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A New Efficient High Order Four-Step Multiderivative Method for the Numerical Solution of Second-Order IVPs with Oscillating Solutions

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Abstract

In this paper, we present a new high order explicit four-step method of eighth algebraic order for solving second-order linear periodic and oscillatory initial value problems of ordinary differential equations such as undamped Duffing's equation. Numerical stability and phase properties of the new method is analyzed. The main structure of the method is multiderivative, and the combined phases were applied to expand the stability interval and to achieve P-stability. The advantage of the method in comparison with similar methods in terms of efficiency, accuracy, and stability is shown by its implementation in some well-known problems.

Keywords: Phase-lag error, Initial value problems, P-stable, Symmetric multistep methods, Periodicity interval.

2010 Mathematics Subject Classification: 65L05, 65L06, 65L20.

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1. Introduction

In this paper, the symmetric multiderivative methods for solving special class of initial value problems associated with second-order ordinary differential equations

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of the type

$$y'' = f(x, y), \quad y(x_0) = y_0, \quad y'(x_0) = y'_0, \tag{1}$$

in which the first order derivatives do not occur explicitly, are discussed. Such problems are usually encountered in much scientific research, many engineering applications, and so on [9, 11, 18]. If the exact solutions of these equations are not available, the numerical solutions are very important and interesting. The methods for the numerical solution of (1) can be divided into two main categories:

1. Methods with constant coefficients.

2. Methods with coefficients depending on the frequency of the problem.

Moreover, the second class of methods can also be divided into two classes of problems: problems where the frequency ω is given (even approximately) and problems where the frequency ω is not known [34]. Our method in this article was designed for the numerical solution of problems where the frequency ω is given (even approximately). To solve problems with unknown frequency ω, the determination of ω is a critical issue, as was shown by Ramos and Vigo-Aguiar [17]. Knowledge of the estimation of the unknown frequency ω is needed to apply the numerical method efficiently, since its coefficients depend on the value of this parameter.

During the recent decades, on the basis of these classes, varietal methods presented by different people, that we ensign some important types of them. Using higher order derivatives to improve the accuracy and extend the stability region, and finding a highly accurate and highly efficient Obrechkoff method has become an important research field in numerical methods, as in [13,26,27] and other papers, in [12,16,19,29]. Numerical methods for fractional differential equations are obtained in [1, 3, 15, 32]. The methods based on vanishing of phase-lag and some of its derivatives [23,24,25], Runge-Kutta methods [17,20,21], multistep methods [5,6,10], and hybrid methods [14,22,31] are some of the approaches that can be used for solving a second-order differential equation.

In [6], Lambert and Watson claimed that the P-stable methods must be implicit; explicit methods can not be P-stable and all of the linear multistep P-stable methods are implicit. Of course, we know that the implicit methods are not applicable alone and to compute the implicit terms, it is required to use another suitable explicit method. In 2003, Li and Wu [7] designed a explicit P-stable method that had nonlinear form. Following that, some modified explicit P-stable methods were presented but all of them had the same nonlinear structures [8]. But in this paper, we generate a new explicit linear four-step singularly P-stable method. Since the new method is explicit, we do not need the other predictor method; thus it has less computational complexity in the numerical implementations. Moreover, this method can be recognized as a suitable predictor method for other predictor-corrector methods. The other important point about the new method is that, with regard to its linear structure, it can be used directly in the vector form for solving differential equations systems and there is no need for the vector product and quotient that was highly necessary to implement the nonlinear P-stable method of Li and Wu [7] etc.

This paper is arranged at four sections: In Section 2, we discuss about the phase-lag analysis of symmetric multistep methods. The presentation, production, and analysis of the new method (local truncation error, Schrödinger error and stability region of periodicity analysis) is presented in Section 3. Numerical tests which are obtained by the application of the new method to some problems such as the radial time independent Schrödinger equation are presented in Section 4.

