On the Estrada Index of Seidel Matrix
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Abstract
Let $G$ be a simple graph with $n$ vertices and with the Seidel matrix $S$. Suppose $\mu_1, \mu_2, \ldots, \mu_s$ are the Seidel eigenvalues of $G$. The Estrada index of the Seidel matrix of $G$ is defined as $SEE(G) = \sum_{\mu_i} e^{\mu_i}$. In this paper, we compute the Estrada index of the Seidel matrix of some known graphs. Also, some bounds for the Seidel energy of graphs are given.

Keywords: Seidel matrix, Seidel eigenvalue, Estrada index.

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1. Introduction

Here, we recall some definitions that will be used in this paper. Let $G$ be a simple graph with $n$ vertices, $m$ edges and let $A$ denote the adjacency matrix of $G$. The eigenvalues of the graph $G$ are the roots of characteristic polynomial $P_G(\lambda) = \det(\lambda I - A)$, where $I$ is the identity matrix. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of $G$. The rank of the matrix $A$, denoted by $\text{rank}(A)$, is equal to the maximum number of linearly independent columns of $A$. For given graph $G$, its complement is denoted by $\overline{G}$. The graph $G - \{v\}$ is a graph obtaining from $G$ by removing the vertex $v$ with all edges connected to $v$. A complete graph on $n$ vertices is denoted by $K_n$.

In 1966, van Lint and Seidel in [19] introduced a symmetric $(0, -1, 1)$-adjacency matrix for a graph $G$, called the Seidel matrix of $G$, as $S(G) = J - I - 2A$, where $J$ is the matrix with entries 1 in every position. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$ be the distinct Seidel eigenvalues of $G$ with multiplicity $t_1, t_2, \ldots, t_s$, respectively. The multiset $\text{Spec}_S(G) = \{[\mu_1]^{t_1}, [\mu_2]^{t_2}, \ldots, [\mu_s]^{t_s}\}$ is called the Seidel spectrum of $G$. 

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Let $U_1$ and $U_2 = V(G) \setminus U_1$ be the partitioned sets of the vertex set $V(G)$ of a graph $G$. Let $G'$ be the graph obtained from $G$ by deleting all edges between $U_1$ and $U_2$ and inserting all edges between $U_1$ and $U_2$ that are not presented in $G$. Then $G'$ and $G$ are said to be Seidel switching with respect to $U_1$, see [17]. If $G'$ and $G$ are Seidel switching, then $S(G')$ and $S(G)$ are similar and therefore $G'$ and $G$ have the same Seidel eigenvalues, see [4].

The Estrada index of a graph $G$ is defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$, where $\lambda_i$’s are eigenvalues of $G$. This graph invariant was introduced by Ernesto Estrada, which has noteworthy chemical applications, see [6, 7, 8, 9] for details. We encourage the interested readers to consult papers [1, 5, 10, 11, 15] for the mathematical properties of Estrada index. Askari et al. [2] proposed the Estrada index of Seidel matrix of a graph $G$ as $SEE(G) = \sum_{i=1}^{n} e^{\mu_i}$, where $\mu_i$’s are the Seidel eigenvalues of $G$. From the power-series expansion of $e^x$, we have

$$SEE(G) = \sum_{k=0}^{\infty} \frac{S_k(G)}{k!},$$

where $S_k(G) = \sum_{i=1}^{n} \mu_i^k$.

**Lemma 1.1.** [4] For any graph $G$ on $n \geq 2$ vertices, we have

i) $\sum_{i=1}^{n} \mu_i = 0$,

ii) $\sum_{i=1}^{n} \mu_i^2 = n(n-1)$,

iii) $\mu_1 \geq 1$.

