n-Capability of A-Groups

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Abstract
Following P. Hall a soluble group whose Sylow subgroups are all abelian is called A-group. The purpose of this article is to give a new and shorter proof for a criterion on the capability of A-groups of order $p^2q$, where $p$ and $q$ are distinct primes. Subsequently we give a sufficient condition for n-capability of groups having the property that their center and derived subgroups have trivial intersection, like the groups with trivial Frattini subgroup and A-groups. An interesting necessary and sufficient condition for capability of the A-groups of square free order will be also given.

Keywords: n-capable group, Sylow subgroup, Frattini subgroup

1. Introduction and Preliminaries

In 1938, Baer [1] initiated a systematic investigation of the question which conditions a group $G$ must fulfill in order to be the group of inner automorphisms of some group $E$ ($G \cong E/Z(E)$). Following M. Hall and Senior [8] such a group $G$ is called capable. Baer classified capable groups that are direct sums of cyclic groups. His characterization of finitely generated abelian groups that are capable is given in the following theorem.

**Theorem 1.1.** [1]. Let $G$ be a finitely generated abelian group written as $G = \mathbb{Z}_{n_1} \oplus \mathbb{Z}_{n_2} \oplus \cdots \oplus \mathbb{Z}_{n_k}$, such that each integer $n_i + 1$ is divisible by $n_i$, where $\mathbb{Z}_0 = \mathbb{Z}$, the infinite cyclic group. Then $G$ is capable if and only if $k \geq 2$ and $n_{k-1} = n_k$.

In 1940, P. Hall [6] introduced the concept of isoclinism of groups which is one of the most significant methods for classification of groups. He showed that
capable groups play an important role in characterizing finite $p$-groups. Capable groups also have a fascinating property that does not hold for an arbitrary group. Extraspecial $p$-groups show that there is no general upper bound on the index of the center of a finite group in terms of the order of its derived subgroup. However, Isaacs [11] proved that if $G$ is a finite capable group, then $|G/Z(G)|$ is bounded above by a function of $|G'|$. Later, Podoski and Szegedy [13] extended the result and gave the following explicit bound for the index of the center in a capable group as follows:

**Theorem 1.2.** If $G$ is a capable group and $|G'| = n$, then $|G/Z(G)| \leq n^{2 \log_2 n}$.

Also, they showed that for a finite capable group we have $|G : Z(G)| \leq |G'|^2$, whenever $G'$ is a cyclic group.

Let us consider soluble groups whose Sylow subgroups are all abelian. Following P. Hall such groups are called $A$-groups. $A$-groups were investigated thoroughly by P. Hall and D. R. Taunt in [7] and [16]. In an $A$-group $G$, the intersection of the commutator subgroup $G'$ and the center $Z(G)$ is trivial. In 2011, Rashid et al. [14] found a criterion for capability of $A$-groups of order $p^2q$, where $p$ and $q$ are distinct primes. They determine the explicit structure of the Schur multiplier and exterior center to obtain a necessary condition for capability of these groups. Then, they show that this condition is sufficient using the presentation of the groups of order $p^2q$ and also $p^2q$ with the condition that $p < q$. In this paper, we obtain the same necessary condition by a different process and then show that the sufficiency of the condition will follow using the above property of $A$-groups. Moreover, we show that capability and $n$-capability coincide for the groups of order $p^2q$. Also, we can obtain a sufficient condition for $n$-capability of the groups having the property that the intersection of the center subgroup and the commutator subgroup is trivial such as $A$-groups and groups with trivial Frattini subgroup. Finally, we give a criterion for capability of the $A$-groups of square free order.

The following famous results will be used in the article.

**Theorem 1.3.** [6] Let $J$ be a generating system for $G$ and $\Delta_J = \cap_{x \in J} \langle x \rangle$. Then $\Delta_J \subseteq \varphi(Z(E))$ for every central extension $(E, \varphi)$ of $G$.

The join of all subgroups $\Delta_J$, where $J$ varies over all generating system of $G$ will be denoted by $\Delta(G)$. It follows from Theorem 1.3 that a capable group $G$ must satisfy $\Delta(G) = 1$. However, this condition is not sufficient for $G$ in order to be capable. Here is another necessary condition for a group to be capable.

**Theorem 1.4.** [2, Proposition 1.2] If $G$ is capable and the commutator factor group $G/G'$ of $G$ is of finite exponent, then also $Z(G)$ is bounded and the exponent of $Z(G)$ divides that of $G/G'$.

**Corollary 1.5.** [5, Proposition 1] Let $G$ be a finitely generated capable group. Then every central element $z$ in $G$ has order dividing $\exp((G/\langle z \rangle)^{ab})$. 
In 1979, Beyl et al. [2] studied capable groups by focusing on a characteristic subgroup $Z(G)$, called the epicenter of $G$ and it is defined to be the intersection of all normal subgroups $N$ of $G$ such that $G/N$ is capable. In fact, they established a necessary and sufficient condition for a group to be capable in terms of the epicenter.

