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Homotopy Category of Cotorsion Flat Representations of Quivers

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Abstract

Recently in [10], it was proved that over any ring R, there exists a complete cotorsion pair $(\mathbb{K}_{\rm p}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R}))$ in $\mathbb{K}(\text{Flat-}R)$, the homotopy category of complexes of flat R-modules, where $\mathbb{K}_{\rm p}(\text{Flat-}R)$ and $\mathbb{K}(\text{dg-CotF-R})$ are the homotopy categories raised by flat (or pure) and dg-cotorsion complexes of flat R-modules, respectively. This paper aims at recognition of a parallel cotorsion pair in $\mathbb{K}(\text{Flat-}\mathcal{Q})$, the homotopy category of flat representation of certain quivers $\mathcal{Q}$, where $\mathcal{Q}$ may also be infinite. The importance of this result lies in the fact that this homotopy categories do not necessarily raise from the category of modules over some ring. In the other part of this paper, we give a classification of compact objects in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$, the homotopy category of dg-cotorsion complexes of flat representations of certain $\mathcal{Q}$, in terms of the corresponding vertex-complexes.

Keywords: Homotopy category, cotorsion pair, compact object, representations of a quiver.

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1. Introduction

In a recent work [10], the basis of a systematic study of $\mathbb{K}(\text{dg-CotF-R})$, the homotopy category of dg-cotorsion complexes of flat modules over a ring R was founded. In that paper it was proved, in particular, that this homotopy category is compactly generated whenever R is right coherent. We point out that

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the importance of the notion of compact generation is something well-understood in the general theory of triangulated categories [16, 17, 20]. One of the basic tools there to obtain such a result was the discovery of a complete cotorsion pair $(\mathbb{K}_{\mathrm{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R}))$ in $\mathbb{K}(\text{Flat-}R)$, the homotopy category of complexes of flat R-modules, where $\mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$ is the triangulated subcategory of $\mathbb{K}(\text{Flat-}R)$ formed by the so-called *flat* (or *pure*) complexes of R-modules. To see more about cotorsion pairs in the setting of homotopy categories and their applications, the reader is referred to [5, 15].

One of the main objectives in this paper is to extend the aforementioned cotorsion pair to the corresponding homotopy categories raised by representations of certain quivers. To give a more detailed introduction, we must recall that for a finite quiver $\mathcal{Q}$, the category of representations of $\mathcal{Q}$ by R-modules, denoted $\mathrm{Rep}(\mathcal{Q}, R)$, is naturally equivalent to $\mathrm{Mod}\, R\mathcal{Q}$, the category of modules over the path algebra $R\mathcal{Q}$ of $\mathcal{Q}$ over R which is, by definition, a free R-module on the basis of all paths in $\mathcal{Q}$ and whose multiplication is given by the usual combination of consecutive arrows of $\mathcal{Q}$ [3, Theorem 1.6]. However, this is not the case for infinite $\mathcal{Q}$. Nonetheless, $\mathrm{Rep}(\mathcal{Q}, R)$ has proved to show rather the same homological behavior as those displayed by the category of modules over a ring. That's why we have chosen the homotopy category raised by $\mathrm{Rep}(\mathcal{Q}, R)$ to state our results. To see further works in this context, the reader may take a glance at [2, 11].

For a (possibly infinite) quiver $\mathcal{Q}$, let $\mathbb{K}(\text{Flat-}\mathcal{Q})$ be the homotopy category of flat representations of $\mathcal{Q}$ by R-modules. Specifically speaking, what makes the current study reasonable to us is that the aforementioned cotorsion pair is capable of being extended to a complete cotorsion pair $(\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q}), \mathbb{K}(\text{dg-CotF-}\mathcal{Q}))$ in $\mathbb{K}(\text{Flat-}\mathcal{Q})$ while these homotopy categories have not necessarily been raised by a category of modules over a ring; we do this is Section 3 for certain $\mathcal{Q}$. Here $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ is the homotopy category of dg-cotorsion complexes of flat representations of $\mathcal{Q}$ by R-modules; we leave the definitions to Section 2.

Recognition of compact objects is another fundamental task in the theory of triangulated categories [17, 20]. This has many applications in connection with forming certain equivalences between triangulated categories. Our Section 4 is devoted to provide a local characterization of compact objects in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. That is, we show how, for certain quivers $\mathcal{Q}$, compactness of an object in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ impresses the resulting vertex-complexes in $\mathbb{K}(\text{dg-CotF-R})$.

2. Notation and Basic Facts

Throughout the paper, R will always denote a ring with unity and R-modules are intended to be left R-modules; $\mathrm{Mod}\, R$ is reserved to denote the category of all R-modules. Also the category of complexes of R-modules and the chain maps between them is denoted $\mathcal{C}(R)$. Recall that the homotopy category of R, $\mathbb{K}(R)$, has the objects of $\mathcal{C}(R)$ as its objects and the morphisms are given by the homotopy equivalence classes of chain maps [22]. The homotopy categories formed by complexes of projective (resp. flat) R-modules is denoted $\mathbb{K}(\text{Proj-}R)$ (resp. $\mathbb{K}(\text{Flat-}R)$). It is well-known that these are typical examples of triangulated categories where the suspension (or shift) functor is given by Σ^{-1}, the usual shifting of complexes to the left; namely, $\Sigma^{-1} : \mathbb{K}(R) \longrightarrow \mathbb{K}(R)$ is an autoequivalence of categories by means of which these homotopy categories are turned into triangulated categories. We refer the reader to the classic text [19] on triangulated categories and their defining axioms and properties. Basic facts about homotopy categories may be found in [22]. As a convention, for a triangulated category $\mathcal{T}$ with suspension functor $T : \mathcal{T} \longrightarrow \mathcal{T}$, a typical triangle is represented by either of the diagrams

$$X \longrightarrow Y \longrightarrow Z \longrightarrow T(X) \text{ or } X \longrightarrow Y \longrightarrow Z \rightsquigarrow .$$

If $\mathcal{S}$ is a triangulated subcategory of a triangulated category $\mathcal{T}$, then we have the following two triangulated subcategories:

$$\mathcal{S}^{\perp} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(S, X) = 0 \text{ for all } S \in \mathcal{S}\},$$

$$^{\perp}\mathcal{S} = \{X \in \mathcal{T} \mid \text{Hom}_{\mathcal{T}}(X, S) = 0 \text{ for all } S \in \mathcal{S}\}.$$

These are called the right and left orthogonals of $\mathcal{S}$ in $\mathcal{T}$.

