Mathematics Interdisciplinary Research **5** (2020) 321 – 344

# $F$-Hypergroups of Type $U$ on the Right

*Mehdi Farshi, Bijan Davvaz* $^\star$ *and Fatemeh Dehghan*

**Abstract**

In this paper, first we introduce $F$-hypergroups of type $U$ on the right. We will prove that every right scalar identity of an $F$-hypergroup of type $U$ on the right of size $\leq 5$ is also a left identity. Also, we will classify $F$-hypergroups of type $U$ on the right of order 2 or 3 up to an isomorphism. Then, we will study cyclic $F$-semihypergroups and finally by using regular relations we construct right reversible quotient $F$-hypergroups.

Keywords: $F$-hypergroup of type $U$ on the right, regular relation, right reversible, cyclic $F$-semihypergroup.

2010 Mathematics Subject Classification: 08A72, 20N20.

**How to cite this article**
M. Farshi, B. Davvaz and F. Dehghan, $F$-hypergroups of type $U$ on the right, *Math. Interdisc. Res.* **5** (2020) 321–344.

## 1. Introduction and Basic Definitions

In this section, after expressing a short history of hyperstructure theory and fuzzy set theory, we will offer all definitions we require of fuzzy hyperstructures. In 1934, F. Marty has introduced algebraic hyperstructures as a natural extension of classical algebraic structures [18]. He defined hypergroups, investigated their properties and applied them to groups and rational algebraic functions. The principal notion of hypergroup theory and some examples can be found in [1, 2, 4, 20]. In 1984, hypergroups of type $U$ on the right right were introduced in [16] to analyze certain hypergroups obtained as quotient sets. That class includes that of hypergroups of type $C$ on the right, cogroups and that of quotient hypergroups $G/g$ of a group $G$ with respect to a non-normal subgroup $g \subseteq G$ ($D$-hypergroups). In ($D$-hypergroups). In [19], the concept of the category of hypergroups of type $U$ on the right is introduced, and some result already known in the field of homology

$^\star$Corresponding author (E-mail: davvaz@yazd.ac.ir)
Academic Editor: Mohammad Ali Iranmanesh
Received 18 October 2020, Accepted 22 November 2020
DOI: 10.22052/mir.2020.240327.1248

of abelian groups are extended to non-commutative groups, while in the latter, the analysis of relationships existing between hypergroups of type $U$ and hypergroups of double cosets is furthered. Later, this topic studied by De Salvo, Fasino, Freni, Lo Faro, etc (see [6, 7, 8, 10, 11]). The hypergroups $H$ of type $U$ on the right can be classified in terms of the family $P_\epsilon = \{\epsilon x \mid x \in H\}$, where $\epsilon \in H$ is the right scalar identity. If $H$ has size six, then in [5] Davvaz and Bardestani showed that there exist twelve cases for the family $P_\epsilon$.

Following the introduction of fuzzy set by Zadeh in 1965 [21], fuzzy set theory has made remarkable progress. Many mathematicians have used this concept in different branches of mathematics. The notion of fuzzy polygroup ($F$-polygroup) has been introduced by Zahedi and Hasankhani in [22, 23]. In [3], Davvaz introduced the notion of $n$-ary $F$-polygroups which is a generalization of ideas presented by Zahedi and Hasankhani. Afterwards, Farshi and Davvaz, generalized the classical isomorphism theorems of groups to $F^n$-polygroups [13]. In [12], the concept of $F^n$-hypergroups is introduced and some related properties are investigated. Now, we express all definitions that we will use in this article.

Let $H$ be a non-empty set. Each mapping $\mu : H \longrightarrow [0,1]$ is called a *fuzzy subset* of $H$. We define the *support* of $\mu$ by $\mathrm{supp}(\mu) = \{x \in H \mid \mu(x) > 0\}$. An empty fuzzy subset of $H$ denoted by $\emptyset$ is the zero function from $H$ to $[0,1]$. Clearly, we have $\mathrm{supp}(\emptyset) = \emptyset$. The set of all non-empty fuzzy subsets of $H$ will be denoted by $I^*(H)$. If $A \subseteq H$ and $t \in [0,1]$, then by $A_t$ we mean a fuzzy subset of $H$ which is defined as follows:

$$A_t(x) = \begin{cases} t & \text{if } x \in A, \\ 0 & \text{if } x \in H \backslash A. \end{cases}$$

In particular, if $A$ is a singleton set, say $\{a\}$, then $\{a\}_t$ is said to be a *fuzzy point* and is denoted by $a_t$, briefly. In fact $\chi_H$, the characteristic function of $H$, is equal to $H_t$ whenever $t = 1$. For fuzzy subsets $\mu$ and $\eta$ of $H$ we define $(\mu \cup \nu)(x) = \max\{\mu(x), \nu(x)\}$ and $(\mu \cap \nu)(x) = \min\{\mu(x), \nu(x)\}$. Let $\{\mu_\alpha \mid \alpha \in \Lambda\}$ be a collection of fuzzy subsets of $H$, where $\Lambda$ is a non-empty indexed set. Then, we define $(\bigcup\limits_{\alpha \in \Lambda} \mu_\alpha)(x) = \bigvee\limits_{\alpha \in \Lambda} \{\mu_\alpha(x)\}$, where $\bigvee$ denotes supremum. An *$F$-hyperoperation* (or fuzzy hyperoperation) on $H$ is a function $\circ : H \times H \longrightarrow I^*(H)$, i.e., $x \circ y$ is a non-empty fuzzy subset of $H$, for all $x, y \in H$. Let $\mu, \nu \in I^*(H)$ and $x \in H$. Then, $\mu \circ \nu$ denotes $\bigcup\limits_{w \in \mathrm{supp}(\mu), z \in \mathrm{supp}(\nu)} w \circ z$ and $x \circ \nu$ denotes $\bigcup\limits_{y \in \mathrm{supp}(\nu)} x \circ y$. Moreover, for non-empty subsets $A$ and $B$ of $H$, $x \circ A$ denotes $x \circ \chi_A$, $\mu \circ A$ denotes $\mu \circ \chi_A$ and $A \circ B$ denotes $\chi_A \circ \chi_B$. A couple $(H, \circ)$, where $\circ$ is an $F$-hyperoperation on $H$, is called an *$F$-hypergroupoid*. An $F$-hypergroupoid $(H, \circ)$ is called an *$F$-semihypergroup* if $\circ$ is associative, i.e., $x \circ (y \circ z) = (x \circ y) \circ z$, for all $x, y, z \in H$. An $F$-semihypergroup $(H, \circ)$ is called an *$F$-hypergroup* if $\mathrm{supp}(x \circ H) = \mathrm{supp}(H \circ x) = H$, for all $x \in H$. This condition is called the *reproduction axiom*. A non-empty subset $K$ of an $F$-semihypergroup $(H, \circ)$ is called an *$F$-subsemihypergroup* if $\mathrm{supp}(K \circ K) \subseteq K$. In the case that $(H, \circ)$ is an

$F$-hypergroup, $K$ is called an *$F$-subhypergroup* if $\mathrm{supp}(K \circ x) = \mathrm{supp}(x \circ K) = K$, for all $x \in K$. Whenever an $F$-hypergroup $(H, \circ)$ contains an element $e$ with the property that, for all $x \in H$, one has $x \in \mathrm{supp}(x \circ e)$ (resp. $x \in \mathrm{supp}(e \circ x)$), then we say that $e$ is a *right identity* (resp. *left identity*) element of $H$. An identity element is a left and right identity element. If $\mathrm{supp}(x \circ e) = \{x\}$ (resp. $\mathrm{supp}(e \circ x) = \{x\}$), for all $x \in H$, then $e$ is called a *right scalar identity* (resp. *left scalar identity*). Finally, an $F$-semihypergroup $(H, \circ)$ is called an *$F$-polygroup* if the following three conditions are satisfied: (i) there exists $e \in H$ such that $x \in \mathrm{supp}(x \circ e \cap e \circ x)$, for every $x \in H$, (ii) for each $x \in H$, there exists a unique element $x^{-1} \in H$ such that $e \in \mathrm{supp}(x \circ x^{-1} \cap x^{-1} \circ x)$, (iii) $z \in \mathrm{supp}(x \circ y) \Rightarrow x \in \mathrm{supp}(z \circ y^{-1}) \Rightarrow y \in \mathrm{supp}(x^{-1} \circ z)$, for every $x, y, z$ in $H$. Clearly, each $F$-polygroup is an $F$-hypergroup.

## 2. $F$-Semihypergroups of Type $U$ on the Right

In this section we introduce the notion of $F$-semihypergroups of type $U$ on the right, giving several examples that illustrate the importance of this new fuzzy hyperstructure.

**Definition 2.1.** An $F$-semihypergroup $(H, \circ)$ is said to be of type $U$ on the right if it fulfills the following conditions:

(1) $H$ has a right scalar identity element $e$,

(2) $x \in \mathrm{supp}(x \circ y)$ implies that $y = e$, for all $x, y \in H$.

We shall use the notation $(H, \circ, e)$ to say that $e$ is a right scalar identity element.

**Example 2.2.** Let $(H, \circ)$ be an $F$-polygroup in which for all $x \in H$ we have $\mathrm{supp}(x \circ e) = \{x\}$ and $\mathrm{supp}(x^{-1} \circ x) = \{e\}$. Then, $(H, \circ, e)$ is an $F$-hypergroup of type $U$ on the right.

**Example 2.3.** Let $H = \{e, a, b\}$. Then, the following table shows an $F$-polygroup structure on $H$ which is not an $F$-hypergroup of type $U$ on the right.

| $\circ$ | $e$ | $a$ | $b$ |
|---|---|---|---|
| $e$ | $\frac{e}{1}, \frac{a}{0}, \frac{b}{0}$ | $\frac{e}{0}, \frac{a}{1}, \frac{b}{0}$ | $\frac{e}{0}, \frac{a}{0}, \frac{b}{1}$ |
| $a$ | $\frac{e}{0}, \frac{a}{1}, \frac{b}{0}$ | $\frac{e}{0}, \frac{a}{1}, \frac{b}{0}$ | $\frac{e}{1}, \frac{a}{1}, \frac{b}{1}$ |
| $b$ | $\frac{e}{0}, \frac{a}{0}, \frac{b}{1}$ | $\frac{e}{1}, \frac{a}{1}, \frac{b}{1}$ | $\frac{e}{0}, \frac{a}{0}, \frac{b}{1}$ |

Next example is a fuzzy version of an example of [14].

**Example 2.4.** Let $H$ be a set with at least 2 elements and choose an element $e \in H$. Let $t \in (0, 1]$. We define an $F$-hyperoperation $*_t$ on $H$ as follows:

$$(x *_t y)(z) = \begin{cases} t & \text{if} \quad y = e, z = x \\ 0 & \text{if} \quad y \neq e, z = x \\ 0 & \text{if} \quad y = e, z \neq x \\ t & \text{if} \quad y \neq e, z \neq x \end{cases}, \quad \text{for all } x, y, z \in H.$$

It is easy to check that $(H, *_t)$ is an $F$-hypergroup of type $U$ on the right and that for all $y \in H \backslash \{e\}$ and all $x \in H$ we have $\text{supp}(x *_t y) = H \backslash \{x\}$.

