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DE Sinc-Collocation Method for Solving a Class of Second-Order Nonlinear BVPs

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Abstract

In this work, we develop the sinc-collocation method coupled with a double exponential transformation for solving a special class of nonlinear second-order multi-point boundary value problems (MBVPs). This method attains a convergence rate of exponential order. Four numerical examples are also examined to demonstrate the efficiency and functionality of the newly proposed approach.

Keywords: double exponential transformation, collocation points, multi-point boundary value problem, sinc methods.

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1. Introduction

We study the second-order ordinary differential equation in bridge design as

$$y'' + g(x, y) = 0, \ \ 0 \leq x \leq 1, \tag{1}$$

where $y(x)$ denotes the displacement of the bridge from the unloaded position and $g(x, y)$ is a continuous function of its arguments. In the design of small bridges,

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usually two support points are contrived. This leads to the two-point boundary value conditions

$$y(0) = y(1) = 0.$$

The equation (1) appears in the design of a large-size bridge with multipoint supports (see e.g., [6, 15, 16]). In the vicinity of each of the endpoints of the bridge, we can set up two different type of boundary conditions. If we focus on the position of the bridge at supporting points near $x = 0$, we suggest the following boundary condition

$$y(0) = \sum_{i=1}^{m-2} \alpha_i y(\xi_i) + \lambda_1, \tag{2}$$

where $\xi_i \in (0, 1)$, $\alpha_i \geq 0$, $i = 1, 2, \ldots, m-2$, and λ_1 is a parameter. If we are interested in controlling the bridge's angles at support points near $x = 0$, we propose the following boundary condition:

$$y'(0) = \sum_{i=1}^{m-2} \alpha_i y'(\xi_i) + \lambda_1. \tag{3}$$

A similar situation near $x = 1$ holds and the multi-point boundary value conditions can be formulated as

$$y(1) = \sum_{i=1}^{m-2} \beta_i y(\xi_i) + \lambda_2, \tag{4}$$

or

$$y'(1) = \sum_{i=1}^{m-2} \beta_i y'(\xi_i) + \lambda_2, \tag{5}$$

where $\beta_i \geq 0$, $i = 1, 2, \ldots, m-2$.

In the last decade, much attention has been focused on providing an effective numerical method to solve the problem (1) with the multi-point boundary conditions (2)-(5). Based upon shooting method, Zou et al. [16] proposed a numerical method for approximating solutions and fold bifurcation solutions of problem (1) with multi-point boundary conditions (2) and (4). In [14], Wang studied a class of $2n$th-order nonlinear multi-point boundary value problems. By the method of upper and lower solutions, he proved the existence of a solution without any monotone condition on the nonlinear function. The authors of [9] employed the sinc-collocation method with a single exponential transformatin to solve the problem (1)-(5). Other numerical/analytic methods, such as Numerov's Method, Birkhoff-Lagrange-collocation method, Adomian decomposition method, variational iteration method and homotopy perturbation method have also been developed to solve the problem (1)-(5) and have been addressed by [2, 3, 6, 9, 14, 15, 16].

In the present paper, we apply the sinc-collocation method coupled with DE transformation for solving the problem (1)-(5). Our method consists of reducing the solution to a set of algebraic equations by expanding as a combination of modified sinc functions with a special boundary treatment. A variety of numerical methods based on Double Exponential (DE) sinc approximations have been studied mainly by the Japanese researchers during the last two decades. It is widely used in various fields of numerical analysis such as interpolation, quadrature, and solution of integral and ordinary/partial differential equations [1, 4, 5, 8, 10]. This method has been studied extensively and found to be a very effective technique and under some conditions, has convergency of order $\exp\left(-\frac{cN}{\ln(N)}\right)$, where $c > 0$ is a constant and $N \in \mathbb{N}$ depends on number of collocation points. For more technical details about sinc methods see [7, 11, 12] and references cited therein.

