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The Projection Strategies of Gireh on the Iranian Historical Domes

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Abstract

The Gireh is an Islamic geometric pattern which is governed by mathematical rules and conforms to Euclidean surfaces with fixed densities. Since dome-shaped surfaces do not have a fixed density, it is difficult to make use of a structure like Gireh on these surfaces. There are, however, multiple domes which have been covered using Gireh, which can certainly be thought of a remarkable achievement by past architects. The aim of this paper is to discover and classify the strategies employed to spread the Gireh over dome surfaces found in Iran. The result of this research can provide new insights into how Iranian architects of the past were able to extend the use of the Gireh from flat surfaces to dome-shaped elements. The result of this paper reveal that the projection of Gireh on dome surface is based on the following six strategies: 1- Spherical solids, 2- radial gore segments, 3- articulation, 4- changing Gireh without articulating, 5- changing the number of Points in the Gireh based on numerical sequences, and 6- hybrid. Except for the first method, all of the other strategies have been discovered in this study. The radial gore segments strategy is different from the previously-developed methods.

Keywords: Islamic mathematics, geometry, Gireh, domical surface, pattern projection

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1. Introduction

Girehes are among the most complex decorative art forms in Islamic architecture, having been developed along with the advancements in mathematics in different periods. The structural nature of Gireh conforms to the Euclidean plans, which is applicable to flat surfaces. Naturally, such a structure is not compatible with dome-like surfaces. Because a dome-like surface does not have a fixed density, its expansion is incongruent with compression and openness. Therefore, the spreading of Gireh on dome-like surfaces requires special strategy. Despite the existence of many beautiful Gireh-covered domes within Islamic architecture, which illustrates the familiarity of Iranian architects with mathematical concepts, there is no reliable historical source about how to project Gireh on dome-like surfaces. Likewise, contemporary research cannot provide a comprehensive answer how Gireh used to be projected on dome-like surfaces. This study strives to provide an answer to this question.

2. Method

The aim of this research is to discover and classify the strategies and components employed in Gireh-covered domes found throughout Iran. The architects of the Islamic period tried to extend the use of Gireh from 2D surfaces to 3D elements such as domes. However, the lack of fundamental studies and historical resources has made it very difficult to find out how this transition took place. The results of this research can answer the following question: what strategies were employed by Iranian Architects to project Gireh, a tool primarily used for flat surfaces, on domical surface? This study is a historical research that makes use of the descriptive-analytical method, and the necessary information have been collected via library studies and field surveys. First, the domes with historical origins were identified; these were then modeled using the Rhino software for a better understanding of the presented strategies. As mentioned, there are multiple domes that have been decorated with Gireh in many countries, but this research focuses only on Iranian domes to achieve exact results and a comprehensive classification of the following edifices: the northeast dome chamber of the Jameh Mosque of Isfahan, the Jameh Mosque of Golpaygan, the Jameh Mosque of Saveh, the Jameh Mosque of Yazd, The shrine of Emamzadeh Ali ibn Jafar in Qom, the Azam Mosque in Qom, the shrine of Shah Ni'mat Allah Vali in Mahan, the Hakim Mosque in Isfahan, the shrine of Emamzadeh Ebrahim in Zanjan, the tomb of Mushtaq Ali-shah in Kerman, the shrine of Shah-zeyd in Isfahan, The shrine of Emamzadeh Abdolah in Yazd, the tomb of Mirza Rafiha in Isfahan, Tekyeh Nasr al-din in Tehran, and the shrine of Jafar in Yazd.

3. Background Research

The Gireh is one of the main decorating elements found in Islamic architecture which has appealed to Muslim artists. Along with the efforts to build Gireh, there were people who had studied spherical projection during the historical and contemporary periods. In treatise with the title "Kitāb fī māyahtāj ilayh al-Şāni' min al-ámāl al-handasia", Abu al-Wafā' Būzjāni discussed spherical trigonometry, but there was no mention of polyhedrals and their principles. He only considered the position of vertices of polyhedrals on spheres. Būzjāni discussed the principle of matter in another way and instead of surrounding the polyhedral, illustrated that the surface of the sphere can be divided into regular and equal polygons. These polygons on the sphere are similar to the surrounding polyhedral [7] (Figure 1).

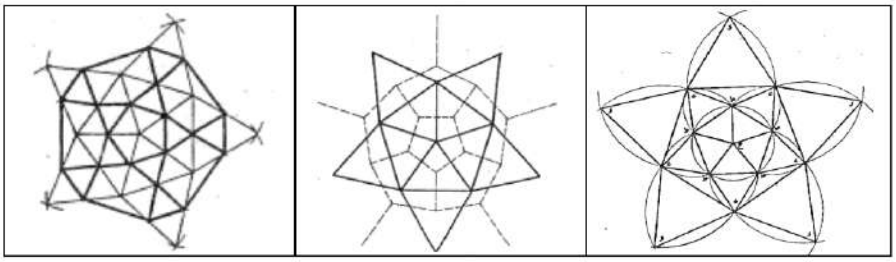

Figure 1: The Spherical Polyhedral of Būzjāni [9].

