On the Entropy Rate of a Random Walk on $t$-Designs

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Abstract
In this paper, a random walk on $t$-designs are considered. We assign a weight to each block and walk randomly on the vertices with a probability proportional to the weight of blocks. This stochastic process is a Markov chain. We obtain a stationary distribution for this process and compute its entropy rate. It is seen that, when the blocks have the same weight, the uniform distribution on the vertices is a stationary distribution and the entropy rate depends only on the number of vertices.

Keywords: random walk, Markov chain, design, entropy rate, stationary distribution.

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1. Introduction
Let $X$ be a discrete random variable with alphabet $\mathcal{X}$ and probability mass function $p(x) = \Pr\{X = x\}, x \in \mathcal{X}$. The entropy $H(X)$ of $X$ is defined as

$$H(X) = -\sum_{x \in \mathcal{X}} p(x) \log p(x),$$

where logarithm is to the base 2 and entropy is expressed in bits. Here, the convention $0 \log 0 = 0$ will be used. The entropy $H(X)$ is a measure of the uncertainty of $X$ and moreover, it is a measure of the amount of information required on the...
average to describe $X$. Let $(X, Y)$ be a pair of discrete random variables with a joint distribution $p(x, y)$, $(x, y) \in \mathcal{X} \times \mathcal{Y}$. The joint entropy $H(X, Y)$ is defined by

$$H(X, Y) = - \sum_{x \in \mathcal{X}} \sum_{y \in \mathcal{Y}} p(x, y) \log p(x, y).$$

Similarly, the entropy of a collection of random variables, such as $H(X_1, X_2, \ldots, X_n)$, is defined.

A stochastic process $\{X_n\}_{n \in \mathbb{N}}$ can be defined as an indexed sequence of random variables. This process is characterized by the probability mass functions

$$\Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\} = p(x_1, x_2, \ldots, x_n),$$

where $(x_1, x_2, \ldots, x_n) \in \mathcal{X}^n$ and $n \in \mathbb{N}$. This process is called to be stationary if $\Pr\{(X_1, X_2, \ldots, X_n) = (x_1, x_2, \ldots, x_n)\}$ is equal to $\Pr\{(X_{l+1}, X_{l+2}, \ldots, X_{l+n}) = (x_1, x_2, \ldots, x_n)\}$, for all $x_1, x_2, \ldots, x_n \in \mathcal{X}$ and every shift $l$. A Markov chain is a stochastic process $\{X_n\}_{n \in \mathbb{N}}$ such that $\Pr\{X_{n+1} = x_{n+1}|X_n = x_n, X_{n-1} = x_{n-1}, \ldots, X_1 = x_1\}$ is equal to $\Pr\{X_{n+1} = x_{n+1}|X_n = x_n\}$, for all $x_1, x_2, \ldots, x_{n+1}$ in $\mathcal{X}$. A Markov chain $\{X_n\}_{n \in \mathbb{N}}$ is called to be time invariant if $\Pr\{X_{n+1} = b|X_n = a\} = \Pr\{X_2 = b|X_1 = a\}$, for all $n \in \mathbb{N}$ and $a, b \in \mathcal{X}$. It is easy to see that a time invariant Markov chain with alphabet $\mathcal{X} = \{1, 2, \ldots, m\}$ can be characterized by an initial state and a probability transition matrix $P = (p_{ij})$, where $p_{ij} = \Pr\{X_{n+1} = j|X_n = i\}$. A distribution $\mu$ on $\mathcal{X}$ is said to be stationary if $\mu P = \mu$.

In other words, $\mu$ is a distribution on the states such that the distributions at the successive times are the same. The entropy rate of a stochastic process $\{X_n\}_{n \in \mathbb{N}}$ is

$$H(\mathcal{X}) = \lim_{n \to \infty} \frac{1}{n} H(X_1, X_2, \ldots, X_n),$$

when the limit exists. Also, a related quantity for entropy rate is defined by

$$H'(\mathcal{X}) = \lim_{n \to \infty} H(X_n|X_{n-1}, X_{n-2}, \ldots, X_1),$$

when the limit exists. These two quantities correspond to different notions. It can be shown that if $\{X_n\}_{n \in \mathbb{N}}$ is a stationary Markov chain then $H(\mathcal{X}) = H'(\mathcal{X}) = H(X_2|X_1)$. See [2, 6, 7] for more details and examples.

In this paper, motivated by a random walk on a weighted graph [2], a random walk on $t$-designs are considered. We assign a weight to each block and walk randomly on the vertices with a probability proportional to the weight of blocks. This stochastic process is a Markov chain. We obtain a stationary distribution for this process and compute its entropy rate. It is seen that, when the blocks have the same weight, the uniform distribution on the vertices is a stationary distribution and the entropy rate depends only on the number of vertices. For more information and some new results on random walks, entropy rates and their applications, please see [3, 5, 8].
2. t-Designs

Let $S = (P, B, \mathcal{I})$ be an incidence structure which consists of point set $P$, block set $B$ and an incidence relation $\mathcal{I} \subseteq P \times B$. The elements of $\mathcal{I}$ are called flags and the notation $pIB$ means that $(p, B) \in \mathcal{I}$. A block $B \in B$ is sometimes identified with the set of points $p$ incident with it. Here, $\mathcal{I}$ is in fact the membership relation $\in$.