2. Basic Theory

For the numerical solution of the initial value problem (1), multistep methods of the form

$$\sum_{i=1}^{k} c_i\left(y_{n+i}+y_{n-i}\right)+c_0 y_n=h^2\left[\sum_{i=1}^{k} b_i\left(f_{n+i}+f_{n-i}\right)+b_0 f_n\right], \qquad (2)$$

with $2k$ steps can be used over the equally spaced intervals $\{x_{n+i}\}_{i=-k}^{k} \in [a,b]$ and $h=|x_{i+1}-x_i|$, $i=-k(1)k-1$. When the symmetric $2k$-step method (2) is applied to the scalar test equation

$$y^{''}(x)=-\omega^2 y(x), \qquad (3)$$

a difference equation $A_0(v)y_n+\sum_{i=1}^{k} A_i(v)\left(y_{n-i}+y_{n+i}\right)=0$ is obtained, where $v=\omega h$, h is the step length and $A_0(v), A_1(v), \ldots, A_k(v)$ are polynomials of v and hence the characteristic equation of (2) will be

$$A_0(v)+\sum_{i=1}^{k} A_i(v)\left(s^{-i}+s^{i}\right)=0.$$

Now, we need the following definitions ([30]).

Definition 2.1. The interval $(0, v_0^2)$ is called the periodicity interval of method (2) if the roots τ_j, $j=1,2,\ldots,2k$, satisfy

$$\tau_{1,2}=\exp(\pm i\theta(v)), \qquad |\tau_j|\leqslant 1, \quad j=3,4,\ldots,2k. \qquad (4)$$

where $\theta(v)$ is a real function of v. A method is called P-stable if its interval of periodicity is equal to $(0,\infty)$.

Definition 2.2. A multistep method is called singularly almost P-stable if its interval of periodicity is equal to $(0,\infty)-S$ where S is a set of distinct points

Definition 2.3. The phase-lag error of method (2) is defined by $PL=v-\theta(v)$. Then if the quantity $PL=O(v^{q+1})$ as $v\to\infty$, the order of phase-lag is q.

Theorem 2.4. *The symmetric $2k$-step method* (2) *has phase-lag order q and phase-lag constant c given by*

$$-cv^{q+2} + O(v^{q+4}) = \frac{\sum_{i=1}^{k} 2A_i(v)\cos(iv) + A_0(v)}{\sum_{i=1}^{k} 2i^2 A_i(v)}. \tag{5}$$

Proof. See [30]. □

3. Development and Analysis

For the numerical integration of (1), we consider four-step, symmetric multiderivative method of the form

$$\begin{aligned} y_{n+2} + y_{n-2} + a_1\left(y_{n+1} + y_{n-1}\right) + a_0 y_n =& h^2\left[b_1\left(f_{n+1} + f_{n-1}\right) + b_0 f_n\right] \\ &+ h^4\left[c_0 f_n^{(2)}\right] \\ &+ h^6\left[d_0 f_n^{(4)}\right], \end{aligned} \tag{6}$$

where a_j, b_j, $j = 0, 1$ and c_0 and d_0 are six arbitrary parameters that must be calculated. Applying (6) to the scalar test Eq. (3), one gets its difference equation

$$A_2(v)(y_{n+2} + y_{n-2}) + A_1(v)(y_{n+1} + y_{n-1}) + A_0(v)y_n = 0, \tag{7}$$

where $A_i(v) = a_i + v^2 b_i - v^4 c_i + v^6 d_i$, $i = 0, 1, 2$ such that $c_1 = c_2 = d_1 = d_2 = 0$, $a_2 = 1$ and $v = \omega h$. So, its corresponding characteristic equation is given by

$$A_2(v)(\lambda^4 + 1) + A_1(v)(\lambda^3 + \lambda) + A_0(v)\lambda^2 = 0. \tag{8}$$

Now, if we assume that $A_1(v) = 0$, then (8) is reduced to $A_2(v)(\lambda^4+1)+A_0(v)\lambda^2 = 0$. In addition, to calculate the phase-lag of the method (6), we apply the direct formula (5) for $k = 2$ and for $A_j(v)$, $j = 0(1)2$. This leads to the following equation:

$$PL = \frac{2\left(d_1 v^6 - c_1 v^4 + v^2 b_1 + a_1\right)\cos(v) + 2\cos(2v) + v^6 d_0 - v^4 c_0 + v^2 b_0 + a_0}{2 d_1 v^6 - 2 c_1 v^4 + 2 v^2 b_1 + 2 a_1 + 8}. \tag{9}$$