Several contemporary researchers have studied this subject. In an article, Bonner [5] describes the transformation of Islamic geometric pattern on domical surfaces using geodesic-conforming polyhedral jitterbug transformations. Bonner [6] has described two historical traditions to apply geometric designs to the surfaces of the domes. In the study, the radial gore segments have been used as radial repeatative units. The author believes that this method was used in the dome of the Jameh Mosque of Golpayegan, a technique that is different from what has been obtained herein. The second method is rarely encountered and projects the polyhedral symmetries of Platonic and Archimedean solids onto the domical surfaces. In another article, Bonner [7] expresses the history of development of Islamic geometric patterns and then illustrates the technique of polyhedral platonic symmetries to project geometric patterns onto the domical surfaces of the northeast dome chamber of the Jameh Mosque of Isfahan. Other researchers who have carried out studies in this area include Kasraei, Nourian, and Mahdavinejad [19] have described Gireh and analyzes three domes. Then, they mentioned that Iranian architects used a dast-gardan pattern to cover the domes. They argue that the dast-gardan pattern generally employs more than two primary star-like motifs within its composition. So, none of the domes analyzed in their article could have a dast-gardan pattern [5]. This study presents a description of multiple interconnected patterns. The dast-gardan method constitutes the manipulating of the structure of Gireh for an incompatible base [20, 21]. Sarhangi [27] described the remaining documents on geometric patterns from the past and the illustrated the Spherical solid developed Būzjāni. Kaplan and Salesin [18] presented the method of Najm, which uses a novel family of tilling techniques used to construct star-like patterns that can cover the surface of spheres. Özdural [25] describes spherical projection in the northeast dome chamber of the Jameh Mosque of Isfahan. Among the research done to examine the projection of Gireh onto spherical surfaces such as domes, an important issue is the lack of comprehensive studies. What separates this research from other studies is its aim to provide a comprehensive classification of strategies of projection of Gireh onto domical surfaces found within Iran.

4. Gireh and Surface

Inspired by mathematical concepts [10], Gireh are complex patterns with regular structures and defined sets of mathematical elements [22]. The evolution of the Gireh was closely tied to that of geometry and mathematics, starting from simple designs based on brick and progressing to very complex concepts until the advent of moaragh tiling [23] (Figure 2).

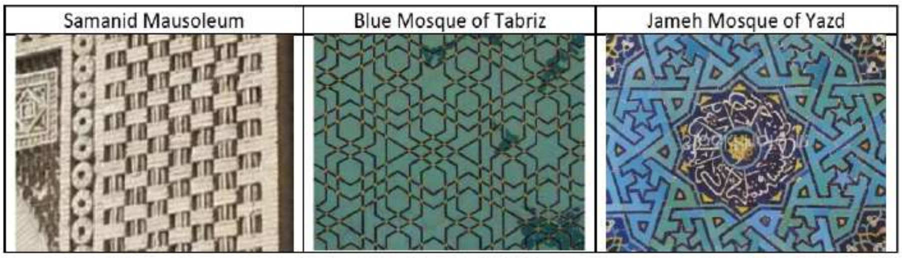

Figure 2: The evolution of Gireh.

The development of geometry in Islamic architecture and art is related to the advancement of science in the Islamic world [24]. Geometrically, the nature of a surface is based on moving a line in the opposite direction along its length [2]. The same is true for the nature of a line, but the action is performed using a point. Therefore, the root of the surface is a set of points that lie next to one another in a particular arrangement. If all points have constant characteristics in a vector, they define a flat surface, which corresponds to a Euclidean plane with a fixed density. Various kinds of surfaces with variable densities can be defined, such as domical and free-form if they do not have constant characteristics in a vector.

4.1. Drawing method of Gireh

In the past, the drawings and calculations of Gireh were preserved by architects through passing down the knowledge to the next generation. Hence, the knowledge of creating and using Gireh is very varied, meaning that different masters used different methods. Also, to avoid disclosure, the master did not write down the entire method, keeping a crucial part of it in his mind [11]. The three main methods in drawing Gireh are the radial, polygonal, and grid methods. The radial method is one of the most common methods among Iranian masters such as Sharabaf, Larzadeh, Heli, and other artists. In this method, Gireh can be easily drawn by a ruler and compass. The artist first draws a radial generating matrix and through the points obtained from the intersection of the radii and arcs, the lines of Gireh are drawn [16, 20, 29, 31]. The polygonal method was first discovered by Hankin [15] while visiting an Indian bath. Based on the structural classification of tessellation works, this discovery was further developed by the work of Jay Bonner. In this method, the Alats are placed based on the basic tessellations. This method is more common among western researchers such as Bonner and Kaplan. According to their structures , each Gireh is based on a type of grid, which can be orthogonal, rhombic (isometric), or triangular. Some artists use these grids to draw Gireh. Artists who have used grids to draw Gireh include Jean-Marc Castera, Ayse Parman, and Issam al-Saeed. The described methods are illustrated in the table below [12] (Figure 3).

