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Gordon-Scantlebury and Platt Indices of Random Plane-oriented Recursive Trees

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Abstract

For a simple graph G, the Gordon-Scantlebury index of G is equal to the number of paths of length two in G, and the Platt index is equal to the total sum of the degrees of all edges in G. In this paper, we study these indices in random plane-oriented recursive trees through a recurrence equation for the first Zagreb index. As $n \to \infty$, the asymptotic normality of these indices are given.

Keywords: Gordon-Scantlebury index, Platt index, the first Zagreb index, plane-oriented recursive tree, asymptotic normality.

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1. Introduction

A graph is a collection of points and lines connecting a subset of them. The points and lines of a graph are also called vertices and edges of the graph, respectively. The vertex and edge sets of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The degree of a vertex v of a graph is the number of edges incident to the vertex v and is denoted by $d(v)$ (or d_v). A path in a graph is a sequence of adjacent edges, which do not pass through the same vertex more than once, and the length of the path is the number of edges in it.

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Trees are defined as connected graphs without cycles, and their properties are basics of graph theory. A recursive tree with n nodes is an unordered rooted tree, where the nodes are labelled by distinct integers from $\{1, 2, 3, ..., n\}$ in such a way that the sequence of labels lying on the unique path from the root node to any node in the tree are always forming an increasing sequence [10]. A plane-oriented recursive tree is a recursive tree with ordered sets of descendants. In recursive trees, the ordering of the immediate descendants of a given node does not matter, as all ordering represent the same tree. A random plane-oriented recursive tree of order n is one chosen with equal probability from the space of all such trees. There is a simple growth rule for the class of plane-oriented recursive trees. In this class, a random tree T_n, of order n, is obtained from T_{n-1}, a random tree of order $n-1$, by choosing a parent in T_{n-1} and adjoining a node labeled n to it. The node n can be adjoined at any of the insertion positions or gaps between the children of the chosen parent since insertion in each gap will give a different ordering. We can describe the plane-oriented recursive tree evolution process which generates random trees (of arbitrary order n) of grown trees. This description is a consequence of the considerations made in:
Step 1: The process starts with the root labelled by 1.
Step $i+1$: At step $i+1$ the node with label $i+1$ is attached to any previous node v (with degree $d(v)$) of the already grown plane-oriented recursive tree of order i with probability

$$p(v) = \frac{d(v)}{2i-1}.$$

Plane-oriented recursive trees, abbreviated as PORTs, were introduced in the literature under a few different names such as heap-ordered trees, nonuniform recursive trees, scale-free trees (see [10] for main results on this tree).

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariant. For a simple graph G, the Gordon-Scantlebury index of G is equal to the number of paths of length two in G [3], and the Platt index is equal to the total sum of the degrees of all edges in G [8]. One of the most important topological indices of a graph is the first Zagreb index. The first Zagreb index is related to the Gordon-Scantlebury and Platt indices. Let $S(G)$, and $P(G)$ be the Gordon-Scantlebury index and the Platt index of the graph G, respectively. We have

$$\begin{aligned} S(G) &= \frac{Z(G)}{2} - |E(G)|, \\ P(G) &= 2S(G), \end{aligned} \tag{1}$$

where $|E(G)|$ is the number of edges of G and the first Zagreb index $Z(G)$ of G is defined as

$$Z(G) = \sum_{v \in V(G)} d_v^2,$$

where $d(v)$ denotes the degree of the vertex v in G [4, 7]. Thus, the first Zagreb index of a graph is defined as the sum of the squares of the degrees of all vertices in the graph. This index introduced by chemists Gutman and Trinajstić [5]. This index reflects the extent of branching of the molecular carbon-atom skeleton, and can thus be viewed as molecular structure-descriptors. Followed by the first Zagreb index, Furtula and Gutman [2] introduced forgotten topological index (also called F-index) of G which was defined as

$$F(G) = \sum_{v \in V(G)} d_v^3.$$

The motivation of studying the indices of trees is multifold (see [9] and refrences therein).

The paper is organized as follows. In Section 2, we give the first two moments (mean and variance) of Gordon-Scantlebury and Platt indices of random plane-oriented recursive trees through a recurrence equation for the first Zagreb index. In Section 3, we give the asymptotic normality of these indices.

2. The First Two Moments

The following lemma is very important for computing the mean and variance of the above topological indices in our tree model.

