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The Fourth and Fifth Laplacian Coefficients of some Rooted Trees

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Abstract

The Laplacian characteristic polynomial of an n-vertex graph G has the form $f(G,x) = x^n + \sum_{i=1}^{n-1} l_i x^{n-i}$. In this paper, the fourth and fifth Laplacian coefficients of $f(T(k,t),x)$ will be computed, where $T(k,t)$ is a rooted tree with degree sequence $k, k, \ldots, k, 1, 1, \ldots, 1$.

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1. Introduction

The mathematical structure of graphs have a valuable feature to help us to visualize, analyze, generalize a situation or problem that we may encounter and, in many cases, assisting us to understand it better and possibly solve it. In this paper, we study one of the types of graphs used in computer science.

A rooted tree is a tree in which one vertex has been designated as root. These trees, often with additional structure such as ordering of the neighbors at each vertex, are a key data structure in computer science. Let G be a simple graph with n vertices and m edges. A matching for G is a set of edges without common

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vertices and a k-matching is a matching with k edges. The number of k-matchings is denoted by $M(G, k)$.

The adjacency matrix of a graph G is a square $n \times n$ matrix A such that A_{ij} is *1* when there is an edge from v_i to v_j and zero otherwise, where $V(G) = \{v_1, ..., v_n\}$. The Laplacian matrix of G is defined as $L(G) = diag(d_1, \ldots, d_n)$-$A(G)$, where d_i is the degree of vertex v_i [2, 3]. The characteristic polynomial of the Laplacian matrix of G, denoted $f(G, x)$, is defined as $f(G, x) = \det(x I_n - L(G)) = x^n + l_1 x^{n-1} + l_2 x^{n-2} + \cdots + l_{n-1} x + l_n$. The coefficient $(-1)^i l_i$ is the sum of the principal minors of $L(G)$ containing i rows and i columns. It is easy to see that $l_n = 0$ and $l_1 = -2m$. A graph H without cycles is called a forest. Such a graph is a union of some components, each of which is a tree. In particular, if H has a unique connected component, then it is a tree. We shall use the symbol $P(H)$ to denote the product of the numbers of vertices in the number of components of H. If H is a tree then $P(H)$ is the number of vertices in H. Our motivation in this paper is explanation of the relation between the coefficients of the characteristic polynomial of the Laplacian matrix of a graph and its subforests. The general case of these relations explained in the following theorem.

Theorem 1.1. [2] The i-th coefficient l_i of $f(G, x)$ is given by the formula $l_i = (-1)^i \sum_H P(H)$, where H is a subforest with i edges.

Theorem 1.2. [8] Let G be a graph with n vertices and m edges. Then the number of 3-matchings in G is

$$\binom{m}{3} - (m-2)\sum_{i=1}^{n}\binom{d_i}{2} + 3\sum_{i=1}^{n}\binom{d_i}{3} + \sum_{ij \in E(G)} (d_i - 1)(d_j - 1) - N_T,$$

where N_T is the number of triangles in G.

Theorem 1.3. [10] Let G be a graph with m edges, A be its adjacency matrix, and $d = (d_1, \ldots, d_n)$ be its non-increasing degree sequence. Then,

$$l_3 = \frac{1}{3}(-4m^3 + 6m^2 + 3m\sum_{i=1}^{n} {d_i}^2 - \sum_{i=1}^{n} {d_i}^3 - 3\sum_{i=1}^{n} {d_i}^2 + tr(A^3)).$$

In this article, our notations are standard and taken mainly from [5–7,9,11,12].

2. Results

The main purpose of this section is to find the coefficients of the Laplacian polynomial of the rooted tree $T(k, t)$ with degree sequence $k, k, \cdots, k, 1, 1, \cdots, 1$ in which t is the distance between the center and a pendent vertex. This tree is depicted in Figure 1.

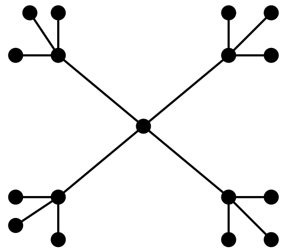

Figure 1: The tree $T(4,2)$.

Lemma 2.1. The number of vertices of the tree $T(k,t)$ is $\frac{k(k-1)^t-2}{k-2}$.