Let $\mathbf{F}$ be in $\mathbb{K}(\text{Flat-}R)$. Following [9], [14], and [20], we say that $\mathbf{F}$ is flat (or pure) provided it lies in $\mathbb{K}(\text{Proj-}R)^{\perp}$, where the orthogonal is taken in $\mathbb{K}(\text{Flat-}R)$. The corresponding homotopy category is denoted $\mathbb{K}_{\text{p}}(\text{Flat-}R)$. In [9] and [20] the objects in $\mathbb{K}_{\text{p}}(\text{Flat-}R)$ are characterized as complexes $\mathbf{F} \in \mathbb{K}(\text{Flat-}R)$ that are exact, that is with trivial homology modules, and have flat syzygy modules.

Recall that an R-module C is called cotorsion if $\text{E}xt^1_R(F, C) = 0$ for every flat R-module F. Complexes $\mathbf{C}$ of cotorsion R-modules in $\mathbb{K}_{\text{p}}(\text{Flat-}R)^{\perp}$ are called dg-cotorsion. These were originally defined in [9] by requiring that $\text{E}xt^1_{\mathcal{C}(R)}(\mathbf{F}, \mathbf{C}) = 0$ for every pure complex $\mathbf{F}$; see [9, Proposition 3.4]. The homotopy category raised by dg-cotorsion complexes of flat R-modules is denoted $\mathbb{K}(\text{dg-CotF-R})$.

Throughout the paper, by a quiver $\mathcal{Q}$ we mean an oriented graph with vertex set V and arrow set E. There exist two functions s and t that correspond to any arrow in E its origin and terminal vertices respectively. A representation $\mathcal{M}$ of $\mathcal{Q}$ comes up by assigning an R-module $\mathcal{M}_v$ to any vertex v of $\mathcal{Q}$ and an R-homomorphism $\mathcal{M}_a : \mathcal{M}_v \longrightarrow \mathcal{M}_w$ to any arrow $a : v \to w$ of $\mathcal{Q}$. A morphism $f : \mathcal{M} \longrightarrow \mathcal{N}$ of representations of $\mathcal{Q}$ is given by a family $f = \{f_v\}_{v \in V}$ of R-homomorphisms $f_v : \mathcal{M}_v \longrightarrow \mathcal{N}_v$ that fulfil the commutativity condition $\mathcal{N}_a \circ f_v = f_w \circ \mathcal{M}_a$ for any arrow $a : v \to w$. It is a classical result that the representations of $\mathcal{Q}$ by R-modules and their morphisms form an abelian category $\text{Rep}(\mathcal{Q}, R)$ which is naturally equivalent to $\text{Mod}\, R\,\mathcal{Q}$, the category of modules over the path algebra $R\,\mathcal{Q}$ of $\mathcal{Q}$ over R whenever $\mathcal{Q}$ is finite, i.e., both V and E are finite. These and more background on the theory of representations of quivers might be found, e.g. in [3].

We need to quote the following construction from [2]. For a quiver $\mathcal{Q}$, we let V_1 be the set of all source vertices in $\mathcal{Q}$, that is, all vertices with no arrow terminating at them. In symbols,

$$V_1 = \{v \in V : \nexists\, a \in E \text{ such that } t(a) = v\}.$$

Suppose β is an ordinal number and V_γ has been defined for any ordinal number $\gamma < \beta$. We let

$$V_\beta = \{v \in V \setminus \cup_{\gamma<\beta} V_\gamma : \ \nexists\, a \in E \setminus \{a : s(a) \in \cup_{\gamma<\beta} V_\gamma\} \text{ such that } t(a) = v\}.$$

Hence, using transfinite induction, we may construct the sets V_β for any ordinal number β. Then $\mathcal{Q}$ is called rooted provided there exists an ordinal number α with $V = \bigcup_{\beta\leq\alpha} V_\beta$; the least possible ordinal α that fulfils this is denoted $\mu(\mathcal{Q})$. According to [2, Proposition 2.8] and [13, Proposition 3.6], this is equivalent to saying that $\mathcal{Q}$ has no subquiver of the form $\cdots \to v_2 \to v_1$. Moreover, $\mathcal{Q}$ is said to be locally finite if the number of input and output arrows is finite for any vertex of $\mathcal{Q}$.

The category $\mathrm{Rep}(\mathcal{Q}, R)$ has been widely studied in the literature; see the papers [2, 7, 8, 13, 21]. In particular, we need to mention the following results that classify flat, projective, and cotorsion representations of rooted $\mathcal{Q}$.

Lemma 2.1. *Let $\mathcal{Q}$ be an arbitrary quiver and let $\mathcal{F}, \mathcal{P}, \mathcal{C} \in \mathrm{Rep}(\mathcal{Q}, R)$.*

- (a) ([13, Theorem 3.7]) *If $\mathcal{F}$ if flat, then for any vertex v of $\mathcal{Q}$,*
  - *$\mathcal{F}_v$ is a flat R-module;*
  - *the R-homomorphism $\coprod_{t(a)=v} \mathcal{F}_{s(a)} \longrightarrow \mathcal{F}_v$ induced from the R-homomorphisms $\mathcal{F}_{s(a)} \longrightarrow \mathcal{F}_v$ is a pure monomorphism.*

  If, further, $\mathcal{Q}$ is rooted, then these conditions are also sufficient.

- (b) ([7, Theorme 3.1]) *If $\mathcal{P}$ if projective, then for any vertex v of $\mathcal{Q}$,*
  - *$\mathcal{P}_v$ is a projective R-module;*
  - *the R-homomorphism $\coprod_{t(a)=v} \mathcal{P}_{s(a)} \longrightarrow \mathcal{P}_v$, as mentioned above, is a split monomorphism.*

  If, further, $\mathcal{Q}$ is rooted, then these conditions are also sufficient.

- (c) ([21, Theorem 6]) *$\mathcal{C}$ is a cotorsion representation of $\mathcal{Q}$ if and only if $\mathcal{C}_v$ is a cotorsion R-module for all v.*

Note that a cotorsion representation of $\mathcal{Q}$ is defined in an obvious way.

For a quiver $\mathcal{Q}$, the subcategory of $\mathrm{Rep}(\mathcal{Q}, R)$ consisting of projective (resp. flat) representations is denoted Proj-$\mathcal{Q}$ (resp. Flat-$\mathcal{Q}$). We must stress that the symbol R has been dropped because it is fixed throughout the paper and gives no ambiguity. The homotopy category formed by all representations of $\mathcal{Q}$ over R is denoted $\mathbb{K}(\mathcal{Q})$ and the triangulated subcategories corresponding to projective and flat representations of $\mathcal{Q}$ are respectively denoted $\mathbb{K}(\text{Proj-}\mathcal{Q})$ and $\mathbb{K}(\text{Flat-}\mathcal{Q})$.