**Lemma 1.2.** [12] Let $S(G)$ be a Seidel matrix of order $n \geq 2$ with the spectrum $\{[\mu_1(G)]^{n-1}, [\mu_2(G)]^{n-1}\}$ for some $t$, $1 \leq t \leq n-1$. Let $S(G')$ be a principal $(n-1) \times (n-1)$ submatrix of $S(G)$. Then the spectrum of $S(G')$ is $\{[\mu_1(G)]^{t-1}, [\mu_2(G)]^{t-1}, [\mu_1(G) + \mu_2(G)]^{1}\}$.

**Lemma 1.3.** [4] Let $G$ be a $k$-regular graph on $n$ vertices. Then the Seidel spectrum of $G$ is $\{n-1-2k, -1-2\lambda_2, \ldots, -1-2\lambda_n\}$, where $\lambda_i$’s ($2 \leq i \leq n$) are eigenvalues of $G$.

2. Main Results

This section is concerned with the use of algebraic techniques in the study of Estrada index of $S(G)$.

It is easy to see that the Estrada index of a graph $G$ is equal to Seidel switching of $G$. We now present an example of a graph on $n$ vertices with two distinct Seidel eigenvalues.
Example 2.1. Let $G$ be a graph on $n \geq 2$ vertices with two distinct Seidel spectra \{[\mu_1]^2, [\mu_2]^2\}. By Lemma 1.1 and Lemma 1.2, we have

$$SEE(G) = t_1 e^{\sqrt{\frac{2}{\pi}}(n-1)} + t_2 e^{-\sqrt{\frac{2}{\pi}}(n-1)},$$

$$SEE(G - \{v\}) = e^{\sqrt{\frac{2}{\pi}}(n-1) - \sqrt{\frac{2}{\pi}}(n-1)} + (t_1 + 1)e^{\sqrt{\frac{2}{\pi}}(n-1)} + (t_2 + 1)e^{-\sqrt{\frac{2}{\pi}}(n-1)},$$

where $t_1 + t_2 = n$.

Theorem 2.2. Let $G$ be a graph of order $n \geq 2$. Then $SEE(G) > n$.

Proof. By Geometric-Arithmetic mean inequality [16], we have

$$\frac{1}{n} SEE(G) \geq \left(\prod_{i=1}^{n} e^{\mu_i}\right)^{\frac{1}{n}} = \sqrt[n]{\sum_{i=1}^{n} \mu_i} = \sqrt[n]{0} = 1,$$

with equality if and only if for all $1 \leq i, j \leq n$, $e^{\mu_i} = e^{\mu_j}$ if and only if for all $1 \leq i, j \leq n$, $\mu_i = \mu_j$. This implies that all $\mu_i$’s are zero. This contradicts the fact that $\mu_1 \geq 1$.

Lemma 2.3. [16] For real positive numbers $x_1, x_2, \ldots, x_n$, we have

i) $\frac{x_1^n + x_2^n + \cdots + x_n^n}{n} + \frac{n(x_1 x_2 \cdots x_n)^{\frac{n}{2}}}{x_1 + x_2 + \cdots + x_n} \geq 2$,

ii) $\frac{x_1^n + x_2^n + \cdots + x_n^n}{x_1 + x_2 + \cdots + x_n} + \frac{(x_1 x_2 \cdots x_n)^{\frac{n}{2}}}{x_1 + x_2 + \cdots + x_n} \geq 1$.

Theorem 2.4. Let $G$ be a graph of order $n$. Then we have

i) $SEE(G) \leq n \left(1 - \sum_{k=0}^{\infty} \frac{1}{k!} n^k S_k(G)\right)^{-1}$,

ii) $SEE(G) \leq n \left(1 - \sum_{k=0}^{\infty} \frac{1}{k!} n^k S_k(G)\right)^{-1}$.