**Theorem 1.6.** [2] A group $G$ is capable if and only if $Z(G) = 1$.

Obviously, the class of all capable groups is neither subgroup closed nor under homomorphic image. But this class is closed under direct product [2, Proposition 6.1]. It follows that $Z(\prod_{i \in I} G_i) \subseteq \prod_{i \in I} Z(G_i)$. One should also notice that the inclusion is proper in general. Beyl et al. [2] gave a sufficient condition forcing equality as follows.

**Theorem 1.7.** Let $G = \prod_{i \in I} G_i$. Assume that for $i \neq j$ the maps $v_i \otimes 1 : Z(G_i) \otimes G_j/G_j' \rightarrow G_i/G_i' \otimes G_j/G_j'$ are zero, where $v_i$ is the natural map $Z(G_i) \rightarrow G_i \rightarrow G_i/G_i'$. Then $Z(G) = \prod_{i \in I} Z(G_i)$.

It follows immediately from Theorem 1.7 that a finite nilpotent group is capable if and only if all of whose Sylow subgroups are capable. Hence, if $G$ is a nilpotent capable group and its order as a product of powers of distinct primes to be $p_1^{\alpha_1}p_2^{\alpha_2} \cdots p_t^{\alpha_t}$, then $\alpha_i$ should be greater than 1, for $1 \leq i \leq t$, since cyclic groups are not capable.

## 2. Main Results

### 2.1 Capable $A$-Groups of Order $p^2q$

The first step shows that a capable group of order $p^2q$ can not be nilpotent.

**Lemma 2.1.** Let $G$ be a capable group of order $p^nq$, for some positive integer $n$ and distinct primes $p$ and $q$. Then $G$ is not a nilpotent group.

**Proof.** It is straightforward. \qed

We recall that a nilpotent group of order $p^2q$ is an abelian group, because all of whose Sylow subgroups are abelian.

**Lemma 2.2.** Let $G$ be a capable group of order $p^nq$, for some positive integer $n$ and distinct primes $p$ and $q$. Then the prime $q$ does not divide the order of $Z(G)$.

**Proof.** If $q$ divides $|Z(G)|$, then $G/Z(G)$ is a finite $p$-group and so $G$ will be nilpotent. It is a contradiction. \qed

The next lemma determines the order of the center of a non-nilpotent group of order $p^3q$. 
Lemma 2.3. Let $G$ be a nonnilpotent group of order $p^2q$. Then the center of $G$ is trivial or of order $p$.

Proof. Using Lemma 2.2, the order of the center is not divisible by the prime $q$. Also, if $|Z(G)| = p^2$, then $G/Z(G)$ is a cyclic group and hence $G$ is an abelian group. It is impossible and the result follows.

It is obvious that if $Z(G) = 1$, then $G$ is capable. Hence, Lemmas 2.1 and 2.3 imply that we should investigate capable groups among nonnilpotent groups in which $|G/Z(G)| = pq$.

Lemma 2.4. Let $H$ be a nonnilpotent group of order $pq$ such that $p > q$. Then $|H'| = p$.

Proof. Let $H$ be a nonnilpotent group of order $pq$. Then $Z(H) = 1$ and so $H$ is a capable group. Using [13, Theorem 7], we have $|H/Z(H)| \leq |H'|^2$. Thus $|H'| = p$, since $p > q$.

Lemma 2.5. Let $G$ be a capable group of order $p^2q$ with nontrivial center subgroup. Then $p < q$.

Proof. Let $p > q$. Since $G/Z(G)$ is a nonnilpotent group of order $pq$, then $|(G/Z(G))'| = p$ by Lemma 2.4. Let $z$ be an element of $G$ such that $Z(G) = \langle z \rangle$. Using Corollary 1.5, the central element $z$ in $G$ has order dividing $exp((G/\langle z \rangle)^{ab})$. It implies that $p|q$ and this is a contradiction.

Lemma 2.5 illustrates that groups of order $p^2q$ with $p > q$ and nontrivial center are not capable.

Lemma 2.6. Let $G$ be a capable group of the order $p^2q$ with nontrivial center subgroup. Then the commutator subgroup of $G$ is of order $q$.

Proof. We see at once that $G$ is a nonnilpotent group whose center has order $p$ with $p < q$. Using Lemma 2.4, we have $|(G/Z(G))'| = q$. Now, since the intersection of the commutator subgroup $G'$ and the center $Z(G)$ in the $A$-group $G$ is trivial, then the result follows.

Rashid et al. [14] using the Schur multiplier and exterior center of groups of order $p^2q$ show that capable groups with nontrivial center can not have the derived factor group isomorphic to a cyclic group of order $p^2$. In the following, we prove this assertion according to the property of capability directly and without the usage of Schur multiplier.