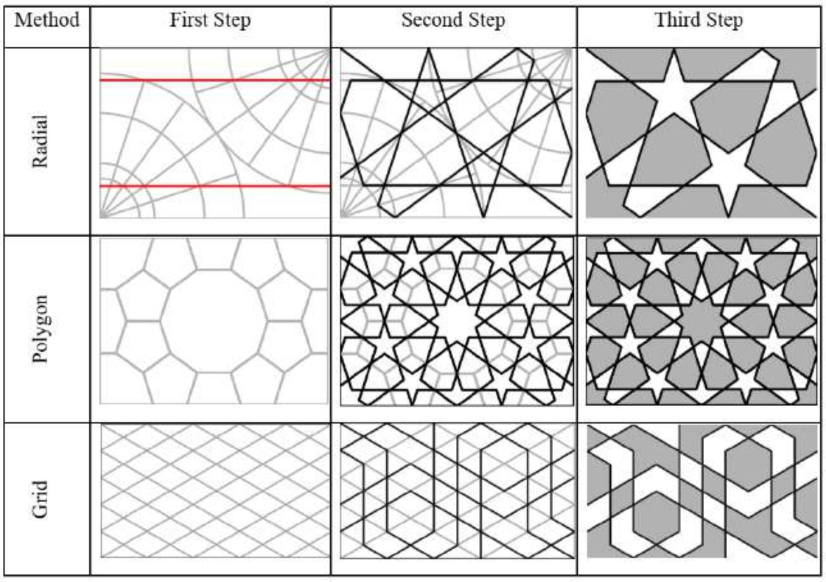

Figure 3: The drawing method of Gireh.

4.2. Extension in Gireh

The most important issue on which all eastern and western scholars are unanimous is that "the Gireh has a multilayered structure which is based on a sub-grid, upon which the motifs are tiled" [8, 26]. Mathematically, each motif is a small planar map [18], which requires a surface that conforms to the Euclidean plane [14]. Mathematically, the Gireh is a highly ordered geometric structure with a set of defined motifs, "Alāt", which can be expanded based on repeating the repeat unit, "Vagireh" [1, 22]. The shape of Gireh can be determined through the type of repeat unit and the shape of repeat unit; for example, the patterns shown in (Figure 4) are obtained from a repeat unit, but the repeating patterns are different. Furthermore, according to the type of employed extension, Gireh can be classified into periodic, non-periodic, and quasi-periodic categories, with periodic expansion itself dividing into wallpaper and frieze groups [3, 17, 28].

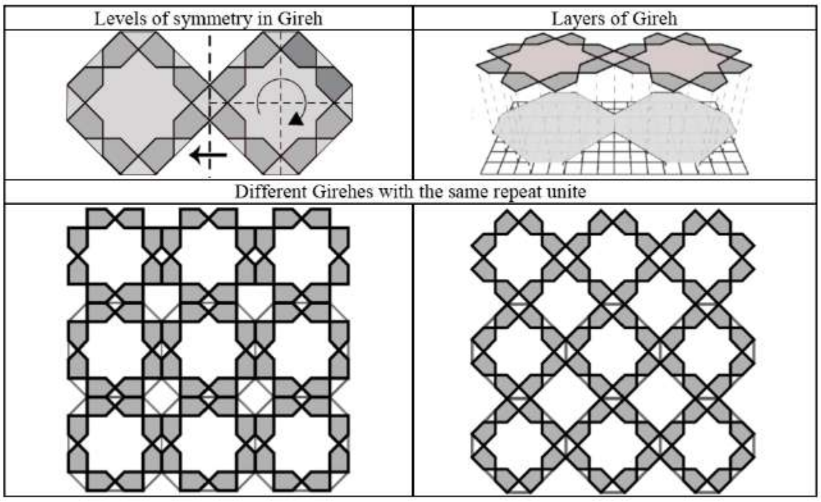

Figure 4: The structure of Gireh.

5. Gireh on Domical Surface

The dome is classified as a closed vault, which is obtained from the rotating of an arch [4]. According to the density of the domical surface, decoration on the domical surface is divided into three parts. The apex surface of the dome: This part has no Gireh, because of the high density and the limitation in the use of material; The median surface of the dome: a collector surface which incorporates a Shāmseh whose number of points is equal to the parts into which the Gireh on the bottom surface of the dome has been subdivided; The bottom surface of the dome: a surface that has Gireh embedded into it. The apex of the dome is like an attractor point, around which the surface expands. This way, the surface extending to the margins becomes wider (Figure 5).

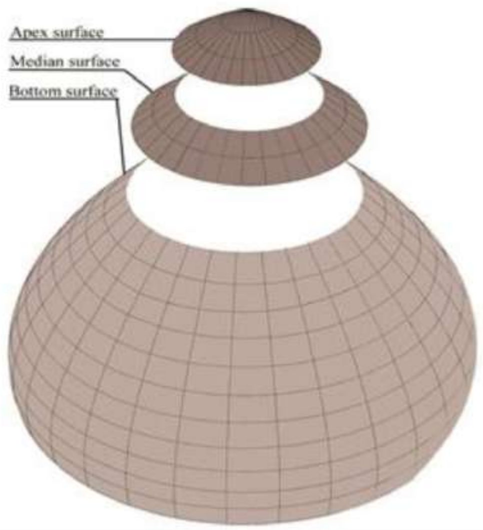

Figure 5: The layer of the domical surface.