We demand that the phase-lag and its first, second, third and fourth derivatives and $A_1(v)$ to be equal to zero. So we have the following system

$$\begin{cases} A_1(v) = 0, \\ PL^{(i)} = 0, \quad \text{i=0,1,2,3,4.} \end{cases}$$

By solving the above system of equations, the coefficients of the new four-step multiderivative method are given by

$$a_0 = \frac{1}{24}\Big(8\left(\cos(v)\right)^3 v^5 - 72\left(\cos(v)\right)^2 \sin(v)\, v^4 - 246\left(\cos(v)\right)^3 v^3 + 4\cos(v)\, v^5$$

$$+ 363\ (\cos(v))^2 \sin(v)\, v^2 + 24\, v^4 \sin(v) + 261\, v\, (\cos(v))^3 + 213\, v^3 \cos(v)$$
$$- 72\ (\cos(v))^2 \sin(v) - 69\, v^2 \sin(v)$$
$$- 45\, v \cos(v) - 144 \sin(v) \Big) \Big/ \Big(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \Big),$$

$$a_1 = \Big(8\ (\cos(v))^2 v^3 - 24 \cos(v) \sin(v)\, v^2 - 30\ (\cos(v))^2 v - 4\, v^3$$
$$+ 15 \sin(v) \cos(v) + 15\, v \Big) Big/ \Big(2\left(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \right) \Big),$$

$$b_0 = -\frac{1}{8} \Big(8\, (\cos(v))^3 v^5 - 56\ (\cos(v))^2 \sin(v)\, v^4 - 142\ (\cos(v))^3 v^3$$
$$+ 4 \cos(v)\, v^5 + 219\ (\cos(v))^2 \sin(v)\, v^2 + 16\, v^4 \sin(v) + 225\, v\, (\cos(v))^3$$
$$+ 149\, v^3 \cos(v) - 120\ (\cos(v))^2 \sin(v) - 81\, v^2 \sin(v) - 105\, v \cos(v) \Big) \Big/$$
$$\Big(v^2 \left(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \right) \Big),$$

$$b_1 = -\frac{1}{2} \Big(8\ (\cos(v))^2 v^3 - 24 \cos(v) \sin(v)\, v^2 - 30\ (\cos(v))^2 v - 4\, v^3$$
$$+ 15 \sin(v) \cos(v) + 15\, v \Big) \Big/ \Big(v^2 \left(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \right) \Big),$$

$$c_0 = -\frac{1}{8} \Big(8\ (\cos(v))^3 v^4 - 40\ (\cos(v))^2 \sin(v)\, v^3 - 70\ (\cos(v))^3 v^2 + 4 \cos(v)\, v^4$$
$$+ 75\ (\cos(v))^2 \sin(v)\, v + 8 \sin(v)\, v^3 + 45\ (\cos(v))^3 + 85 \cos(v)\, v^2$$
$$- 45\, v \sin(v) - 45 \cos(v) \Big) \Big/ \Big(v^3 \left(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \right) \Big),$$

$$d_0 = -\frac{1}{24} \Big(8\ (\cos(v))^3 v^4 - 24\ (\cos(v))^2 \sin(v)\, v^3 - 30\ (\cos(v))^3 v^2$$
$$+ 4 \cos(v)\, v^4 + 27\ (\cos(v))^2 \sin(v)\, v + 9\ (\cos(v))^3 + 21 \cos(v)\, v^2$$
$$- 9\, v \sin(v) - 9 \cos(v) \Big) \Big/ \Big(v^5 \left(v^2 \sin(v) + 3\, v \cos(v) - 3 \sin(v) \right) \Big),$$

