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New Criteria for Univalent, Starlike, Convex and Close-to-Convex Functions on the Unit Disk

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Abstract

In the present paper, we introduce and investigate three interesting superclasses $\mathcal{S}_D$,$\mathcal{S}_D^*$ and $\mathcal{K}_D$ of analytic, normalized, and univalent functions in the open unit disk $\mathbb{D}$. For functions belonging to the classes $\mathcal{S}_D$,$\mathcal{S}_D^*$ and $\mathcal{K}_D$, we derive several properties including the coefficient bounds and growth theorems.Our findings generalize many well-known results. We also obtain a new univalent criterion and some interesting properties for univalent, starlike, convex and close-to-convex functions. Many superclasses studied by various researchers previously are obtained as special cases for our two new superclasses.

Keywords: Univalent functions, Starlike functions, Convex functions, Close-to-convex functions.

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1. Introduction

One of the basic subjects covered by Geometric Function Theory is the study of univalent functions. This subject dates back to the early twentieth century following the works published by Koebe [7], Gromwall [5] and Bieberbach [1].

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There exist many books dedicated to univalent functions, see for instance [2, 3, 4, 6, 8]. In the sequel, we introduce some basic definitions and notations.

Let $\mathbb{D}$ be the open unit disc in the complex plan $\mathbb{C}$, i.e. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. A complex function f is holomorphic on $\mathbb{D}$, if and only if f is derivative for all $z \in \mathbb{D}$. The set of holomorphic functions on $\mathbb{D}$ is denoted by $\mathcal{H}(\mathbb{D})$. In addition $\mathcal{A} \subset \mathcal{H}(\mathbb{D})$ denotes a class of functions f normalized by the conditions $f(0) = f'(0) - 1 = 0$ in $\mathbb{D}$. The set of all univalent (one–to–one) functions f in $\mathbb{D}$ is denoted by $\mathcal{S}$. Let $\mathcal{S}^*$ be the subclass of $\mathcal{S}$ whose members are starlike in $\mathbb{D}$. Analytically, $f \in \mathcal{S}^*$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

$\mathcal{K}$ denotes a class of convex functions in $\mathbb{D}$. Analytically, $f \in \mathcal{K}$ if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > 0 \quad (z \in \mathbb{D}).$$

Then, we have $\mathcal{K} \subset \mathcal{S}^* \subset \mathcal{S}$ [2].

Most of the results presented here based on the theory of univalent functions are classical. However, there are some cases which are relatively new and provide a viewpoint that is slightly different from older results.

2. Main Results

First, we will introduce superclass $\mathcal{S}_D$.

Definition 2.1. The equivalence class $\mathcal{S}_D$ is defined as follows:

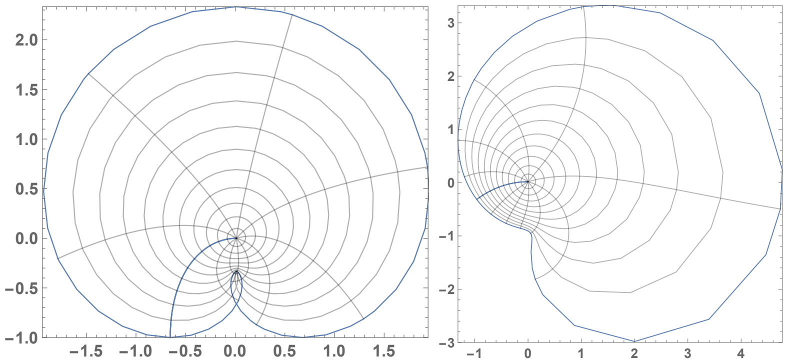
$$\mathcal{S}_D := \left\{ f \in \mathcal{H}(\mathbb{D}) : \ 2f\left(\frac{z}{2-z}\right) \in \mathcal{S} \right\}.$$

Lemma 2.2. *The class $\mathcal{S}$ is a proper subclass of $\mathcal{S}_D$.*

Proof. It is notable that $\mathcal{S}$ is a subset of $\mathcal{S}_D$, and thus we omit the details. To show the class $\mathcal{S}$ is a proper subclass of $\mathcal{S}_D$, consider the function $f_1(z) = z - iz^2 - \frac{1}{3}z^3$. It is a simple exercise where $f_1 \notin \mathcal{S}$. For more details, see Figure 1a. On the other hand,

$$g_1(z) := 2f_1\left(\frac{z}{2-z}\right) = \frac{2z^3(3i+2) - 12z^2(i+2) + 24z}{3(2-z)^3}.$$

A simple calculation gives $g_1 \in \mathcal{S}$ as displayed in Figure 1b. This means that $g_1 \in \mathcal{S}_D$ which completes the proof. □



(a) The boundary curve of $f_1(\mathbb{D})$ (b) The boundary curve of $g_1(\mathbb{D})$

Figure 1: The boundary curve of $f_i(\mathbb{D})$

Theorem 2.3. *If* $f(z) = z + a_2z^2 + a_3z^3 + \cdots \in \mathcal{S}_D$, *then*

$$|a_2 + 1| \leq 4.$$

The above inequality is sharp.

