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# Some Results on the Strong Roman Domination Number of Graphs

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**Abstract**

Let $G = (V, E)$ be a finite and simple graph of order $n$ and maximum degree $\Delta(G)$. A *strong Roman dominating function* on a graph $G$ is a function $f : V(G) \to \{0, 1, \dots, \lceil \frac{\Delta(G)}{2} \rceil + 1\}$ satisfying the condition that every vertex $v$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) \geq 1 + \lceil \frac{1}{2} |N(u) \cap V_0| \rceil$, where $V_0 = \{v \in V \mid f(v) = 0\}$. The minimum of the values $\sum_{v \in V} f(v)$, taken over all strong Roman dominating functions $f$ of $G$, is called the *strong Roman domination number* of $G$ and is denoted by $\gamma_{StR}(G)$. In this paper we continue the study of strong Roman domination number in graphs. In particular, we present some sharp bounds for $\gamma_{StR}(G)$ and we determine the strong Roman domination number of some graphs.

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## 1. Introduction

Throughout this paper, $G$ is a simple graph with vertex set $V(G)$ and edge set $E(G)$ (briefly $V, E$). The order $|V|$ of $G$ is denoted by $n = n(G)$. For every vertex

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$v \in V(G)$, the *open neighborhood* of $v$ is the set $N_G(v) = N(v) = \{u \in V(G) \mid uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V$ is $\deg_G(v) = \deg(v) = |N(v)|$. The *minimum* and *maximum degree* of a graph $G$ are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. A *leaf* of $G$ is a vertex with degree one in $G$, the set of all leaves of $G$ is denoted by $\ell = \ell(G)$. A graph $G$ is *regular* if the degrees of all vertices of $G$ are the same. We write $K_n$ for the *complete graph* and $C_n$ for a *cycle* of order $n$. We also denote the *complete bipartite graph* with two parts of sizes $m$ and $n$, by $K_{m,n}$. The *complement* of a graph $G$ is denoted by $\overline{G}$. A graph $G$ is called *self-complementary* if $G \cong \overline{G}$. The double star $DS_{q,p}$, where $q \geq p \geq 1$, is the graph consisting of the union of two stars $K_{1,q}$ and $K_{1,p}$ together with an edge joining their centers. The *distance* $d_G(u, v)$ between two vertices $u$ and $v$ in a connected graph $G$ is the length of a shortest $u - v$ path in $G$. The *diameter* of a graph $G$, denoted by $\operatorname{diam}(G)$, is the greatest distance between two vertices of $G$. For a vertex $v$ in a rooted tree $T$, let $D(v)$ denote the set of descendants of $v$ and $D[v] = D(v) \cup \{v\}$. The *maximal subtree* at $v$ is the subtree of $T$ induced by $D[v]$, and is denoted by $T_v$.

A subset $X$ of the vertices of $G$ is called a *clique* if the induced subgraph on $X$ is a complete graph. The *clique number* of a graph $G$ is the number of vertices in a maximum clique of $G$ and denoted by $cl(G)$. A *subdivision* of an edge $uv$ is obtained by replacing the edge $uv$ with a path $uwv$, where $w$ is a new vertex. A *unicyclic* graph is a connected graph containing exactly one cycle. An edge of $G$ is said to be *contracted* if it is deleted and its ends are identified. The resulting graph has one less edge than $G$. The *corona* $GoK_1$ of a graph $G$ is obtained by attaching one pendant edge at each vertex of $G$.

A subset $S$ of vertices is called a 2-*packing* if $N[u] \cap N[v] = \emptyset$ for every pair of vertices $u, v \in S$. The 2-*packing number* $\rho := \rho_2(G)$ of a graph $G$ is the maximum cardinality of a 2-packing in $G$. More notation and terminology not explicitly given here are conformed with [2].

A subset $S$ of vertices of $G$ is a *dominating set* if $N[S] = V$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of $G$. A dominating set of minimum cardinality of $G$ is called a $\gamma(G)$-set.

*A Roman dominating function* (RDF for short) on a graph $G = (V, E)$ was defined in [4] and [5] as a function $f : V \to \{0, 1, 2\}$ satisfying the condition that every vertex $v$ for which $f(v) = 0$ is adjacent to at least one vertex $u$ for which $f(u) = 2$. The *weight* of an RDF $f$ is the value $\omega(f) = \sum_{v \in V} f(v)$. The *Roman domination number* of a graph $G$, denoted by $\gamma_R(G)$, equals the minimum weight of an RDF on $G$. A $\gamma_R(G)$-function is a Roman dominating function of $G$ with weight $\gamma_R(G)$.

Consider a graph $G$ of order $n$ and maximum degree $\Delta(G)$. Suppose that $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil + 1\}$ is a function that labels the vertices of $G$.

Then, $f$ is a *strong Roman dominating function* (StRDF for short) for $G$ if every $v \in V_0$ has a neighbor $u$ such that $f(u) \geq 1 + \lceil \frac{1}{2} |N(u) \cap V_0| \rceil$. The minimum weight, $\omega(f) = f(V) = \sum_{v\in V} f(v)$, over all the strong Roman dominating functions for $G$, is called the *strong Roman domination number* of $G$ and we denote it by $\gamma_{StR}(G)$. A $StRDF$ of minimum weight is called a $\gamma_{StR}(G)$-function. This concept was announced in [1]. A strong Roman dominating function $f$ can be represented by the ordered partition $(V_0, V_1, \ldots, V_{\lceil\frac{\Delta(G)}{2}\rceil+1})$ of $V(G)$.

Motivated by [1], we are interested to study the strong Roman domination number in graphs. The outline of paper is as follows. In Section 2, we present some bounds for the strong Roman domination number. In Section 3, we investigate the operations on graphs and the strong Roman domination number. In Section 4, it is shown that $\gamma_{StR}(G) < \frac{6n}{7} + 1$, where $G$ is a unicyclic graph of order $n$. Then, among other results, the strong Roman domination number is determined for some classes of graphs.

We make use of the following results in this paper.

**Proposition A.** ([1]) Let $G$ be a connected graph of order $n$. Then $\gamma_{StR}(G) = n$ if and only if $G = K_1$ or $K_2$.

**Proposition B.** ([1]) If $T$ is a tree of order $n \geq 3$, then $\gamma_{StR}(T) \leq \frac{6n}{7}$.

Let $S(K_{1,3})$ (the star $K_{1,3}$ with all its edges subdivided) be rooted in its center $v$ and let $F_m^p$ consist of all the rooted product graphs $T \circ_v S(K_{1,3})$, where $T$ is any tree on $m$ vertices.

**Proposition C.** ([1]) Let $T$ be an $n$-vertex tree. Then $\gamma_{StR}(T) = \frac{6n}{7}$ if and only if $T \in F_m^p$.

**Proposition D.** ([1])Let $G$ be a graph of order $n$. Then $\gamma_{StR}(G) \leq n - \lfloor\frac{\Delta(G)}{2}\rfloor$.

**Proposition E.** ([1]) Let $G$ be a graph of order $n$. Then $\gamma_{StR}(G) \geq \lceil\frac{n+1}{2}\rceil$. Moreover, if $n$ is odd, then the equality holds if and only if $\Delta(G) = n - 1$.

**Proposition F.** ([3]) For paths $P_n$ and cycles $C_n$, $\gamma_R(P_n) = \gamma_R(C_n) = \lceil\frac{2n}{3}\rceil$.

**Proposition G.** ([1]) For any connected graph $G$ with $\Delta(G) \leq 2$, $\gamma_{StR}(G) = \gamma_R(G)$.

## 2. Bounds on the Strong Roman Domination Number

The authors in [1] gave several bounds for the strong Roman domination number. Our goal in this section is to provide some new bounds. Our bounds are not comparable with bounds in [1] in general.