The layout of the paper is as follows. Section 2 is concerned with the basic formulations of the sinc functions required for our subsequent development. In Section 3 our method is used to approximate the solution of model problem (1)-(5). The numerical experiments are implemented in Maple 17 with a 16-digit floating-point arithmetic. The programs are executed on a Notebook System with 2.0 GHz Intel Core 2 Duo processor with 2 GB 533 MHz DDR2 SDRAM.

2. Basic Definitions and Properties of sinc Function

The sinc function for each $x \in \mathbb{R}$ is defined by

$$\operatorname{sinc}(x) = \begin{cases} \frac{\sin(\pi x)}{\pi x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

For $h > 0$, and any integer k, the translated sinc functions with evenly spaced nodes are defined as $S(k,h)(x) := \operatorname{sinc}\left(\frac{x}{h} - k\right)$. These functions form an interpolatory set of functions, as follows:

$$S(k,h)(jh) = \delta_{j,k} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases} \tag{6}$$

If f is a function defined on real line, then for a fixed value $h > 0$, the cardinal function corresponding to f, denoted $C(f,h)(x)$, is defined by

$$C(f,h)(x) := \sum_{k=-\infty}^{\infty} f(kh)S(k,h)(x), \tag{7}$$

whenever the series in (7) converges. Relations (6) and (7) immediately tell us that the cardinal function interpolates f at the points $\{nh\}_{n=-\infty}^{+\infty}$. Throughout this paper, let D_d denote the infinite strip region of width $2d (d > 0)$ in the complex w-plane, i.e.

$$D_d = \{w = t + \mathrm{i}s : |s| < d\}. \tag{8}$$

For problems on a subinterval $\Gamma \subseteq \mathbb{R}$, we employ a conformal map ϕ for which $\phi(\Gamma) = \mathbb{R}$. Let ϕ be a conformal map, with inverse ψ, of the simply-connected domain D, where $(0,1) \subseteq D$, onto D_d. Then on a subinterval $\Gamma = (0,1) = \psi(\mathbb{R})$ with $\phi(0) = -\infty$ and $\phi(1) = +\infty$, we have the following method of interpolation

$$f(x) \simeq \sum_{k=-N}^{N} f(x_k) S_k(x), \tag{9}$$

where, $x_k = \psi(kh)$ and

$$S_k(x) = S(k,h) \circ \phi(x) = \operatorname{sinc}\left(\frac{\phi(x)}{h} - k\right)$$

are defined as sinc grid points and the translated sinc basic functions, respectively. To construct approximations on the interval $(0,1)$, which is used in this paper, the domain $D_{DE}(d)$ defined by

$$D_{DE}(d) = \left\{ z = x + \mathrm{i}y : \left| \arg\left(\frac{1}{\pi}\ln\left(\frac{z}{1-z}\right) + \sqrt{1 + \left(\frac{1}{\pi}\ln\left(\frac{z}{1-z}\right)\right)^2} \right) \right| < d \right\},$$

is mapped conformally onto the infinite strip D_d via

$$w = \phi(z) = \ln\left(\frac{1}{\pi}\ln\left(\frac{z}{1-z}\right) + \sqrt{1 + \left(\frac{1}{\pi}\ln\left(\frac{z}{1-z}\right)\right)^2} \right),$$

and specifically the interval $(0,1)$ is mapped onto $\mathbb{R}$. We depict the domains $D_{DE}(d)$ and D_d in Figure 1 for $d = \frac{\pi}{6}$.

We recall that the inverse map of $w = \phi(z)$ is

$$z = \psi(w) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh w\right). \tag{10}$$

Thus we may define the inverse images of the real line and of the evenly spaced nodes $\{kh\}_{k=-\infty}^{\infty}$ as

$$\Gamma = \{\psi(w) \in D_{DE}(d) : -\infty < w < \infty\} = (0,1),$$

and

$$x_k = \psi(kh) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh(kh)\right), \quad k = 0, \pm 1, \pm 2, \pm 3, \ldots,$$

respectively.