Proof. Since

$$\frac{z}{2-z} = \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n,$$

It follows that

$$\begin{aligned} g(z) = 2f\left(\frac{z}{2-z}\right) &= 2\left(\frac{z}{2-z}\right) + 2a_2\left(\frac{z}{2-z}\right)^2 + \cdots \\ &= 2\left(\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right) + 2a_2\left(\frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \cdots\right)^2 + \cdots \\ &= z + \left(\frac{1}{2} + \frac{a_2}{2}\right)z^2 + \left(\frac{1}{4} + \frac{a_2}{2} + \frac{a_3}{8}\right)z^3 + \cdots. \end{aligned}$$

Since $g \in \mathcal{S}$, by the Bieberbach theorem,

$$\left|\frac{1}{2} + \frac{a_2}{2}\right| \leq 2.$$

Therefore, $| \; a_2 + 1 \; | \leq 4$. For the sharpness, we consider the function

$$f_2(z) = \frac{z(1+z)}{(1-z)^2} \quad (z \in \mathbb{D}).$$

It can be observed that

$$g_2(z) := 2f_2\left(\frac{z}{2-z}\right) = \frac{z}{(1-z)^2} \quad (z \in \mathbb{D}),$$

belongs to the class $\mathcal{S}$. Now, consider Maclaurin series of $f_2(z)$ as

$$f_2(z) = \frac{z(1+z)}{(1-z)^2} = z + 3z^2 + \cdots .$$

This shows that $\mid a_2 + 1 \mid = 4$, and the inequality is sharp. □

Definition 2.4. Let $N \in \mathbb{N} := \{1, 2, 3, \ldots\}$. Consider the equivalence class $\mathcal{S}_D^N$ as follows:

$$\mathcal{S}_D^N := \left\{ f \in \mathcal{H}(\mathbb{D}) : \ \left(1 + \frac{1}{N}\right) f\left(\frac{z}{1 + \frac{1}{N} - \frac{1}{N}z}\right) \in \mathcal{S} \right\}.$$

Lemma 2.5. *For all $N \in \mathbb{N}$, $\mathcal{S}$ is a subset of $\mathcal{S}_D^N$.*

Proof. Let $f \in \mathcal{S}$. Since $\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}$ is univalent, $\left(1 + \frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is univalent and therefore $f \in \mathcal{S}_D^N$. Hence the result. □

Lemma 2.6. *For all $N \in \mathbb{N}$, $\mathcal{S}_D^N$ is a subset of $\mathcal{S}_D$.*

Proof. Suppose that $f \in \mathcal{S}_D^N$. Then $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is univalent and can be claimed that $\left|\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right| < 1$. In fact, if $\left|\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right| \geq 1$, then $|z| \geq \left|1 + \frac{1}{N} - \frac{1}{N}z\right| \geq 1 + \frac{1}{N} - \frac{1}{N}|z|$ and so $(1 + \frac{1}{N})|z| \geq 1 + \frac{1}{N}$, which implies that $|z| \geq 1$ which is a contradiction. In what follows, is demonstrated that

$$\left|\frac{z}{1 + \frac{1}{N} - \frac{1}{N}z}\right| \geq \left|\frac{z}{2-z}\right|.$$

Let

$$z = re^{i\theta} \quad (0 \leq r < 1, 0 \leq \theta \leq 2\pi).$$

Define:

$$F(r) := r^2\left(1 + \frac{1}{N}\right) - 2r\left(2 + \frac{1}{N}\right) + \left(3 + \frac{1}{N}\right) \quad (0 \leq r < 1, N \in \mathbb{N}).$$

As can be seen F is a decreasing function on $(0, 1)$ for all $N \in \mathbb{N}$. Thus

$$F(r) > F(1) = 0 \quad (0 \leq r < 1).$$

On the other hand,

$$3 + \frac{1}{N} + r^2(1 + \frac{1}{N}) - 2r\cos\theta(2 + \frac{1}{N}) > 3 + \frac{1}{N} + r^2(1 + \frac{1}{N}) - 2r(2 + \frac{1}{N}) > 0,$$

which implies

$$(3+\frac{1}{N})+r^2(1+\frac{1}{N})-2r\cos\theta(2+\frac{1}{N})>0.$$

By multiplying $(1-\frac{1}{N})$ on both sides of the above inequality,

$$(1-\frac{1}{N})(3+\frac{1}{N})+r^2(1-\frac{1}{N})(1+\frac{1}{N})-2r\cos\theta(1-\frac{1}{N})(2+\frac{1}{N})\geq 0.$$

So,

$$r^2-\frac{r^2}{N^2}+\frac{2r}{N}\cos\theta+\frac{2r}{N^2}\cos\theta-3r\cos\theta+3+\frac{1}{N}-\frac{3}{N}-\frac{1}{N^2}\geq 0.$$

Hence,

$$4+r^2-4r\cos\theta\geq\frac{1}{N^2}+1+\frac{2}{N}+\frac{r^2}{N^2}-\frac{2r}{N}\cos\theta-\frac{2r}{N^2}\cos\theta.$$

The last inequality implies that

$$|(2-r\cos\theta)-(r\sin\theta)i|\geq\left|(1+\frac{1}{N}-\frac{r}{N}\cos\theta)-(\frac{r}{N}\sin\theta)i\right|,$$

and thus

$$|2-z|\geq\left|1+\frac{1}{N}-\frac{1}{N}z\right|.$$

Finally,

$$\left|\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right|\geq\left|\frac{z}{2-z}\right|.$$

Now, since $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is univalent, thus $f\left(\frac{z}{2-z}\right)$ is also univalent in $\mathbb{D}$. This means that $f\in\mathcal{S}_D$. □

Theorem 2.7. $S=\bigcap_{N=1}^{\infty}S_D^N$.