For small values of $|v|$, these coefficients are subject to heavy cancelations. In this case the following Taylor series expansions should be used:

$$a_0 = -2 + \frac{64}{7} v^2 - \frac{608}{441} v^4 + \frac{2312}{33957} v^6 - \cdots,$$
$$a_1 = -\frac{32}{7} v^2 + \frac{304}{441} v^4 - \frac{1156}{33957} v^6 + \cdots,$$

$$
\begin{aligned}
b_0 &= -\frac{36}{7} - \frac{1408}{441}v^2 + \frac{2344}{3773}v^4 - \frac{902312}{27810783}v^6 - \cdots, \\
b_1 &= \frac{32}{7} - \frac{304}{441}v^2 + \frac{1156}{33957}v^4 - \frac{22226}{27810783}v^6 + \cdots, \\
c_0 &= -\frac{68}{21} + \frac{136}{441}v^2 + \frac{2384}{101871}v^4 - \frac{752824}{139053915}v^6 + \cdots, \\
d_0 &= -\frac{64}{315} + \frac{296}{6615}v^2 - \frac{6568}{1528065}v^4 + \frac{99536}{417161745}v^6 - \cdots,
\end{aligned}
$$

where $v = \omega h$, and the local truncation error of the new method is

$$
LTE_{ex4} = \frac{67h^{10}}{198450}\left[\omega^{10}y + 5\omega^8 y^{(2)} + 10\omega^6 y^{(4)} + 10\omega^4 y^{(6)} + 5\omega^2 y^{(8)} + y^{(10)}\right]. \tag{10}
$$

The behavior of the coefficients are given in Figures 1, 2 and 3. In the related formulas to a_i, b_i, c_i and d_j, $i = 0, 1$, we get to know that the coefficients of the new method in what values of v are smooth or in what values have high volatility, and they may even have some asymptotic in some states (when the denominator of the ratio is targeted zero). Obviously, when the coefficient for every value of the v is an asymptote, or has a high fluctuation, it would be better to use Taylor series. Since the new method is explicit, it is most important to show its stability

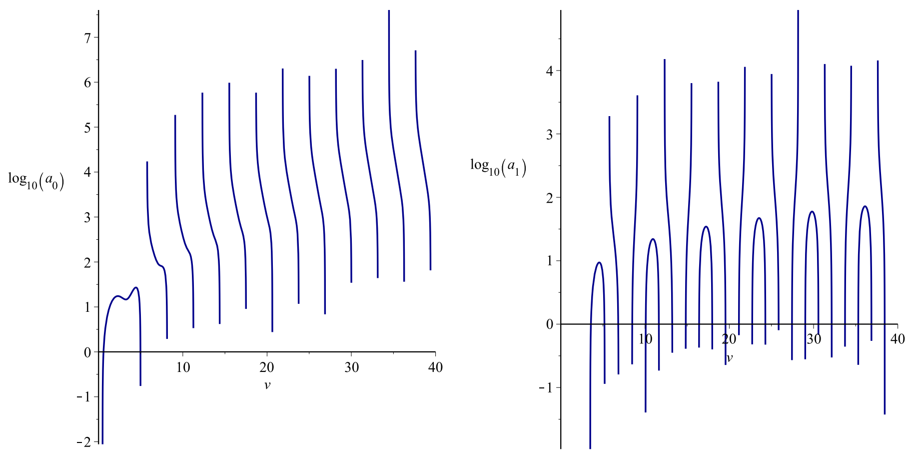

Figure 1: Behavior of the coefficients a_0 and a_1 of the new method.

property. The singularly P-stability of the new method can be demonstrated in two ways. At first, by its figure and at second by the theorem. For this purpose, the application of the new method (6), to the scalar test equation