If we replace each block of $S$ by its complement then we obtain the complement of the structure, denoted by $\overline{S}$. The dual of $S = (P, B, \mathcal{I})$ is the incidence structure $S^\top = (B, P, \mathcal{I}^\top)$, where $B^I^\top p$ if and only if $pIB$. The incidence matrix of $S$ is a matrix $M$ of size $|P| \times |B|$ whose rows and columns are labeled by points and blocks, respectively, such that the entry $(p; B)$ is 1 if and only if $p$ is incident with $B$, and 0 otherwise.

The incidence structure $D = (P, B, \mathcal{I})$ is called a $t$-$(v; k; \lambda)$ design if $|P| = v$, $|B| = k$ for any $B \in B$, and every $t$ distinct points are incident with precisely $\lambda$ blocks. It is known that the number of blocks, denoted by $b$, is equal to $\lambda(v-t\lambda)/v$. The design $D$ is called trivial if $B$ consists of all the $k$-subsets of $P$. If $v = b$ then $D$ is called symmetric. It is well-known that the number of blocks incident with $s$ points $(s \leq t)$, denoted by $\lambda_s$, is independent of the set and $\lambda_s = \lambda(v-s\lambda)/(k-s\lambda)$. Therefore, every $t$-$(v, k, \lambda)$ design is also an $s$-$(v, k, \lambda)$ design, where $s \leq t$. The complement of a $t$-$(v, k, \lambda)$ design $D$ is also a design $D'$ with parameters $t$-$(v, v-k, \lambda')$, where $\lambda' = \sum_{s=0}^{t} (-1)^s \binom{t}{s} \lambda_{v-s}$. The number of blocks incident with any point, $\lambda_1$, is also denoted by $r$ and called the replication number. If $D$ is a $t$-$(v, k, \lambda)$ design then $D^\top$ is a design with $b$ points such that its block size is $r$. If $M$ is the incidence matrix of $D$ then the incidence matrix of $D^\top$ is $M^\top$. It can be shown that if $D$ is a $2$-$(v, k, \lambda)$ design then $bk = vr$ and $\lambda(v - 1) = r(k - 1)$. For more details, see [1, 4].

3. Results

Let the incidence structure $D = (P, B, \mathcal{I})$ be a $t$-$(v, k, \lambda)$ design with the vertex set $\{1, 2, \ldots, v\}$. To each block $B \in B$, we assign a weight $\omega(B) \geq 0$ in $\mathbb{R}$ and set

$$
\omega = \sum_{B \in B} \omega(B),
$$

$$
\omega_i = \sum_{i \in B \in B} \omega(B),
$$

$$
\omega_{ij} = \sum_{i, j \in B \in B} \omega(B),
$$

where $i, j \in P$ and $i \neq j$. In other words, $\omega_i$ is the sum of the weights of all blocks containing the vertex $i$ and $\omega_{ij}$ is also the sum of the weights of all blocks
containing the points \(i\) and \(j\). Note that for any vertex \(i\), we have
\[
\sum_{j \in \mathcal{P}} \omega_{ij} = \sum_{j \in \mathcal{P}} \sum_{B \in \mathcal{B}} \omega(B)
\]
\[
= \sum_{B \in \mathcal{B}} \sum_{j \in \mathcal{P}} \omega(B)
\]
\[
= \sum_{B \in \mathcal{B}} (k - 1)\omega(B)
\]
\[
= (k - 1)\omega_i.
\]
A random walk \(\{X_n\}_{n=1}^\infty\) in \(\mathcal{D}\) is a sequence of points of \(\mathcal{D}\) in such a way that \(X_n = i\) and \(X_{n+1} = j\) if there exists a block \(B\) containing the points \(i\) and \(j\). Moreover, we walk from \(i\) to \(j\) with the probability \(p_{ij} = \omega_{ij} / ((k - 1)\omega_i)\). As it is seen, we walk randomly from the vertex \(i\) to the vertex \(j\) with a probability proportional to the weight of the blocks containing \(i\) and \(j\), and the values \(\{p_{ij}\}_{1 \leq j \leq v}\) form a mass probability function. By definition, this stochastic process is a Markov chain with the probability transition matrix \(P = (p_{ij})_{v \times v}\). Set \(\mu = (\mu_1, \mu_2, \ldots, \mu_v)\), where \(\mu_i = \omega_i / (k\omega)\) for any \(1 \leq i \leq v\). Since
\[
\sum_{i=1}^v \mu_i = \sum_{i=1}^v \frac{\omega_i}{k\omega}
\]
\[
= \frac{1}{k\omega} \sum_{i=1}^v \omega(B)
\]
\[
= \frac{1}{k\omega} \sum_{B \in \mathcal{B}} \sum_{i \in \mathcal{P}} \omega(B)
\]
\[
= \frac{1}{k\omega} \sum_{B \in \mathcal{B}} k\omega(B)
\]
\[
= 1,
\]
\(\mu\) is a probability distribution on the points \(\mathcal{P}\). Moreover, for any \(1 \leq j \leq v\),
\[
\sum_{i=1}^v \mu_i p_{ij} = \sum_{i=1}^v \frac{\omega_i}{k\omega} \frac{\omega_{ij}}{(k - 1)\omega_i}
\]
\[
= \frac{1}{k(k - 1)\omega} \sum_{i=1}^v \omega_{ij}
\]
\[
= \frac{\omega_j}{k\omega}
\]
\[
= \mu_j.
\]
Therefore, μ is also a stationary distribution. Now, the entropy rate of this process is