5.1. The Incompatibility of Gireh with Domical Surface

Making a pattern on a sphere requires a rudimentary grasp of non-Euclidean geometry. In plane geometry, if p is taken as the number of sides of a regular polygon and q as the number of the copies of the same polygon around each vertex, a simple calculation would yield the equation $(p-2)(q-2) = 4$; however, considering that the rules of non-Euclidean geometry allow for the sum of the internal angles of a triangle on a sphere to be greater than 180, the mentioned relationship would take the form $(p-2)(q-2) < 4$ [8]. Covering domes with Gireh was an important challenge for architects [15, 19], and it remains one of their most important achievements during the Seljuk period. The Jameh Mosques of Isfahan and Golpayegan are good examples of Gireh covered domes.

5.2. The Domes Covered with Gireh

The identified examples illustrate how Gireh has evoloved from simple designs to complex concepts. There are many historical domes in Iran, but a small proportion of them have a Gireh covering. It seems that this was due to the special and difficult-to-obtain knowledge needed to project Gireh onto domical surfaces. Also, since the different domes have the same type of Gireh covering, it is possible that they are the work of a single person, confirming that the knowledge was exclusive to only a few individuals (Figure 6).

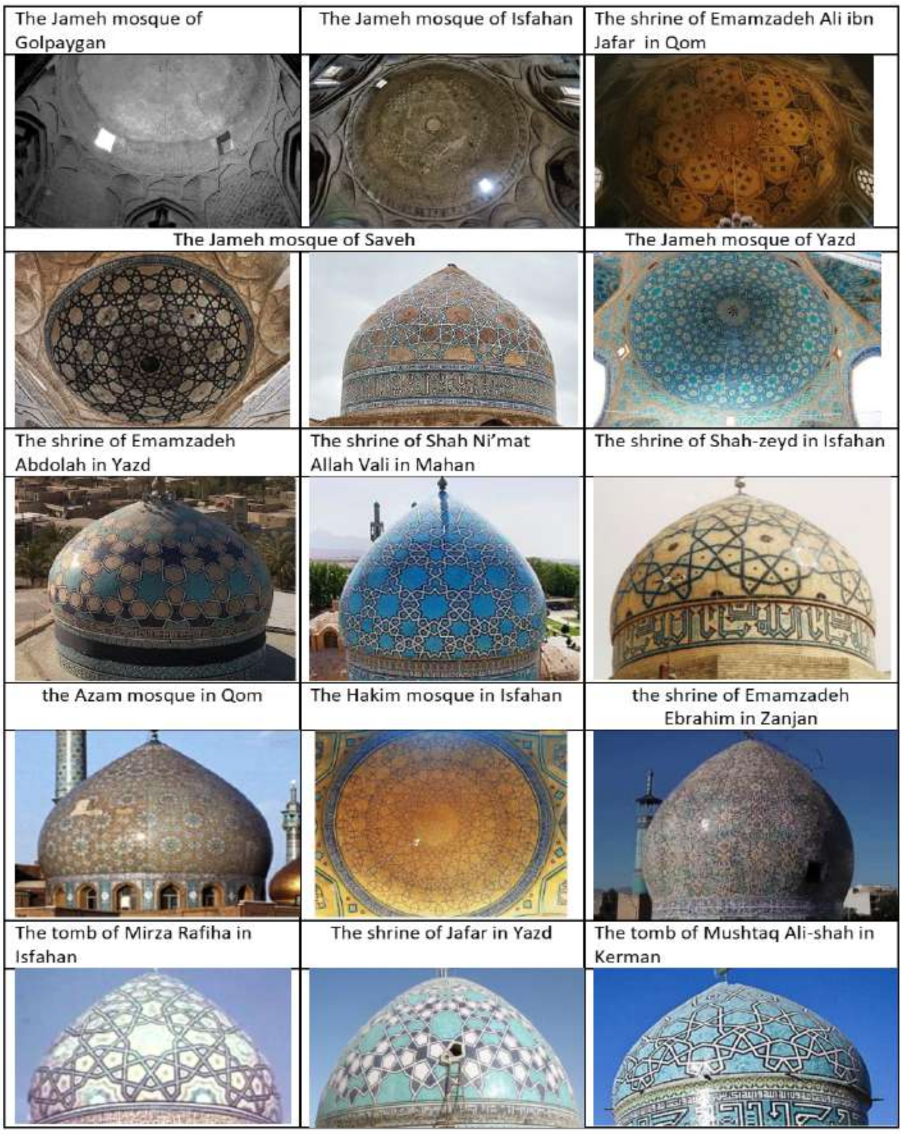

Figure 6: The identified dom.

6. Projecting Gireh on Domical Surface

Geometry cab be divided into the two branches of theoretical and practical. Theoretical geometry is related to lines, surfaces and polyhedrals, and is a pure form of math. Practical geometry, by contrast, is about the geometry of buildings and how they are constructed [13, 30]. Because the domes have various types and have different structures from spheres, the spherical method found in the compilation of mathematicians does not work for all domes. Therefore, projecting Gireh on domical surfaces was done through various strategies, which fall in the practical branch of geometry. Studies on Gireh show that Gireh needs to have a set of properties in order to move from flat surfaces to domical surfaces. In this study, the three components of adaptation to the surface, expansion, and variability have been studied.

