Mathematics Interdisciplinary Research **5** (2020) 43 – 54

On the Estrada Index of Seidel Matrix

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Abstract

Let G be a simple graph with n vertices and with the Seidel matrix S. Suppose $\mu_1, \mu_2, \ldots, \mu_n$ are the Seidel eigenvalues of G. The Estrada index of the Seidel matrix of G is defined as $SEE(G) = \sum_{i=1}^{n} e^{\mu_i}$. In this paper, we compute the Estrada index of the Seidel matrix of some known graphs. Also, some bounds for the Seidel energy of graphs are given.

Keywords: Seidel matrix, Seidel eigenvalue, Estrada index.

2010 Mathematics Subject Classification: 05C50, 05C76.

How to cite this article
M. Hakimi-Nezhaad and M. Ghorbani, On the Estrada index of Seidel matrix, *Math. Interdisc. Res.* **5** (2020) 43 – 54.

1. Introduction

Here, we recall some definitions that will be used in this paper. Let G be a simple graph with n vertices, m edges and let A denote the adjacency matrix of G. The eigenvalues of the graph G are the roots of characteristic polynomial $P_G(\lambda) = \det(\lambda I - A)$, where I is the identity matrix. Suppose that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ are the eigenvalues of G. The rank of the matrix A, denoted by $\mathrm{rank}(A)$, is equal to the maximum number of linearly independent columns of A. For given graph G, its complement is denoted by $\overline{G}$. The graph $G - \{v\}$ is a graph obtaining from G by removing the vertex v with all edges connected to v. A complete graph on n vertices is denoted by K_n.

In 1966, van Lint and Seidel in [19] introduced a symmetric $(0, -1, 1)$-adjacency matrix for a graph G, called the Seidel matrix of G, as $S(G) = J - I - 2A$, where J is the matrix with entries 1 in every position. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_s$ be the distinct Seidel eigenvalues of G with multiplicity $t_1, t_2, \ldots, t_s$, respectively. The multiset $\mathrm{Spec}_S(G) = \{[\mu_1]^{t_1}, [\mu_2]^{t_2}, \ldots, [\mu_s]^{t_s}\}$ is called the Seidel spectrum of G.

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Academic Editor: Mohammad Ali Iranmanesh
Received 17 April 2019, Accepted 28 October 2019
DOI:10.22052/mir.2019.179267.1128

Let U_1 and $U_2 = V(G) \setminus U_1$ be the partitioned sets of the vertex set $V(G)$ of a graph G. Let G' be the graph obtained from G by deleting all edges between U_1 and U_2 and inserting all edges between U_1 and U_2 that are not presented in G. Then G' and G are said to be Seidel switching with respect to U_1, see [17]. If G' and G are Seidel switching, then $S(G')$ and $S(G)$ are similar and therefore G' and G have the same Seidel eigenvalues, see [4].

The Estrada index of a graph G is defined as $EE(G) = \sum_{i=1}^{n} e^{\lambda_i}$, where λ_i's are eigenvalues of G. This graph invariant was introduced by Ernesto Estrada, which has noteworthy chemical applications, see [6, 7, 8, 9] for details. We encourage the interested readers to consult papers [1, 5, 10, 11, 15] for the mathematical properties of Estrada index. Askari et al. [2] proposed the Estrada index of Seidel matrix of a graph G as $SEE(G) = \sum_{i=1}^{n} e^{\mu_i}$, where μ_i's are the Seidel eigenvalues of G. From the power-series expansion of e^x, we have

$$SEE(G) = \sum_{k \geq 0} \frac{S_k(G)}{k!},$$

where $S_k(G) = \sum_{i=1}^{n} \mu_i^k$.

Lemma 1.1. [4] *For any graph G on $n \geq 2$ vertices, we have*

i) $\sum_{i=1}^{n} \mu_i = 0$,

ii) $\sum_{i=1}^{n} \mu_i^2 = n(n-1)$,

iii) $\mu_1 \geq 1$.

