

On n - A -con-cos Groups and Determination of some n - A -con-cos Groups

Ahmad Gholami and Fatemeh Mahmudi*

Abstract

In this paper, we introduce the concept of n - A -con-cos groups for $n \geq 2$, also we mention some properties of them and determine all finite abelian groups with at most two direct factors and dihedral groups, D_{2m} , where m has at most two prime factors which are n - A -con-cos.

Keywords: n^{th} -autocommutator subgroup, finite abelian groups, dihedral groups, n - A -con-cos groups.

2010 Mathematics Subject Classification: 20F28, 20F14, 20K01.

How to cite this article

A. Gholami and F. Mahmudi, On n - A -con-cos groups and determination of some n - A -con-cos groups, *Math. Interdisc. Res.* **x** (20xx) *xx-yy*.

1. Introduction

Let $Aut(G)$ denote the group of automorphisms of a given group G . For any element $g \in G$ and $\alpha \in Aut(G)$, the autocommutator of g and α is defined to be $[g, \alpha] = g^{-1}\alpha(g)$. The absolute centre and autocommutator subgroup of a group G are defined as follows:

$$\begin{aligned} L(G) &= \{g \in G \mid [g, \alpha] = 1, \forall \alpha \in Aut(G)\}, \\ K(G) &= [G, Aut(G)] = \langle [g, \alpha] \mid g \in G, \alpha \in Aut(G) \rangle. \end{aligned}$$

We define the autocommutator of higher weight inductively as follows:

$$[g, \alpha_1, \alpha_2, \dots, \alpha_n] = [[g, \alpha_1, \dots, \alpha_{n-1}], \alpha_n]$$

*Corresponding author (E-mail: a.gholami@qom.ac.ir)

Academic Editor: Ali Reza Ashrafi

Received 23 February 2018, Accepted 05 February 2019

DOI: 10.22052/mir.2019.120428.1093

for all $\alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G)$.

So the autocommutator subgroup of weight $n + 1$ is defined in the following way:

$$\begin{aligned} K_n(G) &= [K_{n-1}(G), \text{Aut}(G)] \\ &= \langle [g, \alpha_1, \alpha_2, \dots, \alpha_n] \mid g \in G, \alpha_1, \alpha_2, \dots, \alpha_n \in \text{Aut}(G) \rangle. \end{aligned}$$

Clearly, $K_n(G)$ is a characteristic subgroup of G for all $n \geq 1$. The following series of subgroups

$$G \supseteq K_1(G) = K(G) \supseteq K_2(G) \supseteq \dots \supseteq K_n(G) \supseteq \dots$$

is called the lower autocentral series of G (see also [3, 5, 7, 8]).

A group G is called A -nilpotent, if the lower autocentral series of G ends in the identity subgroup after a finite number of steps. (see also [6]).

Let G be a group and $a, b \in G$. Then a and b are said to be fused, if there exists $\alpha \in \text{Aut}(G)$ such that $\alpha(a) = b$. (see [4]).

Arora and Karan [1], defined a fusion relation in G as follows:

Two elements a and b are related if they are fused. One can easily check that fusion relation is an equivalence relation. $\overline{cl(a)} = \{\alpha(a) \mid \alpha \in \text{Aut}(G)\}$ denotes the fusion class of a in G . They also defined Auto con-cos groups. We mention the definition of it:

Let G be a group and K be a proper characteristic subgroup of G , then we have two partitions of G , one is coset partition and other is fusion class partition. If these two partitions coincide in $G - K$, that is $\overline{cl(g)} = gK$, for all $g \in G - K$, then we call the group G as Auto con-cos group.

In this paper, we introduce the new notion of n - A -con-cos groups for natural number n where $n \geq 2$ and classify all finite abelian groups with at most two direct factors and dihedral groups, D_{2m} where m has at most two prime factors which are n - A -con-cos groups.

2. Main Results

We begin with the definition of n - A -con-cos groups.

Definition 2.1. A group G would be known as n - A -con-cos, if $K_n(G) < G$ and for all $g \in G - K_n(G)$ and $\alpha_1, \dots, \alpha_{n-1} \in \text{Aut}(G)$ where $[g, \alpha_1, \dots, \alpha_{n-1}] \neq 1$ we have

$$\overline{cl([g, \alpha_1, \dots, \alpha_{n-1}])} = [g, \alpha_1, \dots, \alpha_{n-1}]K_n(G) - 1.$$

The following theorem is useful in our investigation.

Theorem 2.2. Let G be a group and $K_n(G) = K_{n-1}(G) < G$ and $K_{n-1}(G)$ be the union of two fusion classes. Then the group G is n - A -con-cos.