Proof. By Lemma 2.5, $\mathcal{S}\subset\bigcap_{N=1}^{\infty}\mathcal{S}_D^N$. We need to prove $\bigcap_{N=1}^{\infty}\mathcal{S}_D^N\subset\mathcal{S}$. Suppose that $h\in\bigcap_{N=1}^{\infty}\mathcal{S}_D^N$, implying $h\in\mathcal{S}_D^N$ for all $N\in\mathbb{N}$. Therefore,

$$\left(1+\frac{1}{N}\right)h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)\in\mathcal{S}, \tag{1}$$

and $h\in\mathcal{H}(\mathbb{D})$. If we let $N\to\infty$, then $\left(1+\frac{1}{N}\right)h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ tends to $h(z)$ uniformly and by Eq. (1), we get $h\in\mathcal{S}$. Hence the result. □

Theorem 2.8. *Let $f \in \mathcal{A}$. Then*

$$f \in \mathcal{S} \Leftrightarrow \left(1+\frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in S \quad \forall N \in \mathbb{N}.$$

Proof. Based on Lemma 2.5 and Theorem 2.7, the proof is easily obtained. □

Definition 2.9. Consider the equivalence class $\mathcal{S}^*_D$ as follows:

$$\mathcal{S}^*_D := \left\{ f \in \mathcal{H}(\mathbb{D}) : \; 2f\left(\frac{z}{2-z}\right) \in \mathcal{S}^* \right\}.$$

Lemma 2.10. *The class $\mathcal{S}^*$ is a proper subclass of $\mathcal{S}^*_D$.*

Proof. Let $f \in \mathcal{S}^*$. It can be shown that the function

$$\phi(z) := 2f\left(\frac{z}{2-z}\right) \quad (z \in \mathbb{D}),$$

is starlike. A simple calculation gives us

$$\frac{z\phi'(z)}{\phi(z)} = \frac{2}{2-z} \frac{\frac{z}{2-z} f'\left(\frac{z}{2-z}\right)}{f\left(\frac{z}{2-z}\right)} \quad (z \in \mathbb{D}).$$

Since $f \in \mathcal{S}^*$, $|\frac{z}{2-z}| < 1$ and $\mathrm{Re}\{\frac{2}{2-z}\} > \frac{2}{3}$, then $\mathrm{Re}\{\frac{z\phi'(z)}{\phi(z)}\} > 0$. Thus ϕ is a starlike function. It can be observed that $f_3(z) = \frac{4z+4z^2}{(2-z)^2} \notin \mathcal{S}^*$ and

$$g_3(z) = 2f_3\left(\frac{z}{2-z}\right) = \frac{16z}{(4-3z)^2} \in \mathcal{S}^*.$$

This means that $f_3 \in \mathcal{S}^*_D$. Hence $\mathcal{S}^*$ is a proper subset of $\mathcal{S}^*_D$. □

Theorem 2.11. *If $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{S}^*_D$, then $|a_2 + 1| \leq 4$.*

Proof. Since $\frac{z}{2-z} = \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n$,

$$g(z) = 2f\left(\frac{z}{2-z}\right) = z + \left(\frac{1}{2} + \frac{a_2}{2}\right) z^2 + \left(\frac{1}{4} + \frac{a_2}{2} + \frac{a_3}{8}\right) z^3 + \cdots,$$

and since $g \in \mathcal{S}^*$, we have $\left|\frac{1}{2} + \frac{a_2}{2}\right| \leq 2$. Therefore, $|\, a_2 + 1 \,| \leq 4$. For the sharpness consider the function

$$f(z) = \frac{6z(1+z)}{(2-z)^2} \quad (z \in \mathbb{D}).$$

□

Definition 2.12. The equivalence class $\mathcal{S}_D^{*N}$ has been defined as follows:

$$\mathcal{S}_D^{*N} := \left\{ f \in \mathcal{H}(\mathbb{D}) : \ \left(1+\frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in \mathcal{S}^* \right\}.$$

Lemma 2.13. *For all $N \in \mathbb{N}$, $\mathcal{S}^*$ is a subset of $\mathcal{S}_D^{*N}$.*

Proof. Let $f \in \mathcal{S}^*$. Since $\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}$ is starlike, $\left(1+\frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is starlike and therefore $f \in \mathcal{S}_D^{*N}$. □

Lemma 2.14. *For all $N \in \mathbb{N}$, $\mathcal{S}_D^{*N}$ is a subset of $\mathcal{S}_D^*$.*

Proof. Suppose that $f \in \mathcal{S}_D^{*N}$. Then $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is starlike. Following the proof presented for Lemma 2.6,

$$\left|\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right| \geq \left|\frac{z}{2-z}\right|.$$

Now, since $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is starlike, $f\left(\frac{z}{2-z}\right)$ is also starlike in $\mathbb{D}$. This means that $f \in \mathcal{S}_D^*$. □

Theorem 2.15. $\mathcal{S}^* = \bigcap_{N=1}^{\infty} \mathcal{S}_D^{*N}$.