Proof. By Lemma 2.3(i), we yield that

$$2 \leq \frac{\sum_{i=1}^{n} e^{\mu_i} + \sum_{i=1}^{n} e^{\mu_2} + \cdots + \sum_{i=1}^{n} e^{\mu_n}}{e^{\mu_1} + e^{\mu_2} + \cdots + e^{\mu_n}} + \frac{\sum_{i=1}^{n} e^{\mu_1} e^{\mu_2} \cdots e^{\mu_n}}{e^{\mu_1} + e^{\mu_2} + \cdots + e^{\mu_n}}$$

$$= \frac{1}{n} \prod_{i=1}^{n} e^{\mu_i} \sum_{i=1}^{n} e^{\mu_i} + \frac{\sum_{i=1}^{n} e^{\mu_1} e^{\mu_2} \cdots e^{\mu_n}}{SEE(G) \left(\prod_{i=1}^{n} e^{\mu_i}\right)^{\frac{1}{n}}}$$

$$= \frac{1}{n} \sum_{i=1}^{n} e^{\mu_i} + \frac{n}{SEE(G)},$$
where the last equality holds by applying Lemma 1.1(i). Thus, we get
\[ n^2 \geq \text{SEE}(G) \left( 2n - \sum_{i=1}^{n} e^{n\mu_i} \right). \]

Since \( \sum_{i=1}^{n} e^{n\mu_i} = \sum_{i=1}^{n} \sum_{k \geq 0} \frac{(n\mu_i)^k}{k!} = \sum_{k \geq 0} \frac{n^k}{n!} S_k(G) \), we obtain \( \text{SEE}(G) \leq n(2 - \sum_{k \geq 0} \frac{1}{n^k} n^{k-1} S_k(G))^{-1} \). Similar to the last case, by applying Lemma 2.3(ii) we can show that \( \text{SEE}(G) \leq (1 - \sum_{k \geq 0} \frac{1}{n^k} S_k(G))^{-1} \). This yields the proof.

**Theorem 2.5.** Let \( G \) be a graph of order \( n \) and \( a \geq 2 \) be an integer. Then
\[ \text{SEE}(G) \leq e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a} \frac{1}{k!} \left( S_k(G) - \left( \sqrt{n(n-1)} \right)^k \right). \]

**Proof.** We have
\[
\text{SEE}(G) = \sum_{k=0}^{a} \frac{S_k(G)}{k!} + \sum_{k \geq a+1} \frac{1}{k!} \sum_{i=1}^{n} \mu_i^k
\leq \sum_{k=0}^{a} \frac{S_k(G)}{k!} + \sum_{k \geq a+1} \frac{1}{k!} \sum_{i=1}^{n} |\mu_i^k|
\leq \sum_{k=0}^{a} \frac{S_k(G)}{k!} + \sum_{k \geq a+1} \frac{1}{k!} \sum_{i=1}^{n} (\mu_i^2)^{\frac{k}{2}}
= \sum_{k=0}^{a} \frac{S_k(G)}{k!} + \sum_{k \geq a+1} \frac{1}{k!} \sqrt{(n(n-1))^k}
= e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a} \frac{S_k(G)}{k!} - \sum_{k=2}^{a} \frac{1}{k!} \sqrt{(n(n-1))^k}
= e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a} \frac{1}{k!} \left( S_k(G) - \left( \sqrt{n(n-1)} \right)^k \right).
\]

This completes the proof. \( \square \)

**Theorem 2.6.** If \( G \) is a \( k \)-regular graph with \( n \) vertices, then
\[ \text{SEE}(G) \geq e^{n-1-2k} + (n-1)e^{\frac{2k}{n-1}} - 1, \]
with equality if and only if \( G \cong K_n \).
Proof. By Lemma 1.3, we have \( \text{SEE}(G) = e^{n-1-2k} + \sum_{i=2}^{n} e^{-2\lambda_i} \), where \( \lambda_i \)'s (\( 2 \leq i \leq n \)) are eigenvalues of \( G \). By geometric-arithmetic mean inequality, we get

\[
e(\text{SEE}(G) - e^{n-1-2k}) = \sum_{i=2}^{n} e^{-2\lambda_i} \geq (n-1) \left( \prod_{i=2}^{n} e^{-2\lambda_i} \right)^{\frac{1}{n-1}} \sum_{i=2}^{n} \lambda_i
\]

\[
= (n-1)e^{-\frac{2k}{n-1} \sum_{i=2}^{n} \lambda_i}
\]

where the last inequality follows from this fact that \( \sum_{i=2}^{n} \lambda_i = -k \). Therefore

\[
\text{SEE}(G) \geq e^{n-1-2k} + (n-1)e^{\frac{2k}{n-1} - 1},
\]

with equality if and only if \( \lambda_2 = \cdots = \lambda_n \). Hence, \( G \cong K_n \).