Proof. Let $K_n(G) = K_{n-1}(G) = 1 \cup \overline{cl(x)}$, where $1 \neq x \in G$. Hence for every $g \in G - K_n(G)$ and $\alpha_1, \dots, \alpha_{n-1} \in \overline{Aut(G)}$ where $[g, \alpha_1, \dots, \alpha_{n-1}] \neq 1$ we have $[g, \alpha_1, \dots, \alpha_{n-1}] \in K_{n-1}(G) - 1 = \overline{cl(x)}$, therefore

$$\overline{cl([g, \alpha_1, \dots, \alpha_{n-1}])} = \overline{cl(x)} = K_n(G) - 1,$$

also $[g, \alpha_1, \dots, \alpha_{n-1}]K_n(G) - 1 = K_n(G) - 1$, which implies that

$$\overline{cl([g, \alpha_1, \dots, \alpha_{n-1}])} = [g, \alpha_1, \dots, \alpha_{n-1}]K_n(G) - 1.$$

Hence the group G is n - A -con-cos. □

For instance, the group $C_3 \rtimes C_4 = \langle x, y \mid x^4 = y^3 = 1, x^{-1}yx = y^2 \rangle$ is 3- A -con-cos, since

$$K_3(C_3 \rtimes C_4) = K_2(C_3 \rtimes C_4) = \langle y \rangle = 1 \cup \overline{cl(y)}.$$

Theorem 2.3. Let G be an A -nilpotent group where $|G| > 2$ and $K_n(G) = 1$. Then the group G is n - A -con-cos.

Proof. For every $x \in G$ and $\alpha_1, \dots, \alpha_n \in \overline{Aut(G)}$, we have $[x, \alpha_1, \dots, \alpha_n] = 1$, hence $[x, \alpha_1, \dots, \alpha_{n-1}]^{-1}\alpha_n([x, \alpha_1, \dots, \alpha_{n-1}]) = 1$, therefore

$$\overline{cl([x, \alpha_1, \dots, \alpha_{n-1}])} = \{[x, \alpha_1, \dots, \alpha_{n-1}]\},$$

also for every $g \in G - K_n(G)$ and $\alpha_1, \dots, \alpha_{n-1} \in \overline{Aut(G)}$ where $[g, \alpha_1, \dots, \alpha_{n-1}] \neq 1$ we have

$$[g, \alpha_1, \dots, \alpha_{n-1}]K_n(G) - 1 = \{[g, \alpha_1, \dots, \alpha_{n-1}]\}.$$

This implies the result. □

For instance, the cyclic group C_4 is 2- A -con-cos, since C_4 is A -nilpotent and $K_2(C_4) = 1$ and the dihedral group D_8 is 3- A -con-cos since by 1.4 of [6], D_8 is A -nilpotent and $K_3(D_8) = 1$. We have following remark.

Remark 1. Let G be a finite abelian group of odd order. Then by Corollary 2.4 of [6], $K_n(G) = G$, for any natural number n . Hence G is not n - A -con-cos.

The following theorem is one of the main results of this paper.

Theorem 2.4. Let n be a natural number where $n \geq 2$. Then the finite n - A -con-cos abelian groups with at most two direct factors are:

- C_{2^t} , for $1 \leq t \leq n + 1$,

- $C_{2^t} \times C_p$, for $1 \leq t \leq n - 1$,
- $C_{2^t} \times C_2$, for $2 \leq t \leq n + 1$,
- $C_{2^t} \times C_{2^s}$, for $t \leq n + 1$ and $2 \leq s \leq t - 2$,
- $C_{2^t} \times C_{2^s}$, for even number n where $t \leq n - 1$, $s \leq \frac{n}{2}$ and $t = s + 1 \geq 3$,
- $C_{2^t} \times C_{2^s}$, for odd number n where $t \leq n - 1$, $t \leq \frac{n+1}{2}$ and $t = s + 1 \geq 3$, where p is an odd prime number and t, s are natural numbers.

Proof. By Remark 1, we should investigate finite abelian groups with at most two direct factors of even order.

Clearly the group C_{2^t} is n - A -con-cos, for $1 \leq t \leq n - 1$. The group C_{2^n} is n - A -con-cos, since it is A -nilpotent and $K_n(C_{2^n}) = 1$.

Let $C_{2^{n+1}} = \langle x \mid x^{2^{n+1}} = 1 \rangle$, hence by [5, Lemma 2.2],

$$K_n(C_{2^{n+1}}) = C_{2^{n+1}}^{2^n} = \langle x^{2^n} \rangle, \quad K_{n-1}(C_{2^{n+1}}) = C_{2^{n+1}}^{2^{n-1}} = \langle x^{2^{n-1}} \rangle.$$

So for every $g \in C_{2^{n+1}} - K_n(C_{2^{n+1}})$ and $\alpha_1, \dots, \alpha_{n-1} \in \text{Aut}(C_{2^{n+1}})$ where $[g, \alpha_1, \dots, \alpha_{n-1}] \neq 1$. We have

$$[g, \alpha_1, \dots, \alpha_{n-1}] \in K_{n-1}(C_{2^{n+1}}) - 1 = \{x^{2^{n-1}}, x^{2^n}, x^{3 \cdot 2^{n-1}}\}.$$

Clearly

$$\{x^{2^{n-1}}, x^{3 \cdot 2^{n-1}}\} = \overline{cl(x^{3 \cdot 2^{n-1}})} = \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}} K_n(C_{2^{n+1}}) - 1$$

and

$$\{x^{2^n}\} = \overline{cl(x^{2^n})} = x^{2^n} K_n(C_{2^{n+1}}) - 1.$$

Hence the group $C_{2^{n+1}}$ is n - A -con-cos.