**Example 2.5.** Let $t_1, t_2, t_3 \in (0, 1]$. Then, the following tables denote $F$-hypergroups of type $U$ on the right structure on $\mathbb{Z}_2$ and $\mathbb{Z}_3$, respectively.

| $\circ$ | 0 | 1 |
|---|---|---|
| 0 | $\frac{0}{t_1}, \frac{1}{0}$ | $\frac{0}{0}, \frac{1}{t_2}$ |
| 1 | $\frac{0}{0}, \frac{1}{t_2}$ | $\frac{0}{t_1}, \frac{1}{0}$ |

| $\circ$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | $\frac{0}{t_1}, \frac{1}{0}, \frac{2}{0}$ | $\frac{0}{0}, \frac{1}{t_2}, \frac{2}{0}$ | $\frac{0}{0}, \frac{1}{0}, \frac{2}{t_3}$ |
| 1 | $\frac{0}{0}, \frac{1}{t_2}, \frac{2}{0}$ | $\frac{0}{0}, \frac{1}{0}, \frac{2}{t_3}$ | $\frac{0}{t_1}, \frac{1}{0}, \frac{2}{0}$ |
| 2 | $\frac{0}{0}, \frac{1}{0}, \frac{2}{t_3}$ | $\frac{0}{t_1}, \frac{1}{0}, \frac{2}{0}$ | $\frac{0}{0}, \frac{1}{t_2}, \frac{2}{0}$ |

We denote them by $\mathbb{Z}_2(t_1, t_2)$ and $\mathbb{Z}_3(t_1, t_2, t_3)$, respectively.

Next example is a fuzzy version of an example in section 2-2 of [15].

**Example 2.6.** Let $\mathbb{S}_3/\mathbb{S}_2$ be the set of all cosets of the subgroup $\mathbb{S}_2 = \langle(1\ 2)\rangle$ of the symmetric group $\mathbb{S}_3$, i.e., $\mathbb{S}_3/\mathbb{S}_2 = \{\mathbb{S}_2, (1\ 3)\mathbb{S}_2, (2\ 3)\mathbb{S}_2\}$. Let $t_1, t_2$ and $t_3$ be arbitrary elements of $(0, 1]$. Set $e = \mathbb{S}_2$, $x = (1\ 3)\mathbb{S}_2$, and $y = (2\ 3)\mathbb{S}_2$. Then, $\mathbb{S}_3/\mathbb{S}_2$ with the following table is an $F$-hypergroup of type $U$ on the right which we denote by $\mathbb{S}_3/\mathbb{S}_2(t_1, t_2, t_3)$.

| $\circ$ | $e$ | $x$ | $y$ |
|---|---|---|---|
| $e$ | $\frac{e}{t_1}, \frac{x}{0}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{t_2}, \frac{y}{t_3}$ | $\frac{e}{0}, \frac{x}{t_2}, \frac{y}{t_3}$ |
| $x$ | $\frac{e}{0}, \frac{x}{t_2}, \frac{y}{0}$ | $\frac{e}{t_1}, \frac{x}{0}, \frac{y}{t_3}$ | $\frac{e}{t_1}, \frac{x}{0}, \frac{y}{t_3}$ |
| $y$ | $\frac{e}{0}, \frac{x}{0}, \frac{y}{t_3}$ | $\frac{e}{t_1}, \frac{x}{t_2}, \frac{y}{0}$ | $\frac{e}{t_1}, \frac{x}{t_2}, \frac{y}{0}$ |

**Example 2.7.** Let $t$ be an arbitrary element of $(0, 1]$ and $G$ be a group. We define an $F$-hyperoperation $\circ$ on $G$ as follows:

$$(x \circ y)(z) = e_t(xyz^{-1}), \quad \text{for all } x, y, z \in G.$$

It is easy to check that $\circ$ induces an $F$-hypergroup of type $U$ on the right structure on $G$, where $e_t$ is a fuzzy point of $G$.

**Lemma 2.8.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right. Then, for all $x, y, z, t \in H$ the following assertions hold:*

(1) *$supp(e \circ x) = supp(e \circ y)$ implies that $supp(z \circ x) = supp(z \circ y)$.*

(2) *$x \in supp(e \circ y)$ implies that $supp(z \circ x) \subseteq supp(z \circ y)$.*

(3) *If $supp(e \circ x) = \{x\}$ and $z \in supp(x \circ y)$, then $supp(e \circ z) \subseteq supp(x \circ y)$.*

(4) *If $e \in supp(x \circ y)$, then $e \in supp(y \circ x)$.*

(5) *If $y \in supp(x \circ z)$ and $x \in supp(y \circ t)$, then $e \in supp(z \circ t \cap t \circ z)$.*

*Proof.* 1) Let $x, y \in H$ be arbitrary elements and $\text{supp}(e \circ x) = \text{supp}(e \circ y)$. Then, for each $z \in H$ we have

$$\begin{aligned} \text{supp}(z \circ x) = \text{supp}((z \circ e) \circ x) = \text{supp}(z \circ (e \circ x)) &= \text{supp}(z \circ (e \circ y)) \\ &= \text{supp}((z \circ e) \circ y) \\ &= \text{supp}(z \circ y). \end{aligned}$$

2) Let $x, y \in H$ be arbitrary elements and $x \in \text{supp}(e \circ y)$. Then, for each $z \in H$ we have $\text{supp}(z \circ x) \subseteq \text{supp}(z \circ (e \circ y)) = \text{supp}((z \circ e) \circ y) = \text{supp}(z \circ y)$.

3) Let $x, y, z \in H$ be arbitrary elements, $supp(e \circ x) = \{x\}$ and $z \in supp(x \circ y)$. Then, we have

$$supp(e \circ z) \subseteq supp(e \circ (x \circ y)) = supp((e \circ x) \circ y) = supp(x \circ y).$$

4) Let $x, y \in H$ be arbitrary elements and $e \in \text{supp}(x \circ y)$. Then, we have

$$y \in \{y\} = \text{supp}(y \circ e) \subseteq \text{supp}(y \circ (x \circ y)) = \text{supp}((y \circ x) \circ y).$$

So, there exists $w \in \text{supp}(y \circ x)$ such that $y \in \text{supp}(w \circ y)$. This implies that $w \in \text{supp}(y \circ x) \subseteq \text{supp}((w \circ y) \circ x) = \text{supp}(w \circ (y \circ x))$. Hence, there exists $w' \in \text{supp}(y \circ x)$ such that $w \in \text{supp}(w \circ w')$. Since $(H, \circ)$ is of type $U$ on the right, we conclude that $w' = e$ and therefore we have $e \in \text{supp}(y \circ x)$.

5) Let $x, y, z, t \in H$ be arbitrary elements, $y \in \text{supp}(x \circ z)$ and $x \in \text{supp}(y \circ t)$. Then, we have $y \in \text{supp}(x \circ z) \subseteq \text{supp}((y \circ t) \circ z) = \text{supp}(y \circ (t \circ z))$ and so there exists $w \in \text{supp}(t \circ z)$ such that $y \in \text{supp}(y \circ w)$. Since $(H, \circ)$ is of type $U$ on the right, we conclude that $w = e$ and therefore we have $e \in \text{supp}(t \circ z)$. In a similar manner we have $e \in \text{supp}(z \circ t)$. This implies that $e \in \text{supp}(t \circ z) \cap \text{supp}(z \circ t) = \text{supp}(t \circ z \cap z \circ t)$. □

**Lemma 2.9.** *Let $(H, \circ)$ be an $F$-hypergroup of type $U$ on the right and $x \in H$. Then, the following assertions are equivalent:*

(1) *$|supp(e \circ x)| = 1$.*

(2) *supp*$(e \circ x) = \{x\}$.

*Proof.* $1 \Rightarrow 2$) Let $\mathrm{supp}(e \circ x) = \{y\}$. Then, we have

$$\mathrm{supp}(e \circ y) = \mathrm{supp}(e \circ (e \circ x)) = \mathrm{supp}(e \circ x) = \{y\}.$$

On the other hand, by reproduction axiom, there exists $w \in H$ such that $y \in \mathrm{supp}(x \circ w)$ and so we have $\{y\} = \mathrm{supp}(e \circ y) \subseteq \mathrm{supp}((e \circ x) \circ w) = \mathrm{supp}(y \circ w)$. Since $(H, \circ)$ is of type $U$ on the right, we conclude that $w = e$ and therefore $y \in \mathrm{supp}(x \circ e) = \{x\}$. Hence, $x = y$.

$2 \Rightarrow 1$) It is trivial. □

**Lemma 2.10.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right. Then, for each $x \in H \backslash \{e\}$ the following assertions are equivalent:*

(1) $x \notin$ *supp*$(e \circ (H \backslash \{x\}))$.

(2) *supp*$(e \circ x) = \{x\}$.

*Proof.* $1 \Rightarrow 2$) Let $x \notin \mathrm{supp}(e \circ (H \backslash \{x\}))$. Thus, $x \in \mathrm{supp}(e \circ x)$. Let $y \in \mathrm{supp}(e \circ x)$ be an arbitrary element. We have to show that $y = x$. By reproduction axiom, there exists $z \in H$ such that $x \in \mathrm{supp}(y \circ z)$. Whence,

$$x \in \mathrm{supp}(y \circ z) \subseteq \mathrm{supp}((e \circ x) \circ z) = \mathrm{supp}(e \circ (x \circ z)).$$

So, there exists $t \in \mathrm{supp}(x \circ z)$ such that $x \in \mathrm{supp}(e \circ t)$. Since $x \notin \mathrm{supp}(e \circ (H \backslash \{x\}))$, we obtain $x = t$. Thus, $x \in \mathrm{supp}(x \circ z)$. Since $(H, \circ)$ is of type $U$ on the right, we conclude that $z = e$. Now, since $x \in \mathrm{supp}(y \circ z)$ we have $y = x$.