Lemma 1.2. [12] *Let $S(G)$ be a Seidel matrix of order $n \geq 2$ with the spectrum $\{[\mu_1(G)]^{n-t}, [\mu_2(G)]^t\}$ for some t, $1 \leq t \leq n-1$. Let $S(G')$ be a principal $(n-1) \times (n-1)$ submatrix of $S(G)$. Then the spectrum of $S(G')$ is*

$$\Big\{[\mu_1(G)]^{n-t-1}, [\mu_2(G)]^{t-1}, [\mu_1(G) + \mu_2(G)]^1\Big\}.$$

Lemma 1.3. [4] *Let G be a k-regular graph on n vertices. Then the Seidel spectrum of G is $\{n-1-2k, -1-2\lambda_2, \ldots, -1-2\lambda_n\}$, where λ_i's $(2 \leq i \leq n)$ are eigenvalues of G.*

2. Main Results

This section is concerned with the use of algebraic techniques in the study of Estrada index of $S(G)$.

It is easy to see that the Estrada index of a graph G is equal to Seidel switching of G. We now present an example of a graph on n vertices with two distinct Seidel eigenvalues.

Example 2.1. Let G be a graph on $n \geq 2$ vertices with two distinct Seidel spectra $\{[\mu_1]^{t_1}, [\mu_2]^{t_2}\}$. By Lemma 1.1 and Lemma 1.2, we have

$$SEE(G) = t_1 e^{\sqrt{\frac{t_2}{t_1}(n-1)}} + t_2 e^{-\sqrt{\frac{t_1}{t_2}(n-1)}},$$

$$SEE(G - \{v\}) = e^{\sqrt{\frac{t_2}{t_1}(n-1)} - \sqrt{\frac{t_1}{t_2}(n-1)}} + (t_1 - 1)e^{\sqrt{\frac{t_2}{t_1}(n-1)}} + (t_2 - 1)e^{-\sqrt{\frac{t_1}{t_2}(n-1)}},$$

where $t_1 + t_2 = n$.

Theorem 2.2. *Let G be a graph of order $n \geq 2$. Then $SEE(G) > n$.*

Proof. By Geometric-Arithmetic mean inequality [16], we have

$$\frac{1}{n} SEE(G) \geq \left(\prod_{i=1}^{n} e^{\mu_i} \right)^{\frac{1}{n}} = \sqrt[n]{e^{\sum_{i=1}^{n} \mu_i}} = \sqrt[n]{e^0} = 1,$$

with equality if and only if for all $1 \leq i, j \leq n$, $e^{\mu_i} = e^{\mu_j}$ if and only if for all $1 \leq i, j \leq n$, $\mu_i = \mu_j$. This implies that all μ_i's are zero. This contradicts the fact that $\mu_1 \geq 1$. $\square$

Lemma 2.3. [16] *For real positive numbers $x_1, x_2, \ldots, x_n$, we have*

i) $\frac{x_1^n + x_2^n + \cdots + x_n^n}{n x_1 x_2 \ldots x_n} + \frac{n(x_1 x_2 \ldots x_n)^{\frac{1}{n}}}{x_1 + x_2 + \cdots + x_n} \geq 2,$

ii) $\frac{x_1^n + x_2^n + \cdots + x_n^n}{x_1 x_2 \ldots x_n} + \frac{(x_1 x_2 \ldots x_n)^{\frac{1}{n}}}{x_1 + x_2 + \cdots + x_n} \geq 1.$

Theorem 2.4. *Let G be a graph of order n. Then we have*

i) $SEE(G) \leq n(2 - \sum_{k \geq 0} \frac{1}{k!} n^{k-1} S_k(G))^{-1},$

ii) $SEE(G) \leq (1 - \sum_{k \geq 0} \frac{1}{k!} n^k S_k(G))^{-1}.$

Proof. By Lemma 2.3(i), we yield that

$$\begin{aligned}
2 &\leq \frac{e^{n\mu_1} + e^{n\mu_2} + \cdots + e^{n\mu_n}}{n e^{\mu_1} e^{\mu_2} \ldots e^{\mu_n}} + \frac{n(e^{\mu_1} e^{\mu_2} \ldots e^{\mu_n})^{\frac{1}{n}}}{e^{\mu_1} + e^{\mu_2} + \cdots + e^{\mu_n}} \\
&= \frac{1}{n \prod_{i=1}^{n} e^{\mu_i}} \sum_{i=1}^{n} e^{n\mu_i} + \frac{n}{SEE(G)} \left(\prod_{i=1}^{n} e^{\mu_i} \right)^{\frac{1}{n}} \\
&= \frac{1}{n} \sum_{i=1}^{n} e^{n\mu_i} + \frac{n}{SEE(G)},
\end{aligned}$$

