

# Numerical Calculation of Fractional Derivatives for the Sinc Functions via Legendre Polynomials

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## Abstract

This paper provides the fractional derivatives of the Caputo type for the sinc functions. It allows to use efficient numerical method for solving fractional differential equations. At first, some properties of the sinc functions and Legendre polynomials required for our subsequent development are given. Then we use the Legendre polynomials to approximate the fractional derivatives of sinc functions. Some numerical examples are introduced to demonstrate the reliability and effectiveness of the introduced method.

**Keywords:** Sinc functions, Fractional derivatives, Collocation method, Caputo derivative Shifted Legendre polynomials.

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## 1. Introduction

Fractional derivatives arise in many physical and engineering problems such as electroanalytical chemistry, viscoelasticity, physics, electric transmission, modeling of speech signals, fluid mechanics and economics [1, 2]. Today, there are many considerable works on the numerical solution of fractional differential equations and fractional integro-differential equations (see for example [3, 4, 5, 6, 7, 8, 9, 10, 11, 12] and the references therein). There are various definitions of a fractional

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derivative of order  $\beta > 0$  [1, 2]. The Caputo fractional derivative is defined as

$$D^\beta f(x) = \begin{cases} \frac{1}{\Gamma(n-\beta)} \int_0^x \frac{f^{(n)}(t)}{(x-t)^{\beta+1-n}} dt, & n-1 < \beta < n, \quad n \in \mathbb{N}, \\ \frac{d^n}{dx^n} f(x), & \beta = n \in \mathbb{N}. \end{cases} \quad (1)$$

where  $\Gamma(\cdot)$  is the Gamma function and  $n = [\beta] + 1$ , with  $[\beta]$  denoting the integer part of  $\beta$ . Here, the Caputo fractional derivative is considered because it allows traditional initial and boundary conditions to be included in the formulation of the problem [2]. For the Caputo's derivative we have [1],

$$D^\beta C = 0, \quad (2)$$

$$D^\beta (a_1 f(x) + a_2 g(x)) = a_1 D^\beta f(x) + a_2 D^\beta g(x), \quad (3)$$

$$D^\beta x^m = \begin{cases} 0, & m < [\beta], \\ \frac{\Gamma(m+1)}{\Gamma(m+1-\beta)} x^{m-\beta}, & m \geq [\beta], \end{cases} \quad m \in \mathbb{N}, \quad (4)$$

where  $C, a_1$  and  $a_2$  are constants. Also,  $[\beta]$  denoting the smallest integer greater than or equal to  $\beta$ .

The sinc method is a powerful numerical tool for finding fast and accurate solutions in various scientific and engineering problems including squeezing flow [13], integro-differential equation [14], boundary value problems [15], Thomas-Fermi equation [16], Troesch's problem [17], fractional convection-diffusion equations [8], fractional differential equations [18, 19], time-fractional diffusion equation [20, 21], time-fractional order telegraph equation [22] and Bagley-Torvik equation [12]. It is worth indicating that, sinc-based approximations are characterized by exponentially decaying errors and rapidly converging solutions [23, 24].

As far as we know, for the first time in 2009, sinc methods appeared in the domain of fractional calculus [25], and later developed by Frank Stenger and his colleagues for solution of some fractional differential and integral equations [26]. In this paper, we extend the applications of the sinc method to find a numerical solution for fractional differential equations. At first, we use the Legendre polynomials to approximate the fractional derivatives of sinc functions. Then a collocation approach using sinc functions is applied to solve fractional differential equations.

The organization of the rest of the paper is as follows: In Section 2, we present some necessary definitions and mathematical preliminaries of sinc functions and Legendre polynomials. In Section 3, the fractional derivative of the Caputo type for the sinc function is obtained. Section 4 is devoted to applying sinc-collocation method for solving fractional differential equations. In Section 5 the proposed method is applied to several examples and is compared with the method existing in the literature. Section 6 completes this report with a brief conclusion. Note that we have computed the numerical results by Maple programming.

## 2. Preliminaries and Notations

In this section, we present some basic definitions and preliminary materials which will be used throughout the paper.