Suppose that $t \geq n + 2$ and $C_{2^t} = \langle x \mid x^{2^t} = 1 \rangle$, we have $K_n(C_{2^t}) = \langle x^{2^n} \rangle$, hence $x \in C_{2^t} - K_n(C_{2^t})$. Consider $\alpha, \beta \in \text{Aut}(C_{2^t})$ with $\alpha(x) = x^{2^{t-n+1}+1}$ and $\beta(x) = x^3$. It is easy to check that $[x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{t-1}}$. By [7, Theorem 2.2], $x^{2^{t-1}} \in L(C_{2^t})$. Hence, $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$ but $x^{2^{t-1}} K_n(C_{2^t}) - 1 = K_n(C_{2^t}) - 1$ has $2^{t-n} - 1$ elements (There are at least 3 elements), so

$$\overline{cl([x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(C_{2^t}) - 1.$$

Thus the group C_{2^t} is not n - A -con-cos, for $t \geq n + 2$.

In following we investigate the group $C_{2^t} \times C_{p^s}$ for natural numbers t, s with the presentation

$$\langle x, y \mid x^{2^t} = y^{p^s} = [x, y] = 1 \rangle.$$

There are five cases:

Case 1: $1 \leq t \leq n - 1$ and $s = 1$. By [5, Lemma 2.1 and Lemma 2.2],

$$K_n(C_{2^t} \times C_p) = K_n(C_{2^t}) \times K_n(C_p) = 1 \times C_p = C_p$$

and $K_{n-1}(C_{2^t} \times C_p) = C_p$. Clearly $C_p = \langle y \rangle = 1 \cup \overline{cl(y)}$. So by Theorem 2.2, the group $C_{2^t} \times C_p$ is n - A -con-cos, for $1 \leq t \leq n - 1$.

Case 2: $t = n$ and $s = 1$. We know $K_n(C_{2^n} \times C_p) = \langle y \rangle$. Thus $x \in (C_{2^n} \times C_p) - K_n(C_{2^n} \times C_p)$. Consider the automorphism α of $C_{2^n} \times C_p$ with $\alpha(x) = x^3$ and $\alpha(y) = y$. We have $[x, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}] = x^{2^{n-1}}$ and $\overline{cl(x^{2^{n-1}})} = \{x^{2^{n-1}}\}$,

but $x^{2^{n-1}} K_n(C_{2^n} \times C_p) - 1$ has p elements. Hence the group $C_{2^n} \times C_p$ is not n - A -con-cos.

Case 3: $t \geq n + 1$ and $s = 1$. We have $K_n(C_{2^t} \times C_p) = \langle x^{2^n} \rangle \times \langle y \rangle$. So $x \in (C_{2^t} \times C_p) - K_n(C_{2^t} \times C_p)$. Consider $\alpha, \beta \in \text{Aut}(C_{2^t} \times C_p)$ with

$$\alpha(x) = x^{2^{t-n+1}+1}, \quad \alpha(y) = y, \quad \beta(x) = x^3, \quad \beta(y) = y.$$

We have $[x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{t-1}}$ and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$ but $x^{2^{t-1}} K_n(C_{2^t} \times C_p) - 1$ has $2^{t-n}p - 1$ elements. Hence the group $C_{2^t} \times C_p$ is not n - A -con-cos, for $t \geq n + 1$.

Case 4: $t = 1$ and $s \geq 2$. We know $K_n(C_2 \times C_{p^s}) = \langle y \rangle$. So $xy \in (C_2 \times C_{p^s}) - K_n(C_2 \times C_{p^s})$. Consider $\alpha \in \text{Aut}(C_2 \times C_{p^s})$ with $\alpha(x) = x, \alpha(y) = y^2$, therefore $[xy, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}] = y$. Clearly $\overline{cl(y)}$ has $\phi(p^s)$ elements, where ϕ is the Euler's phi function, but $y K_n(C_2 \times C_{p^s}) - 1$ has $p^s - 1$ elements. Since $s \geq 2$ we conclude that the group $C_2 \times C_{p^s}$ is not n - A -con-cos, for $s \geq 2$.