In the following we provide an upper bound on the strong Roman domination number of a tree $T$ in terms of its order $n$ and the number of leaves $\ell$.

**Lemma 2.1.** Let $T$ be a double star of order $n \geq 5$. Then $\gamma_{StR}(T) > 3$.

*Proof.* If $n \leq 6$, then $T = DS_{1,2}$ or $DS_{1,3}$ or $DS_{2,2}$ and it is not hard to see that $\gamma_{StR}(T) = 4 > 3$. If $n \geq 7$, then Proposition E shows that $\gamma_{StR}(T) \geq 4 > 3$, as desired. □

**Theorem 2.2.** Let $T$ be a tree of order $n \geq 2$. Then

$$\gamma_{StR}(T) \geq \lceil \frac{2(n+2-\ell)}{3} \rceil.$$

This bound is sharp for paths.

*Proof.* We proceed by induction on $n$. The statement holds for all trees of order $n \leq 4$. Suppose $n \geq 5$ and let the result hold for all non-trivial tree $T$ of order less than $n$. Let $T$ be a tree of order $n \geq 5$. If $\operatorname{diam}(T) = 2$, then $T$ is a star, which yields $\gamma_{StR}(T) = \lceil \frac{n+1}{2} \rceil > \lceil \frac{2(n+2-\ell)}{3} \rceil = 2$. If $\operatorname{diam}(T) = 3$, then $T$ is a double star and by Lemma 2.1 we have $\gamma_{StR}(T) > 3 = \lceil \frac{2(n+2-\ell)}{3} \rceil$. In this case, the inequality holds. Henceforth we assume that $\operatorname{diam}(T) \geq 4$. Let $v_1 v_2 \ldots v_k$ be a diametral path in $T$ and root $T$ in $v_k$. Let $f$ be a $\gamma_{StR}(T)$-function. We consider the following cases.

**Case 1.** $\deg_T(v_2) = t \geq 3$.
Let $T' = T - T_{v_2}$. Clearly $\gamma_{StR}(T) \geq \gamma_{StR}(T') + \lceil \frac{t}{2} \rceil$, $\ell(T) - (t-1) \leq \ell(T') \leq \ell(T) - (t-2)$ and we conclude from the induction hypothesis that

$$\begin{aligned}
\gamma_{StR}(T) &\geq \gamma_{StR}(T') + \lceil \frac{t}{2} \rceil \\
&\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + \lceil \frac{t}{2} \rceil \\
&\geq \lceil \frac{2((n-t) + 2 - \ell + t - 2)}{3} \rceil + \lceil \frac{t}{2} \rceil \\
&\geq \lceil \frac{2(n-\ell)}{3} \rceil + \lceil \frac{t}{2} \rceil \\
&> \lceil \frac{2(n+2-\ell)}{3} \rceil,
\end{aligned}$$

as desired.

**Case 2.** $\deg_T(v_2) = 2$.
If $\deg_T(v_3) \geq 3$, then let $T' = T - \{v_1, v_2\}$. Clearly $\gamma_{StR}(T) \geq \gamma_{StR}(T') + 1$,

$\ell(T') = \ell(T) - 1$ and we conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{StR}(T) &\geq \gamma_{StR}(T') + 1\\ &\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + 1\\ &\geq \lceil \frac{2((n-2) + 2 - \ell + 1)}{3} \rceil + 1\\ &> \lceil \frac{2((n + 2 - \ell)}{3} \rceil,\end{aligned}$$

as desired.

If $\deg_T(v_3) = 2$, then let $T' = T - \{v_1, v_2, v_3\}$. Clearly $\gamma_{StR}(T) \geq \gamma_{StR}(T') + 2$, $\ell(T) - 1 \leq \ell(T') \leq \ell(T)$ and we conclude from the induction hypothesis that

$$\begin{aligned}\gamma_{StR}(T) &\geq \gamma_{StR}(T') + 2\\ &\geq \lceil \frac{2(n(T') + 2 - \ell(T'))}{3} \rceil + 2\\ &\geq \lceil \frac{2((n-3) + 2 - \ell)}{3} \rceil + 2\\ &\geq \lceil \frac{2((n + 2 - \ell)}{3} \rceil.\end{aligned}$$

This completes the proof. □

The following proposition bounds the strong Roman domination number in terms of the clique number.

**Proposition 2.3.** Let $G$ be a graph of order $n$. Then $\gamma_{StR}(G) \leq n - \lfloor \frac{cl(G)-1}{2} \rfloor$.

*Proof.* Suppose that $v_1, \ldots, v_{cl(G)}$ are the vertices in the maximum clique of $G$. Define $f : V(G) \to \{0, 1, \ldots, \lceil \frac{\Delta(G)}{2} \rceil + 1\}$ by $f(v_1) = \lceil \frac{cl(G)+1}{2} \rceil$, $f(v_i) = 0$ for $i = 2, \ldots, cl(G)$ and $f(v) = 1$ otherwise. Clearly $f$ is a StRDF of $G$ and so we have

$$\gamma_{StR}(G) \leq \omega(f) = (n - cl(G)) + \lceil \frac{cl(G) + 1}{2} \rceil = n - \lfloor \frac{cl(G) - 1}{2} \rfloor.$$

□

The next result gives an upper bound for the strong Roman domination number using 2-packing number.

**Proposition 2.4.** Let $G$ be a graph of order $n$ with minimum degree $\delta$. Then

$$\gamma_{StR}(G) \leq n - \rho \lfloor \frac{\delta}{2} \rfloor.$$

*Proof.* Suppose that $S = \{v_1, \dots, v_\rho\}$ is a 2-packing set of $G$. Define $f$ as follows:

$$f(x) = \begin{cases} 1 + \lceil \frac{\deg_G(x)}{2} \rceil & x \in S \\ 0 & x \in \bigcup_{i=1}^{\rho} N(v_i) \\ +1 & \text{otherwise.} \end{cases}$$

It is easily seen that $f$ is a strong Roman dominating function for $G$. Since $S$ is a 2-packing set, one has

$$\begin{aligned} \gamma_{StR}(G) \le \omega(f) &= \sum_{x \in S} f(x) + (n - |S| - |\bigcup_{i=1}^{\rho} N(v_i)|) \\ &= (1 + \lceil \frac{\deg_G(v_1)}{2} \rceil + \dots + 1 + \lceil \frac{\deg_G(v_\rho)}{2} \rceil) \\ &+ (n - \rho - \deg_G(v_1) - \dots - \deg_G(v_\rho)) \\ &= n - \lfloor \frac{\deg_G(v_1)}{2} \rfloor - \dots - \lfloor \frac{\deg_G(v_\rho)}{2} \rfloor \\ &\le n - \rho \lfloor \frac{\delta}{2} \rfloor \end{aligned}$$

as desired. □

Next we present a bound for strong Roman domination number with regard to the diameter. When $\delta \ge 2$ this bound is better than the bound given in [1, Proposition 11].

**Proposition 2.5.** Let $G$ be a graph of order $n$ with minimum degree $\delta$. Then

$$\gamma_{StR}(G) \le n - (1 + \lfloor \frac{diam(G)}{3} \rfloor) \lfloor \frac{\delta}{2} \rfloor.$$

*Proof.* Suppose that $S = v_0, v_1, \dots, v_d$ is a diametral path, $d = 3t + r$ with integers $t \ge 0$ and $0 < r \le 2$. It is easy to see that $A = \{v_0, v_3, \dots, v_{3t}\}$ is a 2-packing set of $G$ such that $|A| = 1 + \lfloor \frac{diam(G)}{3} \rfloor$. Then we have $\rho \ge |A|$. So by Proposition 2.4, one has

$$\gamma_{StR}(G) \le n - \rho \lfloor \frac{\delta}{2} \rfloor \le n - (1 + \lfloor \frac{diam(G)}{3} \rfloor) \lfloor \frac{\delta}{2} \rfloor.$$

□

For graphs with diameter 2, one can find bounds in terms of $\delta$ and $\Delta(G)$.