Let $\mathcal{Q}$ be an arbitrary quiver and let $\mathcal{Q}'$ be a subquiver. It follows from the definitions above that the natural embedding $\mathcal{Q}' \subseteq \mathcal{Q}$ induces a functor $e^{\mathcal{Q}'} : \mathrm{Rep}(\mathcal{Q}, R) \longrightarrow \mathrm{Rep}(\mathcal{Q}', R)$ called the restriction functor, that restricts any representation of $\mathcal{Q}$ to the vertices of $\mathcal{Q}'$. It is shown in [2] that $e^{\mathcal{Q}'}$ has left and right adjoint functors, respectively denoted $e^{\mathcal{Q}'}_{\lambda}$ and $e^{\mathcal{Q}'}_{\rho}$. We refrain from giving the explicit constructions here, and refer the reader to [2] for the details. In particular, for any vertex v of $\mathcal{Q}$, there corresponds to the natural embedding $\{v\} \subseteq \mathcal{Q}$ a functor $e^v : \mathrm{Rep}(\mathcal{Q}, R) \longrightarrow \text{Mod R}$, called the evaluation at v, with left and right adjoints e^v_{λ} and e^v_{ρ}. For the ease of reader, we hint the rule for e^v_{λ}: For any R-module X, $e^v_{\lambda}(X)$ is the representation of $\mathcal{Q}$ in which the vertex w is represented by the R-module $\coprod_{\mathcal{Q}(v,w)} X$ where $\mathcal{Q}(v,w)$ is the set of all paths in $\mathcal{Q}$ from v to w. Moreover, the arrows are represented by natural injections. There is a dual definition for e^v_{ρ} [2]. Hence, in this notation, one is given the adjoint pairs of functors $(e^{\mathcal{Q}'}_{\lambda}, e^{\mathcal{Q}'})$, $(e^{\mathcal{Q}'}, e^{\mathcal{Q}'}_{\rho})$, and in particular, (e^v_{λ}, e^v), and (e^v, e^v_{ρ}).

As a direct consequence of Lemma 2.1, it follows readily that every projective representation $\mathcal{P}$ of a rooted quiver $\mathcal{Q}$ is of the form $\coprod_{v\in V} e^v_{\lambda}(P^v)$ where the projective R-module P^v is the cokernel of the split monomorphism $\coprod_{t(a)=v} \mathcal{P}_{s(a)} \longrightarrow \mathcal{P}_v$. Also it turns out that the functors e^v_{λ} preserve flatness and projectivity.

What should be pointed out is that the aforementioned functors can naturally be extended to triangulated functors over corresponding homotopy categories. We use the symbols $k^{\mathcal{Q}'}$, $k^{\mathcal{Q}'}_{\lambda}$, $k^{\mathcal{Q}'}_{\rho}$, k^v, k^v_{λ}, and k^v_{ρ} to denote them. This extension procedure goes ahead in a so-called degreewise manner; see [2] for the details. We are therefore equipped with the adjoint pairs of triangulated functors $(k^{\mathcal{Q}'}_{\lambda}, k^{\mathcal{Q}'})$, $(k^{\mathcal{Q}'}, k^{\mathcal{Q}'}_{\rho})$, and in particular, (k^v_{λ}, k^v), and (k^v, k^v_{ρ}) over homotopy categories. A particular case also arises when the subquiver $\mathcal{Q}'$ of a rooted quiver $\mathcal{Q}$ is supposed to be full and has $\bigcup_{\beta\leq\delta} V_{\beta}$, $\delta \leq \mu(\mathcal{Q})$, as its set of vertices. The aforementioned functors are then denoted k^{δ} and k^{δ}_{λ} in this case.

Suppose $\mathrm{Proj}^{\mathrm{op}}$-$\mathcal{Q}$ is the full subcategory of $\mathrm{Rep}(\mathcal{Q}, R)$ consisting those representations $\mathcal{P}$ where $\mathcal{P}_v$ is projective, for any v, but the R-homomorphisms assigned to the arrows are split epimorphisms. It is proved in [2, Theorem 3.8] that there exists an equivalence of triangulated categories

$$\widehat{\ } : \mathbb{K}(\text{Proj-}\mathcal{Q}) \longrightarrow \mathbb{K}(\text{Proj}^{\mathrm{op}}\text{-}\mathcal{Q})$$

whenever the quiver is finite without oriented cycles. The key ingredient of this equivalence is that, as stated above, in this case, every projective representation of $\mathcal{Q}$ can be described as a finite direct sum of projective representation of the form $e^v_\lambda(P^v)$ for various vertices v and projective modules P^v. We now want to generalize this idea to flat representations. For, we need the following definition.

Definition 2.2. A (possibly infinite) rooted quiver $\mathcal{Q}$ is said to *satisfy the property* $(*)$ if every flat representation of $\mathcal{Q}$ is a finite direct sum of flat representations of the form $e^v_\lambda(F^v)$ for various vertices v and flat R-modules F^v.

With regards to this definition, we notice that every $e^v_\lambda(F^v)$ is itself a flat representation according to Lemma 2.1 and the description of the functors e^v_λ given above. Let also $\mathrm{Flat}^{\mathrm{op}}\text{-}\mathcal{Q}$ be the subcategory of $\mathrm{Rep}(\mathcal{Q}, R)$ whose objects are representations $\mathcal{F}$ in which a flat module is assigned to every vertex and the R-homomorphism $\mathcal{F}_v \longrightarrow \mathcal{F}_w$ is a split epimorphism for every arrow $v \to w$.

Lemma 2.3. *Suppose $\mathcal{Q}$ is a locally finite quiver satisfying the property $(*)$. Then there exists an equivalence of triangulated categories* $\widehat{\ }: \mathbb{K}(\mathrm{Flat}\text{-}\mathcal{Q}) \longrightarrow \mathbb{K}(\mathrm{Flat}^{\mathrm{op}}\text{-}\mathcal{Q})$.

Proof. In view of the assumptions on $\mathcal{Q}$, this follows by pursuing the argument given in [2, pages 111-112]. □

3. Extending a Cotorsion Pair

Let us start with recalling the definition of a cotorsion pair in triangulated categories.