\[\square\]

**Corollary 2.7.** If \( G \) is a \( k \)-regular graph with \( n \) vertices, then

\[
\text{SEE}(G) \geq e^{1+2k-n} + (n-1)e^{1+\frac{2k}{n-1}},
\]

with equality if and only if \( G \cong K_n \).

**Theorem 2.8.** Let \( G \) be a \( k \)-regular bipartite graph of order \( n \). Then

\[
\text{SEE}(G) < e^{n-1-2k} + \frac{1}{e}(\text{EE}(G) - e^{-k})^2.
\]

**Proof.** By Lemma 1.3, we have

\[
\text{SEE}(G) = e^{n-1-2k} + \frac{1}{e} \sum_{i=2}^{n} (e^{-\lambda_i})^2
\]

\[
\leq e^{n-1-2k} + \frac{1}{e} \left( \sum_{i=2}^{n} e^{-\lambda_i} \right)^2
= e^{n-1-2k} + \frac{1}{e}(\text{EE}(G) - e^{-k})^2,
\]

where the last equality follows from \( \sum_{i=1}^{n} e^{\lambda_i} = \sum_{i=1}^{n} e^{-\lambda_i} \). Since \( G \) is bipartite, by [3] the eigenvalues of \( G \) are symmetric around zero. The inequality is attained if and only if \( \lambda_1 = \cdots = \lambda_n \) and this is equivalent to \( G \cong K_n \), which is impossible. \(\square\)
The first Zagreb index, one of the oldest vertex degree based structure descriptors, is defined as [18]

\[ M_1 = M_1(G) = \sum_{u \in V(G)} d_u^2, \]

where \( d_u \) denotes the degree (number of first neighbors) of vertex \( u \) in \( G \). Let \( T = [t_{ij}] \in M_n(\mathbb{R}) \). We recall that \( \text{tr}(T) = \sum_{i=1}^n t_{ii} \).

For any real \( x \), one can see that \( e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \) and equality holds if and only if \( x = 0 \).

**Lemma 2.9.** Let \( G \) be a graph of order \( n \) with \( m \) edges and \( t \) triangles. Then

\[ \sum_{i=1}^n \mu_i^3(G) = n(n-1)(n-2) + 12(M_1 - nm - 4t). \]

**Proof.** Suppose that \( A = A(G), S = S(G) \) and \( \mu_1, \ldots, \mu_n \) are the Seidel eigenvalues of \( G \). We know that \( S = A(K_n) - 2A \) and \( \text{tr}(S^3) = \sum_{i=1}^n \mu_i^3 \). Then

\[ \text{tr}(S^3) = \text{tr}(A(K_n) - 2A)^3 \]
\[ = \text{tr}(A^3(K_n)) - 8 \text{tr}(A^3) + 12 \text{tr}(A^2 \cdot A(K_n)) - 6 \text{tr}(A \cdot A^2(K_n)). \]

By [3, Proposition 2.3], we have \( \text{tr}(A^3) = 6t \). Also we have \( \text{tr}(A^3(K_n)) = n(n-1)(n-2) \), \( \text{tr}(A^2 \cdot A(K_n)) = \sum_{u \in V(G)} d_u^2 - 2m \) and \( \text{tr}(A \cdot A^2(K_n)) = 2m(n-2) \).