### 2.1 Sinc functions

The sinc functions and their properties are discussed in [23, 24]. For each integer  $k$  and the mesh size  $h > 0$ , the translated sinc functions with equidistant space nodes  $kh$  are given as

$$S(k, h)(x) = \text{sinc}\left(\frac{x - kh}{h}\right), \quad (5)$$

where the sinc function is defined on  $\mathbb{R}$ , by

$$\text{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

If a function  $f$  is defined on  $\mathbb{R}$ , then for mesh size  $h > 0$  the Whittaker cardinal expansion of  $f$  is as follows

$$C(f, h)(x) = \sum_{k=-\infty}^{\infty} f(kh) \text{sinc}\left(\frac{x - kh}{h}\right),$$

whenever this series converges. To construct approximations on the interval  $(0, 1)$ , we choose the one-to-one conformal mapping

$$\phi(x) = \ln\left(\frac{x}{1-x}\right),$$

which maps the eye-shaped region

$$D_E = \left\{ z = x + iy : \left| \arg\left(\frac{z}{1-z}\right) \right| < d \leq \frac{\pi}{2} \right\},$$

onto the infinite strip domain

$$D_S = \left\{ w = t + is : |s| < d \leq \frac{\pi}{2} \right\}.$$

The basis functions on  $(0, 1)$  are taken to be the composite translated sinc functions,

$$S_k(x) = S(k, h) \circ \phi(x) = \text{sinc}\left(\frac{\phi(x) - kh}{h}\right), \quad (6)$$

where  $S(k, h) \circ \phi(x)$  is defined by  $S(k, h)(\phi(x))$ . The sinc grid points  $x_k \in (0, 1)$  will be defined as the inverse images of the equispaced grids as

$$x_k = \phi^{-1}(kh) = \frac{e^{kh}}{1 + e^{kh}}, \quad k = 0, \pm 1, \pm 2, \dots \quad (7)$$

Also, we may define the inverse image of the real line as

$$\Gamma = \{\psi(t) = \phi^{-1}(t) \in D_E : -\infty < t < \infty\} = (0, 1),$$

Let  $B(D_E)$  denote the family of all functions  $f$  that are analytic in  $D_E$  and satisfy

$$\int_{\psi(t+L)} |f(z)dz| \longrightarrow 0, \quad \text{as } t \longrightarrow \pm\infty,$$

where  $L = \{iv : |v| < d \leq \frac{\pi}{2}\}$ , and on the boundary of  $D_E$ , (denoted  $\partial D_E$ ), satisfy

$$N(f) = \int_{\partial D_E} |f(z)dz| < \infty.$$

The next theorem guarantees the exponential convergence of the sinc approximation in  $B(D_E)$ .

**Theorem 2.1.** ([24]) *If  $\phi'f \in B(D_E)$ , then for all  $x \in \Gamma$*

$$\begin{aligned} \left| f(x) - \sum_{k=-\infty}^{\infty} f(x_k)S(k, h) \circ \phi(x) \right| &\leq \frac{N(f\phi')}{2\pi d \sinh(\pi d/h)} \\ &\leq \frac{2N(f\phi')}{\pi d} e^{-\pi d/h}. \end{aligned}$$

Moreover, if  $|f(x)| \leq Ce^{-\alpha|\phi(x)|}$ ,  $x \in \Gamma$ , for some positive constants  $C$  and  $\alpha$ , and if the selection  $h = \sqrt{\pi d/\alpha N} \leq 2\pi d/\ln 2$ , then

$$\left| f(x) - \sum_{k=-N}^N f(x_k)S(k, h) \circ \phi(x) \right| \leq C_2 \sqrt{N} \exp(-\sqrt{\pi d \alpha N}), \quad x \in \Gamma,$$

where  $C_2$  depends only on  $f$ ,  $d$  and  $\alpha$ .