As mentioned previously, tiling of Gireh has structural rules, which are not compatible with domical surfaces. However, architects had adapted it to be applied to domical surfaces. Adaptation is the first and main component in the study of Gireh on domical surface. The second component is expansion which is formed through applying symmetry to a repeat unit. As long as the pattern expands, it is also stable. Symmetry has various types which depend on the employed strategies in adapting Gireh to be used for domical surfaces; the third component is the variability of strategy. Each strategy of projecting has characteristics based on the surface on which it is used. Some strategies have limitations on are not compatible with using different types of Gireh. In the study of the domes covered with Girehes by these three components, six methods were identified. These are 1- Spherical polyhedral, 2- Radial gore segments, 3- Articulation, 4- Changing Gireh without articulating, 5- Changing the number of Points of Gireh based on numerical sequences, and 6- hybrid.

6.1. Spherical Polyhedral

Mathematicians tried to project Gireh to domical surfaces by using a spherical polyhedral as a sub-grid on the dome. In this strategy, the surface of the dome is divided into a Platonic or Archimedean polyhedral, and the dome surface is similar to a multi-level surface, and Gireh is projected to each level until they join up in a single unit. "There are very few instances of non-Euclidean geometric designs in Islamic architecture, which employ polyhedral symmetry as their repetitive pattern"[6]. The dome chamber on northeast side of the Jameh Mosque of Isfahan incoporates a dodecahedron for projecting Gireh onto surface of the dome. Because the sphere and the dodecahedron have the same center and the vertices are tangential to the sphere, the dodecahedron matches entirely on the spherical surface using the curve vertices. Therefore, each face is considered as an independent surface, and Gireh is projected from these faces to the entirety of the domical surface. Alpay Özdural believed that this dome chamber was designed by Khayam, a prominent Iranian mathematician and poet. He was familiar with the knowledge of spherical projection and lived in Isfahan during the construction of the dome [25].

Expansion of Gireh is based on the type of employed polyhedral. Because all faces of the polyhedral are equidistant from the center, the limitation of projecting Gireh on the apex surface of the dome is eliminated using this strategy. On the other hand, this strategy also includes the limitation that the domical surface should match the hemisphere. In this strategy, all repeat units cannot match the domical surface. Therefore, the repeat units should be adapted to the faces of polyhedral (Figure 7).

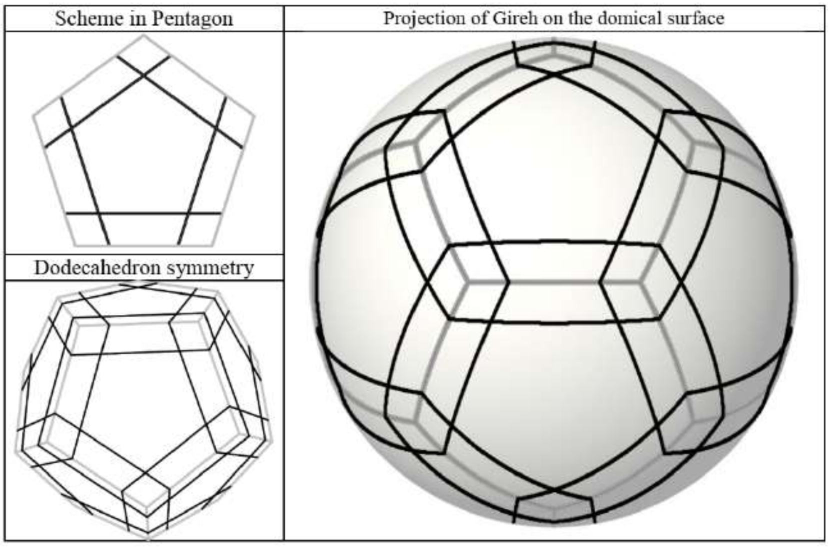

Figure 7: The northeast dom chamber of the Jameh Mosque of Isfahan.

6.2. Radial Gore Segments

As previously mention , the structural framework of Gireh is formed based on a sub-grid, and it is mapped on the sub-grids points which make the segments of Gireh. In this strategy, a grid template is built from a gore segment of the dome. The grid itself is built from a base-curvature which forms the dome. Traditional Iranian masters named this the boat grid, based on which the forming points of Gireh are projected on the domical surface. The radial gore segments can be subdivided into four sections, which follow the symmetry of the dome chamber. The type of expansion is wallpaper, and using the gore segments, a congruence pattern is created which matches the base-curvature forming the dome. In this type of transform, the angle sizes are maintained, but the size of the line segments are changed. This strategy has been employed in the Azam Mosque in Qom and the shrine of Emamzadeh Ebrahim in Zanjan. These two domes have the same type of Gireh work. The similar designs and the chronological proximity of these two domes suggest the possibility that the Emamzadeh Ebrahim in Zanjan was also worked by the same architect, master Lorzadeh. Of course, a simpler way to go about this strategy is to place a Shāmseh onto a dome, and a number of Shāmseh are placed in the lower row, which are attached to the main Shāmseh. Since the Gireh is formed based on a sub-grid, each Gireh is matches an orthogonal grid, and can be project on the domical surface (Figure 8).