**Proposition 2.6.** Let $G$ be a graph of order $n$ and $diam(G) = diam(\overline{G}) = 2$. Then

$$\gamma_{StR}(G) \le \delta(1 + \lceil \frac{\Delta(G)}{2} \rceil), \quad \gamma_{StR}(\overline{G}) \le \lceil \frac{n + 1 + \delta}{2} \rceil.$$

*Proof.* For every vertex $x \in V$, $N(x)$ dominates all vertices of $G$. Choose $v \in V$ such that $\deg_G(v) = \delta$ and define $f : V(G) \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G)}{2} \rceil\}$ by $f(t) = 1 + \lceil \frac{\Delta(G)}{2} \rceil$ for every $t \in N(v)$ and $f(x) = 0$ otherwise. It is clear that $f$ is a strong Roman dominating function which in turn implies that

$$\gamma_{StR}(G) \leq (1 + \lceil \frac{\Delta(G)}{2} \rceil)|N(v)| \leq (1 + \lceil \frac{\Delta(G)}{2} \rceil)\delta.$$

Now we show that $\gamma_{StR}(\overline{G}) \leq \lceil \frac{n+1-\delta}{2} \rceil$. Let $N_G(v) = \{v_1, \ldots, v_k\}$. It follows from $diam(\overline{G}) = 2$ that $d_{\overline{G}}(v, v_i) = 2$ and so $v$ and $v_i$ have a common neighbor $u_i \in X = V(G) - N_G[v]$ for each $1 \leq i \leq k$. Thus the function $f : V(\overline{G}) \to \{0, 1, \ldots, \lceil \frac{\Delta(\overline{G})}{2} \rceil + 1\}$ defined by $f(v) = 1 + \lceil \frac{\Delta(\overline{G})}{2} \rceil$ and $f(u) = 1$ if $u \in N_G(v)$ and $f(x) = 0$ otherwise, is a StRDF of $\overline{G}$, and so

$$\gamma_{StR}(\overline{G}) \leq w(f) = (1 + \lceil \frac{\Delta(\overline{G})}{2} \rceil) + \delta \leq (1 + \lceil \frac{\Delta(\overline{G})}{2} \rceil) + \delta = \lceil \frac{n + 1 + \delta}{2} \rceil.$$

□

Applying the proposition above on regular graphs, we can bound the strong Roman domination number with regard to the order and regularity.

**Corollary 2.7.** Let $G$ be an $r$-regular graph of order $n$. Then

$$\gamma_{StR}(G) \leq \max\{n + 1 - r, r(1 + \lceil \frac{r}{2} \rceil)\}.$$

*Proof.* Suppose first that there exist non-adjacent vertices $u$ and $v$ of $G$ with $N(u) \bigcap N(v) = \emptyset$. Define $f : V(G) \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G)}{2} \rceil\}$ by $f(u) = f(v) = 1 + \lceil \frac{r}{2} \rceil$, $f(x) = 0$ for $x \in N(u) \bigcup N(v)$ and $f(t) = 1$ otherwise. Clearly $f$ is a strong Roman dominating function; so that we have $\gamma_{StR}(G) \leq \omega(f) = 2(1 + \lceil \frac{r}{2} \rceil) + n - 2 - 2r \leq n - r + 1$. Suppose now that $N(u) \bigcap N(v) \neq \emptyset$ for all non-adjacent vertices $u$ and $v$ of $G$. One can see that $diam(G) = 2$. So by Proposition 2.6, one has $\gamma_{StR}(G) \leq r(1 + \lceil \frac{r}{2} \rceil)$ and this completes the proof. □

Alvarez-Ruiz et al. [1] established the following relationship between the domination number and the strong Roman domination number of a graph: For every graph $G$,

$$\gamma(G) \leq \gamma_{StR}(G) \leq (1 + \lceil \frac{\Delta(G)}{2} \rceil)\gamma(G). \tag{1}$$

Now, we present next a trivial necessary and sufficient condition for a graph $G$ such that $\gamma_{StR}(G) = \gamma(G)$ and $\gamma_{StR}(G) = (1 + \lceil \frac{\Delta(G)}{2} \rceil)\gamma(G)$.

**Proposition 2.8.** Let $G$ be a connected graph of order $n$. Then $\gamma_{StR}(G) = \gamma(G)$ if and only if $G = K_1$.

*Proof.* One side is clear. Let $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ be a $\gamma_{StR}(G)$-function. Then $\bigcup_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i$ is a dominating set of $G$ and so we have

$$\gamma(G) \leq \sum_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} |V_i| \leq \sum_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} |V_i| + \sum_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} (i-1)|V_i| = \gamma_{StR}(G) = \gamma(G).$$

Then one has $\sum_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} (i-1)|V_i| = 0$, implying that $\bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i = \emptyset$ and so $\gamma_{StR}(G) = n$. Hence, by Proposition A we have $G = K_1$ or $K_2$. If $G = K_2$, then $\gamma_{StR}(G) = 2\gamma(G)$ which is a contradiction. Hence $G = K_1$, as desired. □

**Proposition 2.9.** Let $G$ be a graph of order $n$. Then $\gamma_{StR}(G) = (\lceil \frac{\Delta(G)}{2} \rceil + 1)\gamma(G)$ if and only if there exists a $\gamma_{StR}(G)$-function $f = (V_0, V_1, \ldots, V_{1 + \lceil \frac{\Delta(G)}{2} \rceil})$ such that $\bigcup_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$.

*Proof.* Let $\gamma_{StR}(G) = (\lceil \frac{\Delta(G)}{2} \rceil + 1)\gamma(G)$ and $S$ be an arbitrary $\gamma(G)$-set. Then the function $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ that assigns a weight of $\lceil \frac{\Delta(G)}{2} \rceil + 1$ to each vertex of $S$ and a weight of 0 to all remaining vertices of $G$ is a StRD-function on $G$, and so

$$\begin{aligned} \gamma_{StR}(G) &\leq f(V(G)) = (\lceil \frac{\Delta(G)}{2} \rceil + 1)|V_{\lceil \frac{\Delta(G)}{2} \rceil + 1}| \\ &\leq (\lceil \frac{\Delta(G)}{2} \rceil + 1)|S| = (\lceil \frac{\Delta(G)}{2} \rceil + 1)\gamma(G) = \gamma_{StR}(G). \end{aligned}$$

Hence, we have equality throughout this inequality chain. In particular,

$$\gamma_{StR}(G) = f(V(G)) = (\lceil \frac{\Delta(G)}{2} \rceil + 1)|V_{\lceil \frac{\Delta(G)}{2} \rceil + 1}|.$$

implying that $f$ is a $\gamma_{StR}(G)$-function satisfying $\bigcup_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$.