Case 5: $t \geq 2$ and $s \geq 2$. $K_n(C_{2^t} \times C_{p^s}) = \langle y \rangle$ for $t \leq n$ and $K_n(C_{2^t} \times C_{p^s}) = \langle x^{2^n} \rangle \times \langle y \rangle$ for $t \geq n + 1$. Hence $xy \in (C_{2^t} \times C_{p^s}) - K_n(C_{2^t} \times C_{p^s})$. Consider $\alpha \in \text{Aut}(C_{2^t} \times C_{p^s})$ with $\alpha(x) = x$ and $\alpha(y) = y^2$. We have $[xy, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}] = y$. Clearly $\overline{cl(y)}$ has $\phi(p^s)$ elements but $y K_n(C_{2^t} \times C_{p^s}) - 1$ has $p^s - 1$ elements for $t \leq n$ and $2^{t-n}p^s - 1$ elements for $t \geq n + 1$, which implies that the group $C_{2^t} \times C_{p^s}$ is not n - A -con-cos, for $t, s \geq 2$.

Now we investigate finite abelian 2-groups with two direct factors. Let

$$C_{2^t} \times C_{2^s} = \langle x, y \mid x^{2^t} = y^{2^s} = [x, y] = 1 \rangle$$

for natural numbers t, s .

There are six cases:

Case 1: $t \geq 2$ and $s = 1$. It is easy to check that the group $C_4 \times C_2$ is 2- A -con-cos. Also $K_n(C_4 \times C_2) = 1$, for $n \geq 3$. So the group $C_4 \times C_2$ is n - A -con-cos.

The group $C_{2^t} \times C_2$ is n - A -con-cos, for $3 \leq t \leq n$, since $K_n(C_{2^t} \times C_2) = 1$.

If $t = n + 1$, then $K_n(C_{2^{n+1}} \times C_2) = \langle x^{2^n} \rangle$. Clearly $C_8 \times C_2$ is 2- A -con-cos. For $n \geq 3$, we have

$$K_{n-1}(C_{2^{n+1}} \times C_2) = \langle x^{2^{n-1}} \rangle = \{1, x^{2^{n-1}}, x^{2^n}, x^{3 \cdot 2^{n-1}}\}.$$

It is easy to check that

$$\{x^{2^{n-1}}, x^{3 \cdot 2^{n-1}}\} = \overline{cl(x^{3 \cdot 2^{n-1}})} = \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}} K_n(C_{2^{n+1}} \times C_2) - 1$$

and

$$\{x^{2^n}\} = \overline{cl(x^{2^n})} = x^{2^n} K_n(C_{2^{n+1}} \times C_2) - 1.$$

Thus the group $C_{2^{n+1}} \times C_2$ is n - A -con-cos.

If $t \geq n + 2$, then $K_n(C_{2^t} \times C_2) = \langle x^{2^n} \rangle$. So $x \in (C_{2^t} \times C_2) - K_n(C_{2^t} \times C_2)$. Consider $\alpha, \beta \in \text{Aut}(C_{2^t} \times C_2)$ with

$$\alpha(x) = x^{2^{t-n+1}+1}, \quad \alpha(y) = y, \quad \beta(x) = x^3, \quad \beta(y) = y.$$

Thus $[x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{t-1}}$ and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$, but $x^{2^{t-1}} K_n(C_{2^t} \times C_2) - 1$ has $2^{t-n} - 1$ elements. Hence the group $C_{2^t} \times C_2$ is not n - A -con-cos, for $t \geq n + 2$.

Case 2: $t = s$. By [2, Theorem 3.1(ii)], $K_n(C_{2^t} \times C_{2^t}) = C_{2^t} \times C_{2^t}$. Hence the group $C_{2^t} \times C_{2^t}$ is not n - A -con-cos.

Case 3: $t > s \geq 2$ and $t \leq n - 1$. If $t \geq s + 2$, then the group $C_{2^t} \times C_{2^s}$ is n - A -con-cos, since $K_n(C_{2^t} \times C_{2^s}) = 1$.

If $t = s + 1$ and n is even, then by [2, Corollary 3.2],

$$K_n(C_{2^t} \times C_{2^{t-1}}) = \langle x^{2^{\frac{n}{2}}} \rangle \times \langle y^{2^{\frac{n}{2}}} \rangle.$$

If $\frac{n}{2} \geq t$, then the group $C_{2^t} \times C_{2^{t-1}}$ is n - A -con-cos, since $K_n(C_{2^t} \times C_{2^{t-1}}) = 1$.

For $\frac{n}{2} = t - 1$ we have $K_n(C_{2^t} \times C_{2^{t-1}}) = \langle x^{2^{t-1}} \rangle$ and

$$K_{n-1}(C_{2^t} \times C_{2^{t-1}}) = \langle x^{2^{t-1}} \rangle \times \langle y^{2^{t-2}} \rangle = \{1, x^{2^{t-1}}, y^{2^{t-2}}, x^{2^{t-1}}y^{2^{t-2}}\}.$$

Clearly

$$\{x^{2^{t-1}}\} = \overline{cl(x^{2^{t-1}})} = x^{2^{t-1}}K_n(C_{2^t} \times C_{2^{t-1}}) - 1$$

and

$$\{x^{2^{t-1}}y^{2^{t-2}}, y^{2^{t-2}}\} = \overline{cl(x^{2^{t-1}}y^{2^{t-2}})} = \overline{cl(y^{2^{t-2}})} = y^{2^{t-2}}K_n(C_{2^t} \times C_{2^{t-1}}) - 1$$

which implies that if n is even and $\frac{n}{2} = t - 1 = s \geq 2$, then the group $C_{2^t} \times C_{2^s}$ is n - A -con-cos.