$$
y''(x) = -\phi^2 y(x), \tag{11}
$$

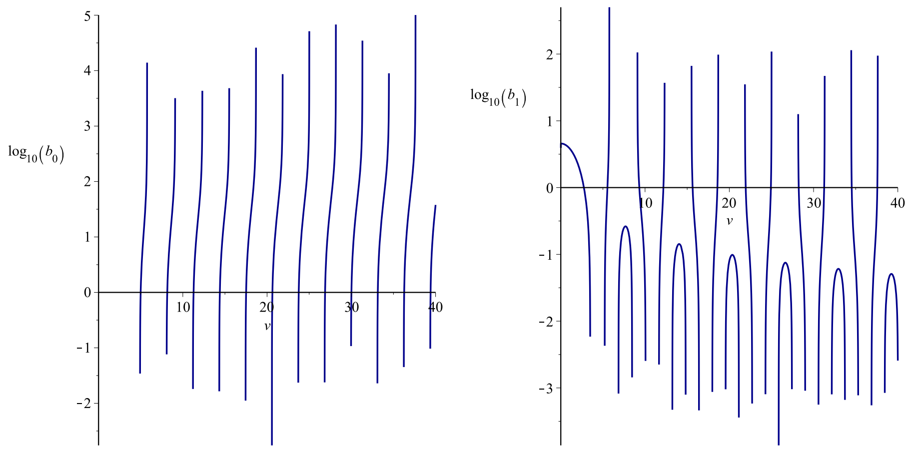

Figure 2: Behavior of the coefficients b_0 and b_1 of the new method.

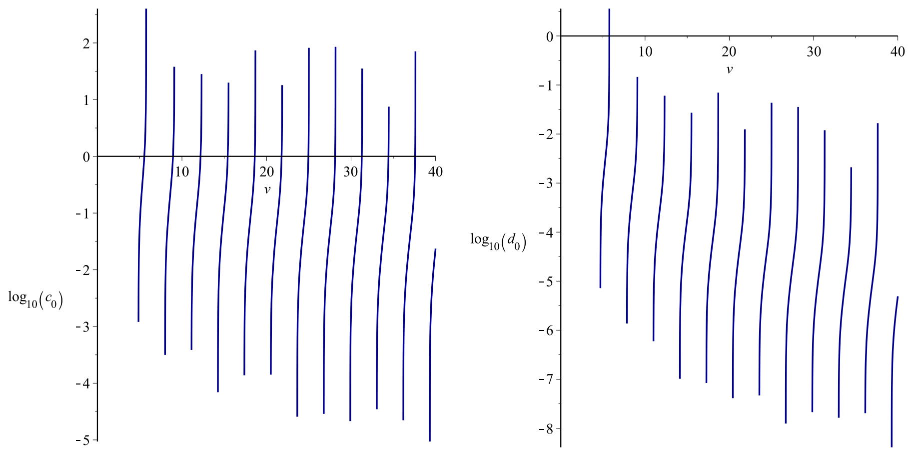

Figure 3: Behavior of the coefficients c_0 and d_0 of the new method.

leads to the following difference equation

$$A_2(s,v)\left(y_{n+2}+y_{n-2}\right)+A_1(s,v)\left(y_{n+1}+y_{n-1}\right)+A_0(s,v)y_n=0, \qquad (12)$$

where $A_0(s,v) = -\frac{1}{24v^5}\frac{A_{00}}{A}$, $A_1(s,v) = -\frac{1}{2v^2}\frac{A_{10}}{A}$, $A_2(s,v) = 1$ where $v = \omega h$, $s = \phi h$ and A_{i0}, $i = 0, 1$ and

