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On Vertex-Uniprimitive Non-Cayley Graphs of Order pq

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Abstract

Let p and q be distinct odd primes. Let $\Gamma = (V(\Gamma), E(\Gamma))$ be a non-Cayley vertex-transitive graph of order pq. Let $G \leq \mathrm{Aut}(\Gamma)$ acts primitively on the vertex set $V(\Gamma)$. In this paper, we show that G is uniprimitive which is primitive but not 2-transitive and we obtain some information about p, q and the minimality of the Socle $T = \mathrm{Soc}(G)$.

Keywords: uniprimitive group, non-Cayley graph, socle

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1. Introduction

Let $\Gamma = (V(\Gamma), E(\Gamma))$ or simply Γ be a finite, simple and undirected graph. We denote by $\mathrm{Aut}(\Gamma)$, the automorphism group of Γ and we say that Γ is vertex-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $V(\Gamma)$. The cardinality of $V(\Gamma)$ is called the order of the graph. For a group G and a subset S of G such that $1_G \notin S$ and $S^{-1} = S$, the Cayley graph $\Gamma = \mathrm{Cay}(G, S)$ is defined to have vertex set G and for $g, h \in G, \{g, h\}$ is an edge if and only if $gs = h$ foe some $s \in S$. Every Cayley graph is a vertex-transitive graph and conversely, see [1], a vertex-transitive graph is isomorphic to a Cayley graph for some group if and only if its automorphism group has a regular subgroup on vertices.

By $\mathcal{NC}$ we denote the set of positive integers n for which there exists a vertex-transitive non-Cayley graph on n vertices which is not a Cayley graph. In [9],

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Marušič proposed the determination of $\mathcal{NC}$ and in [10], he proved that $p, p^2, p^3 \notin \mathcal{NC}$ for prime p. Mckay and Praeger settled the status of all non-square-free numbers n are in $\mathcal{NC}$, with the exceptions $n = 12, n = p^2, n = p^3$, p prime and they finished the characterization of non-Cayley numbers which are the product of two distinct primes (see [7, 8]). The following theorem characterize all numbers $n = pq \in \mathcal{NC}$ where p, q are distinct primes.

Theorem 1.1. [8, Theorem 1] *Let $p < q$ be primes. Then $pq \in \mathcal{NC}$ if and only if one of the following holds.*

(1) p^2 *divides* $q - 1$;

(2) $q = 2p - 1$ *or* $q = \frac{p^2+1}{2}$;

(3) $q = 2^t + 1$ *and either* p *divides* $2^t - 1$ *or* $p = 2^{t-1} - 1$;

(4) $q = 2^t - 1$ *and* $p = 2^{t-1} + 1$;

(5) $pq = 7.11$.

For characterization of non-Cayley number which are the product of three distinct primes we refer to [4, 5, 13].

Suppose that a group G acting on a set V transitively. We say that a partition Σ of V is G-invariant if G permutes the blocks of Σ. A transitive action of G on V is said to be primitive if all G-invariant partitions of V are trivial. A primitive permutation group which is not 2-transitive is said to be uniprimitive. Also a transitive permutation group is said to be minimal transitive if all of its proper subgroups are intransitive. Suppose that Γ is a graph of order pq. In this paper we prove that if $G \leq \mathrm{Aut}(\Gamma)$ acts primitively on the vertex set of Γ, then either Γ is a Cayley graph or G is uniprimitive and when $pq \notin \mathcal{NC}$ then $T = \mathrm{Soc}(G)$ is not minimal transitive.

2. Primitive Permutation Groups of Degree pq

First, we investigate primitive permutation groups of order pq which are 2-transitive.

Proposition 2.1. *Let p and q be distinct odd primes such that $p < q$ and suppose that G, a subgroup of S_{pq}, is 2-transitive of degree pq with Socle T. Then T is a non-abelian simple group and is one of the group listed below.*

(*i*) $T = A_{pq}$;

(*ii*) $T = \mathrm{PSL}(n, s)$ *and* $pq = \frac{s^n-1}{s-1}$;

(*iii*) $T = \mathrm{PSU}(3, 2^a)$ *and* $p = 2^a + 1, q = 2^{2a} - 2^a + 1$;

(iv) $T = \mathrm{Sz}(8)$ *and* $p = 5, q = 13$*;*

(v) $T = A_7$ *and* $p = 3, q = 5$*.*

Also in case (i) and (ii) for some but not all p *and* q*, we have* $pq \in \mathcal{NC}$*. In case (iii),* $pq \notin \mathcal{NC}$ *except where* $p = 5, q = 13$ *and in case (v),* $pq \notin \mathcal{NC}$ *and in case (iv),* $pq \in \mathcal{NC}$*.*

Proof. By the "O'Nan-Scott Theorem" [12], G is almost simple, that is, G has a unique minimal normal subgroup T which is a non-abelian simple group. All possibilities for the socle of almost 2-transitive groups are given for example in [2]. Note that in case (*iv*) we have the Suzuki groups $\mathrm{Sz}(q)$ with $s = 2^{2a+1} = \frac{r^2}{2} \geq 8$. In this case $pq = s^2+1 = (s+r+1)(s+r-1)$. Thus 5 divides pq and hence $p = 5$ and $q = 1$. From Theorem 1.1 we conclude that in case (*iv*), $pq \in \mathcal{NC}$ and in case (*iii*), $pq \in \mathcal{NC}$ if and only if $a = 2, p = 5$ and $q = 13$. □

Now we deal with the primitive permutation groups of degree pq which are not 2-transitive, that is, the uniprimitive permutation groups of degree pq.