Lemma 2.7. Let $G$ be a capable group of order $p^2q$ with $|Z(G)| = p$. Then $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. 
Proof. Let $G$ be a group with the above property. Then Lemma 2.6 implies that $|G'|=q$ and so $|G/G'|=p^2$. Let $G/G'\cong\mathbb{Z}_p$. Consider $x$ and $y$ in $G$ such that $G/G'=\left\langle xG' \right\rangle$ and $G'=\left\langle y \right\rangle$. Obviously $G$ can be generated by $x$ and $y$. First, we assume that $x^{p^2} \neq 1$. Since $x^{p^2} \in G'$, there exists a natural number $t$ such that $x^{pt} = y^{t} \neq 1$. Then $(x) \cap (y) \neq 1$ and so $\triangle(G) \neq 1$. But this implies that $G$ is not capable which contradicts to the assumption. Therefore $x^{p^2} = y^q = 1$. The generator $xG'$ of $G/G'$ has order $p^2$ and so $x^{p^2} \not\in G'$. On the other hand, since $G' \cap Z(G) = 1$, we have $|G/G'Z(G)| = p$. It follows that $x^p \in Z(G)$. Now, since $G$ is capable, there exists a group $H$ such that $G \cong H/Z(H)$. We denote $H := H/Z(H)$. Under the recent isomorphism, there are corresponding elements $x_1$ and $y_1$ of $H$ such that $x_1^{p^2} = y_1^q = 1$. We shall show that $x_1^p \in Z(H)$ which is a contradiction. Now $x^p \in Z(G)$ implies that $x_1^p \in Z(H)$. Hence $[x_1^p, y_1] = 1$ and so there exists $z_0 \in Z(H)$ such that $x_1^p y_1 = y_1 x_1^p z_0$. This equality deduces that $z_0 = 1$, since $x_1^p \in Z(H)$. Similarly, we conclude that $z_0^q = 1$, since $y_1^q \in Z(H)$. Now $z_0^p = z_0^q = 1$ implies that $z_0 = 1$ and hence $x_1^p y_1 = y_1 x_1^p$. Thus $x_1^p \in Z(H)$. Therefore $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. \qed

The next theorem was proved by Rashid et al. using the presentation of groups of order $p^4q$ and also $p^2q$ with the condition that $p < q$. Here, we intend to give an easy proof for it using the important property of $A$-groups mentioned in Section 1.

Theorem 2.8. Let $G$ be a group of order $p^2q$ such that $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Then $G$ is capable.

Proof. The isomorphism in the assumption implies that $G^{ab}$ is capable. Now, the factor groups $G/Z(G)$ and $G/G'$ are capable. Thus, $Z^*(G) \subseteq G' \cap Z(G)$, by the comment mentioned after Corollary 1.5. On the other hand, the intersection of the commutator subgroup $G'$ and the center $Z(G)$ in the $A$-group $G$ is trivial. Therefore, $G$ must be a capable group using Theorem 1.6. \qed

Finally, we can also conclude a criterion for capability of the groups of order $p^2q$ as follows.

Theorem 2.9. Let $G$ be a group of the order $p^2q$. Then $G$ is capable if and only if either $Z(G) = 1$ or $G^{ab} \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$ and $p < q$.

The alternating group $A_4$ of order 4 is an example of capable group with $Z(G) = 1$ and the group $G$ defined by the presentation $G = \langle a, b, c | a^2 = b^q = c^p = 1, bab^{-1} = a^i, ac = ca, bc = cb \rangle$, where $i^p \equiv 1 \mod q$ and $p|q - 1$, can be considered as an example for the second part of Theorem 2.9 (see [4, Section 59] or [14]).

2.2 n-Capability of A-Groups

Burns and Ellis in [3] and Moghaddam and Kayvanfar in [12] independently introduced the concept of an $n$-capable group as follows:
Definition 2.10. A group $G$ is said to be $n$-capable if there exists a group $H$ such that $G \cong H/Z_n(H)$.

Obviously 1-capability is capability and $G$ is $n$-capable ($n \geq 2$) if and only if it is an inner automorphism group of an $(n-1)$-capable group, that is, $n$-capability implies the 1-capability for a group. The existence of a capable group which is not 2-capable is explained in [3]. Hassanzadeh and the third author [9] referred to an example of P. Hall to construct $n$-capable groups which are not $(n+1)$-capable.

Burns and Ellis [3] proved the following theorem.

Theorem 2.11. A finitely generated abelian group $G$ is $n$-capable if and only if it is capable.

We shall also show this coincidence for the groups of order $p^2q$. For this purpose we need the next theorem.