Definition 3.1. A cotorsion pair in a triangulated category $\mathcal{T}$ is a pair $(\mathcal{A}, \mathcal{B})$ of its full subcategories that satisfy $\mathcal{A}^\perp = \mathcal{B}$ and ${}^\perp\mathcal{B} = \mathcal{A}$. Such a cotorsion pair is said to be complete if any object X of $\mathcal{T}$ fits into a triangle $A \longrightarrow X \longrightarrow B \longrightarrow T(A)$ where $A \in \mathcal{A}$ and $B \in \mathcal{B}$.

There is also a notion of cotorsion pair in abelian categories. Both of these concepts, being rather of the same essence, have proved very useful in studying famous problems in various fields; see, e.g., [1, 6, 9]. The following lemma was discovered in [10].

Lemma 3.2. ([10, Theorem 3.7]) *For any ring R, there exists a complete cotorsion pair*

$$(\mathbb{K}_{\mathrm{p}}(\mathrm{Flat}\text{-}R), \mathbb{K}(\mathrm{dg\text{-}CotF\text{-}R})),$$

in $\mathbb{K}(\mathrm{Flat}\text{-}R)$.

In this section we generalize this cotorsion pair to the homotopy category $\mathbb{K}(\mathrm{Flat}\text{-}\mathcal{Q})$ of flat representations of certain quivers $\mathcal{Q}$.

Definition 3.3. A complex $\mathcal{F} \in \mathbb{K}(\mathrm{Flat}\text{-}\mathcal{Q})$ is called flat (or pure) provided $\mathcal{F} \in \mathbb{K}(\mathrm{Proj}\text{-}\mathcal{Q})^\perp$, the orthogonal being taken in $\mathbb{K}(\mathrm{Flat}\text{-}\mathcal{Q})$.

The full subcategory of $\mathbb{K}(\text{Flat-}\mathcal{Q})$ consisting of pure complexes is denoted $\mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$. In order to obtain local characterizations for the objects in $\mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$, we make use of the following observation, made in [2, Lemma 2.10]. Any complex $\mathcal{P}$ in $\mathbb{K}(\text{Proj-}\mathcal{Q})$, where $\mathcal{Q}$ is rooted, fits into a short exact sequence

$$0 \longrightarrow \varinjlim_{\delta<\mu(\mathcal{Q})} k_\lambda^\delta k^\delta(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k_\lambda^v(\mathbf{P}^v) \longrightarrow 0,$$

of complexes where, for each vertex v, $\mathbf{P}^v$ is the cokernel of the degree-wise split monomorphism $\oplus_{t(a)=v}\mathcal{P}_{s(a)} \longrightarrow \mathcal{P}_v$. Indeed this comes up as an upshot of Lemma 2.1. Further this sequence splits at the representations level and thus turns into a triangle

$$\varinjlim_{\delta<\mu(\mathcal{Q})} k_\lambda^\delta k^\delta(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k_\lambda^v(\mathbf{P}^\mathbf{v}) \rightsquigarrow ,$$

in $\mathbb{K}(\text{Proj-}\mathcal{Q})$ by [18, Lemma 2.15]. Now if we assume instead that the quiver satisfies the property $(*)$, then the same argument would apply to show that the same short exact sequence exists for any $\mathcal{F} \in \mathbb{K}(\text{Flat-}\mathcal{Q})$; we use this freely throughout the paper.

The following proposition provides a local classification of pure complexes of representations of a quiver in terms of the resulting vertex complexes.

Proposition 3.4. *Let $\mathcal{Q}$ be a rooted quiver and let $\mathcal{F} \in \mathbb{K}(\text{Flat-}\mathcal{Q})$. Then $\mathcal{F} \in \mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$ if and only if $\mathcal{F}_v \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$, for every vertex v of $\mathcal{Q}$.*

Proof. $(\Rightarrow)$ It is known from Lemma 2.1 that for any vertex v of $\mathcal{Q}$, $\mathcal{F}_v$ is a complex of flat R-modules. Hence, according to definitions given before, we only need to show that for any $\mathbf{P} \in \mathbb{K}(\text{Proj-}R)$, $\text{Hom}_{\mathbb{K}(R)}(\mathbf{P}, \mathcal{F}_v) = 0$. But, the adjoint situation (k_λ^v, k^v) implies that the latter is isomorphic to $\text{Hom}_{\mathbb{K}(\mathcal{Q})}(k_\lambda^v(\mathbf{P}), \mathcal{F})$ and this vanishes by the assumption since, as stated above, the functor e_λ^v preserves projectivity and, consequently, $k_\lambda^v(\mathbf{P})$ lies in $\mathbb{K}(\text{Proj-}\mathcal{Q})$.

$(\Leftarrow)$ We exploit transfinite induction on $\mu(\mathcal{Q})$ to show that $\mathcal{F} \in \mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$. For $\mu(\mathcal{Q}) = 1$, there is nothing to prove since in this case, the quiver is a discrete one. Assume we are done for all rooted quivers $\mathcal{Q}'$ with $\mu(\mathcal{Q}') < \mu(\mathcal{Q})$. Take $\mathcal{P} \in \mathbb{K}(\text{Proj-}\mathcal{Q})$ and consider the triangle

$$\varinjlim_{\delta<\mu(\mathcal{Q})} k_\lambda^\delta k^\delta(\mathcal{P}) \longrightarrow \mathcal{P} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k_\lambda^v(\mathbf{P}^\mathbf{v}) \rightsquigarrow , \quad (*)$$

in $\mathbb{K}(\text{Proj-}\mathcal{Q})$ provided by [2, Lemma 2.10], in which for any vertex v, $\mathbf{P}^v$ fulfills the degree-wise split exact sequence

$$0 \longrightarrow \oplus_{t(a)=v}\mathcal{P}_{s(a)} \longrightarrow \mathcal{P}_v \longrightarrow \mathbf{P}^v \longrightarrow 0, \quad (**)$$

in the category of complexes of projective R-modules. Firstly, the adjoint pair (k_λ^v, k^v) besides the assumption gives that

$$\text{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k_\lambda^v(\mathbf{P}^v), \mathcal{F}) \simeq \text{Hom}_{\mathbb{K}(\text{Flat-}R)}(\mathbf{P}^v, \mathcal{F}_v) = 0,$$