Proof. By Lemma 2.13, $\mathcal{S}^* \subset \bigcap_{N=1}^{\infty} \mathcal{S}_D^{*N}$. Now suppose that $h \in \bigcap_{N=1}^{\infty} \mathcal{S}_D^{*N}$, implying that $h \in \mathcal{S}_D^{*N}$ for all $N \in \mathbb{N}$. Therefore,

$$\left(1+\frac{1}{N}\right) h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in \mathcal{S}^*, \tag{2}$$

and $h \in \mathcal{H}(\mathbb{D})$. If we let $N \to \infty$, then $\left(1+\frac{1}{N}\right) h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ tends to $h(z)$ uniformly and by Eq. (2), $h \in \mathcal{S}^*$. □

Theorem 2.16. *Let $f \in \mathcal{A}$. Then,*

$$f \in \mathcal{S}^* \Leftrightarrow \left(1+\frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in \mathcal{S}^* \quad \forall N \in \mathbb{N}.$$

Proof. By Lemma 2.13 and Theorem 2.15, the proof easily obtained. □

Theorem 2.17. *Let $f \in \mathcal{H}(\mathbb{D})$ and $f(0) = 0$. If*

$$\operatorname{Re}\left\{\left(\frac{zf'(z)}{f(z)}\right)'\right\} > 0 \quad (z \in \mathbb{D}),$$

then f is starlike.

Proof. Suppose that f fails to be starlike. According to Theorem 2.16, there is sequence $\{p_{_N}\} \subseteq \mathbb{N}$ such that the function

$$g_{p_{_N}}(z) = \left(1 + \frac{1}{p_{_N}}\right) f\left(\frac{z}{1 + \frac{1}{p_{_N}} - \frac{z}{p_{_N}}}\right),$$

is not starlike for all $N \in \mathbb{N}$. Since f is not starlike on $\mathbb{D}$, there exists a $u_1 \in \mathbb{D}$ such that

$$\operatorname{Re}\left\{\frac{u_1 f'(u_1)}{f(u_1)}\right\} \leq 0. \tag{3}$$

Let r_0 be the radius of starlikeness of f. Then for every $u_2 \in \{z : \ |z| \leq r_0\}$ we have

$$\operatorname{Re}\left\{\frac{u_2 f'(u_2)}{f(u_2)}\right\} > 0. \tag{4}$$

Now, consider

$$\varphi(t) = \operatorname{Re}\left\{\frac{(tu_2 + (1-t)u_1) f'(tu_2 + (1-t)u_1)}{f(tu_2 + (1-t)u_1)}\right\} \quad (0 \leq t \leq 1).$$

As can be seen, φ is a continuous function on $[0,1]$ and according to Eqs. (3) and (4), $\varphi(0) \leq 0$ and $\varphi(1) > 0$. By the intermediate theorem, there is $u \in \mathbb{D}$ such that

$$\operatorname{Re}\left\{\frac{uf'(u)}{f(u)}\right\} = 0. \tag{5}$$

Similarly, since $g_{p_{_N}}(z)$ is not starlike for all $p_{_N} \in \{p_N\}$, there exists $u_{p_N} \in \mathbb{D}$ such that

$$\operatorname{Re}\left\{\frac{u_{p_N} g'_{p_N}(u_{p_N})}{g_{p_N}(u_{p_N})}\right\} = 0. \tag{6}$$

Besides, according to Eqs. (5) and (6) there exist real numbers $c = c(u)$ and $d = d(u_{p_N})$ such that

$$\frac{uf'(u)}{f(u)} = ci, \quad \text{and} \quad \frac{u_{p_N} f'\left(\frac{u_{p_N}}{1+\frac{1}{p_N}-\frac{u_{p_N}}{p_N}}\right)}{\left(1 + \frac{1}{p_N} - \frac{u_{p_N}}{p_N}\right)^2 f\left(\frac{u_{p_N}}{1+\frac{1}{p_N}-\frac{u_{p_N}}{p_N}}\right)} = di.$$

Therefore,

$$\frac{uf'(u)}{f(u)} - \frac{v_{p_N}\left(1 + \frac{1}{p_N}\right) f'(v_{p_N})}{\left(1 + \frac{1}{p_N} - \frac{v_{p_N}}{p_N}\right) f(v_{p_N})} = (c-d)i, \tag{7}$$

where $v_{p_N} = \frac{u_{p_N}}{1+\frac{1}{p_N}-\frac{u_{p_N}}{p_N}}$. Also by Eq. (7) we have

$$\mathrm{Re}\left\{\frac{uf'(u)}{f(u)} - \frac{v_{p_N}\left(1+\frac{1}{p_N}\right)f'(v_{p_N})}{\left(1+\frac{1}{p_N}-\frac{v_{p_N}}{p_N}\right)f(v_{p_N})}\right\} = 0.$$