Also, the derivatives of sinc basis functions evaluated at the nodes will be needed [24]:

$$\delta_{k,j}^{(0)} = [S(k, h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 1, & k = j, \\ 0, & k \neq j. \end{cases} \quad (8)$$

$$\delta_{k,j}^{(1)} = h \frac{d}{d\phi} [S(k, h) \circ \phi(x)]|_{x=x_j} = \begin{cases} 0, & k = j, \\ \frac{(-1)^{j-k}}{j-k}, & k \neq j. \end{cases} \quad (9)$$

$$\delta_{k,j}^{(2)} = h^2 \frac{d^2}{d\phi^2} [S(k, h) \circ \phi(x)]|_{x=x_j} = \begin{cases} \frac{-\pi^2}{3}, & k = j, \\ \frac{-2(-1)^{j-k}}{(j-k)^2}. & k \neq j. \end{cases} \quad (10)$$

## 2.2 Legendre Polynomials

Legendre polynomials are defined on the interval  $[-1, 1]$  and they satisfy the following recurrence formulae [5]:

$$L_{i+1}(z) = \frac{2i+1}{i+1}zL_i(z) - \frac{i}{i+1}L_{i-1}(z), \quad i = 1, 2, \dots,$$

where  $L_0(z) = 1$  and  $L_1(z) = z$ . In order to use these polynomials on the interval  $x \in [0, 1]$  we define the so-called shifted Legendre polynomials by introducing the change of variable  $z = 2x - 1$ . Let the shifted Legendre polynomials  $L_i(2x - 1)$  be denoted by  $P_i(x)$ . Then  $P_i(x)$  can be obtained as follows :

$$P_{i+1}(x) = \frac{(2i+1)(2x-1)}{i+1}P_i(x) - \frac{i}{i+1}P_{i-1}(x), \quad i = 1, 2, \dots, \quad (11)$$

where  $P_0(x) = 1$  and  $P_1(x) = 2x - 1$ . The analytic closed form of the shifted Legendre polynomial  $P_i(x)$  of degree  $i$  is given by [5]

$$P_i(x) = \sum_{k=0}^i (-1)^{i+k} \frac{(i+k)!}{(i-k)! (k!)^2} x^k. \quad (12)$$

The orthogonality condition is

$$\int_0^1 P_i(x)P_j(x)dx = \begin{cases} \frac{1}{2i+1} & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases} \quad (13)$$

A function  $y(x)$ , square integrable in  $[0, 1]$ , may be expressed in terms of shifted Legendre polynomials as

$$y(x) = \sum_{j=0}^{\infty} c_j P_j(x),$$

where the coefficients  $c_j$  are given by

$$c_j = (2j+1) \int_0^1 y(x)P_j(x)dx, \quad j = 1, 2, \dots$$

Now we introduce the Legendre-Gauss quadrature rule that will be used in the sequel. Let  $z_i$ ,  $1 \leq i \leq m$  be the  $m$  roots of the Legendre polynomial  $L_m(z)$ . Clearly,  $z_i \in (-1, 1)$ . Let [27]

$$\omega_i = \frac{2}{(1-z_i^2)(L'_m(z_i))^2}, \quad i = 1, \dots, m, \quad (14)$$

which are the  $m$  weights in the Legendre-Gauss quadrature formula associated with  $m$  roots. The  $m$ -point Legendre-Gauss quadrature rule can be used to approximate the integral of a function over the range  $[-1, 1]$  as

$$\int_{-1}^1 f(x)dx \simeq \sum_{i=1}^m \omega_i f(z_i). \quad (15)$$

Also, the error is

$$\frac{2^{2m+1}(m!)^4}{(2m+1)[(2m)!]^3} f^{(2m)}(\xi),$$

where  $-1 < \xi < 1$ . Thus, the  $m$ -point Legendre-Gauss quadrature rule is exact when  $f$  is any polynomial of degree  $2m - 1$  or less.