where the last equality holds by applying Lemma 1.1(i). Thus, we get

$$n^2 \geq SEE(G)\Big(2n - \sum_{i=1}^{n} e^{n\mu_i}\Big).$$

Since $\sum_{i=1}^{n} e^{n\mu_i} = \sum_{i=1}^{n}\sum_{k\geq 0}\frac{(n\mu_i)^k}{k!} = \sum_{k\geq 0}\frac{n^k}{k!}S_k(G)$, we obtain $SEE(G) \leq n(2-\sum_{k\geq 0}\frac{1}{k!}n^{k-1}S_k(G))^{-1}$. Similar to the last case, by applying Lemma 2.3(ii) we can show that $SEE(G) \leq (1-\sum_{k\geq 0}\frac{1}{k!}n^k S_k(G))^{-1}$. This yields the proof. □

Theorem 2.5. *Let G be a graph of order n and $a \geq 2$ be an integer. Then*

$$SEE(G) \leq e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a}\frac{1}{k!}\Big(S_k(G) - \big(\sqrt{n(n-1)}\big)^k\Big).$$

Proof. We have

$$\begin{aligned}
SEE(G) &= \sum_{k=0}^{a}\frac{S_k(G)}{k!} + \sum_{k\geq a+1}\frac{1}{k!}\sum_{i=1}^{n}\mu_i^k \\
&\leq \sum_{k=0}^{a}\frac{S_k(G)}{k!} + \sum_{k\geq a+1}\frac{1}{k!}\sum_{i=1}^{n}|\mu_i^k| \\
&\leq \sum_{k=0}^{a}\frac{S_k(G)}{k!} + \sum_{k\geq a+1}\frac{1}{k!}\sum_{i=1}^{n}(\mu_i^2)^{\frac{k}{2}} \\
&= \sum_{k=0}^{a}\frac{S_k(G)}{k!} + \sum_{k\geq a+1}\frac{1}{k!}\sqrt{(n(n-1))^k} \\
&= e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a}\frac{S_k(G)}{k!} - \sum_{k=2}^{a}\frac{1}{k!}\sqrt{(n(n-1))^k} \\
&= e^{\sqrt{n(n-1)}} - \sqrt{n(n-1)} + \sum_{k=2}^{a}\frac{1}{k!}\Big(S_k(G) - \big(\sqrt{n(n-1)}\big)^k\Big).
\end{aligned}$$

This completes the proof. □

Theorem 2.6. *If G is a k-regular graph with n vertices, then*

$$SEE(G) \geq e^{n-1-2k} + (n-1)e^{\frac{2k}{n-1}-1},$$

with equality if and only if $G \cong K_n$.

Proof. By Lemma 1.3, we have $SEE(G) = e^{n-1-2k} + \frac{1}{e}\sum_{i=2}^{n} e^{-2\lambda_i}$, where λ_i's $(2 \leq i \leq n)$ are eigenvalues of G. By geometric-arithmetic mean inequality, we get

$$\begin{aligned} e(SEE(G) - e^{n-1-2k}) = \sum_{i=2}^{n} e^{-2\lambda_i} &\geq (n-1)\left(\prod_{i=2}^{n} e^{-2\lambda_i}\right)^{\frac{1}{n-1}} \\ &= (n-1)e^{-\frac{2}{n-1}\sum_{i=2}^{n}\lambda_i} \\ &= (n-1)e^{\frac{2k}{n-1}}, \end{aligned}$$

where the last inequality follows from this fact that $\sum_{i=2}^{n}\lambda_i = -k$. Therefore

$$SEE(G) \geq e^{n-1-2k} + (n-1)e^{\frac{2k}{n-1}-1},$$

with equality if and only if $\lambda_2 = \cdots = \lambda_n$. Hence, $G \cong K_n$. □

Corollary 2.7. *If G is a k-regular graph with n vertices, then*

$$SEE(\overline{G}) \geq e^{1+2k-n} + (n-1)e^{1+\frac{-2k}{n-1}},$$

with equality if and only if $G \cong K_n$.