Thus,

$$\arg\left\{\frac{uf'(u)}{f(u)} - \frac{v_{p_N}\left(1+\frac{1}{p_N}\right)f'(v_{p_N})}{\left(1+\frac{1}{p_N}-\frac{v_{p_N}}{p_N}\right)f(v_{p_N})}\right\} = \pm\frac{\pi}{2},$$

and

$$\arg\{(u - v_{p_N})\} + \arg\left\{\frac{\frac{uf'(u)}{f(u)} - \frac{v_{p_N}\left(1+\frac{1}{p_N}\right)f'(v_{p_N})}{\left(1+\frac{1}{p_N}-\frac{v_{p_N}}{p_N}\right)f(v_{p_N})}}{u - v_{p_N}}\right\} = \pm\frac{\pi}{2}.$$

But $v_{p_N} \to u$ when $p_N \to \infty$ and $\frac{v_{p_N}\left(1+\frac{1}{p_N}\right)}{1+\frac{1}{p_N}-\frac{v_{p_N}}{p_N}} \to u$. So

$$\arg\left\{(\frac{uf'(u)}{f(u)})'\right\} = \pm\frac{\pi}{2},$$

and

$$\mathrm{Re}\left\{\left(\frac{uf'(u)}{f(u)}\right)'\right\} = 0,$$

which is a contradiction. This completes the proof. □

Example 2.18. The function $f_4(z) = ze^{(z+\frac{1}{8}z^2)}$ is starlike, since

$$\mathrm{Re}\left\{\left(\frac{zf_4'(z)}{f_4(z)}\right)'\right\} = \mathrm{Re}\{1 + \frac{1}{2}z\} > \frac{1}{2} > 0 \quad (z \in \mathbb{D}).$$

The Figure 2 shows the image of $\mathbb{D}$ under the function f_4.

Definition 2.19. Let $f \in H(\mathbb{D})$. It can be said that the function f belongs to the equivalence class $\mathcal{K}_D$, if

$$g(z) := 2f\left(\frac{z}{2-z}\right) \in \mathcal{K}.$$

Lemma 2.20. *$\mathcal{K}$ is a proper subset of $\mathcal{K}_D$.*

Proof. It can be seen that $\mathcal{K}$ is a subset of $\mathcal{K}_D$. We prove that $\mathcal{K}$ is a proper subset of $\mathcal{K}_D$. A simple calculation gives that $f_5(z) := -1 + (1+z)^2 \notin \mathcal{K}$. On the other hand, $g_5(z) := 2f_5(\frac{z}{2-z}) = -2 + \frac{8}{(2-z)^2} \in \mathcal{K}$. This means that $f_5 \in \mathcal{K}_D$. Therefore, $\mathcal{K}$ is a proper subset of $\mathcal{K}_D$. Now, Figure 3 shows the image of the unit disk under the functions f_5 and g_5. □

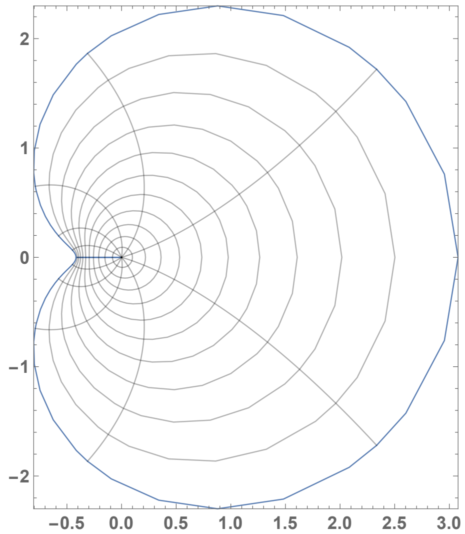
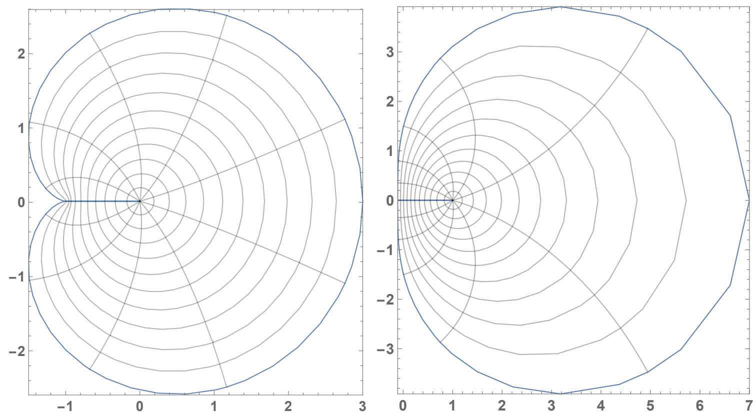


Figure 2: The boundary curve of $f_4(\mathbb{D})$



(a) The boundary curve of $f_5(\mathbb{D})$ (b) The boundary curve of $g_5(\mathbb{D})$

Figure 3: The boundary curve of $f_i(\mathbb{D})$

Theorem 2.21. *If* $f(z) = z + a_2 z^2 + a_3 z^3 + \cdots \in \mathcal{K}_D$, *then* $|a_2 + 1| \leq 2$.