\[ H(\mathcal{X}) = H(X_2|X_1) \]

\[ = - \sum_{i=1}^{v} \mu_i \sum_{j=1}^{v} p_{ij} \log p_{ij} \]

\[ = - \sum_{i=1}^{v} \sum_{j=1}^{v} \frac{\omega_{ij}}{k\omega} \sum_{k=1}^{v} \frac{\omega_{ij}}{(k-1)\omega_i} \log \frac{\omega_{ij}}{(k-1)\omega_i} \]

\[ = - \sum_{i=1}^{v} \sum_{j=1}^{v} \frac{\omega_{ij}}{k(k-1)\omega} \log \frac{\omega_{ij}}{(k-1)\omega_i} \]

\[ = - \sum_{i=1}^{v} \sum_{j=1}^{v} \frac{\omega_{ij}}{k(k-1)\omega} \log \frac{\omega_{ij}}{k(k-1)\omega} + \sum_{i=1}^{v} \sum_{j=1}^{v} \frac{\omega_{ij}}{k(k-1)\omega} \log \frac{\omega_{i}}{k\omega} \]

\[ = - \sum_{i=1}^{v} \sum_{j=1}^{v} \frac{\omega_{ij}}{k(k-1)\omega} \log \frac{\omega_{ij}}{k(k-1)\omega} + \sum_{i=1}^{v} \frac{\omega_{i}}{k\omega} \log \frac{\omega_{i}}{k\omega} \]

\[ = H \left( \cdots, \frac{\omega_{ij}}{k(k-1)\omega}, \cdots \right) - H \left( \cdots, \frac{\omega_{i}}{k\omega}, \cdots \right). \]

So, the following theorem is implied:

**Theorem 3.1.** Let \( \mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathcal{I}) \) be a t-(v, k, λ) design. Assign a non-negative real number \( \omega(B) \) to each block \( B \in \mathcal{B} \) and set

\[ \omega = \sum_{B \in \mathcal{B}} \omega(B), \]

\[ \omega_i = \sum_{i \in B \in \mathcal{B}} \omega(B), \]

\[ \omega_{ij} = \sum_{i,j \in B \in \mathcal{B}} \omega(B), \]

for any \( i \neq j \in \mathcal{P} \). Let \( \{X_n\}_{n=1}^{\infty} \) be a random walk on the points of \( \mathcal{D} \) with the probability transition matrix \( P = (p_{ij})_{v \times v} \), where \( p_{ij} = \omega_{ij}/(k-1)\omega_i \). Set \( \mu_i = \omega_i/(k\omega) \), where \( 1 \leq i \leq v \). Then, \( \{X_n\}_{n=1}^{\infty} \) is a Markov chain with the stationary distribution \( \mu = (\mu_1, \mu_2, \ldots, \mu_v) \) and the entropy rate

\[ H(\mathcal{X}) = H \left( \cdots, \frac{\omega_{ij}}{k(k-1)\omega}, \cdots \right) - H \left( \cdots, \frac{\omega_{i}}{k\omega}, \cdots \right). \]

Note that if all the blocks have equal weight then \( p_{ij} = \lambda_2/(r(k-1)) \) and \( \mu_i = r/(kb) = 1/v \). Also,

\[ \frac{\omega_i}{k\omega} = \frac{r}{kb} = \frac{1}{v}. \]
and
\[
\frac{\omega_{ij}}{k(k-1)\omega} = \frac{\lambda_2}{k(k-1)b} = \frac{1}{v(v-1)}.
\]
Hence, in this case, the uniform distribution on \( P \) is a stationary distribution and the entropy rate is
\[
H(X) = H(\cdots, \frac{1}{v(v-1)}, \cdots) - H(\cdots, \frac{1}{v}, \cdots) = \log(v-1).
\]

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