for any v. Here we use that $\mathbf{P}^v \in \mathbb{K}(\text{Proj-}R)$ by $(**)$. Let $\mathcal{Q}'$ be the full subquiver of $\mathcal{Q}$ with $V(\mathcal{Q}') = \bigcup_{\alpha<\mu(\mathcal{Q})} V_\alpha$ as its set of vertices, where the union is taken over all ordinal numbers strictly less than $\mu(\mathcal{Q})$. Since $\mathcal{Q}$ is a rooted quiver, it is clear that $\mu(\mathcal{Q}') < \mu(\mathcal{Q})$ and it follows from the discussions given in [2, Remark 2.9] that $\varinjlim_{\delta<\mu(\mathcal{Q})} k_\lambda^\delta k^\delta(\mathcal{P}) = k_\lambda^{\mathcal{Q}'} k^{\mathcal{Q}'}(\mathcal{P})$. Hence the adjoint situation $(k_\lambda^{\mathcal{Q}'}, k^{\mathcal{Q}'})$ implies that

$$\begin{aligned}\mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}\big(\varinjlim_{\delta<\mu(\mathcal{Q})} k_\lambda^\delta k^\delta(\mathcal{P}),\, \mathcal{F}\big) &= \mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k_\lambda^{\mathcal{Q}'} k^{\mathcal{Q}'}(\mathcal{P}), \mathcal{F})\\ &\simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q}')}(k^{\mathcal{Q}'}(\mathcal{P}), k^{\mathcal{Q}'}(\mathcal{F})) = 0,\end{aligned}$$

where we make use of the fact that, by Lemma 2.1, $k^{\mathcal{Q}'}(\mathcal{P})$ is a complex of projective representations of the quiver $\mathcal{Q}'$ and that, by the induction hypothesis, $k^{\mathcal{Q}'}(\mathcal{F})$ lies in $\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q}')$ as $\mu(\mathcal{Q}') < \mu(\mathcal{Q})$. Therefore, applying the cohomological functor $\mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(-, \mathcal{F})$ on the triangle $(*)$ gives $\mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(\mathcal{P}, \mathcal{F}) = 0$, as required. □

The following corollary extends the nice characterization of pure complexes given in [20, Theorem 8.6] to the setting of infinite quivers.

Corollary 3.5. *Let $\mathcal{Q}$ be a rooted quiver and $\mathcal{F} \in \mathbb{K}(\mathcal{Q})$. Then $\mathcal{F} \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})$ if and only if it is an exact complex of representations of $\mathcal{Q}$ with flat syzygies.*

Proof. The sufficiency is an immediate consequence of Proposition 3.4 and Lemma 2.1. For the necessity, suppose $\mathcal{F} \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})$. By Proposition 3.4, for any vertex v, $\mathcal{F}_v \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$ and, accordingly, $\mathcal{F}$ is an exact complex of flat representations of $\mathcal{Q}$. Now if $\mathcal{K}$ is the i-th syzygies of $\mathcal{F} : \cdots \to \mathcal{F}^i \xrightarrow{d^i} \mathcal{F}^{i+1} \to \cdots$ then, for any v, there exists a short exact sequence of complexes displayed by the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \coprod_{t(a)=v} \mathcal{K}_{s(a)} & \longrightarrow & \coprod_{t(a)=v} \mathcal{F}^i_{s(a)} & \xrightarrow{\coprod_{t(a)=v} d^i_{s(a)}} & \coprod_{t(a)=v} \mathcal{F}^{i+1}_{s(a)} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{K}_v & \longrightarrow & \mathcal{F}^i_v & \xrightarrow{d^i_v} & \mathcal{F}^{i+1}_v & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & J_v & \longrightarrow & J^i_v & \longrightarrow & J^{i+1}_v & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where the top two rows are exact, the bottom row arises from the cokernels, and the vertical maps are the obvious ones. Hence also the bottom row is exact and therefore the monomorphism $J_v \to J_v^i$ is pure because the other two rows lie in $\mathbb{K}_{\rm p}(\text{Flat-}R)$ as already pointed out; see [20, Corollary 9.4]. This shows that J_v is a flat R-module as J_v^i is. Moreover, Lemma 2.1 combined to the commutativity of the leftmost top square shows that the R-map $\coprod_{t(a)=v} \mathcal{K}_{s(a)} \longrightarrow \mathcal{K}_v$ is a monomorphism. Now Lemma 2.1 gives that $\mathcal{K}$ is indeed a flat representation of $\mathcal{Q}$, completing the proof. □

Definition 3.6. A complex $\mathcal{X} \in \mathbb{K}(\mathcal{Q})$ of cotorsion representations of a quiver $\mathcal{Q}$ is said to be dg-cotorsion if it satisfies $\mathrm{Ext}^1_{\mathcal{Q}}(\mathcal{F}, \mathcal{X}) = 0$ for any pure complex $\mathcal{F}$ of representations of $\mathcal{Q}$.

The following proposition provides a local classification of dg-cotorsion flat complexes of representations of quivers.

Proposition 3.7. *Let $\mathcal{Q}$ be a rooted quiver and $\mathcal{X} \in \mathbb{K}(\text{Flat-}\mathcal{Q})$. If $\mathcal{X} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$, then $\mathcal{X}_v \in \mathbb{K}(\text{dg-CotF-R})$, for any vertex v of $\mathcal{Q}$. Moreover, if $\mathcal{Q}$ satisfies the property $(*)$, then also the converse holds.*

Proof. From Corollary 3.5, it is easy to see that the functors k^v_λ preserve purity. So the first statement follows again from the adjoint situation (k^v_λ, k^v) in view of Lemma 2.1. To see the converse, we need to settle that for any $\mathcal{F} \in \mathbb{K}_{\rm p}(\text{Flat-}\mathcal{Q})$, $\mathrm{Ext}^1_{\mathcal{Q}}(\mathcal{F}, \mathcal{X}) = 0$, for which we establish an induction on $\mu(\mathcal{Q})$. Consider the short exact sequence

$$0 \longrightarrow \varinjlim_{\delta<\mu(\mathcal{Q})} k^\delta_\lambda k^\delta(\mathcal{F}) \longrightarrow \mathcal{F} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k^v_\lambda(\mathbf{F}^v) \longrightarrow 0, \qquad (\dagger)$$

where, for any v, $\mathbf{F}^v$ is defined by the short exact sequence

$$0 \longrightarrow \oplus_{t(a)=v} \mathcal{F}_{s(a)} \longrightarrow \mathcal{F}_v \longrightarrow \mathbf{F}^v \longrightarrow 0,$$

