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# Note on the Sum of Powers of Normalized Signless Laplacian Eigenvalues of Graphs

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**Abstract**

In this paper, for a connected graph $G$ and a real $\alpha \neq 0$, we define a new graph invariant $\sigma_\alpha(G)$-as the sum of the $\alpha$th powers of the normalized signless Laplacian eigenvalues of $G$. Note that $\sigma_{1/2}(G)$ is equal to Randić (normalized) incidence energy which have been recently studied in the literature [5,15]. We present some bounds on $\sigma_\alpha(G)$ $(\alpha \neq 0, 1)$ and also consider the special case $\alpha = 1/2$.

Keywords: Normalized signless Laplacian eigenvalues, Randić (normalized) incidence energy, bound.

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## 1. Introduction

Let $G$ be a simple connected graph with $n$ vertices and $m$ edges and let $V(G) = \{v_1, v_2, \ldots, v_n\}$ denote the set of vertices of $G$. Let $d_i$ be the degree of the vertex $v_i \in V(G)$, for $i = 1, 2, \ldots, n$.

Let $A(G)$ be the $(0,1)$-adjacency matrix of a graph $G$. The eigenvalues of $G$ are the eigenvalues of $A(G)$ [9] and denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Let $D(G)$ be the diagonal matrix of vertex degrees of $G$. The Laplacian matrix of $G$ is the matrix $L(G) = D(G) - A(G)$ with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_n = 0$. The matrix $Q(G) = D(G) + A(G)$ is called as the signless Laplacian matrix of $G$. Let

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$q_1 \geq q_2 \geq \cdots \geq q_n$ be the eigenvalues of $Q(G)$. The eigenvalues of the matrices $L(G)$ and $Q(G)$ are said to be the Laplacian and signless Laplacian eigenvalues of $G$, respectively. For more details on the spectral theory of $L(G)$ and $Q(G)$, see [11–13, 28, 29].

The energy of a graph $G$ is defined as the sum of absolute values of its eigenvalues, i.e., [16]

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| .$$

This concept is originated from theoretical chemistry where it is closely associated with the total $\pi$-electron energy of a molecule [17, 18]. There is an extensive literature on $E(G)$. For more details see the book [23] and the references cited therein.

The graph energy concept was extended to energy of any matrix in the following manner [32]. The singular values of any (real) matrix $M$ are equal to the square roots of the eigenvalues of $MM^T$, where $M^T$ is the transpose of $M$. Then the energy of the matrix $M$ is defined as the sum of its singular values [32]. Evidently, $E(A(G)) = E(G)$.

The incidence matrix $I(G)$ of a graph $G$ with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$ is the matrix whose $(i, j)$-entry is 1 if the vertex $v_i$ is incident with the edge $e_j$ and is 0 otherwise. In the light of the paper [32], Jooyandeh et al. [22] introduced the incidence energy of $G$, denoted by $IE(G)$, as the sum of singular values of $I(G)$. Since $Q(G) = I(G) I(G)^T$, it was discovered that [19]

$$IE = IE(G) = \sum_{i=1}^{n} \sqrt{q_i}.$$

For the basic properties and the details of $IE$, see [4, 19, 20, 22, 35].

In [27], Liu and Liu defined the Laplacian energy-like invariant as

$$LEL = LEL(G) = \sum_{i=1}^{n-1} \sqrt{\mu_i}.$$

For survey and more inforation on the quantity $LEL$, see [21, 26]. Since the Laplacian and signless Laplacian eigenvalues of bipartite graphs coincide [10, 28, 29], $LEL$ is equal to $IE$ for bipartite graphs [19].

The Randić matrix $R(G)$ of a graph $G$ is the matrix whose $(i, j)$-entry is $1/\sqrt{d_i d_j}$ if the vertices $v_i$ and $v_j$ are adjacent and is 0 otherwise [2]. Since $G$ is connected, $D(G)$ is non-singular, then the Randić matrix of $G$ is also defined as $R(G) = D(G)^{-1/2} A(G) D(G)^{-1/2}$ [9]. The Randić eigenvalues of $G$ are the eigenvalues of its Randić matrix and denoted by $\rho_1 = 1 \geq \rho_2 \geq \cdots \geq \rho_n$ [2, 9, 25].