$2 \Rightarrow 1$) By way of contradiction, suppose that there exists $z \in H \backslash \{x\}$ such that $x \in \mathrm{supp}(e \circ z)$. By reproduction axiom, there exists $y \in H \backslash \{e\}$ such that $z \in \mathrm{supp}(x \circ y)$ and so,

$$\mathrm{supp}(e \circ z) \subseteq \mathrm{supp}(e \circ (x \circ y)) = \mathrm{supp}((e \circ x) \circ y) = \mathrm{supp}(x \circ y).$$

Whence, $x \in \mathrm{supp}(x \circ y)$. This implies that $y = e$ and from $z \in \mathrm{supp}(x \circ y)$ it follows that $z = x$, which is a contradiction. □

**Lemma 2.11.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right with at least two elements. Then, the following assertions hold:*

(1) *If supp$(e \circ y) = H \backslash \{e\}$ for some $y \in H$, then supp$(x \circ y) = H \backslash \{x\}$, for all $x \in H$.*

(2) *If there exists $y \in H \backslash \{e\}$ such that supp$(x \circ y) = H \backslash \{x\}$ for some $x \in H$, then $e \in$ supp$(y \circ z)$, for all $z \in H \backslash \{e\}$.*

*Proof.* 1) Let $x, y \in H$ be arbitrary elements and $\text{supp}(e \circ y) = H\backslash\{e\}$. Then, we have

$$\text{supp}(x \circ y) = \text{supp}((x \circ e) \circ y) = \text{supp}(x \circ (e \circ y)) = \text{supp}(x \circ (H\backslash\{e\})).$$

This implies that $x \notin \text{supp}(x \circ y)$. On the other hand, by using reproduction axiom we have

$$\begin{aligned}(H\backslash\{x\}) \cup \{x\} = H = \text{supp}(x \circ H) &= \text{supp}(x \circ ((H\backslash\{e\}) \cup \{e\})) \\ &= \text{supp}(x \circ (H\backslash\{e\})) \cup \text{supp}(x \circ e) \\ &= \text{supp}(x \circ (H\backslash\{e\})) \cup \{x\} \\ &= \text{supp}(x \circ y) \cup \{x\}.\end{aligned}$$

This implies that $\text{supp}(x \circ y) = H\backslash\{x\}$.

2) Let $\text{supp}(x \circ y) = H\backslash\{x\}$, where $x \in H$ and $y \in H\backslash\{e\}$. By way of contradiction, suppose that there exists $z \in H\backslash\{e\}$ such that $e \notin \text{supp}(y \circ z)$. Then,

$$x \notin \text{supp}(x \circ (y \circ z)) = \text{supp}((x \circ y) \circ z) = \text{supp}((H\backslash\{x\}) \circ z).$$

By reproduction axiom, we have $H = \text{supp}((H\backslash\{x\}) \circ z) \cup \text{supp}(x \circ z)$. This implies that $x \in \text{supp}(x \circ z)$. So, we have $z = e$ which is a contradiction. □

**Theorem 2.12.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right with $|H| < 6$. Then, $e$ is a left identity element.*

*Proof.* In the case that $|H| = 1$ we have $H = \{e\}$ and obviously $e$ is a left scalar identity element. Let $H = \{e, x\}$. By way of contradiction, suppose that $x \notin \text{supp}(e \circ x)$. By reproduction axiom, we have $\text{supp}(e \circ x) \cup \text{supp}(x \circ x) = H$. This implies that $x \in \text{supp}(x \circ x)$. Since $(H, \circ)$ is of type $U$ on the right, we have $x = e$ which is a contradiction. Now, assume that $H = \{e, x, y\}$. By reproduction axiom, we have $\text{supp}(e \circ e) \cup \text{supp}(e \circ x) \cup \text{supp}(e \circ y) = H$. This implies that $x \in \text{supp}(e \circ y)$ and $y \in \text{supp}(e \circ x)$. Hence, $x \in \text{supp}(e \circ y) \subseteq \text{supp}(e \circ (e \circ x)) = \text{supp}(e \circ x)$, which is a contradiction. Let $H = \{e, x, y, z\}$. By way of contradiction, suppose that $x \notin \text{supp}(e \circ x)$. By using reproduction axiom, $x \in \text{supp}(e \circ y)$ or $x \in \text{supp}(e \circ z)$. Without loss of generality, we can assume that $x \in \text{supp}(e \circ y)$. Thus, $\text{supp}(e \circ y) \nsubseteq \text{supp}(e \circ x)$ and so $y \notin \text{supp}(e \circ x)$. By way of contradiction, let $y \in \mathit{supp}(e \circ x)$.Then, by Lemma 2.8 (2), $\mathit{supp}(e \circ y) \subseteq \mathit{supp}(e \circ x)$ which is a contradiction. Thus, $\text{supp}(e \circ x) = \{z\}$ and $\text{supp}(e \circ z) = \text{supp}(e \circ (e \circ x)) = \text{supp}(e \circ x) = \{z\}$. By reproduction axiom, there exists $w \in H$ such that $x \in \text{supp}(z \circ w)$ and therefore we have

$$z \in \text{supp}(e \circ x) \subseteq \text{supp}(e \circ (z \circ w)) = \text{supp}((e \circ z) \circ w) = \text{supp}(z \circ w).$$

Since $(H, \circ)$ is an $F$-hypergroup of type $U$ on the right, we have $w = e$. This implies that $x \in \text{supp}(z \circ e) = \{z\}$ which is a contradiction. Finally, assume that $H = \{e, x, y, z, t\}$ and by way of contradiction, let $x \notin \text{supp}(e \circ x)$. By using reproduction axiom, $x \in \text{supp}(e \circ y)$ or $x \in \text{supp}(e \circ z)$ or $x \in \text{supp}(e \circ t)$. Without loss of generality, we can assume that $x \in \text{supp}(e \circ y)$. Thus, $\text{supp}(e \circ y) \not\subseteq \text{supp}(e \circ x)$ and so by Lemma 2.8 (2), $y \notin \text{supp}(e \circ x)$. Therefore, $\text{supp}(e \circ x) \subseteq \{z, t\}$. If $\text{supp}(e \circ x) = \{z\}$, then we have $\text{supp}(e \circ z) = \text{supp}(e \circ (e \circ x)) = \text{supp}(e \circ x) = \{z\}$. On the other hand, by using reproduction axiom, there exists $w \in H$ such that $x \in \text{supp}(z \circ w)$ and therefore we have

$$z \in \text{supp}(e \circ x) \subseteq \text{supp}(e \circ (z \circ w)) = \text{supp}((e \circ z) \circ w) = \text{supp}(z \circ w).$$

Since $(H, \circ)$ is an $F$-hypergroup of type $U$ on the right, we have $w = e$. This implies that $x \in \text{supp}(z \circ e) = \{z\}$ which is a contradiction. In a similar manner, in the case that $\text{supp}(e \circ x) = \{t\}$ we will have a contradiction. So, $\text{supp}(e \circ x) = \{z, t\}$. From

$$\text{supp}(e \circ z) \cup \text{supp}(e \circ t) = \text{supp}(e \circ \{z, t\}) = \text{supp}(e \circ (e \circ x)) = \text{supp}(e \circ x) = \{z, t\},$$

and reproduction axiom, it follows that $y \in \text{supp}(e \circ y)$. Moreover, since $x \in \text{supp}(e \circ y)$, by Lemma 2.8 (2), we have $\{z, t\} = \text{supp}(e \circ x) \subseteq \text{supp}(e \circ y)$ which implies that $\text{supp}(e \circ y) = \{x, y, z, t\}$. Thus, by Lemma 2.11 (1), we have $\text{supp}(x \circ y) = H \backslash \{x\}$ and $\text{supp}(y \circ y) = H \backslash \{y\}$. So,

$$\begin{aligned} y \in \text{supp}(x \circ y) &\subseteq \text{supp}((e \circ y) \circ y) = \text{supp}(e \circ (y \circ y)) \\ &\subseteq \{e\} \cup \text{supp}(e \circ x) \cup \text{supp}(e \circ z) \cup \text{supp}(e \circ t) = \{e, z, t\}, \end{aligned}$$

which is a contradiction. □

In the next example, which is a fuzzy version of Remark 4.1 of [6] , we will offer a right identity element of an $F$-hypergroup of type $U$ on the right which is not a left identity element.

**Example 2.13.** Let $H = \{e, a, b, c, d, f\}$ and $t \in (0, 1]$. Then, $H$ with the following table is an $F$-hypergroup of type $U$ on the right. It is easy to check that $e$ is not a left identity element.

| $\circ$ | $e$ | $a, b, c$ | $d, f$ |
|---|---|---|---|
| $e$ | $e_t$ | $\{c, d\}_t$ | $\{a, b, c, d, f\}_t$ |
| $a$ | $a_t$ | $\{e, d, f\}_t$ | $\{e, b, c, d, f\}_t$ |
| $b$ | $e_t$ | $\{e, a, c, d\}_t$ | $\{e, a, c, d, f\}_t$ |
| $c$ | $e_t$ | $\{e, a, b, f\}_t$ | $\{e, a, b, d, f\}_t$ |
| $d$ | $e_t$ | $\{e, a, f\}_t$ | $\{e, a, b, c, f\}_t$ |
| $f$ | $e_t$ | $\{e, a, d\}_t$ | $\{e, a, b, c, d\}_t$ |

**Theorem 2.14.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right and $P \subset H$. Let $(P, \circ, e)$ be an $F$-polygroup such that supp$(x^{-1} \circ x \cap x \circ x^{-1}) = \{e\}$, for all $x \in P$. Then,*

(1) *supp$((H \backslash P) \circ P) = H \backslash P$,*

(2) *supp$((H \backslash P) \circ x) = H \backslash P$, for all $x \in P$,*

(3) *supp$(x \circ P \cap x \circ (H \backslash P)) = \emptyset$, for all $x \in H$,*

(4) *$|$supp$(x \circ y)| = 1$, for all $x \in H \backslash P$ and all $y \in P$.*

*Proof.* 1) It is obvious that $H \backslash P = \text{supp}((H \backslash P) \circ e) \subseteq \text{supp}((H \backslash P) \circ P)$. Conversely, let $z \in H \backslash P$ and $y \in P$ be arbitrary elements. We prove that $\text{supp}(z \circ y) \subseteq H \backslash P$ which will imply that $\text{supp}((H \backslash P) \circ P) \subseteq H \backslash P$. By way of contradiction, suppose that $\text{supp}(z \circ y) \nsubseteq H \backslash P$. Then, there exists $x \in P$ such that $x \in \text{supp}(z \circ y)$ and so we have

$$\text{supp}(x \circ y^{-1}) \subseteq \text{supp}((z \circ y) \circ y^{-1}) = \text{supp}(z \circ (y \circ y^{-1})) = \text{supp}(z \circ e) = \{z\}.$$

Hence, $z \in P$ which is a contradiction.

2) Let $x \in P$ be an arbitrary element. By using reproduction axiom we have

$$\begin{aligned} H = \text{supp}(H \circ x) &= \text{supp}((H \backslash P) \circ x \cup P \circ x) \\ &= \text{supp}((H \backslash P) \circ x) \cup \text{supp}(P \circ x) \\ &= \text{supp}((H \backslash P) \circ x) \cup P. \end{aligned}$$

Hence, we have $H \backslash P \subseteq \text{supp}((H \backslash P) \circ x)$. On the other hand, by using (1) we have $\text{supp}((H \backslash P) \circ x) \subseteq \text{supp}((H \backslash P) \circ P) = H \backslash P$.