### 3. Fractional Derivative for $S_k(x)$ at the Sinc Nodes

First of all, we approximate  $S_k(x)$  by  $(m + 1)$  terms of shifted Legendre series as

$$S_k(x) \simeq \sum_{j=0}^m c_{kj} P_j(x), \quad (16)$$

where

$$c_{kj} = (2j + 1) \int_0^1 S_k(x) P_j(x) dx.$$

Employing the  $m$ -point Legendre-Gauss quadrature rule (15), we get

$$c_{kj} \simeq \frac{(2j + 1)}{2} \sum_{i=1}^m \omega_i S_k \left( \frac{z_i + 1}{2} \right) P_j \left( \frac{z_i + 1}{2} \right), \quad (17)$$

where  $\omega_i$  is given in Eq. (14).

**Theorem 3.1.** *Let  $S_k(x)$  be approximated by the shifted Legendre polynomials as (16) and also suppose  $\beta > 0$  then*

$$D^\beta S_k(x_i) \simeq \sum_{j=\lceil\beta\rceil}^m \sum_{\ell=\lceil\beta\rceil}^j \frac{c_{kj} (-1)^{(j+\ell)} (j+\ell)! x_i^{\ell-\beta}}{(j-\ell)! (\ell)! \Gamma(\ell+1-\beta)}, \quad (18)$$

where  $c_{kj}$  is given in Eq. (17).

*Proof.* Since the Caputo's fractional differentiation is a linear operation we have

$$D^\beta S_k(x) \simeq \sum_{j=0}^m c_{kj} D^\beta (P_j(x)). \quad (19)$$

Employing Eqs. (2), (3) and (4) in Eq. (12) we obtain

$$D^\beta P_j(x) = 0, \quad j = 0, 1, \dots, \lceil\beta\rceil - 1. \quad (20)$$

Also, for  $j = \lceil\beta\rceil, \dots, m$ , by using Eqs. (3),(4) and (12) we get

$$D^\beta P_j(x) = \sum_{\ell=0}^j \frac{(-1)^{j+\ell} (j+\ell)!}{(j-\ell)! (\ell!)^2} D^\beta (x^\ell) = \sum_{\ell=\lceil\beta\rceil}^j \frac{(-1)^{j+\ell} (j+\ell)!}{(j-\ell)! (\ell)! \Gamma(\ell-\beta+1)} x^{\ell-\beta}. \quad (21)$$

A combination of Eqs. (19), (20) and (21) leads to the desired result.  $\square$

**Corollary 3.2.** Let  $u(x)$  be approximated by the sinc basis functions as

$$u(x) \simeq u_M(x) = \sum_{k=-N}^N u_k S_k(x), \quad (22)$$

where  $u_k = u(x_k)$  and  $M = 2N + 1$ . Also suppose  $\beta > 0$  then

$$D^\beta u_M(x_i) \simeq \sum_{k=-N}^N u_k \left\{ \delta_{ki}^{(\beta)} \right\}, \quad i = -N, \dots, N, \quad (23)$$

where  $\delta_{ki}^{(\beta)}$  is given by

$$\delta_{ki}^{(\beta)} = \sum_{j=\lceil\beta\rceil}^m \sum_{\ell=\lceil\beta\rceil}^j \frac{c_{kj} (-1)^{(j+\ell)} (j+\ell)! x_i^{\ell-\beta}}{(j-\ell)! (\ell)! \Gamma(\ell+1-\beta)} \quad (24)$$

*Proof.* Immediately obtained from Eq. (18).  $\square$

Now, we define the  $M \times M$  matrix  $\mathbf{D}^{(\beta)} = [\delta_{ki}^{(\beta)}]$ , i.e., the matrix whose  $ki$ -entry is given by  $\delta_{ki}^{(\beta)}$ . Then, the approximation of the fractional derivative of order  $\beta$  can be written as

$$\vec{u}^{(\beta)} \approx \mathbf{D}^{(\beta)} \vec{u}. \quad (25)$$

where,  $\vec{u} = [u_{-N}, \dots, u_N]^T$  and  $\vec{u}^{(\beta)} = [D^\beta u_M(x_{-N}), \dots, D^\beta u_M(x_N)]^T$ .