Theorem 2.8. *Let G be a k-regular bipartite graph of order n. Then*

$$SEE(G) < e^{n-1-2k} + \frac{1}{e}(EE(G) - e^{-k})^2.$$

Proof. By Lemma 1.3, we have

$$\begin{aligned} SEE(G) &= e^{n-1-2k} + \frac{1}{e}\sum_{i=2}^{n}(e^{-\lambda_i})^2 \\ &\leq e^{n-1-2k} + \frac{1}{e}\Big(\sum_{i=2}^{n} e^{-\lambda_i}\Big)^2 = e^{n-1-2k} + \frac{1}{e}(EE(G) - e^{-k})^2, \end{aligned}$$

where the last equality follows from $\sum_{i=1}^{n} e^{\lambda_i} = \sum_{i=1}^{n} e^{-\lambda_i}$. Since G is bipartite, by [3] the eigenvalues of G are symmetric around zero. The inequality is attained if and only if $\lambda_1 = \cdots = \lambda_n$ and this is equivalent to $G \cong \overline{K}_n$, which is impossible. □

The first Zagreb index, one of the oldest vertex degree based structure descriptors, is defined as [18]

$$\mathrm{M}_1 = \mathrm{M}_1(G) = \sum_{u \in V(G)} d_u^2,$$

where d_u denotes the degree (number of first neighbors) of vertex u in G. Let $T = [t_{ij}] \in M_n(\mathbb{R})$. We recall that $\mathrm{tr}(T) = \sum_{i=1}^{n} t_{ii}$.

For any real x, one can see that $e^x \geq 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$ and equality holds if and only if $x = 0$.

Lemma 2.9. *Let G be a graph of order n with m edges and t triangles. Then*

$$\sum_{i=1}^{n} \mu_i^3(G) = n(n-1)(n-2) + 12(\mathrm{M}_1 - nm - 4t).$$

Proof. Suppose that $A = A(G)$, $S = S(G)$ and $\mu_1, \ldots, \mu_n$ are the Seidel eigenvalues of G. We know that $S = A(K_n) - 2A$ and $\mathrm{tr}(S^3) = \sum_{i=1}^{n} \mu_i^3$. Then

$$\begin{aligned}\mathrm{tr}(S^3) &= \mathrm{tr}(A(K_n) - 2A)^3 \\ &= \mathrm{tr}(A^3(K_n)) - 8\ \mathrm{tr}(A^3) + 12\ \mathrm{tr}(A^2 \cdot A(K_n)) - 6\ \mathrm{tr}(A \cdot A^2(K_n)).\end{aligned}$$

By [3, Proposition 2.3], we have $\mathrm{tr}(A^3) = 6t$. Also we have $\mathrm{tr}(A^3(K_n)) = n(n-1)(n-2)$, $\mathrm{tr}(A^2 \cdot A(K_n)) = \sum_{u \in V(G)} d_u^2 - 2m$ and $\mathrm{tr}(A \cdot A^2(K_n)) = 2m(n-2)$. □

Theorem 2.10. *Let G be graph of order $n \geq 2$ with m edges and t triangles. Then*

$$SEE(G) > \sqrt{\frac{n}{3}\left(n^3 - n + 12\left(\mathrm{M}_1 + 4t - mn + \frac{1}{2}\right)\right)}.$$

Proof. Suppose that $\mu_1, \ldots, \mu_n$ are the Seidel eigenvalues of G. Then we have

$$\begin{aligned}SEE(G)^2 &= \sum_{i=1}^{n}\sum_{j=1}^{n} e^{\mu_i + \mu_j} \\ &\geq \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(1 + \mu_i + \mu_j + \frac{1}{2}(\mu_i + \mu_j)^2 + \frac{1}{6}(\mu_i + \mu_j)^3\Big) \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} \Big(1 + \mu_i + \mu_j + \mu_i\mu_j + \frac{1}{2}(\mu_i^2 + \mu_j^2 + \mu_i^2\mu_j + \mu_i\mu_j^2) + \frac{1}{6}(\mu_i^3 + \mu_j^3)\Big).\end{aligned}$$

By Lemma 2.9, we get

$$\frac{1}{6}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_i^3 + \mu_j^3) = \frac{n}{6}(\mu_i^3 + \mu_j^3) = \frac{n}{3}\big(n(n-1)(n-2) + 12(\mathrm{M}_1 - nm - 4t)\big).$$

By Lemma 1.1(i), it follows that

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\Big(\mu_i+\mu_j+\mu_i\mu_j+\frac{1}{2}(\mu_i^2\mu_j+\mu_i\mu_j^2)\Big)=0. \tag{1}$$

Also, by Lemma 1.1(ii) we have

$$\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}(\mu_i^2+\mu_j^2)=\frac{n}{2}(\mu_i^2+\mu_j^2)=n^2(n-1). \tag{2}$$