Lemma 2.1. *Suppose that*

$$\mathcal{K}(n, j, i) := \frac{\Gamma\left(n + \frac{3+i}{2}\right)}{\Gamma\left(n + \frac{3-j}{2}\right)}, \quad n \geq 3, \ i, j \geq 1,$$

where $\Gamma(\cdot)$ *is the gamma function. Then*

$$\frac{2n-1}{2n-3} = \frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)},$$
$$\frac{2n}{2n-3} = \frac{\mathcal{K}(n-1,2,1)}{\mathcal{K}(n-2,2,1)},$$
$$\frac{\mathcal{K}(n,4,0)}{\mathcal{K}(n-1,4,0)} = 2\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)} - 1.$$

Proof. The proof is obvious and straightforward, since $\Gamma(x) = (x-1)\Gamma(x-1)$. □

Let $\mathcal{F}_n$ be the sigma-field generated by the first n stages of plane-oriented recursive trees and U_n be a randomly chosen node belonging to these trees of order n [1].

Theorem 2.2. *Let* S_n *and* P_n *be the Gordon-Scantlebury index and Platt index of a random plane-oriented recursive tree of order* $n \geq 3$*, respectively. Then*

$$\mathbb{E}(S_n) = \mathcal{K}(n-1,2,0) \sum_{t=1}^{n-1} \frac{1}{\mathcal{K}(t,2,0)} - (n-1),$$

and

$$\mathbb{E}(P_n) = 2\mathcal{K}(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{\mathcal{K}(t,2,0)} - 2(n-1).$$

Proof. We first compute the mean of the fisrt Zagreb index. Using the definition of the first Zagreb index and by the stochastic growth rule of the tree,

$$Z_n = Z_{n-1} + (d_{U_{n-1}}+1)^2 - d^2_{U_{n-1}} + 1 = Z_{n-1} + 2d_{U_{n-1}} + 2. \tag{2}$$

From Lemma 2.1 and [1],

$$\begin{aligned}
\mathbb{E}(Z_n|\mathcal{F}_{n-1}) &= \mathbb{E}(Z_{n-1} + 2d_{U_{n-1}} + 2\,|\,\mathcal{F}_{n-1}) \\
&= Z_{n-1} + 2\mathbb{E}(d_{U_{n-1}}|\mathcal{F}_{n-1}) + 2 \\
&= Z_{n-1} + 2\,\frac{1}{2n-3}\sum_{i=1}^{n-1} d^2_{v_i} + 2 \\
&= Z_{n-1} + \frac{2}{2n-3}Z_{n-1} + 2 \\
&= \frac{2n-1}{2n-3}Z_{n-1} + 2 \\
&= \frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}Z_{n-1} + 2.
\end{aligned}$$

Thus

$$\mathbb{E}(Z_n) = \frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}\mathbb{E}(Z_{n-1}) + 2. \tag{3}$$

By iteration,

$$\mathbb{E}(Z_n) = 2\mathcal{K}(n-1,2,0)\sum_{t=1}^{n-1}\frac{1}{\mathcal{K}(t,2,0)}, \quad Z_1 = 0, Z_2 = 1.$$

Proof is completed by relations (1). □

It is obvious that

$$\mathcal{K}(n,j,i) = n^{\frac{i+j}{2}}(1+\mathcal{O}(n^{-1})).$$

Thus

$$\begin{aligned}
\mathbb{E}(S_n) &= n\log n + \mathcal{O}(n), \\
\mathbb{E}(P_n) &= 2n\log n + \mathcal{O}(n).
\end{aligned}$$

Theorem 2.3. *Let F_n be the F-index of a random plane-oriented recursive tree of order $n \geq 3$. Then*

$$\mathbb{E}(F_n) = \mathcal{K}(n-1,2,1)\sum_{t=1}^{n-1}\frac{\alpha(t)}{\mathcal{K}(t,2,1)},$$

where

$$\alpha(t) = \frac{3}{2t-1}\mathbb{E}(Z_t) + 2, \qquad t \geq 1.$$

Proof. Using the definition of F-index and by the stochastic growth rule of the tree,

$$\begin{aligned} F_n &= F_{n-1} + (d_{U_{n-1}} + 1)^3 - d^3_{U_{n-1}} + 1 \\ &= F_{n-1} + 3d^2_{U_{n-1}} + 3d_{U_{n-1}} + 2. \end{aligned}$$