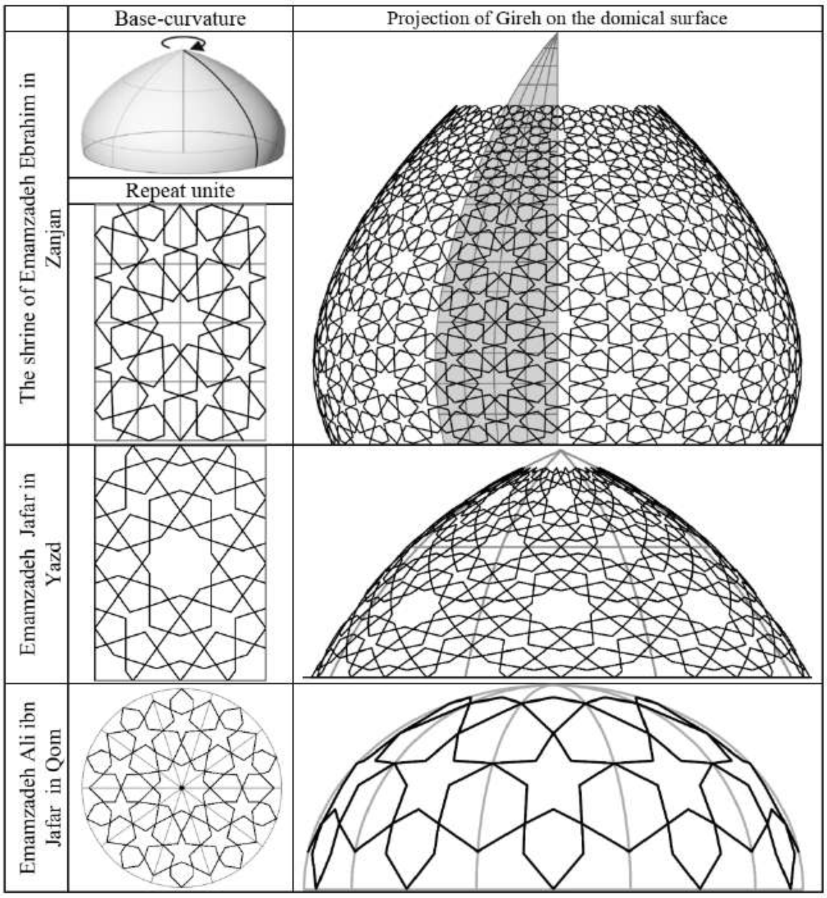

Figure 8: The domes that employed the strategy of radial gore segments.

6.3. Articulation

In previous research on Gireh and its use on domes, this strategy introduced a changing in the number of the points of Shāmseh, and it is believed that this number changes according to the curvature of the dome [8, 19]. Of course, this method has also been used historically and is described below. But the domes studied in previous studies did not employ that strategy. In this strategy, not only are the number points of the Gireh changes , the constituent motifs of the Gireh in each row are also different; therefore, in this strategy, a different kind of Gireh is used, without altering the number of the points. These various Shāmseh are connected to each other by interfaces, shown below in bold gray. Using these interfaces on the surface provides the possibility for each row with Shāmseh to correspond to the curvature of the domical surface. Each of these rows is independent from the other row and each has a different size. Accordingly, if the Gireh employed on the domical surface is projected onto a flat surface, the Shāmseh in the higher rows has fewer elements, but these elements are bigger. As long as it moves to the gore segment of the dome, it's proportion remain proportionate to the size of the Shāmseh in the lower rows. These interfaces often are the motif of the tābl.

Because Gireh expansions are not vertically integrated, and the Gireh in each row is different, frieze expansion is therefore used to develop the design vertically. The architect created visual integration in the overall design by using interfaces. Therefore, this action creates virtual Shāmseh in two rows, shown in pale gray in Figure 9. Another capability would be the resolving of minor inconsistencies in each row with respect to the other rows. These inconsistencies are resolved through changing the shape of interfaces, ultimately creating flexibility. Articulation is applied not only horizontally, but it also vertically in the lower part of the dome so that the surface is more expanded the extension can be controlled. In this strategy, the repeat units can match the interfaces with their edges. For this reason, the different types of Gireh employed in this strategy are very similar. This strategy is the most frequently-used among the domes identified in Iran and they include the shrine of Shah Ni'mat Allah Vali in Mahan, the Jameh Mosque of Saveh, the tomb of Mushtaq Ali-shah in Kerman, the shrine of Shah-zeyd in Isfahan, the tomb of Mirza Rafiha in Isfahan, and Tekyeh Nasr al-din in Tehran (Figure 9).

6.4. Changing Gireh without Articulating

This strategy also follows the frieze-based expansion, similar to the articulation strategy wherein the Gireh is changed in each row to be projected on the domical surface, with virtual Shāmsehes being defined between two rows. It differs from the previous strategy in that the interface does not connect to the Shāmsehes in the other rows. Also, the motif of Gireh in the repeat unit are adjusted in such a way so that the segment lines meet up in one point of the edges of repeat units. Therefore, it requires more precision than the articulation strategy. In terms of variability, each pattern can be adjusted and used in one point of the repeat units. The examples of this strategy include the Hakim Mosque in Isfahan and the Jameh Mosque of Golpaygan (Figure 10).