Proof. Since

$$\frac{z}{2-z} = \sum_{n=1}^{\infty} \left(\frac{z}{2}\right)^n,$$

It follows that

$$g(z) = 2f\left(\frac{z}{2-z}\right) = z + \left(\frac{1}{2} + \frac{a_2}{2}\right)z^2 + \left(\frac{1}{4} + \frac{a_2}{2} + \frac{a_3}{8}\right)z^3 + \cdots .$$

and since $g \in \mathcal{K}$, we have $\left|\frac{1}{2} + \frac{a_2}{2}\right| \leq 1$. Therefore, $\mid a_2 + 1 \mid \leq 2$. For the sharpness consider the function

$$f(z) = \frac{z}{1-z} \quad (z \in \mathbb{D}).$$

It is obvious that:

$$g(z) := 2f\left(\frac{z}{2-z}\right) = \frac{z}{1-z} \quad (z \in \mathbb{D}),$$

belongs to the class $\mathcal{K}$. However

$$f(z) = \frac{z}{1-z} = z + z^2 + \cdots .$$

This shows that $\mid a_2 + 1 \mid = 2$, and the inequality is sharp. □

Definition 2.22. Consider the equivalence class $\mathcal{K}_D^N$ as follows:

$$\mathcal{K}_D^N := \left\{ f \in \mathcal{H}(\mathbb{D});\ \left(1 + \frac{1}{N}\right) f\left(\frac{z}{1 + \frac{1}{N} - \frac{1}{N}z}\right) \in \mathcal{K} \right\}.$$

Lemma 2.23. *For all $N \in \mathbb{N}$, $\mathcal{K}$ is a subset of $\mathcal{K}_D^N$.*

Proof. Let $f \in \mathcal{K}$. Since $\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}$ is convex, $\left(1 + \frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is convex and therefore $f \in \mathcal{K}_D^N$. □

Lemma 2.24. *For all $N \in \mathbb{N}$, $\mathcal{K}_D^N$ is a subset of $\mathcal{K}_D$.*

Proof. Suppose that $f \in \mathcal{K}_D^N$. Then $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is convex. Following the proof presented for Lemma 2.6,

$$\left|\frac{z}{1 + \frac{1}{N} - \frac{1}{N}z}\right| \geq \left|\frac{z}{2-z}\right|.$$

Now, since $f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ is convex, $f\left(\frac{z}{2-z}\right)$ is also convex in $\mathbb{D}$. This means that $f \in \mathcal{K}_D$. □

Theorem 2.25. $\mathcal{K} = \bigcap_{N=1}^{\infty} \mathcal{K}_D^N$.

Proof. By Lemma 2.23, $\mathcal{K} \subset \bigcap_{N=1}^{\infty} \mathcal{K}_D^N$. Now suppose that $h \in \bigcap_{N=1}^{\infty} \mathcal{K}_D^N$, implying that $h \in \mathcal{K}_D^N$ for all $N \in \mathbb{N}$. Therefore,

$$\left(1+\frac{1}{N}\right) h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in \mathcal{K}, \tag{8}$$

and $h \in \mathcal{H}(\mathbb{D})$. If we let $N \to \infty$, then $\left(1+\frac{1}{N}\right) h\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right)$ tends to $h(z)$ uniformly and by Eq. (8), $h \in \mathcal{K}$. □

Theorem 2.26. *Let $f \in \mathcal{H}(\mathbb{D})$ and $f(0) = f'(0) - 1 = 0$. Then*

$$f \in \mathcal{K} \Leftrightarrow \left(1+\frac{1}{N}\right) f\left(\frac{z}{1+\frac{1}{N}-\frac{1}{N}z}\right) \in \mathcal{K}, \quad \forall N \in \mathbb{N}.$$

Proof. By Lemma 2.23 and Theorem 2.25, the proof is easily obtained. □

Theorem 2.27. *Let $f \in \mathcal{H}(\mathbb{D})$ and $f(0) = 0$. If $F(z) = z(1-z)f'(z)$ is convex on $\mathbb{D}$, then, f is convex on $\mathbb{D}$.*

Proof. By contradiction, we suppose that f is not convex. Thus, there exist the number $0 < r < 1$ and real numbers θ, φ and γ such that

$$f(e^{i\theta}) = rf(e^{i\varphi}) + (1-r)f(e^{i\gamma}). \tag{9}$$

Since f is not convex, by Theorem 2.26, there exists $\{p_n\}$ for which $g_{p_n}(z) = (1+\frac{1}{p_n})f(\frac{z}{1+\frac{1}{p_n}-\frac{z}{p_n}})$ is not convex on $\mathbb{D}$. Hence, there exist $0 < r < 1$ and real numbers θ_n, φ_n and γ_n such that

$$g_{p_n}(e^{i\theta_n}) = r_n g_{p_n}(e^{i\varphi_n}) + (1-r_n)g_{p_n}(e^{i\gamma_n}). \tag{10}$$