Conversely, suppose there exists a $\gamma_{StR}(G)$-function $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ such that $\bigcup_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil} V_i = \emptyset$. Since $\bigcup_{i=1}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i = V_{\lceil \frac{\Delta(G)}{2} \rceil + 1}$ is a dominating set of $G$, we have

$$\gamma(G) \leq |V_{\lceil \frac{\Delta(G)}{2} \rceil + 1}| = \frac{1}{\lceil \frac{\Delta(G)}{2} \rceil + 1} \gamma_{StR}(G)$$

and so $\gamma_{StR}(G) \geq (\lceil \frac{\Delta(G)}{2} \rceil + 1)\gamma(G)$. Hence by (1), we have $\gamma_{StR}(G) = (\lceil \frac{\Delta(G)}{2} \rceil + 1)\gamma(G)$. □

**Proposition 2.10.** Let $G$ be a graph with $diam(G) \geq 3$. Then $\gamma_{StR}(\overline{G}) \leq 2(1 + \lceil \frac{\Delta(\overline{G})}{2} \rceil)$.

*Proof.* Let $P = v_1v_2\dots v_q$, $q = diam(G) + 1$, be a diametral path in $G$ where $q \geq 4$. Since $diam(G) \geq 3$, $\{v_1, v_q\}$ is a dominating set for $\overline{G}$. Hence by (1), we have

$$\gamma_{StR}(\overline{G}) \leq (1 + \lceil\frac{\Delta(\overline{G})}{2}\rceil)\gamma(\overline{G}) \leq 2(1 + \lceil\frac{\Delta(\overline{G})}{2}\rceil).$$

□

**Proposition 2.11.** Let $G$ be a graph with at least one cut-edge and minimum degree $\delta$. Then $\gamma_{StR}(G) \leq n + 2 - \delta$.

*Proof.* Let $e = xy$ be a cut-edge of $G$. Suppose that $G_1$ and $G_2$ are two components of $G \setminus e$ with minimum degree $\delta_1$ and $\delta_2$, respectively. Define $f_1$ and $f_2$ as follows:

$$f_1(u) = \begin{cases} 1 + \lceil\frac{\deg_{G_1}(x)}{2}\rceil & u = x \\ 0 & u \in N(x) \\ +1 & \text{otherwise.} \end{cases}$$

$$f_2(u) = \begin{cases} 1 + \lceil\frac{\deg_{G_2}(y)}{2}\rceil & u = y \\ 0 & u \in N(y) \\ +1 & \text{otherwise.} \end{cases}$$

Obviously, $f_1$ and $f_2$ are strong Roman dominating functions for $G_1$ and $G_2$, respectively. Now define $g$ as follows:

$$g(u) = \begin{cases} f_1(u) & u \in G_1 \\ f_2(u) & u \in G_2. \end{cases}$$

It follows immediately that $g$ is a strong Roman dominating function for $G$ which yields that:

$$\begin{aligned}\gamma_{StR}(G) \leq \omega(g) &= 1 + \lceil\frac{\deg_{G_1}(x)}{2}\rceil + (n_1 - 1 - \deg_{G_1}(x)) \\ &+ 1 + \lceil\frac{\deg_{G_2}(y)}{2}\rceil + (n_2 - 1 - \deg_{G_2}(y)) \\ &= n_1 - \lfloor\frac{\deg_{G_1}(x)}{2}\rfloor + n_2 - \lfloor\frac{\deg_{G_2}(y)}{2}\rfloor \leq n_1 + n_2 - \lfloor\frac{\delta_1}{2}\rfloor - \lfloor\frac{\delta_2}{2}\rfloor \\ &\leq n_1 + n_2 - \lfloor\frac{\delta - 1}{2}\rfloor - \lfloor\frac{\delta - 1}{2}\rfloor \leq n + 2 - \delta\end{aligned}$$

as desired. □

Concluding this section, we prepare a lower and upper bound for the strong Roman domination number of a self-complementary graph. To this end, we need the next proposition.

**Proposition 2.12.** Let $G$ be a graph of order $n$. Then $n + 1 \leq \gamma_{StR}(G) + \gamma_{StR}(\bar{G}) \leq \lceil\frac{3n+1}{2}\rceil$. Moreover, If $G$ is a $r$-regular graph such that $\gamma_{StR}(G) + \gamma_{StR}(\bar{G}) = \lceil\frac{3n+1}{2}\rceil$, then $\gamma_{StR}(G) = n - \lfloor\frac{r}{2}\rfloor$ and $\gamma_{StR}(\bar{G}) = n - \lfloor\frac{n-r-1}{2}\rfloor$.

*Proof.* Using Propositions D and E, we have:

$$\begin{aligned} n+1 &\leq \lceil \frac{n+1}{2} \rceil + \lceil \frac{n+1}{2} \rceil \leq \gamma_{StR}(G) + \gamma_{StR}(\bar{G}) \\ &\leq (n - \lfloor \frac{\Delta(G)}{2} \rfloor) + (n - \lfloor \frac{\Delta(\bar{G})}{2} \rfloor) \\ &= (n - \lfloor \frac{\Delta(G)}{2} \rfloor) + (n - \lfloor \frac{n-1-\delta}{2} \rfloor) \\ &\leq \lceil \frac{3n+1}{2} \rceil - \lfloor \frac{\Delta(G)}{2} \rfloor + \lfloor \frac{\delta}{2} \rfloor \\ &\leq \lceil \frac{3n+1}{2} \rceil. \end{aligned}$$

Suppose that $G$ is a $r$-regular graph such that $\gamma_{StR}(G)+\gamma_{StR}(\bar{G}) = \lceil \frac{3n+1}{2} \rceil$. Since there is no $r$-regular $n$-vertex graph such that $r, n$ are odd, by above inequality we have $\gamma_{StR}(G) = n - \lfloor \frac{r}{2} \rfloor$ and $\gamma_{StR}(\bar{G}) = n - \lfloor \frac{n-r-1}{2} \rfloor$. □

Proposition 2.12 yields the following corollary.

**Corollary 2.13.** Let $G$ be a self-complementary graph. Then $\frac{n+1}{2} \leq \gamma_{StR}(G) \leq \lceil \frac{3n+1}{4} \rceil$.

# 3. Strong Roman Domination Number under some Graph Operations

This section is devoted to verify the behaviour of the strong Roman domination number of a graph whenever a vertex or an edge was omitted. We also investigate the strong Roman domination number under contraction and subdivision of an edge.

We begin our investigation with removing a vertex of a graph.

**Proposition 3.1.** Let $G$ be a graph and $v$ be a vertex of $G$. Then $\gamma_{StR}(G) - 1 \leq \gamma_{StR}(G')$, where $G' = G - v$.

*Proof.* Suppose that $f$ is a $\gamma_{StR}(G')$-function. Define $g$ on graph $G$ as follows:

$$g(x) = \begin{cases} 1 & x = v \\ f(x) & \text{otherwise.} \end{cases}$$

It is obvious that $g$ is a strong Roman dominating function over $G$. So we have $\gamma_{StR}(G) \leq \omega(g) = \omega(f) + 1$. Hence $\gamma_{StR}(G) - 1 \leq \gamma_{StR}(G')$. □

Iterating the removing, one can derive the next corollary.

**Corollary 3.2.** Let $G$ be a graph and suppose that $v_1, \ldots, v_k$ are vertices of $G$. Then $\gamma_{StR}(G) - k \leq \gamma_{StR}(G^k)$, where $G^k = G - \{v_1, \ldots, v_k\}$.

Proposition 3.1 yields the following corollary.

**Corollary 3.3.** Let $G$ be a graph with a vertex $v$ of degree $n-1$. Then $\gamma_{StR}(G') \geq \lceil \frac{n-1}{2} \rceil$, where $G' = G - v$.

*Proof.* Since $\gamma(G) = 1$, using [1, Proposition 8] and Proposition E, we have $\gamma_{StR}(G) = \lceil \frac{n+1}{2} \rceil$. The result is therefore immediate from Proposition 3.1. □

The strong Roman domination number of a graph when removing an edge changes as follows.