**Theorem 2.10.** Let \( G \) be graph of order \( n \geq 2 \) with \( m \) edges and \( t \) triangles. Then

\[ \text{SEE}(G) > \sqrt{\frac{n}{3} \left( n^3 - n + 12 \left( M_1 + 4t - mn + \frac{1}{2} \right) \right)}. \]

**Proof.** Suppose that \( \mu_1, \ldots, \mu_n \) are the Seidel eigenvalues of \( G \). Then we have

\[ \text{SEE}(G)^2 = \sum_{i=1}^n \sum_{j=1}^n e^{\mu_i + \mu_j} \]
\[ \geq \sum_{i=1}^n \sum_{j=1}^n \left( 1 + \mu_i + \mu_j + \frac{1}{2}(\mu_i + \mu_j)^2 + \frac{1}{6}(\mu_i + \mu_j)^3 \right) \]
\[ = \sum_{i=1}^n \sum_{j=1}^n \left( 1 + \mu_i + \mu_j + \mu_i \mu_j + \frac{1}{2}(\mu_i^2 + \mu_j^2 + \mu_i \mu_j + \mu_i \mu_j) + \frac{1}{6}(\mu_i^3 + \mu_j^3) \right). \]

By Lemma 2.9, we get

\[ \frac{1}{6} \sum_{i=1}^n \sum_{j=1}^n (\mu_i^3 + \mu_j^3) = \frac{n}{6}(\mu_i^3 + \mu_j^3) = \frac{n}{3} \left( n(n-1)(n-2) + 12(M_1 - nm - 4t) \right). \]
By Lemma 1.1(i), it follows that
\[
\sum_{i=1}^{n} \sum_{j=1}^{n} \left( \mu_i + \mu_j + \mu_i \mu_j + \frac{1}{2} (\mu_i^2 + \mu_j^2) \right) = 0. \tag{1}
\]
Also, by Lemma 1.1(ii) we have
\[
\frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} (\mu_i^2 + \mu_j^2) = \frac{n}{2} (\mu_i^2 + \mu_j^2) = n^2 (n - 1). \tag{2}
\]
By applying Eq.(1) and Eq.(2), we get
\[
\text{SEE}(G) \geq \sqrt{\frac{n}{3} \left( n^3 - n + 12 \left( M_1 + 4t - mn + \frac{1}{2} \right) \right)}.
\]
Equality holds if and only if all $\mu_i$’s are zero which is impossible.

In [14] Haemers defined the Seidel energy of a graph $G$ as $E_S(G) = \sum_{i=1}^{n} |\mu_i|$, where $\mu_i$’s are the Seidel eigenvalues of $G$. Two graphs $G$ and $G'$ are said to be Seidel equienergetic if $E_S(G) = E_S(G')$, see [17]. In a trivial manner, co-spectral graphs are equienergetic. If the Seidel eigenvalues of a graph $G$ are $\mu_i$’s, $(1 \leq i \leq n)$, then the Seidel eigenvalues of $\overline{G}$ are $-\mu_i$’s, $(1 \leq i \leq n)$ and so $E_S(G) = E_S(\overline{G})$, see [14].

In the following theorem, the relationship between the Estrada index of Seidel matrix and the Seidel energy of graphs is investigated.

**Theorem 2.11.** Suppose $G$ is a graph of order $n$. Let $t_1$, $t_2$ and $t_3$ be respectively the numbers of Seidel eigenvalues which are greater than, equal with or less than zero. Then
\[
\text{SEE}(G) \geq t_1 e^{\frac{E_S(G)}{2t_1}} + t_2 + t_3 e^{\frac{-E_S(G)}{2t_3}}.
\]
Equality holds if and only if $G$ is either