According to Eqs. (9) and (10),

$$g_{p_n}(e^{i\theta_n}) - f(e^{i\theta}) = [r_n g_{p_n}(e^{i\varphi_n}) - rf(e^{i\varphi})] + [(1-r_n)g_{p_n}(e^{i\gamma_n}) - (1-r)f(e^{i\gamma})].$$

Then,

$$\begin{aligned}(1+\frac{1}{p_n})f\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}\right) - f(e^{i\theta}) &= \left[(1+\frac{1}{p_n})r_n f\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}\right) - rf(e^{i\varphi})\right] \\ &+ \left[(1+\frac{1}{p_n})(1-r_n)f\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}\right) - (1-r)f(e^{i\gamma})\right].\end{aligned}$$

So,

$$\begin{aligned}f\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}\right) - \frac{1}{(1+\frac{1}{p_n})}f(e^{i\theta}) &= \left[r_n f\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}\right) - r\frac{1}{(1+\frac{1}{p_n})}f(e^{i\varphi})\right] \\ &+ \left[(1-r_n)f\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}\right) - (1-r)\frac{1}{(1+\frac{1}{p_n})}f(e^{i\gamma})\right].\end{aligned}$$

Hence,

$$\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}-e^{i\theta}\right)\frac{f\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}\right)-\frac{1}{(1+\frac{1}{p_n})}f(e^{i\theta})}{\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}-e^{i\theta}}$$

$$=\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}-e^{i\varphi}\right)\frac{r_n f\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}\right)-r\frac{1}{(1+\frac{1}{p_n})}f(e^{i\varphi})}{\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}-e^{i\varphi}}$$

$$+\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}-e^{i\gamma}\right)\frac{(1-r_n)f\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}\right)-(1-r)\frac{1}{(1+\frac{1}{p_n})}f(e^{i\gamma})}{\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}-e^{i\gamma}}.$$

If $p_n \to \infty$, then

$$\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}-e^{i\theta_n}\right)\frac{f\left(\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}\right)-\frac{1}{(1+\frac{1}{p_n})}f(e^{i\theta})}{\frac{e^{i\theta_n}}{1+\frac{1}{p_n}-\frac{e^{i\theta_n}}{p_n}}-e^{i\theta}}$$

$$=\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}-e^{i\varphi_n}\right)\frac{r_n f\left(\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}\right)-r\frac{1}{(1+\frac{1}{p_n})}f(e^{i\varphi})}{\frac{e^{i\varphi_n}}{1+\frac{1}{p_n}-\frac{e^{i\varphi_n}}{p_n}}-e^{i\varphi}}$$

$$+\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}-e^{i\gamma_n}\right)\frac{(1-r_n)f\left(\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}\right)-(1-r)\frac{1}{(1+\frac{1}{p_n})}f(e^{i\gamma})}{\frac{e^{i\gamma_n}}{1+\frac{1}{p_n}-\frac{e^{i\gamma_n}}{p_n}}-e^{i\gamma}}.$$

Since $\theta_n \to \theta$, $\varphi_n \to \varphi$, $\gamma_n \to \gamma$ and $r_n \to r$, a simple calculation gives us

$$e^{i\theta}(1-e^{i\theta})f'(e^{i\theta}) = re^{i\varphi}(1-e^{i\varphi})f'(e^{i\varphi}) + (1-r)e^{i\gamma}(1-e^{i\gamma})f'(e^{i\gamma}).$$

Thus $F(z) = z(1-z)f'(z)$ is not a convex function on $\mathbb{D}$, which is a contradiction. □

Corollary 2.28. *Consider the integral operator $G(z)$ given by*

$$G(z) = \int_0^z \frac{g(t)}{t(1-t)}dt \quad (z \in \mathbb{D}). \tag{11}$$

If g is a convex function, then G is a convex function, too.

Example 2.29. If we put the convex function $g(z) = \frac{z}{1-z}$ in Eq. (11), then

$$G(z) = \frac{1}{1-z} - 1 \quad (z \in \mathbb{D}),$$

is a convex function.

Definition 2.30. Let $f \in \mathcal{H}(\mathbb{D})$ and $f(0) = 0$. Consider the integral operator $g(z)$ as follows,

$$g(z) := \int_0^z \left(\frac{f(t)}{t(1-t)} \right)^\alpha dt \quad (z \in \mathbb{D}), \tag{12}$$

where α is a real number.

Theorem 2.31. *Suppose that $f \in \mathcal{H}(\mathbb{D})$, $f(0) = 0$ and $f(1) \neq 0$. If f is starlike of order β on $\mathbb{D}$, then*

- *if $\beta \geq \frac{3}{2}$ and $\alpha \geq 0$, then g is close-to-convex,*
- *if $\beta < \frac{3}{2}$ and $0 \leq \alpha \leq \frac{1}{\frac{3}{2}-\beta}$, then g is close-to-convex.*

The result is sharp for every α.