The normalized Laplacian and the normalized signless Laplacian matrices of a connected graph $G$ are defined as [7]

$$\mathcal{L}^{-}(G) = D(G)^{-1/2} L(G) D(G)^{-1/2} = I_n - R(G) \tag{1}$$

and

$$\mathcal{L}^{+}(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2} = I_n + R(G) \tag{2}$$

respectively. In here, $I_n$ is the $n \times n$ unit matrix. Let $\gamma_1^- \geq \gamma_2^- \geq \cdots \geq \gamma_n^- = 0$ be the eigenvalues of $\mathcal{L}^-(G)$ and $\gamma_1^+ \geq \gamma_2^+ \geq \cdots \geq \gamma_n^+$ be the eigenvalues of $\mathcal{L}^+(G)$. These eigenvalues are called as the normalized Laplacian and normalized signless Laplacian eigenvalues of $G$, respectively. For more details, see [7].

From the Equations (1) and (2), it follows that [15,25]

$$\gamma_i^- = 1 - \rho_{n-i+1} \text{ and } \gamma_i^+ = 1 + \rho_i, \text{ for } i = 1, 2, \ldots, n. \tag{3}$$

Considering Randić matrix and incidence matrix, Gu et al. [15] defined the $n \times m$ Randić incidence matrix $I_R(G)$ of $G$ whose $(i,j)$-entry is $1/\sqrt{d_i}$ if the vertex $v_i$ is incident with the edge $e_j$ and is 0 otherwise. The Randić incidence energy of $G$, denoted by $I_R E(G)$, is defined as the sum of singular values of its Randić incidence matrix [15]. In [15], It was shown that $\mathcal{L}^+(G) = I_R(G) I_R(G)^T$. Then, by full analogy with the incidence energy [19], the authors also defined the Randić incidence energy as [15]

$$I_R E = I_R E(G) = \sum_{i=1}^{n} \sqrt{\gamma_i^+}.$$

This quantity is studied under the name normalized incidence energy in [5].

By analogy to Laplacian energy-like invariant [27], the Laplacian incidence energy of $G$ is defined as [33]

$$LIE = LIE(G) = \sum_{i=1}^{n-1} \sqrt{\gamma_i^-}.$$

For a connected graph $G$ and a real number $\alpha \neq 0$, the sum of the $\alpha$th powers of the non-zero normalized Laplacian eigenvalues of $G$ is defined as the following [3]

$$s_\alpha = s_\alpha(G) = \sum_{i=1}^{n-1} \left(\gamma_i^-\right)^\alpha.$$

The case $\alpha = 1$ is trivial as $s_1 = n$. In [1,3,8,24], some bounds on $s_\alpha$ was given and the case $\alpha = -1$ was discussed since $2ms_{-1}$ is equal to the degree Kirchoff index [6]. Furher note that, $s_{1/2} = LIE$ which was recently studied in the literature [30,33].

For a connected graph $G$ and a real number $\alpha \neq 0$, we now introduced the sum of the $\alpha$th powers of the normalized signless Laplacian eigenvalues of $G$ as the following

$$\sigma_\alpha = \sigma_\alpha(G) = \sum_{i=1}^{n} \left(\gamma_i^+\right)^\alpha .$$

Note that the case $\alpha = 1$ is trivial as $\sigma_1 = n$. Furthermore, for $\alpha = 1/2$, $\sigma_{1/2} = I_R E$.

In this paper, we present some upper and lower bounds on $\sigma_\alpha(G)$ $(\alpha \neq 0, 1)$ and also consider the special case $\alpha = 1/2$.

## 2. Lemmas

Let $\overline{G}$ and $t = t(G)$ denote the complement and the number of spanning tress of a graph $G$, respectively. Let $G_1 \times G_2$ be the cartesian product of the graphs $G_1$ and $G_2$ [9]. Throughout this paper, for a graph $G$, we use the following auxiliary quantity,

$$t_1 = t_1(G) = \frac{2t(G \times K_2)}{t(G)}. \tag{4}$$

**Lemma 2.1.** If $G$ is a bipartite graph, then the eigenvalues of $\mathcal{L}^-(G)$ and $\mathcal{L}^+(G)$ coincide.

*Proof.* From the Equation (3), we have $\gamma_i^- = 1 - \rho_{n-i+1}$ and $\gamma_i^+ = 1 + \rho_i$, for $1 \leq i \leq n$ [15,25]. Note that Randić eigenvalues of a bipartite graph are symmetric with respect to the zero point of the real axis, i.e., $\rho_i = -\rho_{n-i+1}$, for $1 \leq i \leq n$ [9] (p. 109). Then, we get the required result. □

By Lemma 2.1, we directly have:

**Lemma 2.2.** If $G$ is a bipartite graph, then $\sigma_\alpha$ coincide with $s_\alpha$. Especially, for bipartite graphs, $\sigma_{1/2} = I_R E = LIE = s_{1/2}$.