## 4. Application of the Matrix $\mathbf{D}^{(\beta)}$

In this section, we apply the matrix  $\mathbf{D}^{(\beta)}$  to solve linear two-point fractional boundary value problem

$$a_2(x)u''(x) + a_1(x)u'(x) + \lambda(x)D^\beta u(x) + a_0(x)u(x) = f(x), \quad x \in [0, 1], \quad 0 < \beta < 2, \quad (26)$$

$$u(0) = u(1) = 0, \quad (27)$$

where  $a_0(x), a_1(x), a_2(x)$  and  $\lambda(x)$  are known function, and  $u(x)$  is an unknown function.

To solve this problem, we first use Eq. (22) to approximate  $u(x)$  by  $u_M(x)$ . Note that  $u_M(x) = 0$  when  $x$  tends to 0 or 1. Let us define the first and second derivatives of vector  $\vec{u}$  as  $\vec{u}' = [u'_M(x_{-N}), \dots, u'_M(x_N)]^T$  and  $\vec{u}'' = [u''_M(x_{-N}), \dots, u''_M(x_N)]^T$  respectively. As said in [12], the approximation of the first and second derivatives can be written as

$$\vec{u}' \simeq \left\{ \frac{1}{h} \mathbf{I}^{(1)} \mathbf{E}(\phi') \right\} \vec{u} \equiv \mathbf{D}^{(1)} \vec{u}. \quad (28)$$

$$\vec{u}'' \simeq \left\{ \frac{1}{h} \mathbf{I}^{(1)} \mathbf{E}(\phi'') + \frac{1}{h^2} \mathbf{I}^{(2)} \mathbf{E}(\phi'^2) \right\} \vec{u} \equiv \mathbf{D}^{(2)} \vec{u}, \quad (29)$$

where

$$\mathbf{I}^{(1)} = \begin{pmatrix} 0 & -1 & \dots & \frac{(-1)^{M-1}}{M-1} \\ 1 & & \dots & \vdots \\ \vdots & \vdots & \ddots & -1 \\ \frac{(-1)^{1-M}}{1-M} & \dots & \dots & 0 \end{pmatrix}_{M \times M},$$

$$\mathbf{I}^{(2)} = \begin{pmatrix} \frac{-\pi^2}{3} & 2 & \dots & \frac{-2(-1)^{M-1}}{(M-1)^2} \\ 2 & & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2(-1)^{M-1}}{(M-1)^2} & \dots & \dots & \frac{-\pi^2}{3} \end{pmatrix}_{M \times M},$$

$$\mathbf{E}(p) = \begin{pmatrix} p(x_{-N}) & 0 & \dots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & p(x_N) \end{pmatrix}_{M \times M}, \quad p \text{ is an arbitrary function.}$$

It is worth to mention here that, the Toeplitz matrix  $\mathbf{I}^{(1)} = [\delta_{kj}^{(1)}]$  is a skew-symmetric matrix, i.e.,  $\mathbf{I}_{kj}^{(1)} = -\mathbf{I}_{jk}^{(1)}$ , the Toeplitz matrix  $\mathbf{I}^{(2)} = [\delta_{kj}^{(2)}]$  is a symmetric matrix, i.e.,  $\mathbf{I}_{kj}^{(2)} = \mathbf{I}_{jk}^{(2)}$  and also  $\mathbf{E}(p)$  is a diagonal matrix.

We are now ready to solve problem (26)-(27). Substituting  $u_M(x)$  in Eq. (26) we obtain

$$a_2(x)u_M''(x) + a_1(x)u_M'(x) + \lambda(x)D^\beta u_M(x) + a_0(x)u_M(x) = f(x), \quad (30)$$

A collocation scheme is defined in Eq. (30) by evaluating the result at the sinc points  $\{x_k\}_{k=-N}^{k=N}$  given in Eq. (7). Then the discrete sinc-collocation system for (26)-(27) is given by

$$\mathbf{A} \vec{u} = \vec{f}, \quad (31)$$

where,

$$\mathbf{A} = \mathbf{E}(a_2)\mathbf{D}^{(2)} + \mathbf{E}(a_1)\mathbf{D}^{(1)} + \mathbf{E}(\lambda)\mathbf{D}^{(\beta)} + \mathbf{E}(a_0)\mathbf{I}^{(0)}. \quad (32)$$

Here,  $\vec{f} = [f(x_{-N}), \dots, f(x_N)]^T$  is a known vector and also  $\mathbf{I}^{(0)} = [\delta_{kj}^{(0)}]$  is an identity matrix. The linear system (31) can be directly solved for the unknown vector  $\vec{u}$ . Consequently  $u_M(x)$  given in Eq. (22) can be calculated.