Now we review a well-known class of functions which is suitable for sinc interpolation.

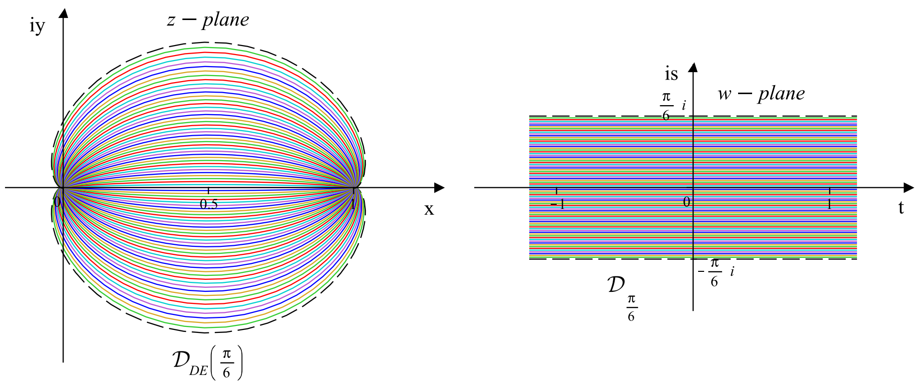

Figure 1: The domains $D_{DE}(\frac{\pi}{6})$ (left) and $D_{\frac{\pi}{6}}$ (right).

Definition 2.1. Let $\mathfrak{D}_{d(\varepsilon)}$ be a rectangular domain defined by

$$\mathfrak{D}_{d(\varepsilon)} = \left\{ z \in \mathbb{C} : |\Re z| < \frac{1}{\varepsilon}, |\Im(z)| < d(1-\varepsilon) \right\}, \quad 0 < \varepsilon < 1. \tag{11}$$

Then $H^1(\mathfrak{D}_d)$ denotes the family of all functions f which are analytic on $\mathfrak{D}_d$ and

$$\mathcal{N}_1(f, \mathfrak{D}_d) = \lim_{\varepsilon \to 0} \oint_{\partial \mathfrak{D}_{d(\varepsilon)}} |f(z)||\mathrm{d}z| < \infty. \tag{12}$$

In the following two theorems the convergence rate of finite sinc approximation over the interval (0, 1) is identified based on the single exponential (SE) transformation

$$\psi(t) = \frac{1}{2}\left(1 + \tanh(\tfrac{t}{2})\right), \tag{13}$$

and DE transformation defined in (10).

Theorem 2.2. [12, 13] *Assume, for some positive constants d and β and with SE variable transformation defined in* (13)*, that the transformed function $f \circ \psi$ be defined on $H^1(\mathfrak{D}_d)$ and $|(f \circ \psi)(t)| = \mathcal{O}\left(e^{-\beta|t|}\right)$ as $t \to \pm\infty$. Then there exists a constant C independent of N, such that*

$$\sup_{0<x<1} |f(x) - \sum_{j=-N}^{N} f(x_j) S_j(x)| \le C\sqrt{N} e^{-\sqrt{\pi d \beta N}}, \tag{14}$$

where

$$h = \sqrt{\frac{\pi d}{\beta N}}. \tag{15}$$

Theorem 2.3. [12, 13] *Assume, with positive constants α, β, γ and d and with DE variable transformation defined in* (10)*, that the transformed function $f \circ \psi$ be defined on $H^1(\mathfrak{D}_d)$ and $f \circ \psi$ decays double exponentially on the real line, that is,*

$$|(f \circ \psi)(t)| \leq \alpha \exp\left(-\beta \exp(\gamma |t|)\right), \quad \text{for all}\, t \in \mathbb{R}. \tag{16}$$