Corollary 2.2. $l_{n-1} = (-1)^{n-1}n$ and

$$l_{n-2} = \frac{k(-1)^{n-2}}{(k-2)^3(k^2-k-1)}[k^2 - k - (k-1)^{t+1}t + (k-1)^{3t+1}k + (k-1)^{t+3}t2(k-1)^t \\ +2(k-1)^{t+2} - (k-1)^{2t+2} + 2(k-1)^{2t}(k-1)^{2t+1} + k(k-1)^{t+2} - k(k-1)^{3t} \\ +k(k-1)^{2t} - (k-1)^{2t+2}k - k(k-1)^{t+1} + t(k-1)^t - t(k-1)^{t+2},$$

where n is the number of vertices in $T(k,t)$.

Proof. The proof is straightforward by [1, p. 67]. □

Corollary 2.3. The coefficient l_2 is equal to

$$-\frac{k}{2(k-2)^2}[k^3(k-1)^{t-1} - 3k^2(k-1)^{t-1} - 2k^2 - 2k + 4 \\ +2k(k-1)^{t-1} - 4k(k-1)^{2t} - 12k(k-1)^t - 8(k-1)^t].$$

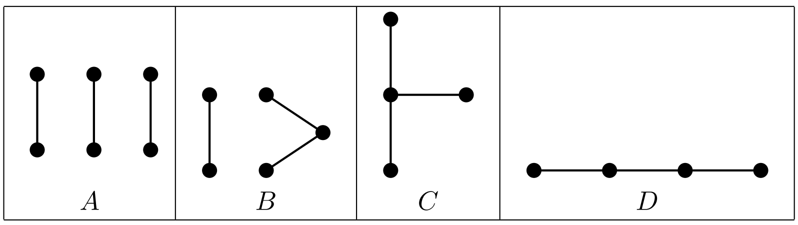

Figure 2: Subforests with 3 edges.

Now we are ready to compute the third coefficient of the characteristic polynomial of $T(k,t)$ tree.

Lemma 2.4. The number of subgraphs of type C and D in $T(K,t)$ which is shown in Figure 2, is $\binom{k}{3}(n - k(k-1)^{t-1})$ and $(k-1)^2(n - k(k-1)^{t-1})$, respectively.

Proof. The first one is the number of star graphs with 4 vertices started from a vertex. The second one is the number of path P_4 in $T(k,t)$. □

Lemma 2.5. The number of subgraphs of type B in $T(K,t)$, Figure 2, is

$$\binom{k}{2}(n-k(k-1)^{t-1})(n-3)-3\binom{k}{3}(n-k(k-1)^{t-1})-2(k-1)^2(n-k(k-1)^{t-1}).$$

Proof. Straightforwardly from [4, p. 117]. □

Lemma 2.6. The number of subgraphs of type A in $T(K,t)$ that is depicted in Figure 2 is

$$\binom{n-1}{3}-(n-3-3\binom{k}{3}+(k-1)^2)(n-k(k-1)^{t-1}).$$

Proof. The proof is an easy consequence of Theorem 1.2. □

Corollary 2.7. The coefficient l_3 is equal to

$$\begin{array}{l} 8+20/3n-8n^2-8k(-\frac{-kn+n+(k-1)^t}{k-1})^2-10k^2n+61/3kn \\ +3k^2n^2-3kn^2-7/3k^3n+8k(k-1)^{t-1}+10k^3(k-1)^{t-1}-61/3k^2(k-1)^{t-1} \\ +7/3k^3(k-1)^{t-1}-3k^3n(k-1)^{t-1}+3k^2n(k-1)^{t-1}-8n(n-k(k-1)^{t-1}) \\ +24\binom{k}{3}(n-k(k-1)^{t-1})+16k(n-k(k-1)^{t-1})+4/3n^3. \end{array}$$

Proof. The proof is straightforward by Theorem 1.1. □

Now, we are ready to compute the fourth coefficient of the characteristic polynomial of $T(k,t)$ tree by above Lemmas.

Lemma 2.8. The number of subgraphs isomorphic to P_5 in $T(k,t)$ is

$$\frac{k(k-1)^{t-2}-2}{k-2}\times\frac{k(k-1)^3}{2}.$$

Proof. The number of vertices that can be the central vertex of a path P_5 is $\frac{k(k-1)^{t-2}-2}{k-2}$. By selecting a central vertex, we can choose two neighborhoods of this vertex by $k(k-1)/2$ ways and two neighborhoods of them by $(k-1)^2$ ways. □