**Proposition 3.4.** Let $G$ be a graph with an edge $e = xy$. Then

$$-1 \leq \gamma_{StR}(G) - \gamma_{StR}(G') \leq 1,$$

where $G' = G - e$.

*Proof.* Suppose that $f' = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G')}{2} \rceil + 1})$ is a $\gamma_{StR}(G')$-function. If $\{f'(x), f'(y)\} \subseteq V_0 \bigcup V_1$, then define $g : V(G) \to \{0, 1, 2, \ldots, \lceil \frac{\Delta(G)}{2} \rceil + 1\}$ by $g(t) = f'(t)$. Obviously $g$ is a strong Roman dominating function over $G$. In the case $\{f'(x), f'(y)\} \subseteq \bigcup_{i=2}^{\lceil \frac{\Delta(G')}{2} \rceil + 1} V_i$, we can define a new function on $G$ similarly. Assume now that $f'(x) = 0$ and $f'(y) \in \bigcup_{i=2}^{\lceil \frac{\Delta(G')}{2} \rceil + 1} V_i$. Without lose of generality, define $h : V(G) \to \{0, 1, 2, \ldots, \lceil \frac{\Delta(G)}{2} \rceil + 1\}$ by $h(x) = 1$ and $h(t) = f'(t)$ otherwise. It is clear that $h$ is a strong Roman dominating function over $G$. Therefore we have $\gamma_{StR}(G) \leq \gamma_{StR}(G') + 1$.

Now, let $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ be a $\gamma_{StR}(G)$-function. If $\{f(u), f(v)\} \subseteq V_0 \cup V_1$, the function $g : V(G') \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G')}{2} \rceil\}$ defined by $g(x) = f(x)$ for $x \in V(G')$ is a StRDF of $G'$. If $\{f(u), f(v)\} \subseteq \bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i$, then we can define a new function on $G'$ similarly. Assume now that $f(u) = 0$ and $f(v) \in \bigcup_{i=2}^{\lceil \frac{\Delta(G}{2} \rceil + 1} V_i$. The function $h : V(G') \to \{0, 1, 2, \ldots, \lceil \frac{\Delta(G')}{2} \rceil + 1\}$ defined by $h(u) = 1$ and $h(x) = f(x)$ otherwise. It is clear that $h$ is a StRDF of $G'$. Therefore we have $\gamma_{StR}(G') - \gamma_{StR}(G) \leq 1$. □

We shall now describe the behaviour of the strong Roman domination number under the contracting an edge of a graph.

**Proposition 3.5.** Let $G$ be a graph with edge $e = uv$. Then $\gamma_{StR}(G') \leq \gamma_{StR}(G) + \lceil \frac{\Delta(G)-1}{2} \rceil$, where $G'$ obtained from $G$ by contracting $e$.

*Proof.* Suppose that $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ is a strong Roman dominating function over $G$. If $f(u) = f(v) = 0$ ($f(u) = f(v) = 1$), then define $f_1 : V(G') \to \{0, 1, 2, \ldots, 1 + \lceil \frac{\Delta(G')}{2} \rceil\}$ by $f_1(u = v) = f(u) = 0$ ($f_1(u = v) = f(v) = 1$) and $f_1(x) = f(x)$ otherwise. Obviously $f_1$ is a strong Roman dominating function over $G'$. Assume now that $f(u) = 0$ and $f(v) = 1$. Define $f_2 : V(G') \to \{0, 1, 2, \ldots, 1 + \lceil \frac{\Delta(G')}{2} \rceil\}$ by $f_2(u = v) = f(v) = 1$ and $f_2(x) = f(x)$ otherwise. It is easy to see that $f_2$ is a strong Roman dominating function over $G'$. Next suppose that $f(u) = 0$ and $f(v) \in \bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i$ and define $f_3 : V(G') \to \{0, 1, 2, \ldots, 1 + \lceil \frac{\Delta(G')}{2} \rceil\}$ by $f_3(u = v) = f(v) + \lceil \frac{\Delta(G) - 1}{2} \rceil$ and $f_3(x) = f(x)$ otherwise. It is easy to see that $f_3$ is a strong Roman dominating function over $G'$. Without lose of generality, assume that $f(u) \leq f(v) \in \bigcup_{i=2}^{\lceil \frac{\Delta(G)}{2} \rceil + 1} V_i$. Define $f_4 : V(G') \to \{0, 1, 2, \ldots, 1 + \lceil \frac{\Delta(G')}{2} \rceil\}$ by $f_4(u = v) = f(v) + \lceil \frac{\Delta(G) - 1}{2} \rceil$ and $f_4(x) = f(x)$ otherwise. Again it is easy to see that $f_4$ is a strong Roman dominating function over $G'$. Therefore, for $i = 1, 2, 3, 4$, we have

$$\gamma_{StR}(G') \leq \omega(f_i) \leq \omega(f) + \lceil \frac{\Delta(G) - 1}{2} \rceil = \gamma_{StR}(G) + \lceil \frac{\Delta(G) - 1}{2} \rceil.$$

□

We now deal with the subdivision.

**Proposition 3.6.** Let $G$ be a graph of order $n \geq 3$. If $G'$ is obtained from $G$ by subdividing the edge $e = xy$, then $\gamma_{StR}(G) \leq \gamma_{StR}(G')$.

*Proof.* Let us to subdivide the edge $e = xy$ with $z$. Suppose that $f' = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G')}{2} \rceil + 1})$ is a $\gamma_{StR}(G')$-function. If $f'(z) \in V_0 \bigcup V_1$, then $f'_{|G}$ is a strong Roman dominating function of $G$. Assume that $f'(z) \in \bigcup_{i=2}^{\lceil \frac{\Delta(G')}{2} \rceil + 1} V_i$. It is easy to see that $f'(x) = 0$ or $f'(y) = 0$. We also have $f'(z) = 2$. Suppose first that $f'(x) = f'(y) = 0$. Define $f : V(G) \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G)}{2} \rceil\}$ by $f(x) = f(y) = 1$ and $f(t) = f'(t)$ for each $t \in V(G) \setminus \{x, y\}$. Clearly, $f$ is a strong Roman dominating function of $G$ and $\omega(f) = \omega(f')$. Suppose now that $f'(x) = 0$ and $f'(y) \neq 0$. Define $g : V(G) \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G)}{2} \rceil\}$ by $g(x) = 1$ and $g(t) = f'(t)$ for each $t \in V(G) \setminus \{x\}$. Clearly, $g$ is a strong Roman dominating function of $G$ and $\omega(g) = \omega(f') - 1$. Therefore we conclude that $\gamma_{StR}(G) \leq \gamma_{StR}(G')$. □

# 4. Strong Roman Domination Number for some Classes of Graphs

The authors in [1] gave an upper bound for $\gamma_{StR}(T)$, where $T$ is a tree of order $n \geq 3$. They conjectured that, for any graph of order $n \geq 3$, $\gamma_{StR}(G) \leq \frac{6n}{7}$. In this section we give an upper bound for unicyclic graph. Also, we determine strong Roman domination number for some classes of graphs.