Now we investigate the group $C_{2^t} \times C_{2^s}$ for $\frac{n}{2} < t - 1$ and $t = n - 1$.

We have $x \in (C_{2^{n-1}} \times C_{2^{n-2}}) - K_n(C_{2^{n-1}} \times C_{2^{n-2}})$. Consider the automorphisms α, β, γ of $C_{2^{n-1}} \times C_{2^{n-2}}$ with

$$\alpha(x) = x^3, \quad \alpha(y) = y, \quad \beta(x) = xy, \quad \beta(y) = y, \quad \gamma(x) = x, \quad \gamma(y) = x^2y.$$

Thus $[x, \underbrace{\alpha, \dots, \alpha}_{n-3\text{-times}}, \beta, \gamma] = x^{2^{n-2}}$. Clearly $\overline{cl(x^{2^{n-2}})} = \{x^{2^{n-2}}\}$ but

$$x^{2^{n-2}}K_n(C_{2^{n-1}} \times C_{2^{n-2}}) - 1 = K_n(C_{2^{n-1}} \times C_{2^{n-2}}) - 1$$

has $2^{n-3} - 1$ elements. Hence this group is not n - A -con-cos.

Similarity, we can show that the group $C_{2^t} \times C_{2^s}$ is not n - A -con-cos, for $\frac{n}{2} < t - 1$ and $t < n - 1$.

If $t = s + 1$ and n is odd, then $K_n(C_{2^t} \times C_{2^{t-1}}) = \langle x^2 \frac{n+1}{2} \rangle \times \langle y^2 \frac{n-1}{2} \rangle$. For $\frac{n+1}{2} \geq t$, we have $\frac{n-1}{2} \geq t - 1$, so the group $C_{2^t} \times C_{2^{t-1}}$ is n - A -con-cos, since $K_n(C_{2^t} \times C_{2^{t-1}}) = 1$.

For $\frac{n+1}{2} < t$ and $t = n - 1$, we have $x \in (C_{2^{n-1}} \times C_{2^{n-2}}) - K_n(C_{2^{n-1}} \times C_{2^{n-2}})$. Consider the automorphisms α, β, γ of $C_{2^{n-1}} \times C_{2^{n-2}}$ with

$$\alpha(x) = x^3, \quad \alpha(y) = y, \quad \beta(x) = xy, \quad \beta(y) = y, \quad \gamma(x) = x, \quad \gamma(y) = x^2y.$$

It is easy to check that

$$\overline{cl([x, \underbrace{\alpha, \dots, \alpha}_{n-3\text{-times}}, \beta, \gamma])} \neq [x, \underbrace{\alpha, \dots, \alpha}_{n-3\text{-times}}, \beta, \gamma]K_n(C_{2^{n-1}} \times C_{2^{n-2}}) - 1.$$

Therefore this group is not n - A -con-cos.

Similarly, we can show that the group $C_{2^t} \times C_{2^{t-1}}$ is not n - A -con-cos, for $\frac{n+1}{2} < t$ and $t < n-1$.

Case 4: $t > s \geq 2$ and $t = n$. If $t \geq s+2$, then the group $C_{2^n} \times C_{2^s}$ is n - A -con-cos, since $K_n(C_{2^n} \times C_{2^s}) = 1$.

If $t = s+1$ and n is even, then the group $C_{2^n} \times C_{2^{n-1}}$ is not n - A -con-cos, since

$$K_n(C_{2^n} \times C_{2^{n-1}}) = \langle x^{\frac{n}{2}} \rangle \times \langle y^{\frac{n}{2}} \rangle.$$

Thus $x \in (C_{2^n} \times C_{2^{n-1}}) - K_n(C_{2^n} \times C_{2^{n-1}})$ and for the automorphism α of $C_{2^n} \times C_{2^{n-1}}$ with $\alpha(x) = x^3$, $\alpha(y) = y$, we have $[x, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}] = x^{2^{n-1}}$ and $\overline{cl(x^{2^{n-1}})} = \{x^{2^{n-1}}\}$ but $x^{2^{n-1}}K_n(C_{2^n} \times C_{2^{n-1}}) - 1$ has $2^{n-1} - 1$ elements.