**Lemma 2.3.** [9] Let $G$ be a connected graph with $n$ vertices, $m$ edges and $t$ spanning trees. Then, $\prod_{i=1}^{n-1} \gamma_i^- = \frac{2mt}{\prod_{i=1}^{n} d_i}$.

**Lemma 2.4.** [12] Let $G$ be a connected non-bipartite graph with $n$ vertices. Then, $\det Q(G) = \prod_{i=1}^{n} q_i = t_1$.

**Lemma 2.5.** If $G$ is a connected bipartite graph with $n$ vertices, $m$ edges and $t$ spanning trees, then $\prod_{i=1}^{n-1} \gamma_i^- = \prod_{i=1}^{n-1} \gamma_i^+ = \frac{2mt}{\prod_{i=1}^{n} d_i}$. If $G$ is a connected non-bipartite graph with $n$ vertices, then $\prod_{i=1}^{n} \gamma_i^+ = \frac{t_1}{\prod_{i=1}^{n} d_i}$.

*Proof.* By Lemmas 2.1 and 2.3, for connected bipartite graphs, one can directly get that

$$\prod_{i=1}^{n-1} \gamma_i^- = \prod_{i=1}^{n-1} \gamma_i^+ = \frac{2mt}{\prod_{i=1}^{n} d_i}.$$

Since $\mathcal{L}^+(G) = D(G)^{-1/2} Q(G) D(G)^{-1/2}$, taking the determinant of both of two sides, we obtain that

$$\det \mathcal{L}^+(G) = \prod_{i=1}^{n} \gamma_i^+ = \frac{\det Q(G)}{\prod_{i=1}^{n} d_i}.$$

Considering this with Lemma 2.4, we get the required result for connected non-bipartite graphs. □

**Lemma 2.6.** [14] Let $G$ be a connected graph with $n > 2$ vertices. Then $\gamma_2^- = \gamma_3^- = \cdots = \gamma_{n-1}^-$ if and only if $G \cong K_n$ or $G \cong K_{p,q}$ $(p + q = n)$.

The proof of the following lemma can be found in the proof of Theorem 2.2 in [15].

**Lemma 2.7.** [15] Let $G$ be a graph of order $n \geq 2$ without isolated vertices. Then $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$ if and only if $G \cong K_n$.

**Lemma 2.8.** [34] Let $a_1, a_2, \ldots, a_N$ be non-negative real numbers. Then

$$\begin{aligned} N\left[\frac{1}{N}\sum_{i=1}^{N} a_i - \left(\prod_{i=1}^{N} a_i\right)^{1/N}\right] &\leq N\sum_{i=1}^{N} a_i - \left(\sum_{i=1}^{N}\sqrt{a_i}\right)^2 \\ &\leq N(N-1)\left[\frac{1}{N}\sum_{i=1}^{N} a_i - \left(\prod_{i=1}^{N} a_i\right)^{1/N}\right]. \end{aligned} \tag{5}$$

Moreover, the equality holds on both sides of (5) if and only if $a_1 = a_2 = \cdots = a_N$.

**Lemma 2.9.** [31] Let $a_i > 0$, $i = 1, 2, \ldots, p$ be $p$ real numbers. Then

$$p(A_p - G_p) \geq (p-1)(A_{p-1} - G_{p-1}), \tag{6}$$

where $A_p = \frac{\sum_{i=1}^{p} a_i}{p}$ and $G_p = \left(\prod_{i=1}^{p} a_i\right)^{1/p}$.

# 3. Main Results

In this section, we present some bounds on $\sigma_\alpha(G)$ $(\alpha \neq 0, 1)$ and also discuss the special case $\alpha = 1/2$.