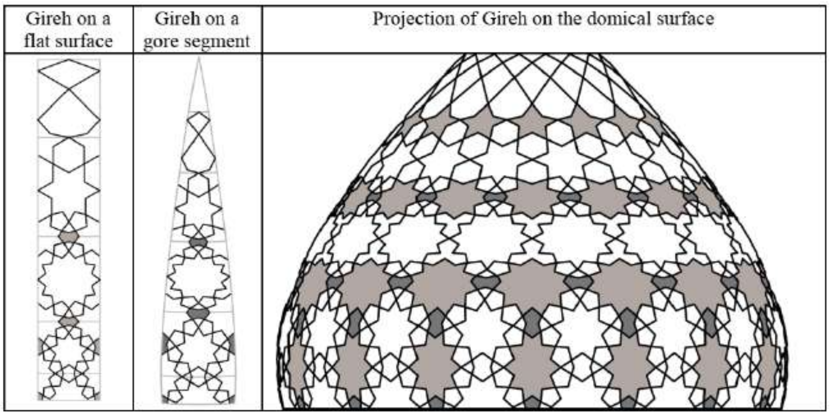

Figure 9: The Gireh on the dome of the shrine of Shah Nimat Allah Vali in Mahan.

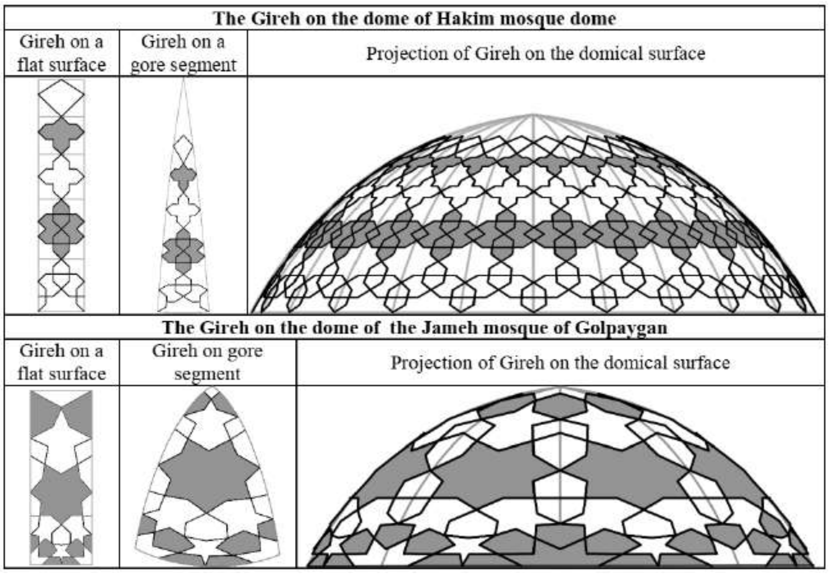

Figure 10: The domes that employed the strategy of changing Gireh without articulating.

6.5. Changing the Number of Points of Gireh based on Numerical Sequences

This method has not already been studied by previous researchers. However, Jay Bonner identified this strategy as a pattern employed on flat surface. The theory behind this stratgy is that single stars with a rotational symmetry of n-folds function quite well in making repeated patterns. Then patterns with $(n\pm1)-$fold can be utilized to make more patterns [6]. Of course, the author mentions this for the dome of the shrine of Shah Ni'mat Allah Vali in Mahan, but as mention previously, but the articulation strategy has been employed to project Gireh on the domical surface of this dome. In this strategy, the tessellation of Gireh on the domical surface is changed to control the increase in the density of the surface toward the apex of the dome. This way, the number of tessellation sides is decreased in a sequentially $(n-1)$ toward the apex of the dome. Reducing the number of sides causes the area to decrease, so that in the two polygons with equal number of sides, the polygon with fewer sides has a smaller area. Hence, the sizes of the components of the units are in relatively equal proportions, thereby creating visual balance in the overall design. The dome of the shrine of Emamzadeh Abdolah in Yazd is the only identified example which employs this strategy. In this dome, the points of Shāmseh are decreased from seven points to five in the numerical sequence. The Gireh expanded using a frieze with a 16-fold symmetry, which is formed based on underlying the polygonal tessellation. This strategy does not have much variability because the constituent motif of this tessellation is limited (Figure 11).

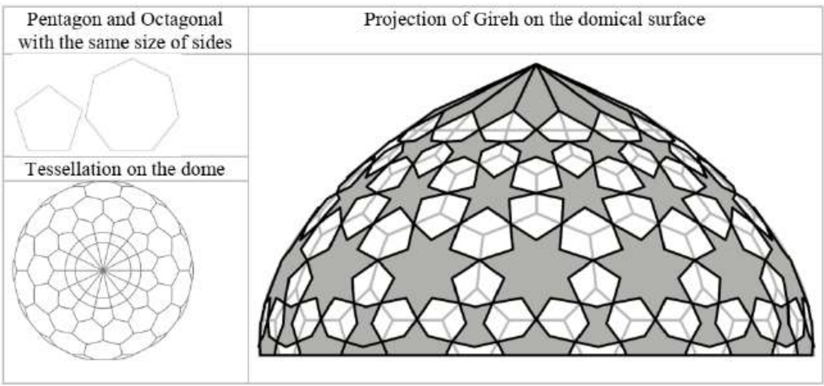

Figure 11: The Gireh on the dome of the Emamzadeh Abdolah in Yazd.