**Remark.** It is important to note that, if instead of homogeneous boundary conditions (27) we have the following nonhomogeneous boundary conditions

$$u(0) = a, \quad u(1) = b,$$

then we reformulate the problem (26)-(27) by applying the transformation  $y(x) = u(x) + (a - b)x - a$  that makes the boundary conditions become homogeneous.

## 5. Numerical Examples

To show the efficiency of the method described above, we present some examples. These examples are chosen such that there exist exact solutions for them. In all examples we choose  $\alpha = 1/2$  and  $d = \pi/2$  which leads to  $h = \pi/\sqrt{N}$ . Also, we choose  $m = 12$ .

**Example 5.1.** As the first example, we consider the following fractional differential equation [19]

$$u''(x) - xu'(x) + D^{0.5}u(x) = f(x),$$

with the nonhomogeneous boundary conditions

$$u(0) = 0, \quad u(1) = 2,$$

where

$$f(x) = -3x^3 - 2x^2 + 6x + 2 + \frac{6}{\Gamma(3.5)}x^{2.5} + \frac{2}{\Gamma(2.5)}x^{1.5}.$$

It can be easily verified that the exact solution is  $u(x) = x^3 + x^2$ . In Table 1, we compare the absolute error of our method with  $N = 16, 32$  and  $64$  together with the result obtained by using sinc-Galerkin method given in [19]. This table shows that our approximate solution is in good agreement with the exact values and also the present method is clearly reliable if compared with the sinc-Galerkin method.

**Example 5.2.** Consider the following Bagley-Torvik equation [11, 12]

$$u''(x) + \frac{8}{17}D^{1.5}u(x) + \frac{13}{51}u(x) = \frac{x^{-1/2}}{89250\sqrt{\pi}} (48p(x) + 7\sqrt{\pi x}q(x)),$$

with the homogeneous boundary conditions

$$u(0) = u(1) = 0,$$

where  $p(x) = 16000x^4 - 32480x^3 + 21280x^2 - 4746x$  and  $q(x) = 3250x^5 - 9425x^4 + 264880x^3 - 448107x^2 + 233262x - 34578$ . The exact solution is

$$u(x) = x^5 - \frac{29}{10}x^4 + \frac{76}{25}x^3 - \frac{339}{250}x^2 + \frac{27}{125}x.$$

To make a comparison, in Table 2 we compare absolute error of the new method with  $N = 32$  and  $64$  together with the result obtained by using the sinc operational matrix method given in [12].

Table 1: Comparison of absolute error for Example 5.1.

$x$	sinc-Galerkin [19]	Present method		
	$N = 100$	$N = 16$	$N = 32$	$N = 64$
0.1	4.32 e-05	1.79 e-04	5.21 e-06	3.16 e-08
0.2	8.58 e-05	1.88 e-04	5.49 e-06	3.33 e-08
0.3	1.30 e-04	1.98 e-04	5.75 e-06	3.49 e-08
0.4	1.69 e-04	2.06 e-04	6.01 e-06	3.65 e-08
0.5	2.01 e-04	2.15 e-04	6.28 e-06	3.80 e-08
0.6	2.24 e-04	2.23 e-04	6.50 e-06	3.95 e-08
0.7	2.48 e-04	2.31 e-04	6.74 e-06	4.09 e-08
0.8	2.40 e-04	2.39 e-04	6.96 e-06	4.23 e-08
0.9	1.61 e-04	2.46 e-04	7.18 e-06	4.36 e-08

Table 2: Comparison of absolute error for Example 5.2.