Lemma 2.9. The number of subgraphs isomorphic to the star graph S_5 in $T(k,t)$ is

$$\binom{k}{4}(n-k(k-1)^{t-1}).$$

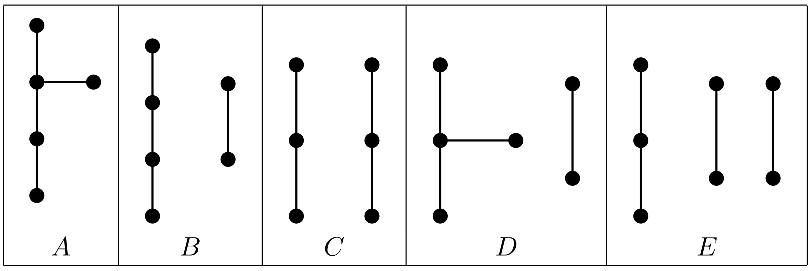

Figure 3: Subforests with 4 edges.

Proof. This goes ahead through the same lines of the proof of Lemma 2.6. □

Lemma 2.10. The number of subgraphs of $T(k,t)$ isomorphic to A, Figure 3, is $(k-1)^2(k-2)(n-k(k-1)^{t-1})$.

Proof. The number of subgraphs of $T(k,t)$ isomorphic to A is $N(P_4)-(n-4)-$ $-2N(P_5)$. It is now enough to apply Lemmas 2.4 and 2.8. □

Lemma 2.11. The number of subgraphs isomorphic to B in $T(k,t)$, Figure 3, is $N(P_4)(n-4)-2N(A)-2N(P_5)$ in which $N(P_4$ is the number of P_4 subgraphs in $T(k,t)$.

Proof. This goes ahead through the same lines of the proof of Lemma 2.10. □

Lemma 2.12. The number of subgraphs isomorphic to C in $T(k,t)$, Figure 3, is

$$(n-k(k-1)^{t-1})k(k-1)/2-(n-k(k-1)^{t-1})(k-1)(k-2)/2-k(k-2)(k-1)^t/2.$$

Proof. It is enough to note that the number of subgraphs isomorphic to C is equal to the number of $2-$matchings in the line graph of $L(k,t)$. □

Lemma 2.13. The number of subgraphs isomorphic to D in $T(k,t)$, Figure 3, is

$$(n-k(k-1)^{t-1})\left(\begin{array}{c}k\\3\end{array}\right)(n-4)-N(S_5)-3N(A).$$

Proof. The number of subgraphs isomorphic to D is $N(S_4)(n\text{-}4)$ -$N(S_5)$ -$3N(A)$. Now by using Lemma 2.4, the proof is completed. □

Lemma 2.14. The number of subgraphs isomorphic to E in $T(k,t)$, Figure 3, is

$$N(P_3\bigcup P_2)(n-4)-2N(B)-N(D)-2N(C)-2N(P_5)-2N(A).$$

Proof. The proof is similar to the proof of Lemma 2.13. □

Lemma 2.15. Suppose $A=(k-1)^{t-1}, B=(k-1)^{t-2}$ and $C=(k-1)^{t-3}$. Then the number of 4−matching in $T(k,t)$ is

$$\begin{aligned}1 &- 73/12n+4kA+2kB-2/3n^2k^3+59/24n^2+7/4n^2k^2-7/3n^2k\\ &- 55/8nk^2+37/4nk-5/8nk^4+9/4nk^3+1/4k^4n^2-7/4nk^3A\\ &+ 7/3nK^2A-3/4BnK^4+3/4Bnk^3-1/4k^5nA+2/3k^4nA-1/4k^6Bn\\ &+ k^5Bn-nkA+9/2k^3B+1/4k^4A^2-9k^2A+77/12K^3A+1/4k^6A^2\\ &+ 7/12k^5A-1/2k^5A^2-4k^2B_1/2k^4Cn^2-1/4k^3Cn^2+3k^3Cn-3k^4Cn\\ &- 1/4k^5Vn^2+1/4k^4Bn^2n-1/2k^5B^2/(k-1)+1/4k^4B^2/(k-1)\\ &+ 1/2k^6B^2n+k^3B^2/(k-1)+k^3B^2-k^6B^2/(k-1)-k^6B^2\\ &+ 1/4k^2Bn^2-3/4Bnk^2-5/12n^3+1/24n^4-1/4k^4B^2(k-1)\\ &+ 1/2k^5B^2(k-1)-1/4k^6B^2(k-1)-1/4k^3Bn^2-k^2Cn+k^5Cn\\ &- k^5B^2n-2k^4A-5/2k^4B-1/4k^7B^3+1/4k^6B^2/(k-1)-1/4k^5B^3(k-1)\\ &+ 3k^5B^2/(k-1)-1/4k^5B^3-1/4k^7B^3(k-1)+1/2k^6B^3+3k^5B^2\\ &- 3k^4B^2/(k-1)+1/2k^6B^3/(k-1)-3k^4B^2.\end{aligned}$$