Proof. Since f is starlike of order β on $\mathbb{D}$,

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \beta \quad (z \in \mathbb{D}). \tag{13}$$

However,

$$\begin{aligned} \mathrm{Re}\left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} &= \mathrm{Re}\left\{ 1 + \alpha z \frac{z(1-z)f'(z) - (1-2z)f(z)}{z^2(1-z)^2} . \frac{z(1-z)}{f(z)} \right\} \\ &= \mathrm{Re}\left\{ 1 + \alpha \frac{zf'(z)}{f(z)} - \alpha \frac{1-2z}{1-z} \right\} \\ &= \alpha \mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} + (1-2\alpha) + \alpha \mathrm{Re}\left\{ \frac{1}{1-z} \right\}. \end{aligned} \tag{14}$$

Since $\mathrm{Re}\left\{ \frac{1}{1-z} \right\} > \frac{1}{2}$, by Eqs. (13) and (14),

$$\mathrm{Re}\left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} > \alpha\beta + (1-2\alpha) + \frac{1}{2}\alpha = 1 + \alpha\beta - \frac{3}{2}\alpha. \tag{15}$$

Hence,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} d\theta > \int_{\theta_1}^{\theta_2} \left(1 + \alpha\beta - \frac{3}{2}\alpha \right) d\theta = \left(1 + \alpha\beta - \frac{3}{2}\alpha \right) (\theta_2 - \theta_1).$$

If $\beta \geq \frac{3}{2}$ and $\alpha \geq 0$, then

$$\left(1 + \alpha\beta - \frac{3}{2}\alpha \right) (\theta_2 - \theta_1) > 0.$$

Therefore,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left\{ 1 + \frac{zg''(z)}{g'(z)} \right\} d\theta > -\pi.$$

By Kaplan theorem, it can be concluded that g is close-to-convex. By the same argument, if $\beta < \frac{3}{2}$, then $0 \le \alpha \le \frac{1}{\frac{3}{2}-\beta}$ and therefore g is close-to-convex.

In order to prove the sharpness, if $\alpha < 0$, then according to Eq. (14),

$$\mathrm{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} = \alpha\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} + 1 - 2\alpha + \alpha\mathrm{Re}\left\{\frac{1}{1-z}\right\}.$$

Since $f(1) \neq 0$ and $\lim_{z\to\bar{1}} \mathrm{Re}\left\{\frac{1}{1-z}\right\} = +\infty$, $\lim_{z\to\bar{1}} \mathrm{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} = -\infty$. Moreover, based on the Kaplan theorem, we conclude that g is not close-to-convex. □

Example 2.32. Since $f(z) = \frac{1}{2}z - \frac{1}{16}z^4$ is starlike function,

$$F(z) = \int_0^z \left(\frac{\frac{1}{2} - \frac{1}{16}t^3}{1-t}\right)^\alpha dt,$$

is close-to-convex when $0 \le \alpha \le \frac{2}{3}$.

Corollary 2.33. *Consider the integral operator $g(z) = \int_0^z \frac{f(t)}{t(1-t)}dt$. If f is starlike of order β, $\beta \ge \frac{3}{2}$, then g is convex of order $(\beta - \frac{1}{2})$.*

Proof. By Eq. (15),

$$\mathrm{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} > 1 + \alpha\beta - \frac{3}{2}\alpha.$$

If we let $\alpha = 1$, then

$$\mathrm{Re}\left\{1+\frac{zg''(z)}{g'(z)}\right\} > \beta - \frac{1}{2}.$$

Hence the function g is convex of order $(\beta - \frac{1}{2})$. □

Corollary 2.34. *Let $f \in \mathcal{H}(\mathbb{D})$ and $f'(1) \neq 0$. Then*

- *if $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta$, $\beta < \frac{3}{2}$ and $0 \le \alpha \le \frac{1}{\frac{3}{2}-\beta}$, then $g(z) = \int_0^z \left(\frac{f'(t)}{1-t}\right)^\alpha dt$ is close-to-convex. The result is sharp,*
- *if $\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \beta$, $\beta \ge \frac{3}{2}$ and $\alpha \ge 0$, then $g(z) = \int_0^z \left(\frac{f'(t)}{1-t}\right)^\alpha dt$ is close-to-convex. The result is sharp.*

Proof. Let $h(z) = zf'(z)$. Then

$$\mathrm{Re}\left\{\frac{zh'(z)}{h(z)}\right\} = \mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}.$$

So $\operatorname{Re}\left\{\frac{zh'(z)}{h(z)}\right\} > \beta$. Consequently, h is starlike of order β. On the other hand, $h(0) = 0$, $h(1) = f'(1) \neq 0$ and

$$\int_0^z \left(\frac{h(t)}{t(1-t)}\right)^{\alpha} dt = \int_0^z \left(\frac{f'(t)}{1-t}\right)^{\alpha} dt.$$

By Theorem 2.31, we demonstrate that $g(z) = \int_0^z \left(\frac{f'(t)}{1-t}\right)^{\alpha} dt$ is close-to-convex on $\mathbb{D}$. □

Example 2.35. The function $f(z) = -\ln(1-z)$ is convex of order $\frac{1}{2}$. Thus $g(z) = \int_0^z (1-t)^{-2\alpha} dt$ is close-to-convex function when $0 \leq \alpha \leq 1$.

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