Proposition 2.2. *Let* p *and* q *be distinct odd primes such that* $p < q$*. Suppose that* G*, is a uniprimitive permutation subgroup of* S_{pq}*, of degree* pq *with Socle* T*. Then* T *is a non-abelian simple group and Table* 1 *contains all the possibilities for* T, p *and* q*.*

Proof. A similar argument to the proof of Proposition 2.1, we conclude that T is a non-abelian simple group. All possibilities for the socle of uniprimitive permutation groups of degree pq are given in [11] based on [6] which are listed in Table 1. □

Table 1: Socle of uniprimitive groups of degree pq.

T	q	p	T	q	p
A_q	$q \geq 5$	$\frac{q-1}{2}$	$\mathrm{PSL}(2,19)$	19	3
A_{q+1}	$q \geq 5$	$\frac{q+1}{2}$	$\mathrm{PSL}(2,29)$	29	7
A_7	7	5	$\mathrm{PSL}(2,59)$	59	29
$\mathrm{PSL}(4,2)$	7	5	$\mathrm{PSL}(2,61)$	61	31
$\mathrm{PSL}(5,2)$	31	5	$\mathrm{PSL}(2,23)$	23	11
$\mathrm{PSp}(4,2^a)$	$2^{2a}+1$	2^a+1	$\mathrm{PSL}(2,11)$	11	5
$\Omega^{\pm}(2n,2)$	$2^n \mp 1 > 7$	$2^{n-1} \pm 1$	$\mathrm{PSL}(2,13)$	13	7
$\mathrm{PSL}(2,q)$	$q \geq 13$	$\frac{q+1}{2}$	M_{23}	23	11
$\mathrm{PSL}(2,q)$	$q \geq 7$	$\frac{q-1}{2}$	M_{22}	11	7
$\mathrm{PSL}(2,p^2)$	$q = \frac{p^2+1}{2}$	p	M_{11}	11	5

3. Uniprimitive Graphs of Order pq

In this section we prove that there is no any non-Cayley graph of order pq which admits a 2-transitive subgroup of automorphisms.

Theorem 3.1. *Let p and q be distinct odd primes and $\Gamma = (V(\Gamma), E(\Gamma))$ be a vertex-transitive graph of order pq. Suppose that there exists a subgroup G of $\mathrm{Aut}(\Gamma)$ which acts 2-transitively on $V(\Gamma)$. Then Γ is a Cayley graph.*

Proof. Let Γ be a vertex-transitive graph of order pq and suppose that $G \leq \mathrm{Aut}(\Gamma)$ is 2-transitive on $V(\Gamma)$. If Γ be an empty graph of order pq, then obviously Γ is a Cayley graph. So we may assume that there exist $x, y \in V(\Gamma)$, such that $\{x, y\} \in E(\Gamma)$. Since G is doubly transitive on $V(\Gamma)$, there exists $g \in G$, such that $x^g = x'$ and $y^g = y'$. Thus $\{x, y\}^g = \{x', y'\} \in E(\Gamma)$. This shows that Γ is a complete graph and the proof is complete. □

Corollary 3.2. *Let Γ be a non-Cayley vertex-transitive graph of order pq. If $G \leq \mathrm{Aut}(\Gamma)$ acts primitively on $V(\Gamma)$, then G is uniprimitive and $T = \mathrm{Soc}(G)$ is one of the group listed in Table 2. Also if $pq \notin \mathcal{NC}$, then T is not minimal transitive.*

Proof. The first part is direct consequence of Theorem 3.1. If $pq \notin \mathcal{NC}$, then T is one of the groups in Table 2. For each line we see that T has a transitive subgroup isomorphic to a Frobenius group of order pq. This can be shown by using [3]. □

Table 2: Socle of uniprimitive groups of degree pq where $pq \notin \mathcal{NC}$.

T	q	p
A_q	$q \geq 5$	$\frac{q-1}{2}$
$\mathrm{PSL}(5,2)$	31	5
$\mathrm{PSL}(2,q)$	$q \geq 7$	$\frac{q-1}{2}$
$\mathrm{PSL}(2,29)$	29	7
$\mathrm{PSL}(2,59)$	59	29
$\mathrm{PSL}(2,23)$	23	11
$\mathrm{PSL}(2,11)$	11	5
M_{23}	23	11
M_{11}	11	5

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