This implies that

$$\begin{aligned} \mathbb{E}(F_n|\mathcal{F}_{n-1}) &= F_{n-1} + 3\mathbb{E}(d^2_{U_{n-1}}|\mathcal{F}_{n-1}) + 3\mathbb{E}(d_{U_{n-1}}|\mathcal{F}_{n-1}) + 2 \\ &= F_{n-1} + \frac{3}{2n-3}\sum_{k=i}^{n-1} d^3_{v_i} + \frac{3}{2n-3}\sum_{i=1}^{n-1} d^2_{v_i} + 2 \\ &= F_{n-1} + \frac{3}{2n-3}\ F_{n-1} + \frac{3}{2n-3}Z_{n-1} + 2 \\ &= \frac{2n}{2n-3}F_{n-1} + \frac{3}{2n-3}Z_{n-1} + 2 \\ &= \frac{\mathcal{K}(n-1,2,1)}{\mathcal{K}(n-2,2,1)}F_{n-1} + \frac{3}{2n-3}Z_{n-1} + 2. \end{aligned}$$

Thus

$$\mathbb{E}(F_n) = \frac{\mathcal{K}(n-1,2,1)}{\mathcal{K}(n-2,2,1)}\mathbb{E}(F_{n-1}) + \alpha(n-1).$$

By iteration, proof is completed since $F_1 = 0$. □

Lemma 2.4. *The sequences*

$$\Big(\frac{S_n - \mathbb{E}(S_n)}{\mathcal{K}(n-1,2,0)}\Big)_{n\geq 1},$$

and

$$\Big(\frac{P_n - \mathbb{E}(P_n)}{\mathcal{K}(n-1,2,1)}\Big)_{n\geq 1},$$

are two martingales relative to the $\mathcal{F}_{n-1}$.

Proof. We have

$$\begin{aligned}
\mathbb{E}\Big(\frac{S_n-\mathbb{E}(S_n)}{\mathcal{K}(n-1,2,0)}|\mathcal{F}_{n-1}\Big) &= \mathbb{E}\Big(\frac{\frac{Z_n}{2}-(n-1)-\frac{\mathbb{E}(Z_n)}{2}+(n-1)}{\mathcal{K}(n-1,2,0)}|\mathcal{F}_{n-1}\Big)\\
&= \frac{1}{2\mathcal{K}(n-1,2,0)}\mathbb{E}(Z_n-\mathbb{E}(Z_n)|\mathcal{F}_{n-1})\\
&= \frac{1}{2\mathcal{K}(n-1,2,0)}\Big(\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}(Z_{n-1}-\mathbb{E}(Z_{n-1}))\Big)\\
&= \frac{1}{2\mathcal{K}(n-2,2,0)}(Z_{n-1}-\mathbb{E}(Z_{n-1}))\\
&= \frac{\frac{Z_{n-1}}{2}-(n-2)-\frac{\mathbb{E}(Z_{n-1})}{2}+(n-2)}{\mathcal{K}(n-2,2,0)}\\
&= \frac{S_{n-1}-\mathbb{E}(S_{n-1})}{\mathcal{K}(n-2,2,0)}.
\end{aligned}$$

The second martingale is obtained by the same method [1]. □

Theorem 2.5. *Let S_n and P_n be the Gordon-Scantlebury index and Platt index of a random plane-oriented recursive tree of order $n\geq 3$, respectively. Then*

$$\mathbb{V}ar(S_n)=\frac{\mathcal{K}(n,4,0)}{4}\sum_{t=1}^{n-1}\frac{\xi(t)}{\mathcal{K}(t+1,4,0)},$$

and

$$\mathbb{V}ar(P_n)=\mathcal{K}(n,4,0)\sum_{t=1}^{n-1}\frac{\xi(t)}{\mathcal{K}(t+1,4,0)},$$

where

$$\xi(t)=\frac{4}{2t-1}\mathbb{E}(F_t)-\Big(\frac{4\mathbb{E}(Z_t)}{2t-1}\Big)^2,\qquad t\geq 1.$$

Proof. We first study the variance of the first Zagreb index. We have

$$\begin{aligned}
\mathbb{E}(Z_n-Z_{n-1}-2)^2 &= 4\mathbb{E}(d_{U_{n-1}})^2\\
&= \frac{4}{2n-3}\sum_{i=1}^{n-1}\mathbb{E}(d_{v_i}^3)\\
&= \frac{4}{2n-3}\mathbb{E}(F_{n-1}). \qquad (4)
\end{aligned}$$