Then there exists a constant C independent of N, such that

$$\sup_{0<x<1} |f(x) - \sum_{j=-N}^{N} f(x_j) S_j(x)| \leq C \exp\left(\frac{-\pi d \gamma N}{\ln(\pi d \gamma N/\beta)}\right), \tag{17}$$

where the mesh size h is taken as following:

$$h = \frac{\ln(\pi d \gamma N/\beta)}{\gamma N}. \tag{18}$$

The sinc-collocation method requires derivatives of composite translated sinc functions evaluated at the nodes. For establishing our numerical scheme, it is convenient to introduce the notation $\delta_{j,k}^{(p)}$ as

$$\delta_{j,k}^{(p)} = h^p \frac{d^p}{d\phi^p} \left[S_j(x)\right]|_{x=x_k}, \quad p = 0, 1, 2. \tag{19}$$

3. Implementation

In this section we only discretize the bridge design corresponding to equation (1) with the multi-point boundary conditions (2) and (4). Thus, we can easily extend our results to other choices of boundary conditions in (2)-(5).

In the sinc-collocation technique, the approximate solution for $y(x)$ in (1) subject to multi-point boundary conditions (2) and (4) is presented by

$$y(x) \simeq y_N(x) = Y(x) + B(x), \tag{20}$$

where

$$Y(x) = \sum_{j=-N}^{N} c_j S_j(x), \tag{21}$$

and

$$B(x) = a_0 + a_1 x + a_2 x^2, \tag{22}$$

defined as the boundary term. The multipliers of boundary term and coefficients $\{c_j\}_{j=-N}^{N}$ are determined by substituting $y_N(x)$ into Equations (1), (2) and (4), and evaluating the results at the collocation points

$$x_k = \psi(kh) = \frac{1}{2} + \frac{1}{2}\tanh\left(\frac{\pi}{2}\sinh(kh)\right), \quad k = -N-1, -N, \ldots, N.$$

To compute $y'_N(x)$, we first differentiate from the translated sinc basic functions as

$$\frac{d}{dx}\left[S_j(x)\right] = \phi'(x)\frac{dS_j(x)}{d\phi}. \tag{23}$$

Now, by using Equations (21)-(23), we obtain

$$y'_N\left(x_k\right) = \sum_{j=-N}^{N} c_j \left\{\frac{1}{\psi'(kh)}\delta_{j,k}^{(1)}\right\} + B'\left(x_k\right). \tag{24}$$

Similarly by taking the second derivative from $S_j(x)$ and using identities (19) we obtain

$$y''_N\left(x_k\right) = \sum_{j=-N}^{N} c_j \left\{-\frac{\psi''(kh)}{\left(\psi'(kh)\right)^3}\delta_{j,k}^{(1)} + \frac{1}{\left(\psi'(kh)\right)^2}\delta_{j,k}^{(2)}\right\} + B''\left(x_k\right). \tag{25}$$

By substituting (20) and (25) into the Equation (1) and the conditions (2) and (4) and evaluating the results at collocation points, we end up with a nonlinear system of algebraic equations as follows:

$$\begin{cases} \sum\limits_{j=-N}^{N} c_j \left\{-\frac{\psi''(kh)}{(\psi'(kh))^3}\delta_{j,k}^{(1)} + \frac{1}{(\psi'(kh))^2}\delta_{j,k}^{(2)}\right\} + B''\left(x_k\right) \\ \quad + g\left(x_k, \sum\limits_{j=-N}^{N} c_j S_j\left(x_k\right) + B\left(x_k\right)\right) = 0, \quad k = -N-1, \ldots, N, \\ B\left(0\right) = \sum\limits_{i=1}^{m-2} \alpha_i \left(\sum\limits_{j=-N}^{N} c_j S_j\left(\xi_i\right) + B\left(\xi_i\right)\right) + \lambda_1, \\ B\left(1\right) = \sum\limits_{i=1}^{m-2} \beta_i \left(\sum\limits_{j=-N}^{N} c_j S_j\left(\xi_i\right) + B\left(\xi_i\right)\right) + \lambda_2. \end{cases} \tag{26}$$

Equations (26) wholly give $2N+4$ algebraic equations which can be solved for the unknown coefficients using Newtons method. Hence $y_N(x)$ given in Equation (20) can be calculated.