Theorem 2.12. [12, Theorem 2.2] Let $N_i$ be a normal subgroup of $G$, and $G/N_i$ an $n$-capable factor group of $G$ ($i \in I$). If $N = \cap_{i \in I} N_i$, then $G/N$ is $n$-capable.

Theorem 2.13. Let $G$ be a group of order $p^2q$. Then $G$ is capable if and only if it is $n$-capable for every $n \in \mathbb{N}$.

Proof. Clearly $G$ is capable if it is an $n$-capable group, for some $n \in \mathbb{N}$. Conversely, let $G$ be a capable group of order $p^2q$. Then either $Z(G) = 1$ or $G/Z(G) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p$. Trivially, $G$ is $n$-capable for every $n \in \mathbb{N}$ if $Z(G) = 1$. If $Z(G) \neq 1$, then $G/Z(G)$ is a non-abelian group of order $pq$. Hence $Z(G/Z(G)) = 1$ and $G/Z(G)$ is $n$-capable for every $n \in \mathbb{N}$. Also, $G/Z(G) \cong \mathbb{Z}_n \oplus \mathbb{Z}_n$ is an $n$-capable group for every $n \in \mathbb{N}$, by Theorems 1.1 and 2.11. Therefore $G/Z(G)$ is $n$-capable using Theorem 2.12. Finally, $G' \cap Z(G) = 1$ implies that $G$ is an $n$-capable group.

In the following we will give a sufficient condition for $n$-capability of a group $G$ for which $G' \cap Z(G) = 1$.

Theorem 2.14. Let $G$ be a group such that $G' \cap Z(G) = 1$. If $G/G'$ is an $n$-capable group, then so is $G$.

Proof. Since $G' \cap Z(G) = 1$, it follows that $G' \cap Z_n(G) = 1$ by induction on $n$. Now, $n$-capability of $G/Z_n(G)$ and $G/G'$ implies that $G$ is $n$-capable using Theorem 2.12.

Theorem 2.15. Let $G$ be a finite group all of whose Sylow subgroups are abelian. If $G/G'$ is an $n$-capable group, then so is $G$.

Proof. Since all of Sylow subgroups of $G$ are abelian, we have $G' \cap Z(G) = 1$ (see [15, 10.1.7]). The result follows using Theorem 2.14.
Example 2.16. Let $G$ be the semidirect product of the normal cyclic subgroup of order $q$ and the elementary abelian $p$-subgroup of rank $t \geq 2$, i.e., $G = (\oplus_q \mathbb{Z}_p) \rtimes \mathbb{Z}_q$, where $p$ and $q$ are two distinct prime. Using Theorem 2.15, one can see that $G$ is an $n$-capable group since all Sylow subgroups of $G$ are abelian and $G/G' \cong \oplus_1^t \mathbb{Z}_p$ is $n$-capable.

Theorem 2.17. Let $G$ be a group with trivial Frattini subgroup. If $G/G'$ is an $n$-capable group, then so is $G$.

Proof. By the assumption, we have $G' \cap Z(G) = 1$. Now the proof is clear similar to Theorem 2.14.

2.3 Capable $A$-Groups of Square-Free Order

Groups of square free order are part of $A$-groups. In this section, we intend to obtain a criterion for capability of groups of order $n$, when $n$ is square-free i.e. $n = p_1 p_2 \ldots p_k$ where the $p_i$'s are distinct primes. For this, we need the next lemma.

Lemma 2.18. [10, 7.9. Corollary] Let $G$ be a group such that $G/Z(G)$ is a finite $\pi$-group, then $G'$ is a finite $\pi$-group.

Theorem 2.19. Let $|G| = p_1 \ldots p_m$, where $p_1, \ldots, p_m$ are distinct primes. Then $G$ is capable if and only if $Z(G) = 1$.

Proof. Let $|Z(G)| = p_i_1 \ldots p_i_t$, where $1 \leq i_1 \leq m$ and $1 \leq j \leq t$. Put $\pi = \{p_1, \ldots, p_i_t\}$. It yields $G/Z(G)$ is a $\pi'$-group. Using Lemma 2.18, the commutator subgroup $G'$ is also a $\pi'$-group. Suppose that $p_{u_1} \ldots p_{u_s}$ where $1 \leq u_s \leq m$ and $1 \leq s \leq r$ be $\pi'$-numbers which do not divide the order of $G'$. Using Couchy’s theorem there are subgroups $H_{u_s}$ of order $p_{u_s}$ in $G$. We can now construct the subgroup $H = G'H_{u_1} \ldots H_{u_r}$ in which $G' \subseteq H$ and $|H| = \frac{|G|}{|Z(G)|}$. Therefore we have $G = H \times Z(G)$, since $H \subseteq G$ and $H \cap Z(G) = 1$. Here is the non capable subgroup $Z(G)$ and so $G$ is not capable, by applying Theorem 1.7.

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