$x$	N=32		N=64	
	Method of [12]	Present method	Method of [12]	Present method
0.1	1.36 e-06	4.76 e-07	3.92 e-09	2.89 e-09
0.2	4.17 e-06	4.18 e-07	4.59 e-09	2.54 e-09
0.3	1.36 e-06	3.57 e-07	4.16 e-09	2.19 e-09
0.4	5.30 e-06	2.99 e-07	4.12 e-09	1.83 e-09
0.5	3.41 e-06	2.39 e-07	3.87 e-09	1.46 e-09
0.6	3.19 e-07	1.79 e-07	3.82 e-09	1.10 e-09
0.7	2.17 e-06	1.21 e-07	4.54 e-09	7.39 e-10
0.8	2.74 e-06	5.75 e-08	4.78 e-09	3.72 e-10
0.9	1.65 e-06	1.23 e-09	3.19 e-09	6.24 e-12

**Example 5.3.** Consider the following linear fractional boundary value problem [18]:

$$u''(x) + x^2u'(x) - D^{0.7}u(x) + u(x) = f(x),$$

with the homogeneous boundary conditions

$$u(0) = u(1) = 0,$$

where

$$f(x) = 5x^6 - 3x^5 - x^4 + 20x^3 - 12x^2 - \frac{120}{\Gamma(5.3)}x^{4.3} + \frac{24}{\Gamma(4.3)}x^{3.3}.$$

The exact solution of this problem is  $u(x) = x^4(x - 1)$ . Figure 1 shows the plot of absolute error with  $N = 32$  and  $N = 64$  using the presented method. The authors of [18] used sinc-collocation method to solve this example. For the purpose of comparison in Table 3 we compare the absolute error of our method with  $N = 5, 50$  together with the absolute error given in [18].

**Example 5.4.** Consider the following singular linear fractional boundary value problem:

$$u''(x) + \left(\frac{1}{x}\right)u'(x) - D^{1.5}u(x) = 6 - \frac{2}{x} - 4\sqrt{\frac{x}{\pi}},$$

subject to the homogeneous boundary conditions

$$u(0) = u(1) = 0.$$

The exact solution of this problem is  $u(x) = x^2 - x$ . Figure 2 shows the plot of absolute error with  $N = 32$  and  $N = 64$  using the presented method.

According to these experiments, we find that the our method can be considered as an efficient method. Also if  $N$  increases, then the errors become smaller quickly.

## 6. Conclusion

In this paper, we use the Legendre polynomials to approximate the fractional derivatives of sinc functions. We derive the matrix  $\mathbf{D}^{(\beta)}$ . This matrix together with the sinc-collocation method are then utilized to reduce the solution of linear fractional differential equations to the solution of a system of linear algebraic equations. The proposed technique is easy to implement. Illustrative examples demonstrate the validity and applicability of the new method.

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**Conflicts of Interest.** The authors declare that there are no conflicts of interest regarding the publication of this article.