$$A = v^2 \sin(v) + 3v\cos(v) - 3\sin(v), \qquad (13)$$

$$
\begin{aligned}
A_{00} =&8\ (\cos(v))^3\, s^6 v^4 - 24\ (\cos(v))^3\, s^4 v^6 \\
&+ 24\ (\cos(v))^3\, s^2 v^8 + 72\ (\cos(v))^2 \sin(v)\, v^9 - 30\ (\cos(v))^3\, s^6 v^2 \\
&+ 210\ (\cos(v))^3\, s^4 v^4 - 426\ (\cos(v))^3\, s^2 v^6 + 4\ \cos(v)\, s^6 v^4 \\
&- 12\ \cos(v)\, s^4 v^6 + 12\ \cos(v)\, s^2 v^8 - 363\ (\cos(v))^2 \sin(v)\, v^7 \\
&- 24\ \sin(v)\, s^4 v^5 + 48\ \sin(v)\, s^2 v^7 - 135\ (\cos(v))^3\, s^4 v^2 \\
&+ 675\ (\cos(v))^3\, s^2 v^4 + 21\ \cos(v)\, s^6 v^2 - 255\ \cos(v)\, s^4 v^4 + 447\ \cos(v)\, s^2 v^6 \\
&+ 72\ (\cos(v))^2 \sin(v)\, v^5 - 9\ \sin(v)\, s^6 v + 135\ \sin(v)\, s^4 v^3 - 243\ \sin(v)\, s^2 v^5 \\
&+ 135\ \cos(v)\, s^4 v^2 - 315\ \cos(v)\, s^2 v^4 - 24\ (\cos(v))^2 \sin(v)\, s^6 v^3 \\
&+ 120\ (\cos(v))^2 \sin(v)\, s^4 v^5 - 168\ (\cos(v))^2 \sin(v)\, s^2 v^7 \\
&+ 27\ (\cos(v))^2 \sin(v)\, s^6 v - 225\ (\cos(v))^2 \sin(v)\, s^4 v^3 \\
&+ 657\ (\cos(v))^2 \sin(v)\, s^2 v^5 - 360\ (\cos(v))^2 \sin(v)\, s^2 v^3 \\
&+ 69\ \sin(v)\, v^7 + 45\ \cos(v)\, v^6 + 144\ \sin(v)\, v^5 \\
&- 8\ (\cos(v))^3\, v^{10} + 246\ (\cos(v))^3\, v^8 - 4\ \cos(v)\, v^{10} - 24\ \sin(v)\, v^9 \\
&+ 9\ (\cos(v))^3\, s^6 - 261\ (\cos(v))^3\, v^6 - 213\ \cos(v)\, v^8 - 9\ \cos(v)\, s^6,
\end{aligned}
$$

and

$$
\begin{aligned}
A_{10} =&\Big(8\,(\cos(v))^2\, v^3 - 24\cos(v)\sin(v)v^2 - 30\,(\cos(v))^2\, v \\
&- 4v^3 + 15\sin(v)\cos(v) + 15v\Big)\left(s^2 - v^2\right). \qquad (14)
\end{aligned}
$$

A linear multistep method is said to be P-stable if the first quadrant of the $s - v$ plane is completely shadowed and it is said to be singularly P-stable if the method is P-stable when $\omega = \phi$ i.e. only when the frequency of the scalar test equation for the stability analysis is equal with the frequency of the scalar test equation for the phase-lag analysis, i.e. the surroundings of the first diagonal of the $s - v$ plane. The stability region ($s - v$ plane) of the new method is plotted in Figure 4. A shadowed area denotes the region where the method is stable, while a white area denotes the region where the method is unstable. According to Figure 4, we can say that the new method is singularly P-stable. Of course, in the following theorem, we prove algebraically that the new method is singularly P-stable.

Theorem 3.1. *The new explicit four-step multiderivative method with vanished phase-lag and its first, second, third and fourth derivatives* (6) *is singularly P-stable.*

Proof. The stability function of the new method is

$$ST = A_2(s,v)\left(\lambda^4 + 1\right) + A_1(s,v)\left(\lambda^3 + \lambda\right) + A_0(s,v)\lambda^2, \qquad (15)$$

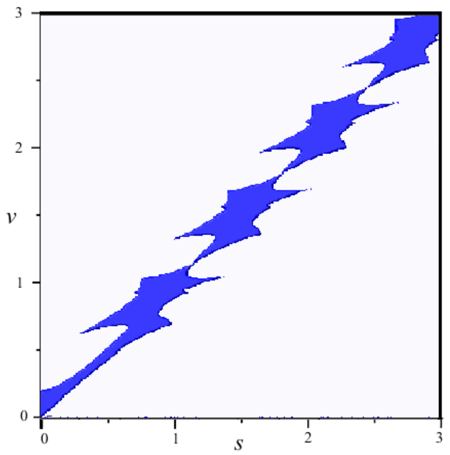

Figure 4: The periodicity region of the new singularly P-stable multiderivative method.

where $A_i(s, v)$, $i = 0, 1, 2$ are mentioned after (12). In the case $s = v$ we have

$$\begin{aligned} A_0 &= 2 - 4\cos^2(v), \\ A_1 &= 0, \\ A_2 &= 1. \end{aligned}$$