6.6. Hybrid

This strategy is a combination of the strategies of articulation and changing the number of points of Gireh based on numerical sequences. This strategy allows for the solving of the combining nuances of the tessellation polygons through via interfaces. It also creates a more complex combination in the overall design. Among the identified examples, the dome of the Jameh Mosque of Yazd is the only one employing this strategy (Figure 12).

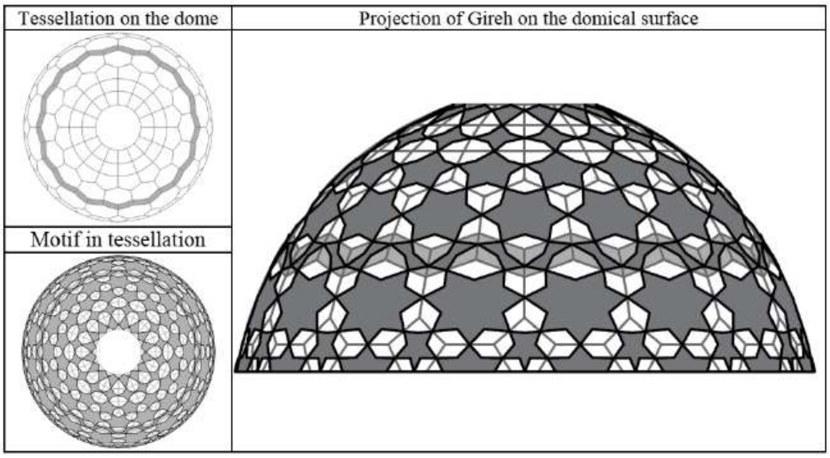

Figure 12: The Gireh on the dome of the Jameh Mosque of Yazd.

7. Comparative Analysis

Studying the identified domes has revealed six strategies incoporating the three mentioned components, which have various properties. To define their properties and specifications, the features of each strategy are gathered based on three components and presented in Table 1.

8. Conclusion

The Gireh is an innovation of the architects and artists of the Islamic period to cover various 2D and 3D surfaces. This is seen from early days of the introduction of spherical elements such as the dome. Due to the nonconformity of the structure of Gireh with domical surfaces, architects or artists employed various strategies for projecting Gireh onto the 3d surface of domes. Understanding and identifying these strategies can help us reuse Gireh in the surfaces of contemporary architecture.

Table 1: Comparative Table.

	Strategy	Adaptation	Expansion	Variability
1	Spherical polyhedral	Generalization based on spherical polyhedral	Based on polyhedral symmetry	The repeat unit should be congruous with faces of polyhedral
2	Radial gore segments	Through formed sub-grid from base-curvature	Wallpaper along with congruence based on the division of dome chamber	Each pattern can be generalized from the orthogonal grid
3	Articulation	Discretization of the pattern by using interfaces	Frieze based on the division of dome chamber	Compatible pattern with interfaces
4	Changing Gireh without articulating	Changing pattern without articulating	Frieze based on the division of dome chamber	Compatible pattern with adjacent patterns
5	Numerical sequences	Decreasing the numbers of the sides of tessellation toward the apex of the dome	Frieze based on the division of dome chamber	Lack of variability
6	Hybrid	Decreasing the numbers of the sides of tessellation toward the apex of the dome discretization by using interfaces	Frieze based on the division of dome chamber	Lack of variability

Due to the lack of comprehensive studies about this element, there is no comprehensive categorization system for these strategies. For this purpose, the components that have made the transition of Gireh projection from 2D planes to domical surfaces possible are identified. Then, through studying 16 historical Iranian domes, six Gireh projection strategies have been identified and classified. One of the previous strategies, the polyhedral strategy, conforms to the strategy of the spherical polyhedral. The strategy of the radial gore segments is different from the previous strategy of the radial gore segments. The illustrated strategy of radial gore segment in this research is based on the congruence it defines using base-curvature. Also, four other new methods have been discovered. The strategy of articulation is based on the discretization of pattern in variable curvature of the dome by using interfaces. This strategy allows each row to be defined individually based the mapping on its surface. The strategy of changing Gireh without articulating is also based on the discretization of patterns in the variable curvature of the dome; the difference is that the line segments in the edges of the repeat units of each row should match the other line segments of other rows. In the strategy of changing the number of points of Gireh based on numerical sequences, the number of sides of the tessellation is decreased based on the numerical sequence $(n-1)$ to control the increase of density toward the apex of the dome. And the hybrid strategy is a combination of the strategies of articulation and changing the number of Points of Gireh based on numerical sequences to solve the combining nuances of the polygons of tessellation on the domical surface. Each of these strategies creates features in the Gireh, one of the most important of which is that except for the strategy of the radial gore segments, they all use frieze expansion rather than wallpaper expansion. Furthermore, the spherical polyhedral uses polyhedral symmetry. In terms of variability, except the strategy of the radial gore segment, other strategies have a lot of limitations in employing different kinds of Gireh on domical surfaces.

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Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

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