i) a strong graph or

ii) a graph of order odd $n$ with Seidel spectrum
\[
\left\{ \left[ -\frac{\sqrt{t_1 t_3 n}}{t_1} \right]^{t_1}, \left[ 0 \right]^{t_2}, \left[ \frac{\sqrt{t_1 t_3 n}}{t_3} \right]^{t_3} \right\},
\]
where $t_1 + t_3 = n - 1$. 
Proof. Let \( \mu_1, \ldots, \mu_t \) be the Seidel eigenvalues of \( G \) greater than zero, and \( \mu_{n-t_3+1}, \ldots, \mu_n \) be the Seidel eigenvalues less than zero. Since the sum of Seidel eigenvalues of a graph \( G \) is zero and

\[
E_S(G) = 2 \sum_{i=1}^{t_1} \mu_i = -2 \sum_{i=n-t_3+1}^{n} \mu_i,
\]

by the geometric-arithmetic mean inequality, we have

\[
\sum_{i=1}^{t_1} e^{\mu_i} \geq t_1 \left( \prod_{i=1}^{t_1} e^{\mu_i} \right)^{1/t_1} = t_1 e^{\frac{1}{t_1}(\mu_1 + \cdots + \mu_{t_1})} = t_1 e^{\frac{E_S(G)}{2t_1}},
\]

\[
\sum_{i=n-t_3+1}^{n} e^{\mu_i} \geq t_3 \left( \prod_{i=n-t_3+1}^{n} e^{\mu_i} \right)^{1/t_3} = t_3 e^{\frac{1}{t_3}(\mu_{n-t_3+1} + \cdots + \mu_n)} = t_3 e^{\frac{E_S(G)}{2t_3}}.
\]

On the other hand, \( \sum_{i=n-t_3+1}^{n} e^{\mu_i} = t_2 \). The equality hold if and only if \( \text{Spec}_S(G) = \{ [\mu_1]^{t_1}, [0]^{t_2}, [\mu_n]^{t_3} \} \). By [14], every Seidel matrix \( S \) of order \( n \) satisfies \( \det(S) \equiv \det(J - I) \equiv n - 1 \) (mod 2). If \( n \) is even, then \( \text{rank}(S) = n \) and if \( n \) is odd, then \( \text{rank}(S) \geq n - 1 \). This implies that the multiplicity of the eigenvalue 0 is at most 1. If \( 0 \notin \text{Spec}_S(G) \), then by Lemma 1.1, we have

\[
\text{Spec}_S(G) = \left\{ \left\lfloor \frac{t_3}{t_1}(n-1) \right\rfloor^{t_1}, \left\lceil \frac{t_3}{t_1}(n-1) \right\rceil^{t_1} \right\},
\]

where \( t_1 + t_3 = n \). Therefore, by [13, Proposition 2], the graph \( G \) is a strong graph. On the other hand, if \( n \) is odd and \( 0 \in \text{Spec}_S(G) \), then it is not difficult to see that

\[
\text{Spec}_S(G) = \left\{ \left\lfloor -\frac{t_1 t_3 n}{t_1} \right\rfloor^{t_1}, [0]^{t_1}, \left\lceil \frac{t_1 t_3 n}{t_3} \right\rceil^{t_3} \right\},
\]

where \( t_1 + t_3 = n - 1 \). This completes the proof. \( \square \)

3. The Estrada index of Seidel matrix of composite graphs

Here, we find two upper bounds for the Estrada index of Seidel matrix of product graphs. Given two graphs \( G \) and \( H \) with vertex sets \( V \) and \( W \), respectively, their Kronecker product \( G \otimes H \) is a graph with vertex set \( V \times W \), where \( (v, w) \) and \( (v', w') \) are adjacent if and only if \( v \) is adjacent with \( v' \) and \( w \) is adjacent with \( w' \). The adjacency matrix of \( G \otimes H \) is the Kronecker product of adjacency matrices of \( G \) and \( H \), namely \( A(G \otimes H) = A(G) \otimes A(H) \), see [4]. Given two graphs \( G \) and \( H \) with vertex sets \( V \) and \( W \), respectively, their Cartesian product \( G \square H \) is the graph with vertex set \( V \times W \), where \( (v, w) \sim (v', w') \) when either \( v = v' \) and \( w \sim w' \) or \( v \sim v' \) and \( w = w' \) [4].
This completes the proof. □

**Lemma 3.1.** Let be $G_1$ and $G_2$ two graphs on $n_1$ and $n_2$ vertices, respectively. Then $S(G_1 \square G_2) = (-2A(G_1) \otimes I_{n_2}) + (I_{n_1} \otimes (I_{n_2} + 2A(G_2))) + J_{n_1n_2}$.