If $t = s+1$ and n is odd, then the group $C_{2^n} \times C_{2^{n-1}}$ is not n - A -con-cos, since

$$K_n(C_{2^n} \times C_{2^{n-1}}) = \langle x^{\frac{n+1}{2}} \rangle \times \langle y^{\frac{n-1}{2}} \rangle$$

and $x \in (C_{2^n} \times C_{2^{n-1}}) - K_n(C_{2^n} \times C_{2^{n-1}})$. Consider $\alpha \in \text{Aut}(C_{2^n} \times C_{2^{n-1}})$ with $\alpha(x) = x^3$, $\alpha(y) = y$. Clearly

$$\overline{cl([x, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}])} \neq [x, \underbrace{\alpha, \dots, \alpha}_{n-1\text{-times}}]K_n(C_{2^n} \times C_{2^{n-1}}) - 1.$$

Case 5: $t > s \geq 2$ and $t = n+1$. If $t \geq s+2$, then $K_n(C_{2^{n+1}} \times C_{2^s}) = \langle x^{2^n} \rangle$. By assumption $n-2 \geq s-1$.

If $n-2 = s-1$, then

$$K_{n-1}(C_{2^{n+1}} \times C_{2^s}) = \langle x^{2^{n-1}} \rangle \times \langle y^{2^{n-2}} \rangle$$

and if $n-2 \geq s$, then

$$K_{n-1}(C_{2^{n+1}} \times C_{2^s}) = \langle x^{2^{n-1}} \rangle.$$

In two cases for every $a \in K_{n-1}(C_{2^{n+1}} \times C_{2^s}) - 1$ we have

$$\overline{cl(a)} = aK_n(C_{2^{n+1}} \times C_{2^s}) - 1$$

and it shows that the group $C_{2^t} \times C_{2^s}$ is n - A -con-cos, for $t = n+1$ and $t \geq s+2 \geq 4$.

Now we investigate the group $C_{2^{n+1}} \times C_{2^n}$, for $t = s + 1$.

We have

$$K_n(C_{2^{n+1}} \times C_{2^n}) = \langle x^{2^{\lfloor \frac{n+1}{2} \rfloor}} \rangle \times \langle y^{2^{\lfloor \frac{n}{2} \rfloor}} \rangle.$$

Thus $x \in (C_{2^{n+1}} \times C_{2^n}) - K_n(C_{2^{n+1}} \times C_{2^n})$. Consider $\alpha, \beta \in \text{Aut}(C_{2^{n+1}} \times C_{2^n})$ with

$$\alpha(x) = x^5, \quad \alpha(y) = y, \quad \beta(x) = x^3, \quad \beta(y) = y.$$

We have

$$\overline{cl([x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(C_{2^{n+1}} \times C_{2^n}) - 1,$$

which implies that this group is not n - A -con-cos.

Case 6: $t > s \geq 2$ and $t \geq n + 2$. If $t \geq s + 2$, then $K_n(C_{2^t} \times C_{2^s}) = \langle x^{2^n} \rangle \times \langle y^{2^{n-1}} \rangle$. So $x \in (C_{2^t} \times C_{2^s}) - K_n(C_{2^t} \times C_{2^s})$. Consider $\alpha, \beta \in \text{Aut}(C_{2^t} \times C_{2^s})$ with

$$\alpha(x) = x^{2^{t-n+1}+1}, \quad \alpha(y) = y, \quad \beta(x) = x^3, \quad \beta(y) = y.$$

We have $[x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{t-1}}$ and $\overline{cl(x^{2^{t-1}})} = \{x^{2^{t-1}}\}$. On the other hand, $x^{2^{t-1}} K_n(C_{2^t} \times C_{2^s}) - 1$ has $2^{t+s-2n+1} - 1$ elements if $s > n - 1$ and has $2^{t-n} - 1$ elements if $s \leq n - 1$, which implies that

$$\overline{cl([x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(C_{2^t} \times C_{2^s}) - 1.$$

Hence the group $C_{2^t} \times C_{2^s}$ is not n - A -con-cos, for $t \geq s + 2 \geq 4$ and $t \geq n + 2$.

If $t = s + 1$, then similar to the previous case we can show that the group $C_{2^t} \times C_{2^s}$ is not n - A -con-cos. So the proof is completed. \square

In following theorem we investigate some dihedral groups.

Theorem 2.5. Let m, n be natural numbers where m has at most two prime factors and $n \geq 2$. Then the dihedral group D_{2m} is n - A -con-cos if $m = 2^{t-1}$, for natural number t , where $3 \leq t \leq n + 1$ or $m = p$ or $m = 2^t p$, for odd prime number p and natural number t where $1 \leq t \leq n - 2$.

Proof. Let $D_{2m} = \langle x, y \mid x^m = y^2 = (xy)^2 = 1 \rangle$ be the dihedral group of order $2m$.

At first we assume that $m = 2^{t-1}$ where t is a natural number and $t \geq 2$.