4. Numerical experiments

In this section, we show some examples to illustrate how the sinc-collocation method is treated when incorporated with the DE transformation (10) on interval $[0,1]$. In all examples, we heuristically choose $d = \frac{\pi}{6}$, $\gamma = 1$ and $\beta = \frac{\pi}{2}$, which leads, according to (18), to $h = \frac{1}{N}\ln\left(\frac{N\pi}{3}\right)$. We also define the maximum absolute errors e_N and $E(N)$ as $e_N = \|y(x) - y_N(x)\|_\infty$ and $E(N) = \max\limits_{-N\le k\le N} \left|y\left(x_k\right) - y_N\left(x_k\right)\right|$, respectively. Moreover, the superscripts SE and DE refer to SE sinc-collocation and DE sinc-collocation methods.

Example 4.1. [6, 9] *Consider the nonlinear MBVP*

$$\begin{cases} y''(x) + x\,y^2(x) = -2 + x^3 - 2x^4 + x^5, & 0 \le x \le 1, \\ y(0) = \frac{1}{5} y\left(\frac{1}{5}\right) + \frac{2}{5} y\left(\frac{4}{5}\right) - 0.096, \\ y(1) = \frac{3}{10} y\left(\frac{1}{5}\right) + \frac{1}{2} y\left(\frac{4}{5}\right) - 0.128. \end{cases}$$

with the exact solution $y(x) = x - x^2$.

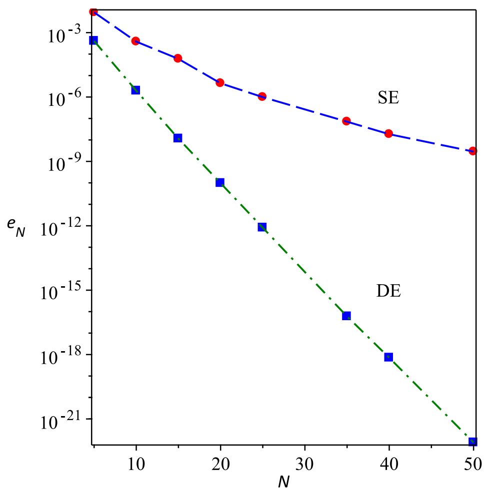

Figure 2: Plot of the absolute error functions in logarithmic mode, with respect to N incorporated with DE and SE sinc-collocation methods for example 4.1.

An illustration of the high accuracy available from the DE sinc-collocation method in comparison with SE sinc-collocation method for this problem is provided in Figure 2. As shown in this Figure, moderate values of N obtain relatively accurate solution.

Example 4.2. As another example we choose the following bridge design problem which documented in [6] and [9]:

$$\begin{cases} y''(x) + \left(x^3 + x + 1\right) y^2(x) = f(x), & 0 \le x \le 1, \\ y(0) = \frac{1}{6} y\left(\frac{2}{9}\right) + \frac{1}{3} y\left(\frac{7}{9}\right) - 0.0286634, \\ y(1) = \frac{1}{5} y\left(\frac{2}{9}\right) + \frac{1}{2} y\left(\frac{7}{9}\right) - 0.0401287, \end{cases}$$

where

$$f(x) = \frac{1}{9}\left[-6\cos\left(x^2 - x\right) + \sin\left(x - x^2\right)\left(-3(1-2x)^2 + \left(1 + x + x^3\right)\sin\left(x - x^2\right)\right)\right],$$

and its exact solution is $y(x) = \frac{1}{3}\sin\left(x - x^2\right)$.

Table 1: The maximum absolute error in solution of example 4.2 for different values of N.

N	e_N^{DE}
3	6.32×10^{-04}
5	1.64×10^{-05}
7	3.07×10^{-06}
10	5.77×10^{-07}
13	4.83×10^{-08}
14	3.76×10^{-08}

Table 1 reports the maximum error e_N for different values of N. This Table shows that our method can achieve high accuracy with moderate computational effort. The results of this table show that our numerical results are acceptable and a slight increase in N, significantly improves the DE-sinc solutions.