**Theorem 3.1.** Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices and let $t_1$ be given by (4) and $\alpha \neq 0, 1$ be a real number. Then

$$\sigma_\alpha(G) \geq 2^\alpha + \sqrt{\sigma_{2\alpha} - 4^\alpha + (n-1)(n-2)\left(\frac{t_1}{2\prod_{i=1}^{n} d_i}\right)^{2\alpha/(n-1)}} \tag{7}$$

and

$$\sigma_\alpha(G) \leq 2^\alpha + \sqrt{(n-2)(\sigma_{2\alpha} - 4^\alpha) + (n-1)\left(\frac{t_1}{2\prod_{i=1}^{n} d_i}\right)^{2\alpha/(n-1)}}. \tag{8}$$

Moreover, equalities in (7) and (8) hold if and only if $G \cong K_n$.

*Proof.* By replacing $N$ with $n-1$ and taking $a_i = \left(\gamma_i^+\right)^{2\alpha}$, $i = 2, 3, ..., n$, in Lemma 2.8, we get

$$T \leq (n-1)\sum_{i=2}^{n}\left(\gamma_i^+\right)^{2\alpha} - \left(\sum_{i=2}^{n}\left(\gamma_i^+\right)^{\alpha}\right)^2 \leq (n-2)\,T$$

where $T = (n-1)\left[\frac{1}{n-1}\sum_{i=2}^{n}\left(\gamma_i^+\right)^{2\alpha} - \left(\prod_{i=2}^{n}\left(\gamma_i^+\right)^{2\alpha}\right)^{1/(n-1)}\right]$. Note that $\gamma_1^+ = 2$ [15] and $\sum_{i=1}^{n}\left(\gamma_i^+\right)^{2\alpha} = \sigma_{2\alpha}$, then we obtain

$$T \leq (n-1)(\sigma_{2\alpha} - 4^\alpha) - (\sigma_\alpha - 2^\alpha)^2 \leq (n-2)\,T \tag{9}$$

and

$$\begin{aligned} T &= (n-1)\left[\frac{1}{n-1}\sum_{i=2}^{n}\left(\gamma_i^+\right)^{2\alpha} - \left(\prod_{i=2}^{n}\left(\gamma_i^+\right)^{2\alpha}\right)^{1/(n-1)}\right] \\ &= (n-1)\left[\frac{1}{n-1}(\sigma_{2\alpha} - 4^\alpha) - \left(\prod_{i=2}^{n}\gamma_i^+\right)^{2\alpha/(n-1)}\right] \\ &= (\sigma_{2\alpha} - 4^\alpha) - (n-1)\left(\frac{t_1}{2\prod_{i=1}^{n} d_i}\right)^{2\alpha/(n-1)}, \text{by Lemma 2.5.} \end{aligned} \tag{10}$$

Combining (9) and (10), we arrive at the inequalities (7) and (8). Now we assume that the equalities hold in (7) and (8). Then, by Lemma 2.8, $\gamma_2^+ = \gamma_3^+ = \cdots = \gamma_n^+$. From Lemma 2.7, this implies that $G \cong K_n$.

Conversely, one can easily see that the equalities in (7) and (8) hold for $G \cong K_n$. □

For $\alpha = 1/2$ in Theorem 3.1, we have the following result on Randić incidence energy of connected non-bipartite graphs.

**Corollary 3.2.** Let $G$ be a connected non-bipartite graph with $n \geq 3$ vertices and let $t_1$ be given by (4). Then

$$I_R E\left(G\right) \geq \sqrt{2} + \sqrt{n - 2 + \left(n-1\right)\left(n-2\right)\left(\frac{t_1}{2\prod_{i=1}^{n} d_i}\right)^{1/(n-1)}} \tag{11}$$

and

$$I_R E\left(G\right) \leq \sqrt{2} + \sqrt{\left(n-2\right)^2 + \left(n-1\right)\left(\frac{t_1}{2\prod_{i=1}^{n} d_i}\right)^{1/(n-1)}}. \tag{12}$$

Moreover, equalities in (11) and (12) hold if and only if $G \cong K_n$.