Clearly the group $D_4 = C_2 \times C_2$ is not n - A -con-cos. If $3 \leq t \leq n-1$, then the group D_{2^t} is n - A -con-cos, since by [2, Theorem 1.1], $K_t(D_{2^t}) = \langle x^{2^{t-1}} \rangle = 1$ and $K_n(D_{2^t}) \subseteq K_t(D_{2^t})$.

If $t = n$ and $n \geq 3$, then D_{2^n} is A -nilpotent and $K_n(D_{2^n}) = 1$. Hence by Theorem 2.3, the group D_{2^n} is n - A -con-cos, for $n \geq 3$.

If $t = n+1$, then $K_n(D_{2^{n+1}}) = \langle x^{2^{n-1}} \rangle = \{1, x^{2^{n-1}}\}$ and

$$K_{n-1}(D_{2^{n+1}}) = \langle x^{2^{n-2}} \rangle = \{1, x^{2^{n-2}}, x^{2^{n-1}}, x^{3 \cdot 2^{n-2}}\}.$$

So for every $g \in D_{2^{n+1}} - K_n(D_{2^{n+1}})$ and $\alpha_1, \dots, \alpha_{n-1} \in \text{Aut}(D_{2^{n+1}})$ where

$[g, \alpha_1, \dots, \alpha_{n-1}] \neq 1$ we have $[g, \alpha_1, \dots, \alpha_{n-1}] \in \{x^{2^{n-2}}, x^{2^{n-1}}, x^{3 \cdot 2^{n-2}}\}$. Clearly

$$\{x^{2^{n-2}}, x^{3 \cdot 2^{n-2}}\} = \overline{cl(x^{2^{n-2}})} = \overline{cl(x^{3 \cdot 2^{n-2}})} = x^{2^{n-2}} K_n(D_{2^{n+1}}) - 1,$$

$$\{x^{2^{n-1}}\} = \overline{cl(x^{2^{n-1}})} = x^{2^{n-1}} K_n(D_{2^{n+1}}) - 1,$$

which implies that the group $D_{2^{n+1}}$ is n - A -con-cos.

If $t \geq n+2$, then $K_n(D_{2^t}) = \langle x^{2^{n-1}} \rangle$, so $x \in D_{2^t} - K_n(D_{2^t})$. Consider $\alpha, \beta \in \text{Aut}(D_{2^t})$ with

$$\alpha(x) = x^{2^{t-n}+1}, \quad \alpha(y) = y, \quad \beta(x) = x^3, \quad \beta(y) = y.$$

We have $[x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{t-2}}$. Clearly $\overline{cl(x^{2^{t-2}})} = \{x^{2^{t-2}}\}$. On the other hand, $x^{2^{t-2}} K_n(D_{2^t}) - 1 = \langle x^{2^{n-1}} \rangle - 1$ has $2^{t-n} - 1$ elements (There are at least 3 elements). Therefore

$$\overline{cl([x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [x, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(D_{2^t}) - 1,$$

which implies that the group D_{2^t} is not n - A -con-cos, for $t \geq n+2$.

Now we investigate the case that $m = p^t$, for odd prime number p and natural number t .

The group D_{2p} is n - A -con-cos, since by [2, Theorem 1.1], $K_{n-1}(D_{2p}) = K_n(D_{2p})$. Also $\langle x \rangle = 1 \cup \overline{cl(x)}$, hence the claim follows from 2.2.

If $t \geq 2$, then $K_n(D_{2p^t}) = \langle x \rangle$, therefore $y \in D_{2p^t} - K_n(D_{2p^t})$. Consider $\alpha, \beta \in \text{Aut}(D_{2p^t})$ with

$$\alpha(x) = x, \quad \alpha(y) = x^{p^t-1}y, \quad \beta(x) = x^2, \quad \beta(y) = y.$$

We have $[y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x$ and $\overline{cl(x)}$ has $\phi(p^t)$ elements. On the other hand, $xK_n(D_{2p^t}) - 1 = \langle x \rangle - 1$ has $p^t - 1$ elements, since $t \geq 2$ we have

$$\overline{cl([y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}]K_n(D_{2p^t}) - 1.$$

Thus the group D_{2p^t} is not n - A -con-cos, for $t \geq 2$.

We assume that m has two distinct prime factors.

Let $m = p^t q^s$ where p, q are distinct odd prime numbers and t, s are natural numbers.

We know $K_n(D_{2p^t q^s}) = \langle x \rangle$, therefore $x^{p^t q^s - 1} y \in D_{2p^t q^s} - K_n(D_{2p^t q^s})$. Consider the automorphisms α and β of $D_{2p^t q^s}$ with

$$\alpha(x) = x, \quad \alpha(y) = x^{p^t q^s - 1} y, \quad \beta(x) = x^2, \quad \beta(y) = y.$$

It is easy to check that

$$\overline{cl([x^{p^t q^s - 1} y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [x^{p^t q^s - 1} y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}]K_n(D_{2p^t q^s}) - 1.$$

Hence the group $D_{2p^t q^s}$ is not n - A -con-cos.