Also

$$\begin{aligned}
&\mathbb{E}((Z_n-\mathbb{E}(Z_n)-Z_{n-1}+\mathbb{E}(Z_{n-1}))(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2))\\
&=(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2)\times\mathbb{E}(Z_n-\mathbb{E}(Z_n)-Z_{n-1}+\mathbb{E}(Z_{n-1}))\\
&=(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2)\times 0\\
&=0,
\end{aligned}$$

and by Lemma 2.4,

$$\begin{aligned}
&\mathbb{E}((Z_n-\mathbb{E}(Z_n))(Z_{n-1}-\mathbb{E}(Z_{n-1})))\\
&=\mathbb{E}(\mathbb{E}((Z_n-\mathbb{E}(Z_n))(Z_{n-1}-\mathbb{E}(Z_{n-1})))|\mathcal{F}_{n-1}))\\
&=\mathbb{E}((Z_{n-1}-\mathbb{E}(Z_{n-1}))\mathbb{E}(Z_n-\mathbb{E}(Z_n)|\mathcal{F}_{n-1}))\\
&=\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}\mathbb{E}((Z_{n-1}-\mathbb{E}(Z_{n-1}))(Z_{n-1}-\mathbb{E}(Z_{n-1})))\\
&=\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}\mathbb{V}(Z_{n-1}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}(Z_n-Z_{n-1}-2)^2\\
&=\mathbb{E}(Z_n-\mathbb{E}(Z_n)-Z_{n-1}+\mathbb{E}(Z_{n-1})+\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2)^2\\
&=\mathbb{E}(Z_n-\mathbb{E}(Z_n)-Z_{n-1}+\mathbb{E}(Z_{n-1}))^2\\
&+\mathbb{E}(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2)^2\\
&+2\mathbb{E}((Z_n-\mathbb{E}(Z_n)-Z_{n-1}+\mathbb{E}(Z_{n-1}))(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2))\\
&=\mathbb{E}(Z_n-\mathbb{E}(Z_n))^2+\mathbb{E}(Z_{n-1}-\mathbb{E}(Z_{n-1}))^2\\
&-2\mathbb{E}((Z_n-\mathbb{E}(Z_n))(Z_{n-1}-\mathbb{E}(Z_{n-1})))\\
&+\mathbb{E}(\mathbb{E}(Z_n)-\mathbb{E}(Z_{n-1})-2)^2\\
&=\mathbb{V}ar(Z_n)+\Big(1-2\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}\Big)\mathbb{V}ar(Z_{n-1})+\Big(\frac{4\mathbb{E}(Z_{n-1})}{2n-3}\Big)^2. \qquad (5)
\end{aligned}$$

From (4) and then (5),

$$\frac{4}{2n-3}\mathbb{E}(F_{n-1})=\mathbb{V}ar(Z_n)+\Big(1-2\frac{\mathcal{K}(n-1,2,0)}{\mathcal{K}(n-2,2,0)}\Big)\mathbb{V}ar(Z_{n-1})+\Big(\frac{4\mathbb{E}(Z_{n-1})}{2n-3}\Big)^2.$$

By Lemma 2.1,

$$\mathbb{V}ar(Z_n)=\frac{\mathcal{K}(n,4,0)}{\mathcal{K}(n-1,4,0)}\mathbb{V}ar(Z_{n-1})+\xi(n-1),\quad \mathbb{V}ar(Z_1)=0.$$

By iteration, relation (1) and this fact that for each random variable X and $a,b\in\mathbb{R}$, $\mathbb{V}ar(aX+b)=a^2\mathbb{V}ar(X)$ proof is completed. □

Corollary 2.6. *We have*

$$\mathbb{V}(Z_n)=n\log n+\mathcal{O}(n).$$

3. Asymptotic Normality

We use the notation $\xrightarrow{D}$ to denote convergence in distribution. The standard random variable $\mathrm{N}(\mu, \sigma^2)$ appear in the following theorem for the normal distributed with mean μ and variance σ^2.

Theorem 3.1. *Let S_n and P_n be the Gordon-Scantlebury index and Platt index of a random plane-oriented recursive tree of order $n \geq 3$, respectively. Then as $n \to \infty$,*

$$S_n^* = \frac{2S_n - 2n\log n}{\sqrt{n\log n}} \xrightarrow{D} N(0,1),$$

and

$$P_n^* = \frac{P_n - 2n\log n}{\sqrt{n\log n}} \xrightarrow{D} N(0,1).$$

Proof. Let

$$U_j = \frac{Z_j - \mathbb{E}(Z_j)}{\mathcal{K}(j-1,2,0)} - \frac{Z_{j-1} - \mathbb{E}(Z_{j-1})}{\mathcal{K}(j-2,2,0)}, \quad j \geq 2,$$

with $U_1 = 0$. Thus $(U_n)_{n\geq 1}$ is a martingale difference sequence. Then $\mathbb{E}(U_j|\mathcal{F}_{j-1}) = 0$ and