By applying Eq.(1) and Eq.(2), we get

$$SEE(G)\geq\sqrt{\frac{n}{3}\left(n^3-n+12\left(\mathrm{M}_1+4t-mn+\frac{1}{2}\right)\right)}.$$

Equality holds if and only if all μ_i's are zero which is impossible. □

In [14] Haemers defined the Seidel energy of a graph G as $E_S(G)=\sum_{i=1}^{n}|\mu_i|$, where μ_i's are the Seidel eigenvalues of G. Two graphs G and G' are said to be Seidel equienergetic if $E_S(G)=E_S(G')$, see [17]. In a trivial manner, co-spectral graphs are equienergetic. If the Seidel eigenvalues of a graph G are μ_i's, $(1\leq i\leq n)$, then the Seidel eigenvalues of $\overline{G}$ are $-\mu_i$'s, $(1\leq i\leq n)$ and so $E_S(G)=E_S(\overline{G})$, see [14].

In the following theorem, the relationship between the Estrada index of Seidel matrix and the Seidel energy of graphs is investigated.

Theorem 2.11. *Suppose G is a graph of order n. Let t_1, t_2 and t_3 be respectively the numbers of Seidel eigenvalues which are greater than, equal with or less than zero. Then*

$$SEE(G)\geq t_1e^{\frac{E_S(G)}{2t_1}}+t_2+t_3e^{-\frac{E_S(G)}{2t_3}}.$$

Equality holds if and only if G is either

i) *a strong graph or*

ii) *a graph of order odd n with Seidel spectrum*

$$\left\{\left[-\frac{\sqrt{t_1t_3n}}{t_1}\right]^{t_1},[0]^1,\left[\frac{\sqrt{t_1t_3n}}{t_3}\right]^{t_3}\right\},$$

where $t_1+t_3=n-1$.

Proof. Let $\mu_1, \ldots, \mu_{t_1}$ be the Seidel eigenvalues of G greater than zero, and μ_{n-t_3+1}, $\ldots$, μ_n be the Seidel eigenvalues less than zero. Since the sum of Seidel eigenvalues of a graph G is zero and

$$E_S(G) = 2\sum_{i=1}^{t_1} \mu_i = -2 \sum_{i=n-t_3+1}^{n} \mu_i,$$

by the geometric-arithmetic mean inequality, we have

$$\sum_{i=1}^{t_1} e^{\mu_i} \geq t_1 \left(\prod_{i=1}^{t_1} e^{\mu_i}\right)^{\frac{1}{t_1}} = t_1 e^{\frac{1}{t_1}(\mu_1+\cdots+\mu_{t_1})} = t_1 e^{\frac{E_S(G)}{2t_1}},$$

$$\sum_{i=n-t_3+1}^{n} e^{\mu_i} \geq t_3 \left(\prod_{i=n-t_3+1}^{n} e^{\mu_i}\right)^{\frac{1}{t_3}} = t_3 e^{\frac{1}{t_3}(\mu_{n-t_3+1}+\cdots+\mu_n)} = t_3 e^{-\frac{E_S(G)}{2t_3}}.$$

On the other hand, $\sum_{i=t_1+1}^{n-t_3} e^{\mu_i} = t_2$. The equality hold if and only if $\mathrm{Spec}_S(G) = \{[\mu_1]^{t_1}, [0]^{t_2}, [\mu_n]^{t_3}\}$. By [14], every Seidel matrix S of order n satisfies $\det(S) \equiv \det(J - I) \equiv n - 1 \pmod 2$. If n is even, then $\mathrm{rank}(S) = n$ and if n is odd, then $\mathrm{rank}(S) \geq n - 1$. This implies that the multiplicity of the eigenvalue 0 is at most 1. If $0 \notin \mathrm{Spec}_S(G)$, then by Lemma 1.1, we have

$$\mathrm{Spec}_S(G) = \left\{ \left[\sqrt{\frac{t_3}{t_1}(n-1)}\right]^{t_1}, \left[-\sqrt{\frac{t_1}{t_3}(n-1)}\right]^{t_3} \right\},$$

where $t_1 + t_3 = n$. Therefore, by [13, Proposition 2], the graph G is a strong graph. On the other hand, if n is odd and $0 \in \mathrm{Spec}_S(G)$, then it is not difficult to see that

$$\mathrm{Spec}_S(G) = \left\{ \left[-\frac{\sqrt{t_1 t_3 n}}{t_1}\right]^{t_1}, [0]^1, \left[\frac{\sqrt{t_1 t_3 n}}{t_3}\right]^{t_3} \right\},$$