Finally we investigate the group $D_{2^{t+1} p^s}$ where p is an odd prime number and t, s are natural numbers.

We have three cases:

Case 1: $2 \leq t + 1 \leq n - 1$ and $s = 1$. We have

$$K_{n-1}(D_{2^{t+1} p}) = K_n(D_{2^{t+1} p}) = \langle x^{2^{n-1}} \rangle = 1 \cup \overline{cl(x^{2^{n-1}})}$$

So by 2.2, the group $D_{2^{t+1} p}$ is n - A -con-cos, for $2 \leq t + 1 \leq n - 1$.

Case 2: $2 \leq t + 1 \leq n - 1$ and $s \geq 2$. We have $K_n(D_{2^{t+1} p^s}) = \langle x^{2^{n-1}} \rangle$, so $y \in D_{2^{t+1} p^s} - K_n(D_{2^{t+1} p^s})$. Consider $\alpha, \beta \in \text{Aut}(D_{2^{t+1} p^s})$ with

$$\alpha(x) = x, \quad \alpha(y) = xy, \quad \beta(x) = x^{2^t p^s - 1}, \quad \beta(y) = y.$$

We have $[y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{(-1)^{n-1} 2^{n-2}}$.

If n is odd, then $[y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{2^{n-2}}$ and $\overline{cl(x^{2^{n-2}})}$ has $p^s - p^{s-1}$ elements but $x^{2^{n-2}} K_n(D_{2^{t+1} p^s}) - 1 = \langle x^{2^{n-1}} \rangle - 1$ has $p^s - 1$ elements.

If n is even, then $[y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] = x^{-2^{n-2}}$ and $\overline{cl(x^{-2^{n-2}})} = \overline{cl(x^{2^{n-2}})}$ has $p^s - p^{s-1}$ elements but $x^{-2^{n-2}} K_n(D_{2^{t+1}p^s}) - 1$ has $p^s - 1$ elements. Thus the group $D_{2^{t+1}p^s}$ is not n - A -con-cos, for $2 \leq t+1 \leq n-1$ and $s \geq 2$.

Case 3: $t+1 \geq n$ and $s \geq 1$. We have $K_n(D_{2^{t+1}p^s}) = \langle x^{2^{n-1}} \rangle$, so $y \in D_{2^{t+1}p^s} - K_n(D_{2^{t+1}p^s})$. Consider the automorphisms α and β of $D_{2^{t+1}p^s}$ with

$$\alpha(x) = x, \quad \alpha(y) = xy, \quad \beta(x) = x^{2^t p^s - 1}, \quad \beta(y) = y.$$

It is easy to check that $\overline{cl([y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])}$ has $2^{t-n+1}p^{s-1}(p-1)$ elements but $[y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(D_{2^{t+1}p^s}) - 1$ has $2^{t-n+1}p^s$ elements, therefore

$$\overline{cl([y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}])} \neq [y, \alpha, \underbrace{\beta, \dots, \beta}_{n-2\text{-times}}] K_n(D_{2^{t+1}p^s}) - 1.$$

Hence the group $D_{2^{t+1}p^s}$ is not n - A -con-cos, for $t+1 \geq n$ and $s \geq 1$. Which completes the proof. \square

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

- [1] H. Arora and R. Karan, A Note on Autocamina Groups, *Note Math.* **35**(2) (2015) 39 – 50.
- [2] A. Gholamian and M. M. Nasrabadi, On m^{th} -autocommutator subgroup of finite abelian groups, *J. Linear Topol. Algebra* **5**(2) (2016) 135 – 144.
- [3] P. Hegarty, The absolute centre of a group, *J. Algebra* **169**(3) (1994) 929–935.
- [4] C. H. Li and C. E. Praeger, Finite groups in which any two elements of the same order are either fused or inverse-fused, *Comm. Algebra* **25**(10) (1997) 3081 – 3118.
- [5] M. R. R. Moghaddam, F. Parvaneh and M. Naghshineh, The lower autocentral series of abelian groups, *Bull. Korean Math. Soc.* **48**(1) (2011) 79 – 83.
- [6] M. M. Nasrabadi and A. Gholamian, On A -nilpotent abelian groups, *Proc. Indian Acad. Sci. Math. Sci.* **124**(4) (2014) 517 – 525.

- [7] F. Parvaneh, Some properties of autonilpotent groups, *Ital. J. Pure Appl. Math.* (35) (2015) 1 – 8.
- [8] F. Parvaneh and M. R. R. Moghaddam, Some properties of autosoluble groups, *J. Math. Ext.* 5(1) (2010) 13 – 19.

Ahmad Gholami
Department of Mathematics,
Faculty of Science,
University of Qom,
Qom, I. R. Iran
E-mail: a.gholami@qom.ac.ir

Fatemeh Mahmudi
Department of Mathematics,
Faculty of Science,
University of Qom,
Qom, I. R. Iran
E-mail: mahmodi.fateme64@gmail.com