**Proof.** By [4], if $A(G_1)$ and $A(G_2)$ are the adjacency matrices of $G_1$ and $G_2$, then $A(G \square H) = A(G_1) \otimes I_{n_2} + I_{n_1} \otimes A(G_2)$. Thus

$$S(G_1 \square G_2) = J_{n_1n_2} - I_{n_1n_2} - 2(A(G_1) \otimes I_{n_1} + I_{n_2} \otimes A(G_2))$$

$$+ (I_{n_1} \otimes J_{n_2}) - (I_{n_1} \otimes I_{n_2}) - (2A(G_1) \otimes I_{n_2}) - (I_{n_1} \otimes 2A(G_2))$$

$$= (J_{n_1} - I_{n_1} - 2A(G_1)) \otimes I_{n_2} - (J_{n_1} \otimes I_{n_2})$$

$$+ (J_{n_2} \otimes J_{n_1}) - (I_{n_1} \otimes 2A(G_2))$$

$$= (J_{n_1} - I_{n_1} - 2A(G_1)) \otimes I_{n_2} + I_{n_1} \otimes (J_{n_2} - I_{n_2} - 2A(G_2))$$

$$+ (J_{n_1} \otimes J_{n_2}) + (I_{n_1} \otimes I_{n_2}) - (I_{n_1} \otimes J_{n_2})$$

$$= (-2A(G_1) \otimes I_{n_2}) - (I_{n_1} \otimes (I_{n_2} + 2A(G_2))) + J_{n_1n_2}.$$

This completes the proof. □

**Theorem 3.2.** Let $G_1$ be a $k_1$-regular and $G_2$ be a $k_2$-regular graph on respectively $n_1$ and $n_2$ vertices. Then

$$SEE(G_1 \square G_2) \geq \frac{1}{e} EE(G_1)^{-2} EE(G_2)^{-2} - \frac{1 + e^{n_1n_2}}{e^{2(k_1+k_2)+1}}.$$

**Proof.** Suppose $\lambda_1(G_1) \geq \lambda_2(G_1) \geq \cdots \geq \lambda_{n_1}(G_1)$ are the eigenvalues of $G_1$ and $\lambda_1(G_2) \geq \lambda_2(G_2) \geq \cdots \geq \lambda_{n_2}(G_2)$ are the eigenvalues of $G_2$. Then by Lemma 3.1, the Seidel eigenvalues of $G_1 \square G_2$ are $[n_1n_2 - 2(k_1 + k_2) - 1]^1$, $[-2(k_1 + \lambda_j(G_2)) - 1]^1$ for $(2 \leq j \leq n_2)$ and $[-2(\lambda_i(G_1) + \lambda_j(G_2)) - 1]^1$ for $(2 \leq i \leq n_1)$ and $(1 \leq j \leq n_2)$. Hence,

$$SEE(G_1 \square G_2) = \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} e^{-2(\lambda_i(G_1) + \lambda_j(G_2)) - 1} + e^{n_1n_2 - 2(k_1 + k_2) - 1} - e^{-1 - 2(k_1 + k_2)}$$

$$\geq \frac{1}{e} EE(G_1)^{-2} EE(G_2)^{-2} - \frac{1 - e^{n_1n_2}}{e^{2(k_1+k_2)+1}},$$

where the last non-equality follows from $\sum_{i=1}^{n_1} e^{-2\lambda_i(G_i)} \geq \left( \sum_{i=1}^{n_1} e^{\lambda_i(G_i)} \right)^{-2}$, $(j = 1, 2)$. □