where $t_1 + t_3 = n - 1$. This completes the proof. □

3. The Estrada index of Seidel matrix of composite graphs

Here, we find two upper bounds for the Estrada index of Seidel matrix of product graphs. Given two graphs G and H with vertex sets V and W, respectively, their Kronecker product $G \otimes H$ is a graph with vertex set $V \times W$, where (v, w) and (v', w') are adjacent if and only if v is adjacent with v' and w is adjacent with w'. The adjacency matrix of $G \otimes H$ is the Kronecker product of adjacency matrices of G and H, namely $A(G \otimes H) = A(G) \otimes A(H)$, see [4]. Given two graphs G and H with vertex sets V and W, respectively, their Cartesian product $G \square H$ is the graph with vertex set $V \times W$, where $(v, w) \sim (v', w')$ when either $v = v'$ and $w \sim w'$ or $w = w'$ and $v \sim v'$ [4].

Lemma 3.1. *Let be G_1 and G_2 two graphs on n_1 and n_2 vertices, respectively. Then* $S(G_1\Box G_2) = (-2A(G_1)\otimes I_{n_2}) - \Big(I_{n_1}\otimes\big(I_{n_2}+2A(G_2)\big)\Big) + J_{n_1n_2}$.

Proof. By [4], if $A(G_1)$ and $A(G_2)$ are the adjacency matrices of G_1 and G_2, then $A(G\Box H) = A(G_1)\otimes I_{n_2} + I_{n_1}\otimes A(G_2)$. Thus

$$\begin{aligned}
S(G_1\Box G_2) &= J_{n_1n_2} - I_{n_1n_2} - 2\big(A(G_1)\otimes I_{n_1} + I_{n_2}\otimes A(G_2)\big)\\
&= (J_{n_1}\otimes J_{n_2}) - (I_{n_1}\otimes I_{n_2}) - (2A(G_1)\otimes I_{n_2}) - (I_{n_1}\otimes 2A(G_2))\\
&= \big(J_{n_1} - I_{n_1} - 2A(G_1)\big)\otimes I_{n_2} - (J_{n_1}\otimes I_{n_2})\\
&+ (J_{n_2}\otimes J_{n_1}) - (I_{n_1}\otimes 2A(G_2))\\
&= \big(J_{n_1} - I_{n_1} - 2A(G_1)\big)\otimes I_{n_2} + I_{n_1}\otimes\big(J_{n_2} - I_{n_2} - 2A(G_2)\big)\\
&+ (J_{n_1}\otimes I_{n_2}) + (J_{n_1}\otimes J_{n_2}) + (I_{n_1}\otimes I_{n_2}) - (I_{n_1}\otimes J_{n_2})\\
&= (-2A(G_1)\otimes I_{n_2}) - \Big(I_{n_1}\otimes\big(I_{n_2}+2A(G_2)\big)\Big) + J_{n_1n_2}.
\end{aligned}$$

This completes the proof. □

Theorem 3.2. *Let G_1 be a k_1-regular and G_2 be a k_2-regular graph on respectively n_1 and n_2 vertices. Then*

$$SEE(G_1\Box G_2) \geq \frac{1}{e}EE(G_1)^{-2}EE(G_2)^{-2} - \frac{1+e^{n_1n_2}}{e^{2(k_1+k_2)+1}}.$$

Proof. Suppose $\lambda_1(G_1)\geq\lambda_2(G_1)\geq\cdots\geq\lambda_{n_1}(G_1)$ are the eigenvalues of G_1 and $\lambda_1(G_2)\geq\lambda_2(G_2)\geq\cdots\geq\lambda_{n_2}(G_2)$ are the eigenvalues of G_2. Then by Lemma 3.1, the Seidel eigenvalues of $G_1\Box G_2$ are $[n_1n_2-2(k_1+k_2)-1]^1$, $[-2(k_1+\lambda_j(G_2))-1]^1$ for $(2\leq j\leq n_2)$ and $[-2(\lambda_i(G_1)+\lambda_j(G_2))-1]^1$ for $(2\leq i\leq n_1)$ and $(1\leq j\leq n_2)$. Hence,

$$\begin{aligned}
SEE(G_1\Box G_2) &= \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} e^{-2(\lambda_i(G_1)+\lambda_j(G_2))-1} + e^{n_1n_2-2(k_1+k_2)-1} - e^{-1-2(k_1+k_2)}\\
&\geq \frac{1}{e}EE(G_1)^{-2}EE(G_2)^{-2} - \frac{1-e^{n_1n_2}}{e^{2(k_1+k_2)+1}},
\end{aligned}$$

where the last non-equality follows from $\sum_{i=1}^{n_j} e^{-2\lambda_i(G_j)} \geq \Big(\sum_{i=1}^{n_j} e^{\lambda_i(G_j)}\Big)^{-2}$, $(j = 1,2)$. □