3) By way of contradiction, suppose that $\text{supp}(x \circ P \cap x \circ (H \backslash P)) \neq \emptyset$, for some $x \in H$. Then, there exist $y \in P$ and $z \in H \backslash P$ such that $\text{supp}(x \circ y) \cap \text{supp}(x \circ z) \neq z) \neq \emptyset$. Assume that $w \in \text{supp}(x \circ y) \cap \text{supp}(x \circ z)$. Then, we have

$$\text{supp}(w \circ y^{-1}) \subseteq \text{supp}(x \circ y \circ y^{-1}) = \text{supp}(x \circ e) = \{x\}.$$

So, $\{x\} = \text{supp}(w \circ y^{-1}) \subseteq \text{supp}((x \circ z) \circ y^{-1}) = \text{supp}(x \circ (z \circ y^{-1}))$. Thus, by condition (2) of Definition 2.1 we have $e \in \text{supp}(z \circ y^{-1})$. Therefore,

$$\text{supp}(e \circ y) \subseteq \text{supp}((z \circ y^{-1}) \circ y) = \text{supp}(z \circ (y \circ y^{-1})) = \text{supp}(z \circ e) = \{z\}.$$

Consequently, we have $z \in \text{supp}(e \circ y) \subseteq P$ which is a contradiction.

4) Let $x \in H \backslash P$ and $y \in P$ be arbitrary elements. Suppose that $\{t, w\} \subseteq \text{supp}(x \circ y)$. Then, we have

$$\text{supp}(t \circ y^{-1}) \subseteq \text{supp}((x \circ y) \circ y^{-1}) = \text{supp}(x \circ (y \circ y^{-1})) = \text{supp}(x \circ e) = \{x\}.$$

Similarly, we have $\text{supp}(w \circ y^{-1}) \subseteq \{x\}$. Hence, $\text{supp}(t \circ y^{-1}) = \text{supp}(w \circ y^{-1}) = \{x\}$. This implies that

$$\{t\} = \text{supp}(t \circ e) = \text{supp}(t \circ (y^{-1} \circ y)) = \text{supp}((w \circ y^{-1}) \circ y) = \text{supp}(w \circ e) = \{w\}.$$

Therefore, we have $|\text{supp}(x \circ y)| = 1$. □

**Definition 2.15.** Let $(H, \circ)$ be an $F$-hypergroup of type $U$ on the right with at least two elements. Then, an element $x \in H$ is said to be *total* or a $T_F$-element if $\text{supp}(x \circ y) = H \backslash \{x\}$, for all $y \in H \backslash \{e\}$.

**Example 2.16.** Let $(H, \circ)$ be the $F$-hypergroup of type $U$ on the right which is defined in Example 2.6. Then, each element of $H$ is a $T_F$-element while $(\mathbb{Z}_3, \circ)$ defined in Example 2.5 has no $T_F$-element.

**Example 2.17.** Let $H = \{e, a, b, c, d\}$ and $t \in (0, 1]$. Then, $H$ with the following table is an $F$-hypergroup of type $U$ on the right. It is easy to check that $b, c, d$ are $T_F$-elements while $a$ is not a $T_F$- element.

| $\circ$ | $e$ | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|---|
| $e$ | $e_t$ | $\{a, b\}_t$ | $\{a, b\}_t$ | $\{c, d\}_t$ | $\{c, d\}_t$ |
| $a$ | $a_t$ | $\{e, b, c\}_t$ | $\{e, b, c\}_t$ | $\{e, b, c, d\}_t$ | $\{e, b, c, d\}_t$ |
| $b$ | $b_t$ | $\{e, a, c, d\}_t$ | $\{e, a, c, d\}_t$ | $\{e, a, c, d\}_t$ | $\{e, a, c, d\}_t$ |
| $c$ | $c_t$ | $\{e, a, b, d\}_t$ | $\{e, a, b, d\}_t$ | $\{e, a, b, d\}_t$ | $\{e, a, b, d\}_t$ |
| $d$ | $d_t$ | $\{e, a, b, c\}_t$ | $\{e, a, b, c\}_t$ | $\{e, a, b, c\}_t$ | $\{e, a, b, c\}_t$ |

**Proposition 2.18.** *Let $(H, \circ)$ be an $F$-hypergroup of type $U$ on the right such that $|H| \geq 2$. Let $x$ be a $T_F$-element in $H$ and $y, z \in H \backslash \{e\}$. Then, the following assertions hold:*

(1) *$e \in supp(y \circ z)$.*

(2) *If $|H| \geq 3$, then $|supp(y \circ z)| \geq 2$.*

*Proof.* 1) It follows from Lemma 2.11 (2).

2) If $x = e$, then the result follows from Lemma 2.11 (1). So, suppose that $x \in H \backslash \{e\}$. By reproduction axiom, there exists $w \in H$ such that $x \in \text{supp}(w \circ y)$. It is obvious that $w \neq x$. Moreover, we have

$$H \backslash \{x\} = \text{supp}(x \circ z) \subseteq \text{supp}((w \circ y) \circ z) = \text{supp}(w \circ (y \circ z)).$$

If $|\text{supp}(y \circ z)| = 1$, then from the previous point we obtain $\text{supp}(y \circ z) = \{e\}$. Whence, $H \backslash \{x\} = \text{supp}(w \circ e) = \{w\}$. This is absurd because $|H| \geq 3$. Consequently, we have $|\text{supp}(y \circ z)| \geq 2$. □

**Lemma 2.19.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right. Let $K \subset H$ be an $F$-subsemihypergroup such that $|K| \geq 2$. Then, $|H - K| > 1$.*

*Proof.* By way of contradiction, suppose that $H \backslash K = \{x\}$, for some $x \in H$. Since $|K| \geq 2$, there exists an element $a$ in $K \backslash \{e\}$. Now, by using reproduction axiom we have

$$\begin{aligned} x \in H = \text{supp}(H \circ a) &= \text{supp}((K \cup (H \backslash K)) \circ a) \\ &= \text{supp}(K \circ a) \cup \text{supp}((H \backslash K) \circ a) \\ &\subseteq K \cup \text{supp}(x \circ a). \end{aligned}$$

Since $x \notin K$, we have $x \in \text{supp}(x \circ a)$ which implies that $a = e$, a contradiction. □

**Lemma 2.20.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right and $K \subset H$ be an $F$-subhypergroup. If $|K| \geq 3$ and $|H \backslash K| = 2$, then supp$(x \circ y) \neq K \backslash \{x\}$, for every two distinct elements $x, y \in K$.*

*Proof.* By way of contradiction, suppose that there exist two distinct elements $x, y$ in $K$ such that $\text{supp}(x \circ y) = K \backslash \{x\}$. If $y = e$, then $\text{supp}(x \circ e) = \{x\} = K \backslash \{x\}$, which is a contradiction. Thus, $y \neq e$. By hypothesis, there exist $u, v \in H$ such that $H \backslash K = \{u, v\}$. Suppose that $x \neq e$. By reproduction axiom, there exists $w \in H$ such that $u \in \text{supp}(w \circ y)$. If $w \in K$, then we have $u \in \text{supp}(w \circ y) \subseteq K$, which is absurd. Thus, $w \notin K$. On the other hand, since $u \notin \text{supp}(u \circ y)$, we conclude that $w = v$ and therefore $u \in \text{supp}(v \circ y)$. From $x \neq e$ it follows that $v \notin \text{supp}(v \circ x)$ and so we have $\text{supp}(v \circ x) \subseteq H \backslash \{v\}$. We have

$$\begin{aligned} u \in \text{supp}(v \circ y) &\subseteq \text{supp}(v \circ (K \backslash \{x\})) = \text{supp}(v \circ (x \circ y)) = \text{supp}((v \circ x) \circ y) \\ &\subseteq \text{supp}((H \backslash \{v\}) \circ y) = \text{supp}((K \cup \{u\}) \circ y) \\ &= \text{supp}(K \circ y) \cup \text{supp}(u \circ y) = K \cup \text{supp}(u \circ y), \end{aligned}$$

which is a contradiction. Thus, $x = e$ and $\text{supp}(e \circ y) = K \backslash \{e\}$. On the other hand, since $|K| \geq 3$, there exists $z \in H$ such that $\{e, y, z\} \subseteq K$. Therefore,

$$\begin{aligned} K = \text{supp}(z \circ K) = \text{supp}(z \circ e \cup z \circ (K \backslash \{e\})) &= \text{supp}(z \circ e) \cup \text{supp}(z \circ (K \backslash \{e\})) \\ &= \{z\} \cup \text{supp}(z \circ (e \circ y)) \\ &= \{z\} \cup \text{supp}(z \circ y). \end{aligned}$$

Hence, we have $\text{supp}(z \circ y) = K \backslash \{z\}$. By the above argument we have $z = e$ which is a contradiction. □

## 3. Isomorphism of $F$-Hypergrous of Type $U$ on the Right

In this section, we begin with the definition of isomorphism of $F$-hypergroupoids. We use this notion to obtain characterizations of $F$-hypergroups of type $U$ on the right of order 2 or 3.

**Definition 3.1.** Let $(H_1, *)$ and $(H_2, \circ)$ be two $F$-hypergroupoids. A one-to-one and onto mapping $\varphi : H_1 \longrightarrow H_2$ is called an *isomorphism* if there exists a positive real number $r$ such that $(x * y)(a) = r(\varphi(x) \circ \varphi(y))(\varphi(a))$, for every $x, y, a \in H_1$. We say that $H_1$ is *isomorphic* to $H_2$, denoted by $H_1 \cong H_2$, if there exists an isomorphism from $H_1$ to $H_2$.

**Example 3.2.** Let $t \in (0, 1]$ and $H = \{e, x, y\}$ be equipped with the $F$-hyperoperation $*_t$ defined in Example 2.4. If we define the mapping $\varphi : H \longrightarrow \mathbb{S}_3/\mathbb{S}_2(t_1, t_1, t_1)$ as follows:

$$\varphi(e) = \mathbb{S}_2, \quad \varphi(x) = (1\ 3)\mathbb{S}_2 \quad \text{and} \quad \varphi(y) = (2\ 3)\mathbb{S}_2,$$

where $t_1 \in (0, 1]$ and $\mathbb{S}_3/\mathbb{S}_2(t_1, t_1, t_1)$ is the $F$-hypergroup of type $U$ on the right defined in Example 2.6, then $H \cong \mathbb{S}_3/\mathbb{S}_2(t_1, t_1, t_1)$.

**Lemma 3.3.** *Let $(H_1, *)$ and $(H_2, \circ)$ be two $F$-hypergroupoids $F$-hypergroupoids and $\varphi : H_1 \longrightarrow H_2$ be a map. Then, $\varphi$ is an isomorphism if and only if $\varphi$ satisfies the following conditions:*

(1) $\varphi(supp(x * y)) = supp(\varphi(x) \circ \varphi(y))$, *for every* $x, y \in H_1$,

(2) $(x * y)(a)(\varphi(z) \circ \varphi(w))(\varphi(b)) = (\varphi(x) \circ \varphi(y))(\varphi(a))(z * w)(b)$, *for every* $x, y, z, w, a, b \in H_1$.