The line graph $L(G)$ of a graph $G$ is the graph whose vertices correspond to the edges of $G$. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of $G$ have a common vertex. Define $L^r(G) = G$, $L^r(G) = L(L^{r-1}(G))$, $r \geq 1$. A well-known result in graph theory states that the line graph $L^r(G)$ of an $k$-regular $n$-vertex graph $G$ is a $(2^r k - 2^{r+1} + 2)$-regular $n \prod_{i=0}^{r-1} (2^{i-1} k - 2^i + 1)$-vertex graph with exactly $n \prod_{i=0}^{r-1} (2^{i-1} k - 2^i + 1)$ edges, see [3].
Theorem 3.3. Let $G$ be a $k$-regular graph with $n \geq 2$ vertices, then

$$\text{SEE}(L(G)) \geq e^3 \left(\frac{n}{2}(k - 2) + e^{k(\frac{2}{n} - 4)} + \frac{1}{e^{2k}} (EE(G) - e^k)^{-2}\right),$$

with equality holds if and only if $G \cong K_2$.

Proof. By [3, Theorem 3.8], the eigenvalues of $L(G)$ are $[-2]^\frac{n}{2}(k - 2)$ and $[\lambda_i + k - 2]^1$, $(1 \leq i \leq n)$. Since the line graph of $G$ is $(2k - 2)$-regular with $\frac{n}{2}$ vertices, then by Lemma 1.3, the Seidel eigenvalues of $G$ are $[3]^\frac{n}{2}(k - 2)$, $[\frac{n}{2} - 4k + 3]^1$ and $[3 - 2(\lambda_i + k)]^1$, $(2 \leq i \leq n)$. Thus, we have

$$\text{SEE}(L(G)) = \frac{n}{2}(k - 2)e^3 + e^{\frac{n}{2}-4k+3} + \sum_{i=2}^{n} e^{3-2(\lambda_i + k)}$$

$$\geq e^3 \left(\frac{n}{2}(k - 2) + e^{k(\frac{2}{n} - 4)} + \frac{1}{e^{2k}} \left(\sum_{i=2}^{n} e^{\lambda_i}\right)^{-2}\right)$$

$$= e^3 \left(\frac{n}{2}(k - 2) + e^{k(\frac{2}{n} - 4)} + \frac{1}{e^{2k}} (EE(G) - e^k)^{-2}\right).$$

Equality holds if and only if $G \cong K_2$. \hfill \qed

Corollary 3.4. Let $L(G) = L^1(G)$ and $L^{r+1}(G) = L(L^r(G))$. If $G$ is $k$-regular then

$$\text{SEE}(L^r(G)) \geq e^3 \left(\frac{n_r}{2}(k_r - 2) + e^{k_r(\frac{2}{n_r} - 4)} + \frac{1}{e^{2k_r}} (EE(L^r(G)) - e^{k_r})^{-2}\right),$$

where $L^r(G)$ is $k_r$-regular with $n_r$ vertices, $k_r = (k - 2)2^r + 2$ and

$$n_r = \frac{n}{2r} \prod_{i=0}^{r-1} \left(2^i k - 2^{i-1} + 2\right).$$

Conjecture 3.5. Among all graphs on $n$ vertices, the graphs $K_n$ and $K_i \cup K_j$, $(i + j = n)$ has the minimum Estrada index of Seidel matrix.

Conjecture 3.6. Among all graphs on $n$ vertices, the graphs $\overline{K}_n$ and $K_{i,j}$, $(i + j = n)$ has the maximum Estrada index of Seidel matrix.

Conjecture 3.7. Among trees $T$ on $n$ vertices, the path $P_n$ has the minimum and the star $S_n$ has the maximum Estrada index of Seidel matrix. In other words,

$$\text{SEE}(P_n) \leq \text{SEE}(T) \leq \text{SEE}(S_n).$$

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