**Lemma 4.1.** *Let $G = F_m^p$. Then*

$$\gamma_{StR}(G+e) - \gamma_{StR}(G) \leq 0$$

*where $e = uv$ is a new edge which is added to $G$.*

*Proof.* Let $D_3$ be the vertices of degree at least 3 in $G$ and $D_2$ be the vertices of degree 2 in $G$ and $D_1$ be the leaves of $G$. Then the function $f : V(G) \to \{0, 1, \ldots, 1 + \lceil \frac{\Delta(G)}{2} \rceil\}$ defined by $f(x) = 3$ for $x \in D_3$, $f(x) = 1$ for $x \in D_1$ and $f(x) = 0$ for $x \in D_2$, is a StRDF of $G$ of weight $2(3m) = \frac{6n}{7}$. We show that $f$ is a StRDF of $G + e$. If $u, v \in D_3 \cup D_1$ or $u, v \in D_2 \cup D_1$, then clearly $f$ is a StRDF of $G + e$. Without lose of generality, let $u \in D_3$ and $v \in D_2$. Since $1 + |\lceil \frac{N_G(u) \cap V_0}{2} \rceil| = 3$ and $1 + |\lceil \frac{N_{G+e}(u) \cap V_0}{2} \rceil| = 3$, then $f$ is a StRDF of $G + e$. Therefore, $\gamma_{StR}(G + e) - \gamma_{StR}(G) \leq 0$. □

**Theorem 4.2.** *Let $G$ be a unicyclic graph. Then*

$$StR(G) \begin{cases} \leq \frac{6n}{7} & \text{if } n \equiv 0 \pmod 7 \\ < \frac{6n}{7} + 1 & O.W \end{cases}$$

*Proof.* Consider $C$ as a cycle of $G$ and $e \in E(C)$. Note that $G' = G - e$ is a tree. If $G' \in F_m^p$, then by Lemma 4.1 and Proposition C, we have $\gamma_{StR}(G) \leq \frac{6n}{7}$. Suppose $G' \notin F_m^p$. By Proposition 3.4, we have $\gamma_{StR}(G) \leq \gamma_{StR}(G') + 1$. If $n \equiv 0 \pmod 7$, then using Proposition B, $\gamma_{StR}(G) \leq \gamma_{StR}(G') + 1 \leq \frac{6n}{7} - 1 + 1 = \frac{6n}{7}$, otherwise we have

$$\gamma_{StR}(G) \leq \gamma_{StR}(G') + 1 < \frac{6n}{7} + 1.$$

□

**Proposition 4.3.** For $m \geq 3$, $\gamma_{StR}(C_m \circ K_1) \leq \lceil \frac{5n}{6} \rceil$, where $n = 2m$.

*Proof.* Suppose that $f$ is a $\gamma_{StR}(C_m)$-function. Define $g : V(C_m \circ K_1) \to \{0, 1, 2, 3\}$ by $g(x) = f(x)$ for $x \in V(C_m)$ and $g(x) = 1$ otherwise. Clearly $g$ is a strong Roman dominating function for $C_m \circ K_1$ and, hence, we have $\gamma_{StR}(C_m \circ K_1) \leq \omega(g) \leq \omega(f) + m$. Using this in conjunction with Propositions F, one obtains $\gamma_{StR}(C_m \circ K_1) \leq \lceil \frac{2m}{3} \rceil + \frac{n}{2} = \lceil \frac{5n}{6} \rceil$. □

Noting to the proof of Corollary 3.3, one has the following result.

**Corollary 4.4.** For $n \geq 1$, $\gamma_{StR}(K_n) = \lceil \frac{n+1}{2} \rceil$.

The following auxiliary lemma which is interesting in itself is of fundamental importance in finding $\gamma_{StR}(K_{n,m})$. It's simple proof is omitted.

**Lemma 4.5.** Let $G$ be a graph of order $n$ and $f = (V_0, V_1, \ldots, V_{\lceil \frac{\Delta(G)}{2} \rceil + 1})$ be a strong Roman dominating function. Then $|V_0| \leq n - 1$. Moreover, if the equality holds, then $\Delta = n - 1$.

**Proposition 4.6.** For $2 \leq n \leq m$,

$$StR(K_{n,m}) = \begin{cases} \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil & \text{if } m \text{ is odd} \\ \lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil & \text{if } m \text{ is even and } n \geq 3 \\ 2 + \frac{m}{2} & \text{if } m \text{ is even and } n = 2. \end{cases}$$

*Proof.* Let $(X, Y)$ be two parts of the complete bipartite graph $K_{n,m}$ and $X = \{u_1, \dots, u_n\}$ and $Y = \{v_1, \dots, v_m\}$. Using Proposition E, we have

$$\gamma_{StR}(K_{n,m}) \geq \lceil \frac{n+m+1}{2} \rceil.$$

Define $f$ as follows:

$$f(x) = \begin{cases} 1 + \lceil \frac{m-1}{2} \rceil & x = u_1 \\ 1 + \lceil \frac{n-1}{2} \rceil & x = v_1 \\ 0 & \text{otherwise.} \end{cases}$$

It is clear that $f$ is a strong Roman dominating function for $K_{n,m}$ which yields

$$\lceil \frac{n+m+1}{2} \rceil \leq \gamma_{StR}(K_{n,m}) \leq \omega(f) = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m+1}{2} \rceil.$$

Now consider two following cases:
**Case 1.** $m$ is odd. Suppose first that $n$ is even. So in this case we have

$$\lceil \frac{n+m+1}{2} \rceil = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil - 1 \leq \gamma_{StR}(K_{n,m}) \leq \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil.$$

We claim that $\gamma_{StR}(K_{n,m}) = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil$. Suppose, on the contrary, that $\gamma_{StR}(K_{n,m}) = \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil - 1$ and let $g$ be a $\gamma_{StR}(K_{n,m})$-function for $K_{n,m}$. It is straightforward to see that $|V_0| \geq n - 1$. Without lose of generality, assume that $g(u_i) = 0$ for $i = 1, \dots, t$ and $g(v_i) = 0$ for $i = 1, \dots, s$. By Lemma 4.5, we have $t + s \leq m + n - 2$. So we have $\lfloor \frac{t+s}{2} \rfloor \leq \lfloor \frac{m+n}{2} \rfloor - 1$. In this case, there is a vertex $u_j \in X$, for some $1 \leq j \leq n$ such that $g(u_j) \geq 1 + \lceil \frac{s}{2} \rceil$ and a vertex $v_j \in Y$, for some $1 \leq j \leq m$ such that $g(v_j) \geq 1 + \lceil \frac{t}{2} \rceil$. Therefore

$$\begin{aligned} \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil - 1 = \omega(g) &\geq 1 + \lceil \frac{t}{2} \rceil + 1 + \lceil \frac{s}{2} \rceil + (m + n - 2) - t - s \\ &\geq \lceil \frac{t+s}{2} \rceil + m + n - t - s \\ &\geq m + n - \lfloor \frac{t+s}{2} \rfloor \\ &\geq m + n - \lfloor \frac{m+n}{2} \rfloor + 1 \\ &= \lceil \frac{n+1}{2} \rceil + \lceil \frac{m}{2} \rceil. \end{aligned}$$

So we get a contradiction. Suppose now that $n$ is odd. Hence we have

$$\lceil\frac{n+m+1}{2}\rceil = \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil \le \gamma_{StR}(K_{n,m}) \le \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil$$

as desired.