$$\sum_{j=1}^{n} U_j = \frac{Z_n - \mathbb{E}(Z_n)}{\mathcal{K}(n-1,2,0)}.$$

Set

$$X_{n,j} = \frac{\mathcal{K}(n-1,2,0)}{\sqrt{n\log n}} U_j.$$

Thus

$$\sum_{j=1}^{n} X_{n,j} = \frac{Z_n - \mathbb{E}(Z_n)}{\sqrt{n\log n}}.$$

But

$$\sum_{j=1}^{n} \mathbb{E}(X_{n,j}|\mathcal{F}_{j-1}) = 0.$$

By (2) and (3),

$$\begin{aligned} U_j &= \Big(1 - \frac{\mathcal{K}(j-1,2,0)}{\mathcal{K}(j-2,2,0)}\Big)\frac{Z_{j-1}}{\mathcal{K}(j-1,2,0)} + \frac{2d_{U_{j-1}}}{\mathcal{K}(j-1,2,0)} \\ &= 2\frac{d_{U_{j-1}} - \frac{Z_{j-1}}{2j-3}}{\mathcal{K}(j-1,2,0)}. \end{aligned}$$

Then there exists a positive constant c independent of n such that

$$\max_{1\leq j\leq n} |U_j| \leq \frac{c}{\mathcal{K}(n-1,2,0)} = o\Big(\frac{1}{\sqrt{n\log n}\mathcal{K}(n-1,2,0)}\Big),$$

and thus for any $\varepsilon > 0$,

$$\sum_{j=1}^{n} \mathbb{E}(X_{n,j}^2 \mathbb{I}(|X_{n,j}| > \varepsilon)|\mathcal{F}_{j-1}) \xrightarrow{P} 0.$$

It is obvious that

$$\mathbb{E}((Z_j - Z_{j-1} - 2)^2|\mathcal{F}_{j-1}) = \frac{4}{2j-3} F_{j-1},$$

and

$$\begin{aligned}\mathbb{E}((Z_j - Z_{j-1} - 2)^2|\mathcal{F}_{j-1}) &= \mathbb{E}((Z_j - \mathbb{E}(Z_j))^2|\mathcal{F}_{j-1}) \\ &\quad + \Big(1 - 2\frac{\mathcal{K}(j-1,2,0)}{\mathcal{K}(j-2,2,0)}\Big)(Z_{j-1} - \mathbb{E}(Z_{j-1}))^2 \\ &\quad + \mathbb{E}((\mathbb{E}(Z_j) - \mathbb{E}(Z_{j-1}) - 2)^2|\mathcal{F}_{j-1}).\end{aligned}$$

Thus

$$\mathbb{E}((Z_j - \mathbb{E}(Z_j))^2|\mathcal{F}_{j-1}) = \frac{\mathcal{K}(j,4,0)}{\mathcal{K}(j-1,4,0)}(Z_{j-1} - \mathbb{E}(Z_{j-1}))^2 + \eta(j-1), \qquad (6)$$

where

$$\eta(j) = \frac{4}{2j-1} F_j - \Big(\frac{4\mathbb{E}(Z_j)}{2j-1}\Big)^2.$$

Then from (6),

$$\begin{aligned}\sum_{j=1}^{n} \mathbb{E}(X_{n,j}^2|\mathcal{F}_{j-1}) &= \frac{\mathcal{K}(n-1,2,0)^2}{n\log n} \sum_{j=1}^{n} \mathbb{E}(U_j^2|\mathcal{F}_{j-1}) \\ &= \frac{\mathcal{K}(n-1,2,0)^2}{n\log n} \sum_{j=1}^{n} \mathbb{E}\Big(\Big(\frac{Z_j - \mathbb{E}(Z_j)}{\mathcal{K}(j-1,2,0)} - \frac{Z_{j-1} - \mathbb{E}(Z_{j-1})}{\mathcal{K}(j-2,2,0)}\Big)^2 |\mathcal{F}_{j-1}\Big) \\ &= \frac{\mathcal{K}(n-1,2,0)^2}{n\log n} \sum_{j=1}^{n} \Big(\frac{\mathbb{E}((Z_j - \mathbb{E}(Z_j))^2|\mathcal{F}_{j-1})}{\mathcal{K}(j-1,2,0)^2} - \frac{(Z_{j-1} - \mathbb{E}(Z_{j-1})^2}{\mathcal{K}(j-2,2,0)^2}\Big) \\ &\xrightarrow{P} 1.\end{aligned}$$

Now, proof is completed through the martingale central limit theorem [6] and by relation (1). □

Confilcts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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