*Proof.* Let $\varphi : H_1 \longrightarrow H_2$ be an isomorphism and $\varphi(a) \in \varphi(\mathrm{supp}(x * y))$ be an arbitrary element. Since $\varphi$ is one-to-one, we have $a \in \mathrm{supp}(x * y)$ and therefore $(x*y)(a) \neq 0$. Thus, $(\varphi(x)\circ\varphi(y))(\varphi(a)) \neq 0$ which implies that $\varphi(a) \in \mathrm{supp}(\varphi(x)\circ \varphi(y))$. So, we have $\varphi(\mathrm{supp}(x * y)) \subseteq \mathrm{supp}(\varphi(x) \circ \varphi(y))$. To prove the reverse inclusion, suppose that $\varphi(a) \in \mathrm{supp}(\varphi(x) \circ \varphi(y))$ be an arbitrary element. Then, $(\varphi(x) \circ \varphi(y))(\varphi(a)) \neq 0$. So, we have $(x * y)(a) \neq 0$. Consequently, we have $a \in \mathrm{supp}(x * y)$ which implies that $\varphi(a) \in \varphi(\mathrm{supp}(x * y))$. Now, to prove (2), let $x, y, z, w, a, b \in H_1$ be arbitrary elements. If $(x*y)(a) = 0$ or $(z*w)(b) = 0$, then by using (1) we have $(\varphi(x)\circ\varphi(y))(\varphi(a)) = 0$ or $(\varphi(z)\circ\varphi(w))(\varphi(b)) = 0$ and so in this case the desired result holds. So, assume that $a \in \mathrm{supp}(x*y)$ and $b \in \mathrm{supp}(z*w)$. Since $\varphi : H_1 \longrightarrow H_2$ is an isomorphism, there exists a positive real number $r$ such that $(x*y)(a) = r(\varphi(x)\circ\varphi(y))(\varphi(a))$ and $(z*w)(b) = r(\varphi(z)\circ\varphi(w))(\varphi(b))$. Hence, the desired result follows easily.

Conversely, suppose that $\varphi$ satisfies conditions (1) and (2). Let $x, y, a \in H_1$ be arbitrary elements. We choose $z, w, b \in H_1$ such that $b \in \mathrm{supp}(z * w)$. By (1), we have $(\varphi(z) \circ \varphi(w))(\varphi(b)) \neq 0$. We set $r = (z * w)(b)/(\varphi(z) \circ \varphi(w))(\varphi(b))$. By (2), we have $(x * y)(a)(\varphi(z) \circ \varphi(w))(\varphi(b)) = (\varphi(x) \circ \varphi(y))(\varphi(a))(z * w)(b)$ which implies that $(x * y)(a) = r(\varphi(x) \circ \varphi(y))(\varphi(a))$. □

**Corollary 3.4.** *Let $(H_1, *)$ and $(H_2, \circ)$ be two $F$-hypergroupoids $F$-hypergroupoids and $\varphi : H_1 \longrightarrow H_2$ be an $(x*y)(a) = (z*w)(b)$ if and only if $(\varphi(x)\circ\varphi(y))(\varphi(a)) = (\varphi(z) \circ \varphi(w))(\varphi(b))$, for every $x, y, z, w, a, b \in H_1$.*

Next example shows that the converse of Corollary 3.4 does not hold in general.

**Example 3.5.** We equip the sets $H_1 = \{0,1\}$ and $H_2 = \{e,a\}$ with the $F$-hyperoperations $*$ and $\circ$ which are defined in the following tables:

| $*$ | $0$ | $1$ |
|---|---|---|
| $0$ | $\frac{0}{0.4}, \frac{1}{0}$ | $\frac{0}{0}, \frac{1}{0.3}$ |
| $1$ | $\frac{0}{0}, \frac{1}{0.5}$ | $\frac{0}{0.6}, \frac{1}{0}$ |

| $\circ$ | $e$ | $a$ |
|---|---|---|
| $e$ | $\frac{e}{0.8}, \frac{a}{0}$ | $\frac{e}{0}, \frac{a}{0.1}$ |
| $a$ | $\frac{e}{0}, \frac{a}{0.9}$ | $\frac{e}{0.3}, \frac{a}{0}$ |

We define $\varphi : H_1 \longrightarrow H_2$ by $\varphi(0) = e$ and $\varphi(1) = a$. We can see easily that for each $x, y, z, w, a, b \in H_1$ we have

$$(x * y)(a) = (z * w)(b) \text{ if and only if } (\varphi(x) \circ \varphi(y))(\varphi(a)) = (\varphi(z) \circ \varphi(w))(\varphi(b)).$$

But $\varphi$ is not an isomorphism because we have $(0 * 0)(0) = 0.5(\varphi(0) \circ \varphi(0))(\varphi(0))$ and $(1 * 1)(0) = 2(\varphi(1) \circ \varphi(1))(\varphi(0))$.

**Lemma 3.6.** *Let $(H_1, *)$, $(H_2, \circ)$ and $(H_3, \bullet)$ be $F$-hypergroupoids such that $H_1 \cong H_2$ and $H_2 \cong H_3$. Then, $H_1 \cong H_3$.*

*Proof.* It is straightforward. □

**Lemma 3.7.** *Let $(H_1, *)$ and $(H_2, \circ)$ be two $F$-hypergroupoids and $\varphi : H_1 \longrightarrow H_2$ be an isomorphism. Then,*

*(1) $e$ is a right scalar element of $(H_1, *)$ if and only if $\varphi(e)$ is a right scalar element of $(H_2, \circ)$.*

*(2) $\varphi^{-1} : H_2 \longrightarrow H_1$ is an isomorphism.*

*Proof.* 1) Let $e$ be a right scalar element of $(H_1, *)$ and $y \in H_2$ be an arbitrary element. Since $\varphi$ is onto, there exists $x \in H_1$ such that $\varphi(x) = y$. So, by using Lemma 3.3 we have

$$\text{supp}(y \circ \varphi(e)) = \text{supp}(\varphi(x) \circ \varphi(e)) = \varphi(\text{supp}(x * e)) = \{\varphi(x)\} = \{y\}.$$

Therefore, $\varphi(e)$ is a right scalar element of $(H_2, \circ)$. Conversely, let $\varphi(e)$ be a right scalar element of $(H_2, \circ)$. Then, for each element $x$ of $H_1$ we have

$$\varphi(\text{supp}(x * e)) = \text{supp}(\varphi(x) \circ \varphi(e)) = \{\varphi(x)\}.$$

Since $\varphi$ is one-to-one, we have $\text{supp}(x * e) = \{x\}$ and therefore $e$ is a right scalar element of $(H_1, *)$.

2) It is straightforward. □

**Lemma 3.8.** *Let $(H_1, *)$ and $(H_2, \circ)$ be two $F$-semihypergroups and $\varphi : H_1 \longrightarrow H_2$ be an isomorphism. Then, the following assertions are equivalent:*

(1) $(H_1, *, e)$ *is an $F$-hypergroup of type $U$ on the right.*

(2) $(H_2, \circ, \varphi(e))$ *is an $F$-hypergroup of type $U$ on the right.*

*Proof.* 1⇒2) Let $y \in H_2$ be an arbitrary element. Since $\varphi$ is onto, there exists $x \in H_1$ such that $\varphi(x) = y$. By assumption we have $\mathrm{supp}(x*H_1) = \mathrm{supp}(H_1*x) = H_1$. Hence,

$$\mathrm{supp}(y \circ H_2) = \mathrm{supp}(\varphi(x) \circ \varphi(H_1)) = \varphi(\mathrm{supp}(x * H_1)) = \varphi(H_1) = H_2.$$

Similarly, we have $\mathrm{supp}(H_2 \circ y) = H_2$. Therefore, $(H_2, \circ)$ is an $F$-hypergroup. By Lemma 3.7, $\varphi(e)$ is a right scalar element of $(H_2, \circ)$. Thus, condition (1) of Definition 2.1 holds. It is sufficient to show that condition (2) of Definition 2.1 is true. Let $y, z \in H_2$ be arbitrary elements and $y \in \mathrm{supp}(y \circ z)$. Since $\varphi$ is onto, there exist $x, t \in H_1$ such that $\varphi(x) = y$ and $\varphi(t) = z$. Thus, by using Lemma 3.3 we have $\varphi(x) \in \mathrm{supp}(\varphi(x) \circ \varphi(t)) = \varphi(\mathrm{supp}(x * t))$. So, there exists $w \in \mathrm{supp}(x*t)$ such that $\varphi(x) = \varphi(w)$. Since $\varphi$ is one-to-one, we have $x = w$ and therefore $x \in \mathrm{supp}(x * t)$. Since $(H_1, *)$ is of type $U$ on the right we have $t = e$ which implies that $z = \varphi(t) = \varphi(e)$.

2⇒1) Let $(H_2, \circ, \varphi(e))$ be an $F$-hypergroup of type $U$ on the right. By Lemma 3.7 (2), $\varphi^{-1} : H_2 \longrightarrow H_1$ is an isomorphism and therefore by the above argument $(H_1, *, e)$ is an $F$-hypergroup of type $U$ on the right. □

**Theorem 3.9.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right with $|H| = 2$. Then, $H \cong \mathbb{Z}_2(t_1, t_2)$, for some $t_1, t_2 \in (0, 1]$. (See Example 2.5.)*

*Proof.* Let $H = \{e, x\}$. By conditions (1) and (2) of Definition 2.1, $\circ$ has the following table:

| $\circ$ | $e$ | $x$ |
|---|---|---|
| $e$ | $\frac{e}{t_1}, \frac{x}{0}$ | $\frac{e}{0}, \frac{x}{t_2}$ |
| $x$ | $\frac{e}{0}, \frac{x}{t_3}$ | $\frac{e}{t_4}, \frac{x}{0}$ |

where $t_1, t_2, t_3, t_4 \in (0, 1]$. Since $(H, \circ)$ is an $F$-hypergroup of type $U$ on the right, we have

$$t_2 = (e \circ x)(x) = ((x \circ x) \circ x)(x) = (x \circ (x \circ x))(x) = (x \circ e)(x) = t_3$$

and

$$t_4 = (x \circ x)(e) = ((e \circ x) \circ x)(e) = (e \circ (x \circ x))(e) = (e \circ e)(e) = t_1.$$

We define the mapping $\varphi : H \longrightarrow \mathbb{Z}_2$ by $\varphi(e) = 0$ and $\varphi(x) = 1$. Obviously, $\varphi$ is an isomorphism and therefore the desired result holds. □

**Theorem 3.10.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right with $|H| = 3$. Then, either $H \cong \mathbb{Z}_3(t_1, t_2, t_3)$ or $H \cong \mathbb{S}_3/\mathbb{S}_2(t_1, t_2, t_3)$, for some $t_1, t_2, t_3 \in (0, 1]$. (See Examples 2.5 and 2.6.)*

*Proof.* Let $H = \{e, x, y\}$. By condition (2) of Definition 2.1 we have

$$\mathrm{supp}(e \circ x) \cup \mathrm{supp}(e \circ y) \subseteq \{x, y\}.$$

We claim that $|\mathrm{supp}(e \circ x)| = |\mathrm{supp}(e \circ y)|$. If this is not the case, then without loss of generality we can assume that $|\mathrm{supp}(e \circ x)| = 1$ and $|\mathrm{supp}(e \circ y)| = 2$. Thus, we have $\mathrm{supp}(e \circ y) = \{x, y\}$ and by Lemma 2.9 we have $\mathrm{supp}(e \circ x) = \{x\}$. By reproduction axiom, there exists $z \in H$ such that $y \in \mathrm{supp}(x \circ z)$. Hence,

$$x \in \mathrm{supp}(e \circ y) \subseteq \mathrm{supp}(e \circ (x \circ z)) = \mathrm{supp}((e \circ x) \circ z) = \mathrm{supp}(x \circ z),$$

which implies that $z = e$ and so we have $y \in \mathrm{supp}(x \circ e) = \{x\}$, a contradiction. Thus, we have the following two cases.