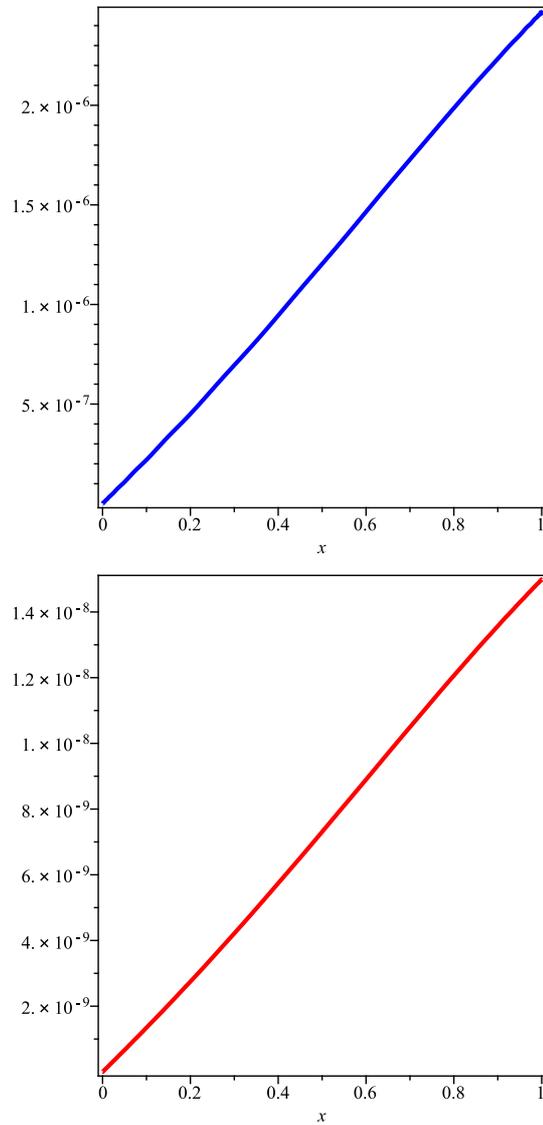


Figure 1: Plot of the absolute error, with  $N = 32$  (upper) and  $N = 64$  (down) for Example 5.3.

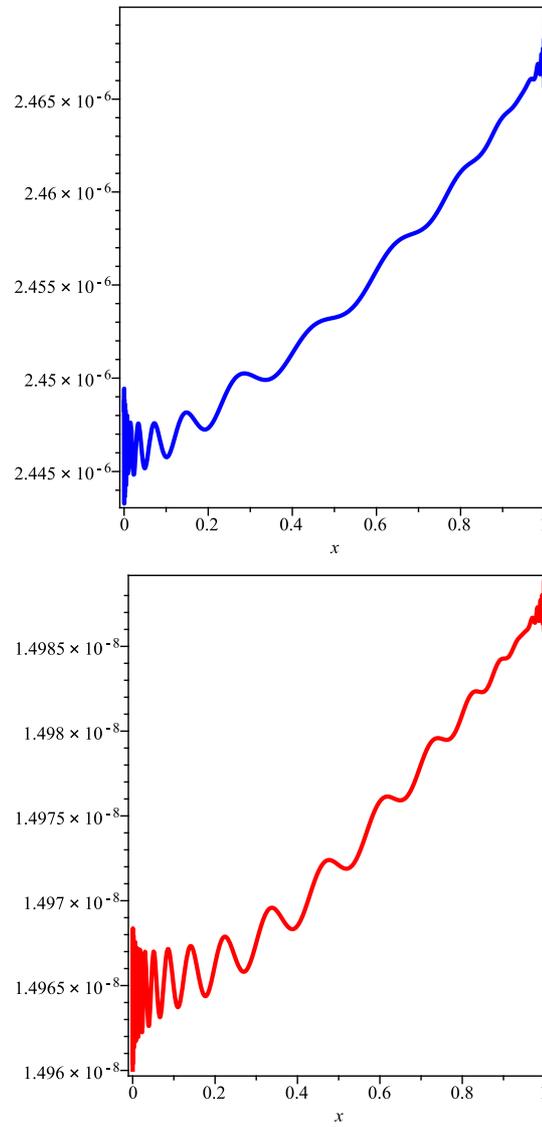


Figure 2: Plot of the absolute error, with  $N = 32$  (upper) and  $N = 64$  (down) for Example 5.4.

Table 3: Comparison of absolute error for Example 3.

$x$	N=5		N=50	
	Method of [18]	Present method	Method of [18]	Present method
0.1	2.82 e-03	4.52 e-04	1.35 e-05	1.02 e-08
0.2	1.73 e-03	2.45 e-05	2.76 e-05	2.09 e-08
0.3	3.31 e-04	7.54 e-04	4.12 e-05	3.21 e-08
0.4	1.15 e-03	1.32 e-03	5.25 e-05	4.37 e-08
0.5	1.75 e-03	1.55 e-03	5.84 e-05	5.57 e-08
0.6	2.36 e-03	1.73 e-03	5.56 e-05	6.78 e-08
0.7	1.49 e-03	1.81 e-03	4.20 e-05	8.00 e-08
0.8	2.66 e-03	1.82 e-03	1.99 e-05	9.19 e-08
0.9	4.88 e-03	2.80 e-03	9.90 e-07	1.03 e-07

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