Example 4.3. In this example we study the following multi-point boundary value problem

$$\begin{cases} y''(x) + \frac{\sin((x+1)y)}{x} = -\frac{2}{(x+1)^3} + \frac{\sin(x)}{x}, \quad 0 \le x \le 1, \\ y\left(\frac{1}{2}\right) - y(0) = \frac{1}{3}, \\ y(1) - 2y\left(\frac{1}{3}\right) = 0. \end{cases}$$

This BVP has the exact solution $y(x) = \frac{x}{x+1}$ and also has an algebraic singularity at the origin.

In Table 2, we compare SE-sinc and DE-sinc solutions on uniform gridpoints $x = 0 : 0.1 : 1$. Both methods are convergent exponentially to the exact solution and they handle the singularity at the origin. Also this table states that via the use of DE transformation (10), it is possible to obtain a more rapid rate of convergence of sinc methods.

Example 4.4. Consider the following BVP over the interval $[0, 1]$:

$$\begin{cases} y''(x) + \frac{y(x)}{x(1+y(x))} = \frac{1}{x} + \frac{\ln(x)}{1+x\ln(x)}, \\ y(0) = y(1) = 0, \end{cases}$$

whose solution is $y(x) = x\ln(x)$. This problem has both an algebraic and logarithmic singularity at $x = 0$.

Table 3 reports the values of $E^{SE}(N)$ and $E^{DE}(N)$ for some values of N. The maximum absolute errors reported in this Table show that for small values of N, the DE sinc-collocation method gives more accurate results than those obtained

Table 2: Comparison of SE-sinc and DE-sinc solutions for $N = 4$ and $N = 8$ for Example 4.3.

x	$y(x)$	SE-sinc solution		DE-sinc solution	
		$N=4$	$N=8$	$N=4$	$N=8$
0	0	-0.003136	0.000146	0.000144	-0.000000
0.1	0.090909	0.086005	0.090985	0.090980	0.090909
0.2	0.166667	0.162252	0.166560	0.166770	0.166665
0.3	0.230769	0.226905	0.230658	0.230861	0.230769
0.4	0.285714	0.282104	0.285626	0.285818	0.285715
0.5	0.333333	0.330197	0.333348	0.333477	0.333333
0.6	0.375000	0.372312	0.375104	0.375179	0.374999
0.7	0.411765	0.408488	0.411729	0.411936	0.411764
0.8	0.444444	0.438306	0.444054	0.444568	0.444445
0.9	0.473684	0.464195	0.473580	0.473829	0.473684
1	0.5	0.492431	0.499779	0.500184	0.500000

In SE-sinc solution, we choose $\beta = \frac{1}{2}$ and $d = \frac{\pi}{2}$.

by the sinc-collocation method incorprated with SE transformation (13). Thus, this method enables us to obtain accurate approximations with a relatively small numbers of evaluation points.

Table 3: The maximum absolute errors $E^{SE}(N)$ and $E^{DE}(N)$ in solution of example 4.4 for different values of spectral resolution.

N	$E^{SE}(N)$	$E^{DE}(N)$
2	5.32×10^{-02}	1.10×10^{-01}
4	4.96×10^{-03}	2.44×10^{-03}
8	9.58×10^{-04}	1.38×10^{-06}
16	4.62×10^{-05}	1.65×10^{-10}
32	4.25×10^{-07}	1.45×10^{-17}
64	4.37×10^{-10}	2.41×10^{-30}

5. Concluding Remarks

In this paper a new impelementation of well-known sinc-collocation method incorporated with a DE transformation used to reduce the MBVP given in (1), (2) and (4) to a nonlinear system of algebraic equations. Because of exponential convergence of DE sinc-collocation, the method exhibits high accuracy as seen from the comparison with the exact solutions. Hence, moderate values of spectral resolution N, leads to acceptable solutions.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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