**Case 2.** $m$ is even. Let us consider two subcases:

**Subcase 2.1.** Suppose first that $n \ge 3$. One has

$$\lceil\frac{n+m+1}{2}\rceil = \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil \le \gamma_{StR}(K_{n,m}) \le \lceil\frac{n+1}{2}\rceil + \lceil\frac{m+1}{2}\rceil.$$

We claim that $\gamma_{StR}(K_{n,m}) = \lceil\frac{n+1}{2}\rceil + \lceil\frac{m+1}{2}\rceil$. Suppose, on contrary, that $\gamma_{StR}(K_{n,m}) = \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil$ and let $g$ be a $\gamma_{StR}(K_{n,m})$-function. Without lose of generality, assume that $g(u_i) = 0$ for $i = 1, \ldots, t$ and $g(v_i) = 0$ for $i = 1, \ldots, s$. By Lemma 4.5, we have $t+s \le m+n-2$. At first, suppose that $t+s = m+n-2$. Since $n \ge 3$, there is exactly a vertex $x \in X$ and $y \in Y$ such that $g(x) \ne 0$ and $g(y) \ne 0$. Also by definition of strong dominating function $g$, one can see $g(x) \ge 1 + \lceil\frac{m-1}{2}\rceil = 1 + \lceil\frac{m}{2}\rceil$ and $g(y) \ge 1 + \lceil\frac{n-1}{2}\rceil = \lceil\frac{n+1}{2}\rceil$. Hence we have

$$\lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil = \omega(g) \ge g(x) + g(y) = 1 + \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil$$

which is a contradiction. Assume that $t+s = m+n-3$. Choose a vertex $u_1 \in X$ and $v_1, v_2 \in Y$ such that $g(u_1) \ne 0$, $g(v_1) \ne 0$ and $g(v_2) \ne 0$. By definition of strong dominating function $g$, without lose of generality, one can suppose that $g(u_1) \ge 1 + \lceil\frac{m-2}{2}\rceil = \lceil\frac{m}{2}\rceil$, $g(v_1) \ge 1 + \lceil\frac{n-1}{2}\rceil = \lceil\frac{n+1}{2}\rceil$ and $g(v_2) \ge 1$. Hence we have

$$\lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil = \omega(g) \ge g(u_1) + g(v_1) + g(u_2) = \lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil + 1$$

which is a contradiction. Suppose now that $t+s \le m+n-4$. So we have $\lfloor\frac{t+s}{2}\rfloor < \lfloor\frac{m+n}{2}\rfloor - 1$. In this case, there is a vertex $u_j \in X$, for some $1 \le j \le n$ such that $g(u_j) \ge 1 + \lceil\frac{s}{2}\rceil$ and a vertex $v_j \in Y$, for some $1 \le j \le m$ such that $g(v_j) \ge 1 + \lceil\frac{t}{2}\rceil$. Therefore

$$\begin{aligned}
\lceil\frac{n+1}{2}\rceil + \lceil\frac{m}{2}\rceil = \omega(g) &\ge 1 + \lceil\frac{t}{2}\rceil + 1 + \lceil\frac{s}{2}\rceil + (m+n-2) - t - s \\
&\ge \lceil\frac{t+s}{2}\rceil + m + n - t - s \\
&\ge m + n - \lfloor\frac{t+s}{2}\rfloor \\
&> m + n - \lfloor\frac{m+n}{2}\rfloor + 1 \\
&= \lceil\frac{m+n}{2}\rceil + 1.
\end{aligned}$$

So we get a contradiction.

**Subcase 2.2.** Suppose now that $n = 2$. Define $h : V(K_{2,m}) \to \{0, 1, \dots, 1 + \lceil \frac{m}{2} \rceil\}$ by $h(u_1) = 1 + \frac{m}{2}$, $h(u_2) = 1$ and $h(v_i) = 0$ for all $1 \le i \le m$. It is easy to see that $h$ is a strong Roman dominating function of $K_{2,m}$. This together with Proposition E gives

$$2 + \frac{m}{2} \le \gamma_{StR}(K_{2,m}) \le \omega(h) = 2 + \frac{m}{2}$$

and the proof has been completed. □

As an immedate consequence one has the following.

**Corollary 4.7.** $\gamma_{StR}(K_{n,n}) = \begin{cases} n+1 & \text{if } n \text{ is odd} \\ n+2 & \text{if } n \text{ is even.} \end{cases}$

The *Cartesian product* of graphs $G$ and $H$ is the graph $G \square H$ with the vertex set $V(G) \square V(H)$ and $(x_1, x_2)(y_1, y_2) \in E(G \square H)$ whenever $x_1 y_1 \in E(G)$ and $x_2 = y_2$, or $x_2 y_2 \in E(H)$ and $x_1 = y_1$. In the follow we determine the strong Roman domination number of $P_2 \square P_n$. We assume the vertices of the $i$-th copy of $P_2$ in $P_2 \square P_n$ are $u_i^1, u_i^2$ for $i = 1, 2, \dots, n$.

**Proposition 4.8.** For $n \ge 1$,

$$\gamma_{StR}(P_2 \square P_n) = \begin{cases} \frac{4n}{3} & \text{if} \quad n \equiv 0 \pmod 3 \\ \frac{4n+2}{3} & \text{if} \quad n \equiv 1 \pmod 3 \\ \frac{4n+1}{3} & \text{if} \quad n \equiv 2 \pmod 3). \end{cases}$$

*Proof.* Define $f : V(P_2 \square P_n) \to \{0, 1, 2, 3\}$ by $f(u_{3i+2}^1) = f(u_{3i+2}^2) = 2$ for $0 \le i \le \lfloor \frac{n-2}{3} \rfloor$ and $f(u) = 0$ otherwise if $n \equiv 0 \pmod 3$, by $f(u_n^1) = f(u_n^2) = 1$, $f(u_{3i+2}^1) = f(u_{3i+2}^2) = 2$ for $0 \le i \le \lfloor \frac{n-2}{3} \rfloor$ and $f(u) = 0$ otherwise if $n \equiv 1 \pmod 3$ and by $f(u_n^1) = f(u_{3i+2}^1) = f(u_{3i+2}^2) = 2$ for $0 \le i \le \lfloor \frac{n-4}{3} \rfloor$, $f(u_{n-1}^2) = 1$ and $f(u) = 0$ otherwise if $n \equiv 2 \pmod 3$. Clearly $f$ is a StRDF of $P_2 \square P_n$ with desired weight and so

$$\gamma_{StR}(P_2 \square P_n) \le \begin{cases} \frac{4n}{3} & \text{if} \quad n \equiv 0 \pmod 3 \\ \frac{4n+2}{3} & \text{if} \quad n \equiv 1 \pmod 3 \\ \frac{4n+1}{3} & \text{if} \quad n \equiv 2 \pmod 3). \end{cases}$$

Now we prove the inverse inequality by induction on $n$. The results is trivial for $n = 1, 2, 3$. Assume that $n \ge 4$ and the result is true for $P_2 \square P_{n'}$ for each $n' < n$. Let

$$G' = P_2 \square P_n - \{u_n^1, u_n^2, u_{n-1}^1, u_{n-1}^2, u_{n-2}^1, u_{n-2}^2\}.$$

Clearly $G' = P_2 \square P_{n-3}$. Assume $f = (V_0, V_1, \dots, V_{\lceil \frac{\Delta}{2} \rceil + 1})$ is a $\gamma_{StR}(P_2 \square P_n)$-function such that $\bigcup_{i=2}^{\lceil \frac{\Delta}{2} \rceil + 1} V_i$ is as small as possible. We consider following cases.

**Case 1.** $f(u_n^1) = f(u_n^2) = 0$.

We have $f(u_{n-1}^1) = f(u_{n-1}^2) = 2$. If $f(u_{n-2}^1) \ge 1$ or $f(u_{n-2}^2) \ge 1$, then the function

$g : V(P_2\Box P_n) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^1_n) = 1$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function which contradicts the choice of $f$. Thus $f(u^1_{n-2}) = f(u^2_{n-2}) = 0$. If $f(u^1_{n-3}) = 3$ (the case $f(u^2_{n-3}) = 3$ is similar), then the function $g : V(P_2\Box P_n) \to \{0,1,2,3\}$ defined by $g(u^1_{n-3}) = g(u^1_{n-4}) = g(u^2_{n-3}) = 1$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function which contradicts the choice of $f$. Then the function $f$, restricted to $G'$ is a StRDF of $G'$ of weight $\gamma_{StR}(P_2\Box P_n) - 4$ and the result follows by the induction hypothesis.