of complexes. Assume that the subquiver $\mathcal{Q}'$ of $\mathcal{Q}$ is defined as in Proposition 3.4. Then $\mu(\mathcal{Q}') < \mu(\mathcal{Q})$ and $\varinjlim_{\delta<\mu(\mathcal{Q})} k^\delta_\lambda k^\delta(\mathcal{F}) = k^{\mathcal{Q}'}_\lambda k^{\mathcal{Q}'}(\mathcal{F})$. We also notice that the adjoint isomorphisms coming up by the adjoint pairs $(k^{\mathcal{Q}'}_\lambda, k^{\mathcal{Q}'})$ and (k^v_λ, k^v) easily extend to isomorphisms between corresponding $\mathrm{Ext}^i(-,-)$ groups. Thus from the induction hypothesis and Proposition 3.4 one obtains that $\mathrm{Ext}^1_{\mathcal{Q}}(k^{\mathcal{Q}'}_\lambda k^{\mathcal{Q}'}(\mathcal{F}), \mathcal{X}) \simeq \mathrm{Ext}^1_{\mathcal{Q}'}(k^{\mathcal{Q}'}(\mathcal{F}), k^{\mathcal{Q}'}(\mathcal{X})) = 0$. Also, from Proposition 3.4 and the latter sequence one gets $\mathbf{F}^v \in \mathbb{K}_{\rm p}(\text{Flat-}R)$. So $\mathrm{Ext}^1_{\mathcal{Q}}(k^v_\lambda(\mathbf{F}^v), \mathcal{X}) \simeq \mathrm{Ext}^1_R(\mathbf{F}^v, \mathcal{X}_v) = 0$ according to the assumption. Now write the long exact sequence of $\mathrm{Ext}_{\mathcal{Q}}(-,-)$ arising from the short exact sequence $(\dagger)$ and use the aforementioned isomorphisms besides Lemma 2.1 to deduce that $\mathrm{Ext}^1_{\mathcal{Q}}(\mathcal{F}, \mathcal{X}) = 0$; that is, $\mathcal{X} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. □

The main result of this section reads as:

Theorem 3.8. *Suppose $\mathcal{Q}$ is a locally finite quiver satisfying the property $(*)$. Then there exists a complete cotorsion pair* $(\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q}), \mathbb{K}(\text{dg-CotF-}\mathcal{Q}))$ *in* $\mathbb{K}(\text{Flat-}\mathcal{Q})$.

Proof. The proof goes ahead in two steps. The first step shows that $\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})^{\perp} = \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$, the orthogonal being taken is $\mathbb{K}(\text{Flat-}\mathcal{Q})$. By definitions, we only need to see that $\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})^{\perp} \subseteq \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. But by Lemma 2.1, this amounts to showing that for any $\mathcal{F} \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})^{\perp}$ and any vertex v of $\mathcal{Q}$, $\mathcal{F}_v$ is locally represented by cotorsion R-modules. Lemma 3.2 yields that we are done if $\mathcal{F}_v \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}R)^{\perp}$. For take some $\mathbf{F}' \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$. Then there is an adjoint isomorphism

$$\mathrm{Hom}_{\mathbb{K}(\text{Flat-}R)}(\mathbf{F}', \mathcal{F}_v) \simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k^v_{\lambda}(\mathbf{F}'), \mathcal{F}) = 0,$$

where the latter Hom vanishes because the functor k^v_{λ} preserves purity.

The next step is to settle that $\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q}) = {}^{\perp}\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ and it suffices to verify only the inclusion ${}^{\perp}\mathbb{K}(\text{dg-CotF-}\mathcal{Q}) \subseteq \mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})$. Choose a vertex v and some $\mathcal{G} \in \mathbb{K}(\text{Flat-}\mathcal{Q})$ such that $\mathcal{G} \in {}^{\perp}\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. Then for any $\mathbf{K} \in \mathbb{K}(\text{dg-CotF-R})$, there exist isomorphisms

$$\mathrm{Hom}_{\mathbb{K}(\text{Flat-}R)}(\hat{\mathcal{G}}_v, \mathbf{K}) \simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat}^{\mathrm{op}}\text{-}\mathcal{Q})}(\hat{\mathcal{G}}, k^v_{\rho}(\mathbf{K})) \simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(\mathcal{G}, k^v_{\lambda}(\mathbf{K})), \quad (*)$$

where we are using Lemma 2.3 and the adjoint isomorphism provided by the adjoint pair (k^v, k^v_{ρ}). The claim now is that $k^v_{\lambda}(\mathbf{K}) \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. That it is a complex of cotorsion flat representations of $\mathcal{Q}$ follows from Lemma 2.1 because the quiver is supposed to be locally finite. On the other hand, if $\mathcal{L} \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})$, then another application of 2.3 yields the isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(\mathcal{L}, k^v_{\lambda}(\mathbf{K})) &\simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat}^{\mathrm{op}}\text{-}\mathcal{Q})}(\hat{\mathcal{L}}, k^v_{\rho}(\mathbf{K})) \\
&\simeq \mathrm{Hom}_{\mathbb{K}(\text{Flat-}R)}(\hat{\mathcal{L}}_v, \mathbf{K}) \\
&= 0,
\end{aligned}$$

where the vanishing of the latter Hom follows from Proposition 3.4 and $\mathbf{K}$ being in $\mathbb{K}(\text{dg-CotF-R})$. By virtue of these computations, we deduce that $k^v_{\lambda}(\mathbf{K}) \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$; therefore $\hat{\mathcal{G}}_v \in \mathbb{K}_{\mathrm{p}}(\text{Flat-}R)$ for any vertex v according to $(*)$ and Lemma 3.2.

Now consider the short exact sequences

$$0 \longrightarrow \oplus_{t(a)=v}\mathcal{G}_{s(a)} \longrightarrow \mathcal{G}_v \longrightarrow \mathbf{G}^v \longrightarrow 0\,,$$

and

$$0 \longrightarrow \mathbf{G}^v \longrightarrow \hat{\mathcal{G}}_v \longrightarrow \oplus_{s(a)=v}\hat{\mathcal{G}}_{t(a)} \longrightarrow 0\,,$$

arising from the definition of the functor $\widehat{\ }$: $\mathbb{K}(\text{Flat-}\mathcal{Q}) \longrightarrow \mathbb{K}(\text{Flat}^{\text{op}}\text{-}\mathcal{Q})$ and note that the second one combined to what we have proved so far gives $\mathbf{G}^v \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$ for all vertices v of $\mathcal{Q}$. Since for any $v \in V_1$, $\mathbf{G}^v = \mathcal{G}_v$, one may apply the first sequence above in conjunction with transfinite induction to show that $\mathcal{G}_v \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$ for any vertex v. Hence, Proposition 3.4 gives $\mathcal{G} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$.

