contexts, see for example [2, 3, 4, 11, 12]. Also, very recently, in analogous to Cayley graphs, *quasi-abelian semi-Cayley graphs* defined and considered in [1]. In this paper, we prove the following theorem:

Theorem A Let G be a finite group. Then all of semi-Cayley graphs over G are quasi-abelian if and only if G is abelian.

2. Proof of Theorem A

Let Γ be a Cayley graph of G. Then $R(G) = \{r_g : G \to G \mid g \in G\}$, where $x^{r_g} = xg$ is a regular subgroup of $\mathrm{Aut}(\Gamma)$ isomorphic to G. Recall that $\mathrm{Cay}(G,S)$ is called *quasi-abelian* whenever $L(G) = \{l_g : G \to G \mid g \in G\}$, where $x^{l_g} = gx$ is a subgroup of $\mathrm{Aut}(\Gamma)$. Also, by [9, Lemma 2], $Cay(G,S)$ is quasi-abelian if and only if $\xi : G \to G$, by the rule $g^{\xi} = g^{-1}$ is an automorphism of $\mathrm{Cay}(G,S)$. Hence, a group G is graphically abelian if and only if for every Cayley graph of G, $L(G)$ is a subgroup of its automorphism group.

Recall that an undirected graph Γ is called a *semi-Cayley graph* over G if $\mathrm{Aut}(\Gamma)$ contains a semi-regular subgroup isomorphic to G with two orbits. Let G be a group and R, L, S be subsets of G such that $R = R^{-1}$, $L = L^{-1}$ and $1 \notin R \cup L$. Consider the undirected graph $SC(G; R, L, S)$ with vertex set $G \times \{1,2\}$ and edge set $\{\{(g,1),(rg,1)\} \mid g \in G, r \in R\} \cup \{\{(g,2),(lg,2)\} \mid g \in G, l \in L\} \cup \{\{(g,1),(sg,2)\} \mid g \in G, s \in S\}$. Then $R_G = \{\rho_g : G \times \{1,2\} \to G \times \{1,2\} \mid g \in G\}$, where $(x,i)^{\rho_g} = (xg,i)$, is a semi-regular subgroup of $\mathrm{SC}(G; R, L, S)$ isomorphic to G with orbits $G \times \{1\}$ and $G \times \{2\}$. It is easy to see that an undirected graph Γ is a semi-Cayley graph over a group G if and only if there exists subsets R, L, S of G with $R = R^{-1}$, $L = L^{-1}$ and $1 \notin R \cup L$ such that $\Gamma = \mathrm{SC}(G; R, L, S)$, see [6, Lemma 2.1] or [2, Lemma 2].

We say that $\Gamma = \mathrm{SC}(G; R, L, S)$ is *quasi-abelian semi-Cayley graph* if $L_G = \{\psi_g : G \times \{1,2\} \to G \times \{1,2\} \mid g \in G\}$, where $(x,i)^{\psi_g} = (gx,i)$ is a subgroup of $\mathrm{Aut}(\Gamma)$. Also we say that a group G is *semi-graphically abelian* if L_G is a subgroup of every semi-Cayley graph over G. In the following lemma, we prove that the set of all graphically abelian groups contain the set of all semi-graphically abelian groups.

Lemma 1. *Every semi-graphically abelian group is a graphically abelian group.*

Proof. Let G be a semi-graphically abelian group, $R = R^{-1}$ be a subset of $G \setminus \{1\}$ and $\Sigma = \mathrm{Cay}(G, R)$. We shall prove that $L(G) \leq \mathrm{Aut}(\Sigma)$. Let $\Gamma = \mathrm{SC}(G; R, \emptyset, \{1\})$. Then the map $\varphi : \mathrm{Aut}(\Sigma) \to \mathrm{Aut}(\Gamma)$ given by $\varphi(\sigma) = \sigma'$, where $\sigma' : V(\Gamma) \to V(\Gamma)$ is the map by the rule $(g, i)^{\sigma'} = (g^{\sigma}, i)$, is a group isomorphism and $L(G)^{\psi} = L_G$. Since G is a semi-graphically abelian group, $L_G \leq \mathrm{Aut}(\Gamma)$. Hence $L(G) = (L_G)^{\varphi^{-1}}$ is a subgroup of $\mathrm{Aut}(\Sigma)$, which means that Σ is a quasi-abelian Cayley graph. This proves that G is a graphically abelian group. □

Lemma 2. *Let $G \cong Z_2^r \times Q_8$, $r \geq 0$. Then G is not a semi-graphically abelian group.*

Proof. Let $Q_8 = \langle a, b \mid a^4 = b^4 = 1, a^2 = b^2, bab^{-1} = a^{-1} \rangle$ and $S = \{a\}$. Let $\Gamma = \mathrm{SC}(G; R, L, S)$ for some subsets $R = R^{-1}$ and $L = L^{-1}$ of G. Suppose, towards a contradiction, that G is semi-graphically abelian. Then $L_G \leq \mathrm{Aut}(\Gamma)$. Since $\{(1, 1), (a, 2)\} \in E(\Gamma)$, applying ψ_b, we have $\{(b, 1), (ba, 2)\} \in E(\Gamma)$. This implies that $bab^{-1} \in S$. Hence $a^{-1} = a$, a contradiction. □

Now we are now ready to prove Theorem A.

Proof of Theorem A If G is abelian then $L_G = R_G$ is a subgroup of the automorphism group of any semi-Cayley graph over G. This proves one direction. Conversely, suppose that G is a semi-graphically abelian group. Then, by Lemma 1, G is graphically abelian. Hence, by [5, Theorem 16], G is abelian or $G \cong \mathbb{Z}_2^r \times Q_8$ for some $r \geq 0$. On the other hand, by Lemma 2, the later is impossible. Hence G is abelian, which completes the proof.

Conflicts of Interest. The author declares that there is no conflicts of interest regarding the publication of this article.

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