*Remark* 1. *For a graph $G$ of order $n \geq 2$ without isolated vertices, Gu et al. obtained that [15]*

$$I_R E\left(G\right) \leq \sqrt{2} + \sqrt{\left(n-1\right)\left(n-2\right)}. \tag{13}$$

*The equality holds in (13) if and only if $G \cong K_n$. By using arithmetic-geometric mean inequality, one can conclude that the upper bound (12) is better than the upper bound (13) for connected non-bipartite graphs.*

Considering $\gamma_1^+ = 2$ [15] and similar arguments in Theorem 3.1 and using Lemmas 2.1, 2.5, 2.6 and 2.8, we have:

**Theorem 3.3.** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t$ spanning trees and let $\alpha \neq 0, 1$ be a real number. Then

$$s_\alpha\left(G\right) = \sigma_\alpha\left(G\right) \geq 2^\alpha + \sqrt{\sigma_{2\alpha} - 4^\alpha + \left(n-2\right)\left(n-3\right)\left(\frac{mt}{\prod_{i=1}^{n} d_i}\right)^{2\alpha/(n-2)}} \tag{14}$$

and

$$s_\alpha\left(G\right) = \sigma_\alpha\left(G\right) \leq 2^\alpha + \sqrt{\left(n-3\right)\left(\sigma_{2\alpha} - 4^\alpha\right) + \left(n-2\right)\left(\frac{mt}{\prod_{i=1}^{n} d_i}\right)^{2\alpha/(n-2)}}. \tag{15}$$

Moreover, equalities in (14) and (15) hold if and only if $G \cong K_{p,q}$ $(p+q=n)$.

For $\alpha = 1/2$ in Theorem 3.3, we obtain the following result.

**Corollary 3.4.** Let $G$ be a connected bipartite graph with $n \geq 3$ vertices, $m$ edges and $t$ spanning trees. Then

$$LIE\left(G\right) = I_R E\left(G\right) \geq \sqrt{2} + \sqrt{n - 2 + \left(n-2\right)\left(n-3\right)\left(\frac{mt}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}} \tag{16}$$

and

$$LIE\left(G\right) = I_R E\left(G\right) \leq \sqrt{2} + \sqrt{\left(n-2\right)\left(n-3\right) + \left(n-2\right)\left(\frac{mt}{\prod_{i=1}^{n} d_i}\right)^{1/(n-2)}}. \tag{17}$$

Moreover, equalities in (16) and (17) hold if and only if $G \cong K_{p,q}$ $(p+q=n)$.

**Theorem 3.5.** [3] Let $G$ be a connected bipartite graph of order $n \geq 3$. If $0 < \alpha < 1$, then

$$s_\alpha(G) = \sigma_\alpha(G) \leq n + 2(2^{\alpha-1} - 1). \tag{18}$$

The equality holds in (18) if and only if $G \cong K_{p,q}$ $(p + q = n)$.

*Remark* 2. *For a bipartite graph $G$ of order $n$ without isolated vertices, Gu et al. obtained that [15]*

$$I_R E(G) \leq n - 2 + \sqrt{2}. \tag{19}$$

*The equality holds in (19) if and only if $G$ is a complete bipartite graph. In fact, for connected bipartite graphs, (19) is a special case of (18) when $\alpha = 1/2$. Furthermore,by arithmetic-geometric mean inequality, we conclude that (17) is better than (19) for connected bipartite graphs.*

As well known in graph theory every tree is bipartite. Furthermore, for a tree $T$, $m = n - 1$ and $t = 1$. Then, from Theorem 3.3, we have:

**Corollary 3.6.** Let $T$ be a tree with $n \geq 3$ vertices and let $\alpha \neq 0, 1$ be a real number. Then

$$s_\alpha(T) = \sigma_\alpha(T) \geq 2^\alpha + \sqrt{\sigma_{2\alpha} - 4^\alpha + (n-2)(n-3)\left(\frac{n-1}{\prod_{i=1}^n d_i}\right)^{2\alpha/(n-2)}} \tag{20}$$

and

$$s_\alpha(T) = \sigma_\alpha(T) \leq 2^\alpha + \sqrt{(n-3)(\sigma_{2\alpha} - 4^\alpha) + (n-2)\left(\frac{n-1}{\prod_{i=1}^n d_i}\right)^{2\alpha/(n-2)}}. \tag{21}$$

Moreover, equalities in (20) and (21) hold if and only if $G \cong K_{1,n-1}$.