Finally, that this cotorsion pair is complete follows from [13, Proposition 4.2] and [14, Corollary 4.10]. □

4. Compact Objects of $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$

As stated before, this section provides a classification of compact objects of the triangulated category $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ in terms of resulting vertex-complexes. Let us recall that an object X in a triangulated category $\mathcal{T}$ with small coproducts is said to be compact if the functor $\text{Hom}_{\mathcal{T}}(X, -)$ commutes with categorical coproducts in $\mathcal{T}$.

The following is a preparatory, independently interesting, lemma. Prior to giving it, we need to recall from [10, Proposition 4.1] that for a family $\{\mathbf{C}_i\}_{i\in I}$ of objects in $\mathbb{K}(\text{dg-CotF-R})$, indexed over a set I, the categorical coproduct of this family exists and is isomorphic, in $\mathbb{K}(\text{dg-CotF-R})$, to an object $\mathbf{C}$ that embeds into a triangle

$$\coprod_{pw} \mathbf{C}_i \xrightarrow{\alpha} \mathbf{C} \xrightarrow{\beta} \mathbf{F} \xrightarrow{\gamma} \Sigma^{-1} \coprod_{pw} \mathbf{C}_i,$$

where $\coprod_{pw} \mathbf{C}_i$ is the pointwise coproduct of the complexes $\mathbf{C}_i$ in $\mathbb{K}(\text{Flat-}R)$, and $\mathbf{F} \in \mathbb{K}_{\text{p}}(\text{Flat-}\mathbf{R})$. Since the existence of this triangle relies on the existence of the complete cotorsion pair

$$(\mathbb{K}_{\text{p}}(\text{Flat-}R), \mathbb{K}(\text{dg-CotF-R})),$$

in $\mathbb{K}(\text{Flat-}R)$, provided by Lemma 3.2, it is possible to pursue the argument given in [10, Proposition 4.1] in conjunction with Theorem 3.8 to deduce that the same statement holds for $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$; namely, if $\mathcal{Q}$ is a locally finite quiver satisfying the property $(*)$, then $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ has categorical coproducts that are given via the same triangle as mentioned above.

Lemma 4.1. *Let $\mathcal{Q}$ be a locally finite quiver satisfying the property $(*)$. Assume $\{\mathbf{C}_i\}_{i\in I}$ is a family of objects in $\mathbb{K}(\text{dg-CotF-R})$, indexed over a set I, and let $\mathbf{C}$ be its categorical coproduct, as mentioned above. Then for any vertex v of $\mathcal{Q}$, $\coprod_i k^v_\lambda(\mathbf{C}_i) \simeq k^v_\lambda(\mathbf{C})$ as objects in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$.*

Proof. Note that $k^v_\lambda(\mathbf{C}_i)$, $i \in I$, and $k^v_\lambda(\mathbf{C})$ both lie in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ by Proposition 3.7. Hence, as mentioned above, one is given two triangles

$$\coprod_{pw} \mathbf{C}_i \xrightarrow{\alpha} \mathbf{C} \xrightarrow{\beta} \mathbf{F} \xrightarrow{\gamma} \Sigma^{-1} \coprod_{pw} \mathbf{C}_i,$$

and

$$\coprod_{pw} k_\lambda^v(\mathbf{C}_i) \xrightarrow{f} \mathcal{W} \xrightarrow{g} \mathcal{T} \xrightarrow{h} \Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i), \qquad (*)$$

respectively in $\mathbb{K}(\text{Flat-}R)$ and $\mathbb{K}(\text{Flat-}\mathcal{Q})$ where $\mathbf{C} \in \mathbb{K}(\text{dg-CotF-R})$, $\mathbf{F} \in \mathbb{K}_{\text{p}}(\text{Flat-}R)$, $\mathcal{W} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$, and $\mathcal{T} \in \mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$. Since k_λ^v is a triangulated functor, the first triangle above provides us with a triangle

$$\coprod_{pw} k_\lambda^v(\mathbf{C}_i) \xrightarrow{k_\lambda^v(\alpha)} k_\lambda^v(\mathbf{C}) \xrightarrow{k_\lambda^v(\beta)} k_\lambda^v(\mathbf{F}) \xrightarrow{k_\lambda^v(\gamma)} \Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i)\ , \qquad (**)$$

in $\mathbb{K}(\text{Flat-}\mathcal{Q})$. Since $\mathcal{W} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ and $k_\lambda^v(\mathbf{F}) \in \mathbb{K}_{\text{p}}(\text{Flat-}\mathcal{Q})$ by Corollary 3.5, Theorem 3.8 via applying the functor $\text{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k_\lambda^v(\mathbf{F}), -)$ on the triangle $(*)$ gives an isomorphism

$$h^* : \text{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k_\lambda^v(\mathbf{F}), \mathcal{T}) \simeq \text{Hom}_{\mathbb{K}(\text{Flat-}\mathcal{Q})}(k_\lambda^v(\mathbf{F}), \Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i)).$$

Hence, corresponding to $k_\lambda^v(\gamma)$ one obtains a map $t : k_\lambda^v(\mathbf{F}) \longrightarrow \mathcal{T}$ with $ht = k_\lambda^v(\gamma)$. Using the axioms of triangulated categories, we may complete this to a triangle

$$k_\lambda^v(\mathbf{F}) \xrightarrow{t} \mathcal{T} \longrightarrow \mathcal{Y} \longrightarrow \Sigma^{-1}\ k_\lambda^v(\mathbf{F}), \qquad (***)$$

in $\mathbb{K}(\text{Flat-}\mathcal{Q})$. Now apply the Octahedral Axiom [19, Proposition 1.4.6] to form the commutative diagram

$$\begin{array}{ccccccc}
\coprod_{pw} k_\lambda^v(\mathbf{C}_i) & \xrightarrow{k_\lambda^v(\alpha)} & k_\lambda^v(\mathbf{C}) & \xrightarrow{k_\lambda^v(\beta)} & k_\lambda^v(\mathbf{F}) & \xrightarrow{k_\lambda^v(\gamma)} & \Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i) \\
\downarrow{=} & & \downarrow & & \downarrow{t} & & \downarrow{=} \\
\coprod_{pw} k_\lambda^v(\mathbf{C}_i) & \xrightarrow{f} & \mathcal{W} & \xrightarrow{g} & \mathcal{T} & \xrightarrow{h} & \Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathcal{Y} & \xrightarrow{=} & \mathcal{Y} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Sigma^{-1} \coprod_{pw} k_\lambda^v(\mathbf{C}_i) & \longrightarrow & \Sigma^{-1} k_\lambda^v(\mathbf{C}) & \longrightarrow & \Sigma^{-1} k_\lambda^v(\mathbf{F}) & \longrightarrow & \Sigma^{-2} \coprod_{pw} k_\lambda^v(\mathbf{C}_i)
\end{array}$$