Case 1: Let $\mathrm{supp}(e \circ x) = \mathrm{supp}(e \circ y) = \{x, y\}$. Then, we have

$$\begin{aligned}\mathrm{supp}(x \circ y) = \mathrm{supp}((x \circ e) \circ y) = \mathrm{supp}(x \circ (e \circ y)) &= \mathrm{supp}(x \circ (e \circ x)) \\ &= \mathrm{supp}(x \circ x).\end{aligned}$$

Since $x \notin \mathrm{supp}(x \circ y)$ we have $\mathrm{supp}(x \circ y) \subseteq \{e, y\}$. In the case that $\mathrm{supp}(x \circ y) = \{e\}$, we have

$$\{e, x\} = \mathrm{supp}(x \circ x) \cup \mathrm{supp}(x \circ y) \cup \mathrm{supp}(x \circ e) = \mathrm{supp}(x \circ H) = H,$$

which is a contradiction. In the case that $\mathrm{supp}(x \circ y) = \{y\}$, we have

$$\{y\} = \mathrm{supp}(x \circ y) = \mathrm{supp}(x \circ (x \circ y)) = \mathrm{supp}((x \circ x) \circ y) = \mathrm{supp}(y \circ y).$$

This implies that $y = e$ which is a contradiction. Thus, $\mathrm{supp}(x \circ y) = \{e, y\}$. In a similar manner we can show that $\mathrm{supp}(y \circ x) = \mathrm{supp}(y \circ y) = \{e, x\}$. So, $\circ$ has the following table in which $t_i$'s are in $(0, 1]$.

| $\circ$ | $e$ | $x$ | $y$ |
|---|---|---|---|
| $e$ | $\frac{e}{t_1}, \frac{x}{0}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{t_2}, \frac{y}{t_3}$ | $\frac{e}{0}, \frac{x}{t_4}, \frac{y}{t_5}$ |
| $x$ | $\frac{e}{0}, \frac{x}{t_6}, \frac{y}{0}$ | $\frac{e}{t_7}, \frac{x}{0}, \frac{y}{t_8}$ | $\frac{e}{t_9}, \frac{x}{0}, \frac{y}{t_{10}}$ |
| $y$ | $\frac{e}{0}, \frac{x}{0}, \frac{y}{t_{11}}$ | $\frac{e}{t_{12}}, \frac{x}{t_{13}}, \frac{y}{0}$ | $\frac{e}{t_{14}}, \frac{x}{t_{15}}, \frac{y}{0}$ |

We have $t_{11} = \bigvee \{(e \circ e)(y), (y \circ e)(y)\} = ((x \circ y) \circ e)(y) = (x \circ y)(y) = t_{10}$. In a can show that $t_1 = t_7 = t_9 = t_{12} = t_{14}$, $t_2 = t_4 = t_6 = t_{13} = t_{15}$ and $t_3 = t_5 = t_8 = t_{11}$. Therefore, in this case we have $H \cong \mathbb{S}_3/\mathbb{S}_2(t_1, t_2, t_3)$.

Case 2: Let $|\mathrm{supp}(e \circ x)| = |\mathrm{supp}(e \circ y)| = 1$. By Lemma 2.9, we have $\mathrm{supp}(e \circ x) = \{x\}$ and $\mathrm{supp}(e \circ y) = \{y\}$. We claim that $\mathrm{supp}(x \circ x) = \{y\}$. If $\mathrm{supp}(x \circ x) = \{e\}$, then as $\mathrm{supp}(e \circ x) \cup \mathrm{supp}(x \circ x) \cup \mathrm{supp}(y \circ x) = \mathrm{supp}(H \circ x) = H$, we have $y \in \mathrm{supp}(y \circ x)$ which implies that $x = e$, a contradiction. If $\mathrm{supp}(x \circ x) = \{e, y\}$, then from the following equalities we conclude that $y \notin \mathrm{supp}(x \circ y)$.

$$\begin{aligned} \{x\} \cup \mathrm{supp}(y \circ x) = \mathrm{supp}(e \circ x) \cup \mathrm{supp}(y \circ x) &= \mathrm{supp}((x \circ x) \circ x) \\ &= \mathrm{supp}(x \circ (x \circ x)) \\ &= \mathrm{supp}(x \circ e) \cup \mathrm{supp}(x \circ y). \end{aligned}$$

On the other hand, by condition (2) of Definition 2.1 we have $x \notin \mathrm{supp}(x \circ y)$. Thus, $\mathrm{supp}(x \circ y) = \{e\}$. Therefore,

$$y \in \mathrm{supp}(e \circ y) \cup \mathrm{supp}(y \circ y) = \mathrm{supp}((x \circ x) \circ y) = \mathrm{supp}(x \circ (x \circ y)) = \{x\},$$

that is a contradiction. Hence, $\mathrm{supp}(x \circ x) = \{y\}$. In a similar manner we have $\mathrm{supp}(y \circ y) = \{x\}$. From $\mathrm{supp}(x \circ y) = \mathrm{supp}(x \circ (x \circ x)) = \mathrm{supp}((x \circ x) \circ x) = \mathrm{supp}(y \circ x)$, $x \notin \mathrm{supp}(x \circ y)$ and $y \notin \mathrm{supp}(y \circ x)$ we conclude that $\{x, y\} \nsubseteq \mathrm{supp}(x \circ y)$ and therefore $\mathrm{supp}(x \circ y) = \{e\}$. So, $\circ$ has the following table in which $t_i$'s are in $(0, 1]$.

| $\circ$ | $e$ | $x$ | $y$ |
|---|---|---|---|
| $e$ | $\frac{e}{t_1}, \frac{x}{0}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{t_2}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{0}, \frac{y}{t_3}$ |
| $x$ | $\frac{e}{0}, \frac{x}{t_4}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{0}, \frac{y}{t_5}$ | $\frac{e}{t_6}, \frac{x}{0}, \frac{y}{0}$ |
| $y$ | $\frac{e}{0}, \frac{x}{0}, \frac{y}{t_7}$ | $\frac{e}{t_8}, \frac{x}{0}, \frac{y}{0}$ | $\frac{e}{0}, \frac{x}{t_9}, \frac{y}{0}$ |

We have $t_1 = (e \circ e)(e) = ((x \circ y) \circ e)(e) = (x \circ (y \circ e))(e) = (x \circ y)(e) = t_6$. In a similar way we can show that $t_1 = t_8$, $t_2 = t_4 = t_9$ and $t_3 = t_5 = t_7$. So, in this case we have $H \cong \mathbb{Z}_3(t_1, t_2, t_3)$. □

Notice that, by Theorem 3.9, there are 2 different $F$-hypergroups of type $U$ on the right of order two up to an isomorphism. One of them is $\mathbb{Z}_2(t_1, t_2)$ with $t_1 = t_2$ and the other one is $\mathbb{Z}_2(t_1, t_2)$ with $t_1 \neq t_2$. Also, by Theorem 3.10, there are 10 different $F$-hypergroups of type $U$ on the right of order three up to an isomorphism depending on whether some $t_i$'s are equal or not.

**Theorem 3.11.** *Let $(H, \circ, e)$ be an $F$-semihypergroup of type $U$ on the right. Assume that $K \subsetneq H$ is an $F$-subsemihypergroup of $H$ isomorphic to $(K, *_t)$, for some $t \in (0, 1]$ (see Example 2.4) and let $e \in K$. Also, let $x \in H \backslash K$ and $a, b \in K \backslash \{e\}$ be arbitrary elements. Then, the following assertions hold:*

(1) *$supp(x \circ a) = supp(x \circ b)$,*

(2) *$supp(x \circ a) \cap K \neq \emptyset$ $K \subseteq supp(x \circ a)$,*

(3) $supp(x \circ a) \neq K$,

(4) $supp(x \circ a) \cap (H \backslash a) \cap (H \backslash K) \neq \emptyset$.

(5) $|supp(x \circ a)| > 1$.

*Proof.* 1) Let $\varphi : (K, *_t) \longrightarrow (K, \circ)$ be an isomorphism. Then, by Lemma 3.8, $(K, \circ)$ is an $F$-hypergroup of type $U$ on the right. By Lemma 3.7, we have $\varphi(e) = e$. Since $\varphi$ is an onto mapping, there exists $c, d \in K$ such that $\varphi(c) = a$ and $\varphi(d) = b$. Therefore we have

$$\text{supp}(e \circ a) = \text{supp}(\varphi(e) \circ \varphi(c)) = \varphi(\text{supp}(e *_t c)) = \varphi(K \backslash \{e\}) = K \backslash \{e\}.$$

Similarly, we have $\text{supp}(e \circ b) = K \backslash \{e\}$. So,

$$\begin{aligned} \text{supp}(x \circ a) &= \text{supp}((x \circ e) \circ a) \\ &= \text{supp}(x \circ (e \circ a)) \\ &= \text{supp}(x \circ (e \circ b)) \\ &= \text{supp}((x \circ e) \circ b) \\ &= \text{supp}(x \circ b). \end{aligned}$$

2) Let $c \in \text{supp}(x \circ a) \cap K$. Since $K$ is an $F$-hypergroup, we have

$$\begin{aligned} K = \text{supp}(c \circ K) &\subseteq \text{supp}((x \circ a) \circ K) \\ &= \text{supp}(x \circ (a \circ K)) \\ &= \text{supp}(x \circ K) \\ &= \text{supp}(x \circ a) \cup \{x\}. \end{aligned}$$

This implies that $K \subseteq \text{supp}(x \circ a)$.

3) By way of contradiction, suppose that $\text{supp}(x \circ a) = K$. By assumption we have

$$\text{supp}(a \circ b) = \text{supp}(\varphi(c) \circ \varphi(d)) = \varphi(\text{supp}(c *_t d)) = \varphi(K \backslash \{c\}) = K - \{a\}.$$

Therefore,

$$\begin{aligned} K = \text{supp}(K \circ b) &= \text{supp}((x \circ a) \circ b) \\ &= \text{supp}(x \circ (a \circ b)) \\ &= \text{supp}(x \circ (K - \{a\}) \\ &= \{x\} \cup \text{supp}(x \circ b) \\ &= \{x\} \cup \text{supp}(x \circ a). \end{aligned}$$

This implies that $x \in K$ which is a contradiction.

4) By way of contradiction, suppose that $\text{supp}(x \circ a) \cap (H \backslash K) = \emptyset$. Then, we have $\text{supp}(x \circ a) \subseteq K$ and so by (2) we have $K \subseteq \text{supp}(x \circ a)$ which implies that $K = \text{supp}(x \circ a)$. By (3), this is a contradiction.