Then the characteristic equation for the new method (6) is given by $ChE = \lambda^4 - 2\left(2\cos^2(v) - 1\right)\lambda^2 + 1$. Now, since $\cos(2v) = 2\cos^2(v) - 1$, we have $\lambda_{1,2} = \exp(\pm iv)$, $\lambda_{3,4} = -\exp(\pm iv)$. So, obviously the interval of periodicity of the new method is $(0, \infty)$, and thus when $s = v$, the new method is P-stable, i.e. the new explicit four-step multiderivative method with vanished phase-lag and some of its derivatives (6) is singularly P-stable. □

4. Numerical Results

In this section, we compare the numerical performance of the new multiderivative method with that of some existing multistep methods proposed in the scientific literature:

- Simos: The 12th order Obrechkoff method of Simos [28].
- Daele: The 12th order Obrechkoff method of Van Daele [33].
- Achar: The 8th order Obrechkoff method of Achar [2].
- Wang: The 12th order Obrechkoff method of Wang [35].

- New: The 8th order multiderivative method developed in this paper.

Example 4.1. *We consider the periodically forced nonlinear problem* (*undamped Duffing's equation*)

$$y'' = -y - y^3 + B\cos(\omega x), \quad y(0) = 0.200426728067, \quad y'(0) = 0, \tag{16}$$

where $B = 0.002$, $\omega = 1.01$ *and* $x \in \left[0, \frac{40.5\pi}{1.01}\right]$. *We use the following exact solution for* (16) *from* [14], $g(x) = \sum_{i=0}^{3} K_{2i+1}\cos((2i+1)\omega x)$, *where*

$$\{K_1, K_3, K_5, K_7\} = \{0.200179477536, 0.246946143 \times 10^{-3}, \\ 0.304016 \times 10^{-6}, 0.374 \times 10^{-9}\}.$$

In order to integrate this equation by a Obrechkoff method, one needs the values of y', *which occur in calculating* $y^{(4)}$. *These higher order derivatives can all be expressed in terms of* $y(x)$ *and* $y'(x)$ *through* (16), *i.e.*

$$y^{(3)}(x) = -(1 + 3y^2(x))y'(x) - B\omega\sin(\omega x),$$
$$y^{(4)}(x) = -(1 + 3y^2(x))y''(x) - 6y(x)y'(x)^2 - B\omega^2\cos(\omega x).$$

h	New	Simos	Daele	Achar	Wang
$\frac{M}{500}$	4.36e-5	3.15e-4	4.06e-5	4.09e-5	4.08e-5
$\frac{M}{1000}$	1.04e-6	1.81e-5	1.87e-6	1.27e-6	1.27e-6
$\frac{M}{2000}$	1.21e-8	1.08e-6	3.83e-8	3.94e-8	3.93e-8
$\frac{M}{3000}$	3.14e-9	2.09e-7	5.13e-9	5.18e-9	5.17e-9
$\frac{M}{4000}$	6.13e-11	6.55e-8	3.19e-9	1.23e-9	1.23e-9
$\frac{M}{5000}$	2.16e-13	2.67e-8	9.89e-10	4.09e-10	4.07e-10

Table 1: Comparison of the end-point absolute error in the approximations obtained by using Methods: New, Simos, Daele, Achar and Wang for Example 4.1.

Example 4.2. *Consider the initial value problem*

$$y'' = -100y + 99\sin(x), \quad y(0) = 1, \quad y'(0) = 11,$$

with the exact solution $y(t) = \sin(t) + \sin(10t) + \cos(10t)$. *This equation has been solved numerically for* $0 \le x \le 10\pi$ *using exact starting values. In the numerical experiment, we take the step lengths* $h = \pi/50$, $\pi/100$, $\pi/200$, $\pi/300$, $\pi/400$, *and* $\pi/500$ *and for this problem we use* $w = 1$.