Proof. By Theorem 3.1 in [2] the proof is straightforward. □

Ashrafi et al. [1] found the forth coefficient of the Laplacian coefficient of all trees. They proved that:

Theorem 2.16. The forth coefficient of the Laplacian polynomial of $T(k,t)$ is $5N(P_5)+8N(A)+8N(B)+9N(C)+8N(D)+12N(E)+16N(F)$, where F is a 4−matching in $T(k,t)$.

Proof. The subforests with four edges are isomorphic to P_5, A, B, C, D, E or F. If $H=P_5$, then $P(H)=5$, if H is isomorphic to A, then $P(H)=5$, if H is isomorphic to B, then $P(H)=8$, if H is isomorphic to C, then $P(H)=9$, if H is isomorphic to D, then $P(H)=8$, if H is isomorphic to E, then $P(H)=12$, and finally if H is isomorphic to F, then $P(H)=16$. Therefore, by Theorem 1.1, $5N(P_5)+8N(A)+8N(B)+9N(C)+8N(D)+12N(E)+16N(F)$. □

Lemma 2.17. The number of subgraphs in $T(k,t)$ isomorphic to (b), (c), (d), (e), (f) and (o), Figure 4, are respectively as follows:
1. $(n-1-k(k-1)^{t-1}-k(k-1)^{t-2})(k-1)^4$,
2. $(n-1-k(k-1)^{t-1}-k(k-1)^{t-2})\binom{k}{2}\binom{k-1}{2}$,
3. $(n-1-k(k-1)^{t-1})\binom{k-1}{2}^2$,
4. $2(n-1-k(k-1)^{t-1})\binom{k-1}{3}$,
5. $(n-k(k-1)^{t-1})\binom{k}{5}$,
6. $3(n-1-k(k-1)^{t-1}-k(k-1)^{t-2})\binom{k}{3}(k-1)^2$.