5) By way of contradiction, suppose that $|\text{supp}(x \circ a)| = 1$. If $\text{supp}(x \circ a) = \{e\}$, then by (2) we have $K \subseteq \text{supp}(x \circ a) = \{e\}$ which is absurd. If $\text{supp}(x \circ a) = \{y\} \neq \{e\}$, then we have

$$\begin{aligned}\text{supp}(y \circ b) &= \text{supp}((x \circ a) \circ b) \\ &= \text{supp}(x \circ (K - \{a\}) \\ &= \{x\} \cup \text{supp}(x \circ b) \\ &= \{x\} \cup \text{supp}(x \circ a) \\ &= \{x, y\}.\end{aligned}$$

This implies that $y \in \text{supp}(y \circ b)$. So, by condition (2) of Definition 2.1 we have $b = e$ which is a contradiction. □

# 4. Cyclic $F$-Semihypergroups

What will happen in this section, is a fuzzy version of some parts of [14]. Let $(H, \circ)$ be an $F$-semihypergroup. Then, the intersection $\bigcap_{i \in \Lambda} S_i$ of a family $\{S_i\}_{i \in \Lambda}$ of $F$-subsemihypergroups of $H$ (if it is non-empty) is an $F$-subsemihypergroup. For every non-empty subset $A$ of $H$, there exists at least an $F$-subsemihypergroup of $H$ containing $A$ ($H$ itself). Hence, the intersection of all $F$-subsemihypergroups of $H$ containing $A$ is an $F$-subsemihypergroup. We denote it by $\check{A}$. It is easy to see that

(1) $A \subseteq \check{A}$;

(2) $\check{A} \subseteq S$, where $S$ is an $F$-subsemihypergroup $H$ containing $A$.

Furthermore, one easily checks that $\check{A} = A \cup \big( \bigcup\limits_{k \geq 2} \text{supp}(x_1 \circ \ldots \circ x_k) \big)$, where $x_i$'s are in $A$. In particular, if $A$ is a singleton set, say $\{x\}$, then $\check{A}$ will be denoted by $\check{x}$ and $\check{x} = \{x\} \cup \big( \bigcup\limits_{k \geq 2} \text{supp}(x^k) \big)$ in which $x^k$ means $\underbrace{x \circ \ldots \circ x}_{k \text{ times}}$. If $|H| = n$, then we have $\check{x} = \bigcup\limits_{k=1}^{n} \text{supp}(x^k)$. It is obvious that $x \in \check{y} \Leftrightarrow \check{x} \subseteq \check{y}$, for every $x, y \in H$.

**Definition 4.1.** Let $H$ be an $F$-semihypergroup. Then, $H$ is called *cyclic* if there exists an element $x \in H$ such that $H = \check{x}$.

**Example 4.2.** Let $G$ be the Klein 4-group. If we equip $G$ with the $F$-hyperoperation $\circ$ defined in Example 2.7, then $(G, \circ)$ is not cyclic while $(H, \circ)$ defined in Example 2.4 is a cyclic $F$-hypergroup of type $U$ on the right.

**Theorem 4.3.** *Let $(H, \circ)$ be a finite $F$-semihypergroup of type $U$ on the right. If there exists an element $x$ in $H$ such that $\check{x} \neq H$ and $\check{x}$ is isomorphic to $\mathbb{S}_3/\mathbb{S}_2(t,t,t)$, for some $t \in (0,1]$, then $|H| \geq 6$.*

*Proof.* Since $\check{x}$ is isomorphic to $\mathbb{S}_3/\mathbb{S}_2(t,t,t)$, for some $t \in (0,1]$, we have $|\check{x}| = 3$ (see Example 2.6). Set $\check{x} = \{e, x, y\}$. By Lemma 3.6, $(\check{x}, \circ)$ is isomorphic to $(\check{x}, *_t)$ (see Example 3.2). From $\check{x} \neq H$ it follows that $|H| \geq 4$. If $|H| = 4$, then we can assume that $H \backslash \check{x} = \{z\}$ and so by Theorem 3.11 (4) , we have $z \in \mathrm{supp}(z \circ x)$. This implies $x = e$ which is a contradiction. If $|H| = 5$, then we set $H = \{e, x, y, w, z\}$. So, we have $H \backslash \check{x} = \{w, z\}$. Since $y \in \check{x}$ we have $\check{y} \subseteq \check{x}$. On the other hand, by assumption we have $x \in \check{y}$ which implies that $\check{x} \subseteq \check{y}$. Thus $\check{x} = \check{y}$. Since $w \notin \mathrm{supp}(w \circ x)$ and $z \notin \mathrm{supp}(z \circ x)$, $z \notin \mathrm{supp}(z \circ x)$, by using (2), (4) and (5) of Theorem 3.11 , we have $\mathrm{supp}(w \circ x) = \check{x} \cup \{z\}$ and $\mathrm{supp}(z \circ x) = \check{x} \cup \{w\}$. Similarly, we have $\mathrm{supp}(w \circ y) = \check{y} \cup \{z\}$ and

$$\begin{aligned}
H = \{w\} \cup \mathrm{supp}(w \circ x) &= \mathrm{supp}(w \circ e) \cup \mathrm{supp}(w \circ x) \\
&= \mathrm{supp}(w \circ \{e, x\}) \\
&= \mathrm{supp}(w \circ (y \circ y)) \\
&= \mathrm{supp}((w \circ y) \circ y) \\
&= \bigcup_{t \in \mathrm{supp}(w \circ y)} \mathrm{supp}(t \circ y) \\
&= \mathrm{supp}(\check{y} \circ y) \cup \mathrm{supp}(z \circ y) \\
&= \check{y} \cup \{w\} = H \backslash \{z\}.
\end{aligned}$$

This contradiction completes the proof. □

# 5. Regular Relations over $F$-Hypergroups

In this section, inspired by [9], after defining the notion of regular $F$-hypergroups containing a right identity element, we define right reversible $F$-hypergroups. Then, by using regular relations on an $F$-hypergroup we construct right reversible quotient $F$-hypergroups.

**Definition 5.1.** Let $(H, \circ, e)$ be an $F$-hypergroup (not necessarily an $F$-hypergroup of type $U$ on the right) where $e$ is a right identity element. Let $x, y \in H$. Then, $y$ is called an *inverse* of $x$ if

$$e \in \mathrm{supp}(x \circ y \cap y \circ x).$$

The set of all inverses of $x$ will be denoted by $x^{-1}$. $(H, \circ, e)$ is called *regular* if $e$ is an identity element and $x^{-1} \neq \emptyset$, for every $x \in H$. A regular $F$-hypergroup $(H, \circ, e)$ is said to be *right reversible* if for every $x, y, z \in H$ with $x \in \mathrm{supp}(y \circ z)$, there exists $t \in z^{-1}$ such that $y \in \mathrm{supp}(x \circ t)$.

**Theorem 5.2.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right. Then, the following assertions are equivalent:*

(1) *$e$ is a left identity element,*

(2) *$(H, \circ, e)$ is right reversible.*

*Proof.* $1 \Rightarrow 2$) Let $x$ be an arbitrary element of $H$. By reproduction axiom, there exists $y$ in $H$ such that $e \in \text{supp}(x \circ y)$. By Lemma 2.8 (4), we $e \in \text{supp}(y \circ x)$. This implies that $(H, \circ, e)$ is regular. Now, let $x, y, z$ be arbitrary elements of $H$ such that $x \in \text{supp}(y \circ z)$. By reproduction axiom, there exists an element $t \in H$ such that $y \in \text{supp}(x \circ t)$. By t)$.ByLemma2.8(5), wehavet \circ z)$. This means that $t \in z^{-1}$ and so there is nothing to prove.

$2 \Longrightarrow 1$) It is trivial. □

**Lemma 5.3.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right such that $P_e = \{supp(e \circ x) \mid x \in H\}$ is a partition of $H$. Then, $e$ is a left identity element.*

*Proof.* Let $x \in H$ be an arbitrary element. By reproduction axiom, there exists $y \in H$ such that $x \in \text{supp}(e \circ y)$ and so we have

$$\text{supp}(e \circ x) \subseteq \text{supp}(e \circ (e \circ y)) = \text{supp}(e \circ y).$$

Since $P_e$ is a partition of $H$, we deduce that $\text{supp}(e \circ x) = \text{supp}(e \circ y)$ and therefore $x \in \text{supp}(e \circ x)$. □

In what follows, we denote by $m_e$ the maximum size of the elements of $P_e$

**Proposition 5.4.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right. Then, the following assertions hold:*

(1) *$m_e = 1$ if and only if $e$ is a left scalar identity element.*

(2) *If $2 \leq m_e < \infty$, then there exist two distinct elements $x, y \in H \setminus \{e\}$ such that $supp(e \circ x) = supp(e \circ y)$ and $|supp(e \circ x)| = |supp(e \circ y)| = m_e$.*

*Proof.* 1) By using Lemma 2.9, the proof is trivial.

2) Let $m_e \geq 2$. Then, there exists $x \in H \setminus \{e\}$ such that $|\text{supp}(e \circ x)| = m_e$. By Lemma 2.10, there exists $y \in H \setminus \{x\}$ such that $x \in \text{supp}(e \circ y)$. Consequently, $\text{supp}(e \circ x) \subseteq \text{supp}(e \circ y)$ and $m_e = |\text{supp}(e \circ x)| \leq |\text{supp}(e \circ y)|$. Since $m_e$ is maximal, we obtain $|\text{supp}(e \circ x)| = |\text{supp}(e \circ y)|$ and therefore $\text{supp}(e \circ x) = \text{supp}(e \circ y)$. □

Let $R$ be a relation on a non-empty set $X$ and $A, B \subseteq X$. Then, $A\overline{R}B$ means that for each $a$ in there exists $b \in B$ such that $aRb$ and for each $b$ in $B$, there exists $a \in A$ such that $bRa$. For an equivalence relation $R$ on $X$, we may use $R(x)$ to denote the equivalence class of $x \in X$. We let $H/R$ denote the family $\{R(x) \mid x \in X\}$ of classes of $R$.

Let $(H, \circ)$ be an $F$-hypergroup of type $U$ on the right. An equivalence relation $R$ on $H$ is called regular if

$$xR\,y \Longrightarrow \text{supp}(z \circ x)\overline{R}\,\text{supp}(z \circ y) \text{ and } \text{supp}(x \circ z)\overline{R}\,\text{supp}(y \circ z),$$

for every $x, y, z \in H$.

**Theorem 5.5.** *Let $(H, \circ, e)$ be an $F$-hypergroup of type $U$ on the right such that $P_e$ is a partition of $H$. Then, the following assertions hold:*

(1) *The relation $R \subseteq H^2$ defined as follows is a regular relation.*

$$xR\,y \Longleftrightarrow supp(e \circ x) = supp(e \circ y).$$

(2) *The set $H/R$ endowed with the following $F$-hyperoperation is right reversible.*

$$R(x) \odot R(y) = \chi_{\{R(t) \ \mid \ t \in supp(x \circ y)\}}.$$

(3) *Let $R(y) = R(u)$ and $R(z) = R(v)$, for some $y, z, u, v \in H$. Then,*

  (a) *the following statements are equivalent, for some $x \in H$:*

    (i) $R(x) \in supp(R(y) \odot R(z))$.