Example 4.3. *Consider the initial value problem*

$$y'' = \frac{8y^2}{1+2x}, \quad y(0) = 1, \quad y'(0) = -2, \quad x \in [0, 4.5],$$

with the exact solution $y(x) = \frac{1}{1+2x}$. *The absolute errors have been calculated at* $x = 4.5$. *For this problem we use* $w = 1$.

h	New	Simos	Daele	Achar	Wang
$\frac{M}{500}$	1.3	1.4	1.5	1.2	1.4
$\frac{M}{1000}$	2.4	2.9	2.9	2.3	2.9
$\frac{M}{2000}$	4.3	6.2	6.3	4.8	6.2
$\frac{M}{3000}$	6.8	9.8	9.7	7.5	9.5
$\frac{M}{4000}$	9.2	13.5	13.3	10	13
$\frac{M}{5000}$	10.1	17	17	12.9	16.5

Table 2: CPU time for the Example 4.1, are calculated for comparison among five methods: New, Simos, Daele, Achar and Wang.

h	New	Simos	Daele	Achar
$\frac{\pi}{50}$	3.21e-14	3.05e-11	1.20e-11	5.79e-13
$\frac{\pi}{100}$	2.18e-16	2.28e-13	7.34e-13	5.79e-13
$\frac{\pi}{200}$	5.17e-16	4.40e-13	8.62e-13	1.32e-12
$\frac{\pi}{300}$	6.14e-15	2.11e-12	2.63e-12	1.96e-12
$\frac{\pi}{400}$	2.68e-15	1.38e-12	2.93e-12	4.78e-12
$\frac{\pi}{500}$	1.91e-15	6.46e-12	2.89e-12	7.50e-12

Table 3: Comparison of the end-point absolute error in the approximations obtained by using Methods: New, Simos, Daele, and Achar for Example 4.2.

h	New	Simos	Daele	Achar
$\frac{\pi}{50}$	0.14	0.17	0.25	0.19
$\frac{\pi}{100}$	0.36	0.51	0.53	0.45
$\frac{\pi}{200}$	0.72	0.86	0.83	0.75
$\frac{\pi}{300}$	0.82	1.14	1.15	0.95
$\frac{\pi}{400}$	1.13	1.39	1.40	1.23
$\frac{\pi}{500}$	1.41	1.70	1.78	1.47

Table 4: CPU time for the Example 4.2, are calculated for comparison among four methods: New, Simos, Daele and Achar.

5. Conclusions

In this paper, we have presented a new explicit singularly P-stable four-step multiderivative method for the numerical solution of periodic or high oscillatory initial value problems. From the numerical test to the well-known problems such as Duffing's equation without damping, we found that the new method has the advantage in simplicity, accuracy, stability and efficiency.

All computations were carried out on a PC(i5@2.67GHz) using Maple 17 using double precision arithmetic with 16 significant digits accuracy (IEEE standard).

h	New	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	2.17e-9	1.24e-7	1.26e-7	1.26e-7	1.24e-7
$\frac{4.5}{1000}$	4.38e-11	3.82e-9	3.90e-9	3.85e-9	3.82e-9
$\frac{4.5}{2000}$	2.92e-13	1.19e-10	1.23e-10	1.20e-10	1.19e-10
$\frac{4.5}{3000}$	2.73e-14	1.92e-11	2.02e-11	1.40e-11	1.92e-11
$\frac{4.5}{4000}$	1.81e-15	7.85e-12	7.85e-12	2.68e-12	7.85e-12
$\frac{4.5}{5000}$	1.05e-16	1.63e-12	1.63e-12	7.47e-14	1.63e-12

Table 5: Comparison of the end-point absolute error in the approximations obtained by using Methods: New, Simos, Daele, Achar and Wang for Example 4.3.

h	New	Simos	Daele	Achar	Wang
$\frac{4.5}{500}$	0.14	0.369	0.34	0.19	0.31
$\frac{4.5}{1000}$	0.49	0.62	0.61	0.76	1.23
$\frac{4.5}{2000}$	0.49	0.62	0.61	0.76	1.23
$\frac{4.5}{3000}$	0.83	1.23	1.92	1.20	1.87
$\frac{4.5}{4000}$	1.36	1.89	2.59	1.62	2.56
$\frac{4.5}{5000}$	2.01	2.59	3.29	2.06	3.24

Table 6: CPU time for the Example 4.3, are calculated for comparison among five methods: New, Simos, Daele, Achar and Wang.

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