**Case 2.** $f(u^1_n) = 0$ and $f(u^2_n) = 1$ (the case $f(u^1_n) = 1$ and $f(u^2_n) = 0$ is similar). We have $f(u^1_{n-1}) \geq 2$ . If $f(u^1_{n-1}) = 3$, then $f(u^2_{n-1}) = f(u^1_{n-2}) = 0$. If $f(u^2_{n-2}) = 1$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. If $f(u^2_{n-2}) \geq 2$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^2_{n-3}) = 1$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. Hence $f(u^2_{n-2}) = 0$ and so the function $f_1 : V(G) \to \{0,1,2,3\}$ defined by $f_1(u^1_{n-1}) = f_1(u^2_{n-1}) = 2$, $f_1(u^1_{n-2}) = f_1(u^2_{n-2}) = f_1(u^1_n) = f_1(u^2_n) = 0$ and $f_1(u) = f(u)$ otherwise, is a StRDF of $P_2\Box P_n$ such that $f_1(u^1_n) = f_1(u^2_n) = 0$ and similar above case, we are done.

Let $f(u^1_{n-1}) = 2$. Then $f(u^2_{n-1}) \geq 1$ or $f(u^1_{n-2}) \geq 1$. Suppose that $f(u^2_{n-1}) \geq 1$. If $f(u^2_{n-1}) = 2$, the function $g : V(P_2\Box P_n) \to \{0,1,2,3\}$ defined by $g(u^2_n) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. Hence, $f(u^2_{n-1}) = 1$. Then the function $h : V(G) \to \{0,1,2,3\}$ defined by $h(u^1_{n-1}) = h(u^2_{n-1}) = 2$, $h(u^1_{n-2}) = h(u^2_{n-2}) = h(u^1_n) = h(u^2_n) = 0$ and $h(u) = f(u)$ otherwise, is a StRDF of $P_2\Box P_n$ of with $h(u^1_n) = h(u^2_n) = 0$ and similar above case, we are done. Now let $f(u^1_{n-2}) \geq 1$. If $f(u^1_{n-2}) = 1$, then the function $h$ defined in above, is a StRDF of $P_2\Box P_n$ of with $h(u^1_n) = h(u^2_n) = 0$ and similar above case, we are done. If $f(u^1_{n-2}) = 2$, the function $f_2 : V(G) \to \{0,1,2,3\}$ defined by $f_2(u^1_{n-3}) = 1$, and $f_2(u) = f_1(u)$ otherwise, is a StRDF of $P_2\Box P_n$ of with $f_1(u^1_n) = f_1(u^2_n) = 0$ and similar above case, we are done.

**Case 3.** $f(u^1_n) = f(u^2_n) = 1$.
If $f(u^1_{n-1}) = f(u^2_{n-1}) = 0$, then to strong Roman dominate $u^1_{n-1}, u^2_{n-1}$ we must have $f(u^1_{n-2}) = f(u^2_{n-2}) = 2$. It follows from the choice of $f$ that $f(u^1_{n-3}) = f(u^2_{n-3}) = 0$. Now the function $f_3 : V(G') \to \{0,1,2,3\}$ defined by $f_3(u^1_{n-3}) = f_3(u^2_{n-3}) = 1$ and $f_3(u) = f(u)$ otherwise, is a StRDF of $G'$ of weight $\gamma_{tR}(P_2\Box P_n)-4$ and the result follows by the induction hypothesis. Hence, we assume without loss of generality that $f(u^1_{n-1}) \geq 1$. Consider the following subcases.

**Subcase 3.1.** $f(u^1_{n-1}) = 2$.
By the choice of $f$, we must have $f(u^1_{n-2}) = f(u^2_{n-1}) = 0$. If $f(u^2_{n-2}) = 0$, then the function $f$ restricted to $G'$ is a StRDF of $G'$ of weight $\gamma_{tR}(P_2\Box P_n)-4$ and the result follows by the induction hypothesis. Suppose $f(u^2_{n-2}) \geq 1$. If $f(u^2_{n-2}) = 1$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-

function of weight less than $f$ which is a contradiction. If $f(u^2_{n-2}) \geq 2$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^2_{n-3}) = 1$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction.

**Subcase 3.2.** $f(u^1_{n-1}) = 1$.
By choise of $f$ we have $f(u^2_{n-1}) \leq 1$. If $f(u^1_{n-2}) \neq 2$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_n) = 2, g(u^2_n) = g(u^1_{n-1}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. If $f(u^1_{n-2}) = 2$, then we must have $f(u^2_{n-2}) = f(u^1_{n-3}) = 0$ and $f(u^2_{n-1}) = 1$. Then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^2_n) = 2, g(u^1_n) = g(u^2_{n-1}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction.

**Case 4.** $f(u^1_n) = 2$ (the case $f(u^2_n) = 2$ is similar).
Then we must have $f(u^1_{n-1}) = f(u^2_n) = 0$. If $f(u^2_{n-1}) \geq 2$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_n) = 1$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. Hence $f(u^2_{n-1}) \leq 1$. If $f(u^2_{n-1}) = 0$, then $f(u^2_{n-2}) \geq 2$. If $f(u^2_{n-2}) = 3$, then the function $f_4 : V(G') \to \{0,1,2,3\}$ defined by $f_4(u^2_{n-3}) = 1$ and $f_4(u) = f(u)$ otherwise, is a StRDF of $G'$ of weight $\gamma_{tR}(P_2\Box P_n) - 4$ and the result follows by the induction hypothesis. If $f(u^2_{n-2}) = 2$ and $f(u^1_{n-2}) = 0$, then the function $f$, restricted to $G'$ is a StRDF of $G'$ of weight $\gamma_{StR}(P_2\Box P_n) - 4$ and the result follows by the induction hypothesis. If $f(u^2_{n-2}) = 2$ and $f(u^1_{n-2}) = 1$, then the function $f_5 : V(G') \to \{0,1,2,3\}$ defined by $f_5(u^2_{n-3}) = 1$ and $f_5(u) = f(u)$ otherwise, is a StRDF of $G'$ of weight $\gamma_{StR}(P_2\Box P_n) - 4$ and the result follows by the induction hypothesis. If $f(u^2_{n-2}) = 2$ and $f(u^1_{n-2}) = 2$, then the function $f_6 : V(G') \to \{0,1,2,3\}$ defined by $f_6(u^1_{n-3}) = f_6(u^2_{n-3}) = 1$ and $f_6(u) = f(u)$ otherwise, is a StRDF of $G'$ of weight $\gamma_{StR}(P_2\Box P_n) - 4$ and the result follows by the induction hypothesis. Let $f(u^2_{n-1}) = 1$. If $f(u^1_{n-2}) = f(u^2_{n-2}) = 1$, then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction. If $f(u^1_{n-2}) = 3$ (similarly $f(u^2_{n-2}) = 3$), then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^1_{n-3}) = 1$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction.

If $f(u^1_{n-2}) = 2$ (similarly $f(u^2_{n-2}) = 2$), then $f(u^2_{n-2}) \geq 1$ or $f(u^1_{n-3}) \geq 1$. Without generality, let $f(u^2_{n-2}) \geq 1$. Then the function $g : V(G) \to \{0,1,2,3\}$ defined by $g(u^1_{n-1}) = g(u^2_{n-1}) = 2$, $g(u^1_{n-3}) = 1$, $g(u^1_n) = g(u^2_n) = g(u^1_{n-2}) = g(u^2_{n-2}) = 0$ and $g(u) = f(u)$ otherwise, is a $\gamma_{StR}(P_2\Box P_n)$-function of weight less than $f$ which is a contradiction.

This completes the proof. □

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