Setting $\alpha = 1/2$ in Corollary 3.6, we obtain:

**Corollary 3.7.** Let $T$ be a tree with $n \geq 3$ vertices. Then

$$LIE(T) = I_R E(T) \geq \sqrt{2} + \sqrt{n - 2 + (n-2)(n-3)\left(\frac{n-1}{\prod_{i=1}^n d_i}\right)^{1/(n-2)}} \tag{22}$$

and

$$LIE(T) = I_R E(T) \leq \sqrt{2} + \sqrt{(n-2)(n-3) + (n-2)\left(\frac{n-1}{\prod_{i=1}^n d_i}\right)^{1/(n-2)}}. \tag{23}$$

Moreover, equalities in (22) and (23) hold if and only if $G \cong K_{1,n-1}$.

**Theorem 3.8.** Let $G$ be a connected graph with $n \geq 3$ vertices, $m$ edges and $t$ spanning trees and let $t_1$ be given by (4) and $\alpha \neq 0, 1$ be a real number.

(i) If $G$ is bipartite, then there exists a real number $\epsilon \geq 0$ such that [24]

$$s_\alpha (G) = \sigma_\alpha (G) \geq 2^\alpha + (n-2) \left( \frac{mt}{\prod_{i=1}^{n} d_i} \right)^{\alpha/(n-2)} + \epsilon. \tag{24}$$

(ii) If $G$ is non-bipartite, then there exists a real number $\epsilon \geq 0$ such that

$$\sigma_\alpha (G) \geq 2^\alpha + (n-1) \left( \frac{t_1}{2\prod_{i=1}^{n} d_i} \right)^{\alpha/(n-1)} + \epsilon. \tag{25}$$

*Proof.* The lower bound (24) has been obtained in [24]. So, we omit its proof. We now only prove the lower bound (25).

Let $p = n-1$ , $a_1 = \left(\gamma_2^+\right)^\alpha$, $a_2 = \left(\gamma_n^+\right)^\alpha$ and $a_i = \left(\gamma_i^+\right)^\alpha$ for $i = 3, ..., n-1$ in Equation (6). Then, from Lemma 2.9, we have

$$(n-1) \left( \frac{\left(\sum_{i=2}^{n} \left(\gamma_i^+\right)^\alpha\right)}{n-1} - \left(\prod_{i=2}^{n} \left(\gamma_i^+\right)^\alpha\right)^{1/(n-1)} \right) \geq \cdots \geq \left( \left(\gamma_2^+\right)^{\alpha/2} - \left(\gamma_n^+\right)^{\alpha/2} \right)^2$$

i.e.,

$$\sigma_\alpha (G) \geq \left(\gamma_1^+\right)^\alpha + (n-1) \left(\prod_{i=2}^{n} \gamma_i^+\right)^{\alpha/(n-1)} + \left( \left(\gamma_2^+\right)^{\alpha/2} - \left(\gamma_n^+\right)^{\alpha/2} \right)^2. \tag{26}$$

Let say $\epsilon = \left( \left(\gamma_2^+\right)^{\alpha/2} - \left(\gamma_n^+\right)^{\alpha/2} \right)^2$. Considering Equation (26) with $\gamma_1^+ = 2$ [15] and Lemma 2.5, we obtain

$$\sigma_\alpha (G) \geq 2^\alpha + (n-1) \left( \frac{t_1}{2\prod_{i=1}^{n} d_i} \right)^{\alpha/(n-1)} + \epsilon.$$

Hence the result holds. □

Taking $\alpha = 1/2$ in Theorem 3.8, we have:

**Corollary 3.9.** Let $G$ be a connected graph with $n \geq 3$ vertices, $m$ edges and $t$ spanning trees and let $t_1$ be given by (4).

(i) If $G$ is bipartite, then there exists a real number $\epsilon \geq 0$ such that [24]

$$LIE (G) = I_R E (G) \geq \sqrt{2} + (n-2) \left( \frac{mt}{\prod_{i=1}^{n} d_i} \right)^{1/2(n-2)} + \epsilon.$$

(ii) If $G$ is non-bipartite, then there exists a real number $\epsilon \geq 0$ such that

$$I_R E (G) \geq \sqrt{2} + (n-1) \left( \frac{t_1}{2\prod_{i=1}^{n} d_i} \right)^{1/2(n-1)} + \epsilon.$$

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