The line graph $L(G)$ of a graph G is the graph whose vertices correspond to the edges of G. Two vertices of $L(G)$ are adjacent if and only if the corresponding edges of G have a common vertex. Define $L^0(G) = G$, $L^r(G) = L(L^{r-1}(G))$, $r\geq 1$. A well-known result in graph theory states that the line graph $L^r(G)$ of an k-regular n-vertex graph G is a $(2^rk-2^{r+1}+2)$-regular $n\prod_{i=0}^{r-1}(2^{i-1}k-2^i+1)$-vertex graph with exactly $n\prod_{i=0}^{r}(2^{i-1}k-2^i+1)$ edges, see [3].

Theorem 3.3. *Let G be a k-regular graph with $n \geq 2$ vertices, then*

$$SEE(L(G)) \geq e^3\Big(\frac{n}{2}(k-2) + e^{k(\frac{n}{2}-4)} + \frac{1}{e^{2k}}(EE(G) - e^k)^{-2}\Big),$$

with equality holds if and only if $G \cong K_2$.

Proof. By [3, Theorem 3.8], the eigenvalues of $L(G)$ are $[-2]^{\frac{n}{2}(k-2)}$ and $[\lambda_i+k-2]^1$, $(1 \leq i \leq n)$. Since the line graph of G is $(2k-2)$-regular with $\frac{nk}{2}$ vertices, then by Lemma 1.3, the Seidel eigenvalues of G are $[3]^{\frac{n}{2}(k-2)}$, $[\frac{nk}{2} - 4k + 3]^1$ and $[3 - 2(\lambda_i + k)]^1$, $(2 \leq i \leq n)$. Thus, we have

$$\begin{aligned} SEE(L(G)) &= \frac{n}{2}(k-2)e^3 + e^{\frac{nk}{2}-4k+3} + \sum_{i=2}^{n} e^{3-2(\lambda_i+k)} \\ &\geq e^3\Big(\frac{n}{2}(k-2) + e^{k(\frac{n}{2}-4)} + \frac{1}{e^{2k}}\Big(\sum_{i=2}^{n} e^{\lambda_i}\Big)^{-2}\Big) \\ &= e^3\Big(\frac{n}{2}(k-2) + e^{k(\frac{n}{2}-4)} + \frac{1}{e^{2k}}(EE(G) - e^k)^{-2}\Big). \end{aligned}$$

Equality holds if and only if $G \cong K_2$. □

Corollary 3.4. *Let $L(G) = L^1(G)$ and $L^{r+1}(G) = L(L^r(G))$. If G is k-regular then*

$$SEE(L^{r+1}(G)) \geq e^3\Big(\frac{n_r}{2}(k_r-2) + e^{k_r(\frac{n_r}{2}-4)} + \frac{1}{e^{2k_r}}(EE(L^r(G)) - e^{k_r})^{-2}\Big),$$

where $L^r(G)$ is k_r-regular with n_r vertices, $k_r = (k-2)2^r + 2$ and

$$n_r = \frac{n}{2^r}\prod_{i=0}^{r-1}(2^i k - 2^{i-1} + 2).$$

Conjecture 3.5. *Among all graphs on n vertices, the graphs K_n and $K_i \cup K_j$, $(i + j = n)$ has the minimum Estrada index of Seidel matrix.*

Conjecture 3.6. *Among all graphs on n vertices, the graphs $\overline{K}_n$ and $K_{i,j}$, $(i+j = n)$ has the maximum Estrada index of Seidel matrix.*

Conjecture 3.7. *Among trees T on n vertices, the path P_n has the minimum and the star S_n has the maximum Estrada index of Seidel matrix. In other words,*

$$SEE(P_n) \leq SEE(T) \leq SEE(S_n).$$

Acknowledgement. This research is partially supported by Shahid Rajaee Teacher Training University.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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