all of whose rows and columns are triangles. Apply now the triangle $(***)$ to deduce that $\mathcal{Y}$ lies in $\mathbb{K}_{\mathrm{p}}(\text{Flat-}\mathcal{Q})$ because the other two entries do. On the other hand, since $k^v_\lambda(\mathbf{C}) \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$, the triangle

$$k^v_\lambda(\mathbf{C}) \longrightarrow \mathcal{W} \longrightarrow \mathcal{Y} \longrightarrow \Sigma^{-1} k^v_\lambda(\mathbf{F}) \ ,$$

comes to our aid to settle that $\mathcal{Y} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. Thus Theorem 3.8 reveals that $\mathcal{Y} \simeq 0$ in $\mathbb{K}(\text{Flat-}\mathcal{Q})$. Therefore $\coprod_i k^v_\lambda(\mathbf{C}_i) = \mathcal{W} \simeq k^v_\lambda(\mathbf{C})$ in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. □

The following theorem is the main result of this section.

Theorem 4.2. *Suppose $\mathcal{Q}$ is a locally finite quiver satisfying the property $(*)$ and let $\mathcal{X} \in \mathbb{K}(\text{dg-CotF-}\mathcal{Q})$. Then $\mathcal{X}$ is compact if and only if for any vertex v of $\mathcal{Q}$, $\mathcal{X}_v$ is compact as an object in $\mathbb{K}(\text{dg-CotF-R})$.*

Proof. We must point out firstly that this theorem makes sense in view of Proposition 3.7. Assume first that $\mathcal{X}$ is a compact object in $\mathbb{K}(\text{dg-CotF-}\mathcal{Q})$ and take a family $\{\mathbf{C}_i\}_{i\in I}$ of objects in $\mathbb{K}(\text{dg-CotF-R})$ indexed over a set I. If $\mathbf{C}$ is the categorical coproduct of this family in $\mathbb{K}(\text{dg-CotF-R})$, then for any vertex v of $\mathcal{Q}$, there exist a sequence of isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathbb{K}(\text{dg-CotF-R})}(\mathcal{X}_v, \mathbf{C}) &\simeq \mathrm{Hom}_{\mathbb{K}(\text{dg-CotF-}\mathcal{Q})}(\mathcal{X}, k^v_\lambda(\mathbf{C})) \\ &\simeq \mathrm{Hom}_{\mathbb{K}(\text{dg-CotF-}\mathcal{Q})}(\mathcal{X}, \coprod_i k^v_\lambda(\mathbf{C}_i)) \\ &\simeq \coprod_i \mathrm{Hom}_{\mathbb{K}(\text{dg-CotF-}\mathcal{Q})}(\mathcal{X}, k^v_\lambda(\mathbf{C}_i)) \\ &\simeq \coprod_i \mathrm{Hom}_{\mathbb{K}(\text{dg-CotF-R})}(\mathcal{X}_v, \mathbf{C}_i), \end{aligned}$$

where the first isomorphism makes sense by Proposition 3.7 and the adjoint situation (k^v_λ, k^v), while the second and the third one follow respectively from Lemma 4.1 and our assumption.

Suppose conversely that $\mathcal{X}_v$ is a compact object in $\mathbb{K}(\text{dg-CotF-R})$ for any vertex v. Consider the short exact sequence

$$0 \longrightarrow \varinjlim_{\delta<\mu(\mathcal{Q})} k^\delta_\lambda k^\delta(\mathcal{X}) \longrightarrow \mathcal{X} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k^v_\lambda(\mathbf{X}^v) \longrightarrow 0, \qquad (*)$$

of complexes of representations of $\mathcal{Q}$ where, for any v, $\mathbf{X}^v$ is defined by the short exact sequence

$$0 \longrightarrow \oplus_{t(a)=v} \mathcal{X}_{s(a)} \longrightarrow \mathcal{X}_v \longrightarrow \mathbf{X}^v \longrightarrow 0. \qquad (**)$$

Note that the sequence $(**)$ in conjunction with locally finiteness of $\mathcal{Q}$ and Lemma 2.1 yields that $\mathbf{X}^v$ is a complex of cotorsion flat R-modules for any vertex v;

in particular, this makes $(**)$ into a degree-wise split sequence of R-complexes. Hence, indeed, $(**)$ turns into a triangle

$$\oplus_{t(a)=v}\mathcal{X}_{s(a)} \longrightarrow \mathcal{X}_v \longrightarrow \mathbf{X}^v \rightsquigarrow ,$$

in $\mathbb{K}$(dg-CotF-R), by [18, Lemma 2.15], from which one obtains that $\mathbf{X}^v$ is a compact object in $\mathbb{K}$(dg-CotF-R) according to our hypothesis. This, in turn, gives that $\oplus_{v\in V_{\mu(\mathcal{Q})}} k^v_\lambda(\mathbf{X}^v)$ lies in the subcategory of compact objects in $\mathbb{K}$(dg-CotF-$\mathcal{Q}$) by the adjoint situation (k^v_λ, k^v). On the other hand, if we set $\mathcal{Q}'$ be the subquiver of $\mathcal{Q}$ as defined in Proposition 3.4, we get that $\varinjlim_{\delta<\mu(\mathcal{Q})} k^\delta_\lambda k^\delta(\mathcal{X}) = k^{\mathcal{Q}'}_\lambda k^{\mathcal{Q}'}(\mathcal{X})$ and $\mu(\mathcal{Q}') < \mu(\mathcal{Q})$. Therefore, a straightforward transfinite induction on $\mu(\mathcal{Q})$ implies that $k^{\mathcal{Q}'}_\lambda k^{\mathcal{Q}'}(\mathcal{X})$ is also a compact object in $\mathbb{K}$(dg-CotF-$\mathcal{Q}$); here we also use Proposition 3.7. But we also get from definition of $\mathcal{Q}'$ and Lemma 2.1 that the sequence $(*)$ splits at the representations level, thus turns into a triangle

$$k^{\mathcal{Q}'}_\lambda k^{\mathcal{Q}'}(\mathcal{X}) \longrightarrow \mathcal{X} \longrightarrow \oplus_{v\in V_{\mu(\mathcal{Q})}} k^v_\lambda(\mathbf{X}^v) \rightsquigarrow ,$$

in $\mathbb{K}$(dg-CotF-$\mathcal{Q}$). This triangle simply gives that $\mathcal{X}$ is also compact in $\mathbb{K}$(dg-CotF-$\mathcal{Q}$). □

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