    (ii) $R(x) \cap supp(u \circ v) \neq \emptyset$.

    (iii) $R(x) \subseteq supp(e \circ u \circ v)$.

  (b) *if $|R(x)| = 1$, for some $x \in H$, then*

$$R(x) \in supp(R(y) \odot R(z)) \iff x \in supp(u \circ v).$$

*Proof.* 1) Obviously, $R$ is an equivalence relation. Let $xR\,y$ and $a \in H$. We show that supp$(x \circ \overline{R}$ supp$(y \circ a)$. Assume that $z$ is an an arbitrary element of supp$(x \circ a)$. By Lemma 5.3 we have $z \in \text{supp}(e \circ z)$. Since

$$\begin{aligned} \text{supp}(e \circ z) &\subseteq \text{supp}(e \circ (x \circ a)) = \text{supp}((e \circ x) \circ a) \\ &= \text{supp}((e \circ y) \circ a) = \text{supp}(e \circ (y \circ a)), \end{aligned}$$

there exists $w \in \text{supp}(y \circ a)$ such that $z \in \text{supp}(e \circ w)$. Since $P_e$ is a partition of $H$, we have $\text{supp}(e \circ z) = \text{supp}(e \circ w)$. Hence, $zR\,w$. In a similar manner, we can show that for each $z \in \text{supp}(y \circ a)$ there exists $w \in \text{supp}(x \circ a)$ such that $zR\,w$. Thus, $\text{supp}(x \circ a)\overline{R}$ $\text{supp}(y \circ a)$. On the other hand, we have

$$\begin{aligned} \text{supp}(a \circ x) = \text{supp}((a \circ e) \circ x) &= \text{supp}(a \circ (e \circ x)) = \text{supp}(a \circ (e \circ y)) \\ &= \text{supp}((a \circ e) \circ y) = \text{supp}(a \circ y). \end{aligned}$$

Hence, $\text{supp}(a \circ x)\overline{R}\,\text{supp}(a \circ y)$ and the desired result follows.

2) By Theorem 3.1 of [12], $(H/R, \odot)$ is an $F$-hypergroup. By Lemma 5.3, $e$ is a left identity element, so we have $x \in \text{supp}(x \circ \circ x)$, for every $x \in H$. Thus, for each $R(x) \in H/R$, we have

$$R(x) \in \text{supp}(R(x) \odot R(e)) \cap \text{supp}(R(e) \odot R(x)).$$

Hence, $R(e)$ is an identity element of $(H/R, \odot)$. Let $R(x)$ be an arbitrary element of $H/R$. By reproduction axiom, there exists $y \in H$ such that $e \in \text{supp}(x \circ y)$ and so by Lemma 2.8 (4) we have $e \in \text{supp}(y \circ x)$. This implies that

$$R(e) \in \text{supp}(R(x) \odot R(y)) \cap \text{supp}(R(y) \odot R(y)) \cap \text{supp}(R(y) \odot R(x)).$$

Hence, $R(y) \in (R(x))^{-1}$. So, $(H/R, \odot)$ is regular. Finally, let $R(x), R(y), R(z)$ be arbitrary elements of $H/R$ such that $R(x) \in \text{supp}(R(y) \odot R(z))$. Then, there exists $t \in \text{supp}(y \circ z)$ such that $R(x) = R(t)$. By using Lemma 5.3 and Theorem 5.2, $(H, \circ, e)$ is right reversible. So, there exists $w \in z^{-1}$ such that $y \in \text{supp}(t \circ w)$. Therefore,

$$R(y) \in supp(R(t) \odot R(w)) = supp(R(x) \odot R(w)).$$

Clearly, $R(w) \in (R(z))^{-1}$. This proves that $(H/R, \odot)$ is right reversible.

3) First, we prove (a).

*i* ⇒ *ii*) Let $R(x) \in \text{supp}(R(y) \odot R(z))$. As $R(y) = R(u)$ and $R(z) = R(v)$, we have $R(x) \in \text{supp}(R(u) \odot R(v))$. Thus, there exists $a \in \text{supp}(u \circ v)$ such that $R(x) = R(a)$. So, $a \in R(x) \cap \text{supp}(u \circ v)$.

*ii* ⇒ *iii*) We claim that $R(x) = \text{supp}(e \circ x)$. For each $w \in R(x)$, by Lemma 5.3, we have $w \in \text{supp}(e \circ w) = \text{supp}(e \circ x)$. So, $R(x) \subseteq \text{supp}(e \circ x)$. Conversely, for each $w \in \text{supp}(e \circ x)$ we have $\text{supp}(e \circ w) \subseteq \text{supp}(e \circ (e \circ x)) = \text{supp}(e \circ x)$. Since $P_e$ is a partition of $H$, we have $\text{supp}(e \circ w) = \text{supp}(e \circ x)$ that is $w \in R(x)$. So, $\text{supp}(e \circ x) \subseteq R(x)$. Therefore, $R(x) = \text{supp}(e \circ x)$. Now, let $a \in R(x) \cap \text{supp}(u \circ v)$. Then,

$$R(x) = R(a) = \text{supp}(e \circ a) \subseteq \text{supp}(e \circ u \circ v).$$

*iii* ⇒ *i*) According to hypothesis, we have $x \in \text{supp}(e \circ u \circ v)$. Thus, there exists $a \in \text{supp}(u \circ v)$ such that $x \in \text{supp}(e \circ \circ a)$. Now, from $R(a) = \text{supp}(e \circ a)$ it $R(x) = R(a)$. On the other hand, $R(a) \in \text{supp}(R(u) \odot R(v))$ which implies that $R(x) \in \text{supp}(R(y) \odot R(z))$.

The proof of (b) is trivial. □

## 6. Conclusion

By an $F$-hypergroup of type $U$ on the right, we mean an $F$-hypergroup $(H, \circ)$ which has a right scalar identity element $e$ such that for all $x, y \in H$, from $x \in \text{supp}(x \circ y)$ it follows that $y = e$. In the resent paper, we classified $F$-hypergroups of type $U$ on the right of order 2 or 3 up to an isomorphism. An interested reader can think about classifying $F$-hypergroups of higher orders and think about ternary $F$-hypergroups of type $U$.

**Conflicts of Interest.** The authors declare that there are no conflicts of interest regarding the publication of this article.

# References

[1] P. Corsini, *Prolegomena of Hypergroup Theory*, Aviani editor, Second edition, 1993.

[2] P. Corsini and V. Leoreanu, *Applications of Hyperstructure Theory*, Advances in Mathematics, Kluwer Academic Publishers, Dordrecht, 2003.

[3] B. Davvaz, Construction of $n$-ary $F$-polygroups, *Inform. Sci.* **275** (2014) 199–212.

[4] B. Davvaz, *Polygroup Theory and Related Systems*, World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2013.

[5] B. Davvaz, F. Bardestani, Hypergroups of type $U$ on the right of size six, *Arab. J. Sci. Eng.* **36** (2011) 487 – 499.

[6] M. De Salvo, D. Fasino, D. Freni and G. Lo Faro, Hypergroups with a strongly unilateral identity, *J. Mult.-Valued Logic Soft Comput.* **21** (1-2) (2013) 165 – 182.

[7] M. De Salvo, D. Fasino, D. Freni and G. Lo Faro, Isomorphism classes of the hypergroups of type $U$ on the right of size five, *Comput. Math. Appl.* **58** (2) (2009) 390 – 402.

[8] M. De Salvo, D. Fasino, D. Freni and G. Lo Faro, On strongly conjugable extensions of hypergroups with scalar identity, *Filomat* **27** (6) (2013) 977 – 994.

[9] M. De Salvo, D. Freni and G. Lo Faro, A new family of hypergroups and hypergroups of type $U$ on the right of size five, *Far East J. Math. Sci.* (*FJMS*) **26** (2) (2007) 393 – 418.

[10] M. De Salvo, D. Freni and G. Lo Faro, Hypergroups of type $U$ on the right of size five. II, *Mat. Vesnik* **60** (1) (2008) 23 – 45.

[11] M. De Salvo, D. Freni and G. Lo Faro, On the hypergroups of type $U$ on the right of size five, with scalar identity, *J. Mult.-Valued Logic Soft Comput.* **17** (5-6) (2011) 425 – 441.

[12] M. Farshi and B. Davvaz, $F^n$-Hypergroups based on fuzzy hyperoperations and fundamental relations, *J. Intell. Fuzzy Systems* **26** (2014) 1453 – 1464.

[13] M. Farshi and B. Davvaz, On isomorphism theorems of $F^n$-polygroups, *J. Sci. I. R. Iran* **24** (3) (2013) 259 – 267.

[14] D. Fasino and D. Freni, Existence of proper semihypergroups of type $U$ on the right, *Discrete Math.* **307** (22) (2007) 2826 – 2836.

[15] D. Fasino and D. Freni, Minimal order semihypergroups of type $U$ on the right, *Mediterr. J. Math.* **5** (3) (2008) 295 – 314.

[16] D. Freni, Structure des hypergroupes quotients et des hypergroupes de type $U$ (Structure of quotient hypergroups and of hypergroups of type $U$), *Ann. Sci. Univ. Clermont-Ferrand II Math.* **22** (1984) 51 – 77.

[17] D. Freni and M. Gutan, Sur les hypergroupes de type $U$, *Mathematica (Cluj)* **36** (59) (1994) 25 – 32.

[18] F. Marty, Sur une generalization de la notion de group, *In 8th Congress Math. Scandenaves* (1934) 45 – 49.

[19] Y. Sureau and D. Freni, Hypergroupes de type $U$ et homologie de complexes, *Algebra Universalis* **35** (1) (1996) 34 – 62.

[20] T. Vougiouklis, *Hyperstructures and Their Representations*, Aviani editor, Hadronic Press, Palm Harbor, USA, 1994.

[21] L. A. Zadeh, Fuzzy sets, *Inf. and Control* **8** (1965) 338 – 353.

[22] M. Zahedi and A. Hasankhani, $F$-polygroups. I, *J. Fuzzy Math.* **4** (3) (1996) 533 – 548.

[23] M. Zahedi and A. Hasankhani, $F$-polygroups. II, *Inform. Sci.* **89** (3-4) (1996) 225 – 243.

Mehdi Farshi
Department of Mathematics,
Yazd University,
Yazd, I. R. Iran
e-mail: m.farshi@yahoo.com

Bijan Davvaz
Department of Mathematics,
Yazd University,
Yazd, I. R. Iran
e-mail: davvaz@yazd.ac.ir

Fatemeh Dehghan
Department of Mathematics,
Yazd University,
Yazd, I. R. Iran
e-mail: fdehghan@stu.yazd.ac.ir