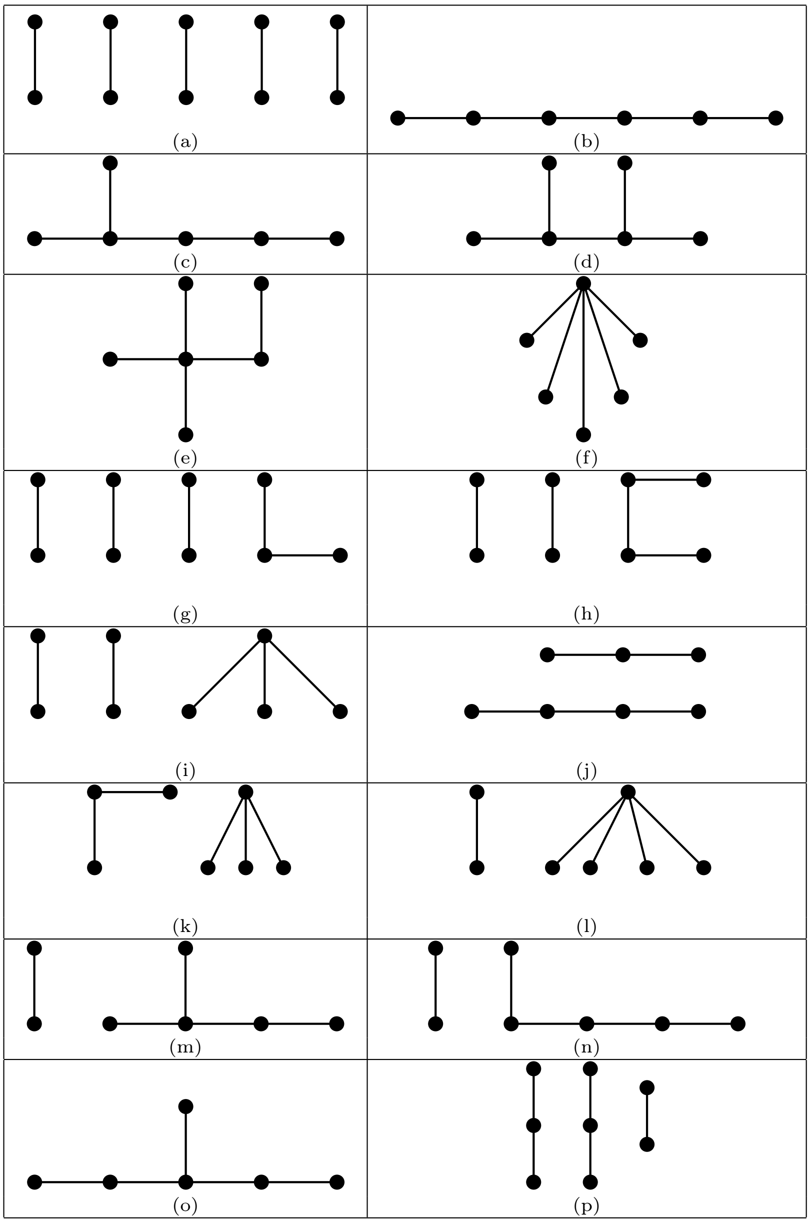

Figure 4: Subforests with 5 edges.

Proof. To count the number of subgraphs isomorphic to (b), we first count the number of middle edges. Since this graph has no cycle, we choose the neighbor of a middle edge with $(k-1)^2$ ways. Thus the number of subgraphs isomorphic to (b) is $(n-1-k(k-1)^{t-1}-k(k-1)^{t-2})(k-1)^4$. Other cases are proven in a similar argument as the first one. □

Lemma 2.18. The number of subgraphs in $T(k,t)$ isomorphic to (a), (g), (h), (i), (j), (k), (l), (m) and (n), Figure 4, are respectively as follows: $M(T(k,t),5)$, $N(H)(n-5)-2N(j)-2N(n)-2N(m)-4N(b)-4N(o)$ where H is a union of P_4 and an edge, $N(H)(n-5)-2N(k)-N(l)-3N(e)$, where H is a union of S_4 and an edge,

$$
\begin{aligned}
k(k-1)^{t-2}(k-1)[(2k-3)\binom{k-1}{2} \quad & + \quad (n-3k+1-(k-1)^{t-1})\binom{k}{2}] \\
& + \quad k(k-1)^{t-2}(k-1)(k-2)(k-2) \\
& . \quad [\binom{k-1}{2}+n-k-k(k-1)^{t-1}\binom{k}{2}] \\
& + \quad (n-k(k-1)^{t-1}-k(k-1)^{t-2})(k-1)^2 \\
& \quad\;\; [(4k-6)\binom{k-1}{2} \\
& + \quad (n-4k+6-k(k-1)^{t-1})\binom{k}{2}] \\
& \quad\;\; \binom{k}{3}(k(k-1)^{t-2}((k-1)\binom{k-1}{2} \\
& + \quad (n-1-k(k-1)^{t-1}-k+1))\binom{k}{2} \\
& + \quad (n-k(k-1)^{t-1}-k(k-1)^{t-2})(k\binom{k}{2} \\
& + \quad (n-k(k-1)^{t-1}-k)\binom{k}{2})) \\
& \quad\;\; 2(n-1-k(k-1)^{t-1})\binom{k-1}{2}(n-5) \\
& - \quad 2N(o)-N(d)-N(c)N(P_4)(n-5) \\
& - \quad 2N(P_5)-2N(c)-N(o)N(H)(n-5) \\
& - \quad 4N(2P_3\cup P_2)-4(N(J)-N(i) \\
& - \quad 2N(h)-8N(n).
\end{aligned}
$$

Proof. The proof is similar to Lemma 2.15. □

Theorem 2.19. The fifth Laplacian coefficient of $T(k,t)$ is

$$\begin{aligned}32N(a) &+ 6N(b)+6N(c)+6N(d)+6n(e)+6N(f)+24N(g)+16N(h)\\ &+ 16N(i)+12N(j)+12N(k)+10N(l)+10N(m)+10N(n)+6N(o)\\ &+ 18N(p),\end{aligned}$$

where F is a $4-$matching in $T(k,t)$.

Proof. All subforests with five edges are depicted in Figure 4. If $H = P_6$ then $P(H) = 6$, if H is isomorphic to (a), then $P(H) = 32$ and so on. Therefore, by Theorem 1.1,

$$\begin{aligned}l_5 &= 32N(a)+6N(b)+6N(c)+6N(d)+6n(e)+6N(f)+24N(g)+16N(h)\\ &+ 16N(i)+12N(j)+12N(k)+10N(l)+10N(m)+10N(n)+6N(o)\\ &+ 18N(p),\end{aligned}$$

proving the result. □

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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