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Commuting Conjugacy Class Graph of G when $\frac{G}{Z(G)} \cong D_{2n}$

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Abstract

Suppose G is a finite non-abelian group and $\Gamma(G)$ is a simple graph with the non-central conjugacy classes of G as its vertex set. Two different non-central conjugacy classes C and B are assumed to be adjacent in $\Gamma(G)$ if and only if there are elements $a \in A$ and $b \in B$ such that $ab = ba$. This graph is called the commuting conjugacy class graph of G. In this paper, the structure of the commuting conjugacy class graph of a group G with this property that $\frac{G}{Z(G)} \cong D_{2n}$ will be determined.

Keywords: Commuting conjugacy class graph, Conjugacy classes, Center, Centralizer, Normalizer, CA-Group.

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1. Introduction

Throughout this paper all groups are finite and graphs are assumed to be simple and undirected. We refer to [6] for our group theory notations and [4] for graph theory notions. Suppose $X = \{\Lambda_1, \ldots, \Lambda_s\}$ is a set of undirected graphs with mutually disjoint vertex sets. The notation $\Lambda_1 \cup \cdots \cup \Lambda_s$ is used for a graph with the vertex set $V(\Lambda_1) \cup \cdots \cup V(\Lambda_s)$ and edge set $E(\Delta_1) \cup \cdots \cup E(\Delta_s)$. In the case that all members of X are isomorphic, we use the notation $s\Lambda_1$ as $\Lambda_1 \cup \cdots \cup \Lambda_s$.

Suppose G is a non-abelian finite group. The **commuting conjugacy class graph**, $\Gamma(G)$, of G is a simple and undirected graph with non-central conjugacy

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classes of G as its vertex set and two distinct vertices C and D are adjacent if and only if there are $a \in C$ and $b \in D$ such that $ab = ba$ [5]. In the mentioned paper, the authors classified all finite groups G with this property that $\Gamma(G)$ is triangle-free.

Set $Cent(G) = \{C_G(x) \mid x \in G\}$, where $C_G(x)$ denotes the centralizer of x in G. The group G is said to be $n-$centralizer if $n = |Cent(G)|$. The finite groups with small number of element centralizers are characterized by Belcastro and Sherman in 1994 [3]. The group G is called a $CA-$group if for each non-central element $x \in G$, $C_G(x)$ is abelian [8]. The following theorem is crucial throughout this paper:

Theorem 1.1. *Let G be a finite group. Then the following are hold:*

1. *Let $\frac{G}{Z(G)}$ be non-abelian, n be an integer and p be a prime. If $\frac{G}{Z(G)} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_p$, then G is a CA-group* (*See Baishya* [2, Lemma 2.10]).

2. *Let $n \geq 2$ be an integer and let G be a finite group such that $\frac{G}{Z(G)} \cong D_{2n}$. Then $|Cent(G)| = n + 2$ and there are $r, s \in G$ such that $Cent(G) = \{G,\ C_G(r),\ C_G(r^i s)\ :\ 1 \leq i \leq n\}$* (*See Abdollahi et al.* [1, Proposition 2.2]).

3. *Let G be a CA-Group. The non-central conjugacy classes x^G and y^G of G are adjacent in $\Gamma(G)$ if and only if $C_G(x)$ and $C_G(y)$ are conjugate in G* (*See Salahshour et al.* [7, Lemma 3.1]).

4. *If G is a CA-group then $\Gamma(G) = \bigcup_{\frac{C_G(x)}{\sim} \in A(G)} K_{n_{\frac{C_G(x)}{\sim}}}$, where $n_{\frac{C_G(x)}{\sim}} = \frac{|C_G(x)|-|Z(G)|}{[N_G(C_G(x)):C_G(x)]}$, $A(G) = \frac{Cent(G)\setminus\{G\}}{\sim}$ and $\sim$ is an equivalence relation on $Cent(G) \setminus \{G\}$ by $C_G(x) \sim C_G(y)$ if and only if $C_G(x)$ and $C_G(y)$ are conjugate in G* (*See Salahshour et al.* [7, Theorem 3.3]).

5. *The commuting conjugacy class graph of dihedral group D_{2n} is as follows:*

$$\Gamma(D_{2n}) = \begin{cases} K_{\frac{n-1}{2}} \cup K_1, & n \text{ is odd}, \\ K_{\frac{n}{2}-1} \cup 2K_1, & n \text{ and } \frac{n}{2} \text{ are even}, \\ K_{\frac{n}{2}-1} \cup K_2, & n \text{ is even and } \frac{n}{2} \text{ is odd}, \end{cases}$$

   (*See Salahshour et al.* [7, Proposition 2.1]).

6. *The commuting conjugacy class graph of dicyclic group T_{4n} is as follows:*

$$\Gamma(T_{4n}) = \begin{cases} K_{n-1} \cup 2K_1, & n \text{ is even}, \\ K_{n-1} \cup K_2, & n \text{ is odd}, \end{cases}$$

   (*See Salahshour et al.* [7, Proposition 2.2]).

7. *The commuting conjugacy class graph of the group V_{8n} is as follows:*

$$\Gamma(V_{8n}) = \begin{cases} K_{2n-2} \cup 2K_2, & 2 \mid n, \\ K_{2n-1} \cup 2K_1, & 2 \nmid n, \end{cases}$$

(*See Salahshour et al.* [7, Proposition 2.4]).

Suppose D_{2n} denotes the dihedral group of order $2n$. The aim of this paper is to calculate the commuting conjugacy class graph of a group G with this property that $\frac{G}{Z(G)} \cong D_{2n}$. Our calculations are done with the aid of GAP [9]. The following theorem is the main result of this paper.

Theorem 1.2. *Let G be a finite group with center Z such that $\frac{G}{Z} \cong D_{2n}$. Then*

$$\Gamma(G) = \begin{cases} K_{\frac{(n-1)|Z|}{2}} \cup 2K_{\frac{|Z|}{2}}, & n \text{ is even}, \\ K_{\frac{(n-1)|Z|}{2}} \cup K_{|Z|}, & n \text{ is odd}. \end{cases}$$

2. Proof of the Main Theorem

The aim of this section is to obtain the structure of the commuting conjugacy class graph of G when $\frac{G}{Z(G)} \cong D_{2n}$. To explain our result, an example is also presented.

Note that $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$. Since $\frac{G}{Z} \cong D_{2n}$, by Theorem 1.1(1.1), G is a CA-group. By Theorem 1.1(1.1),

$$\Gamma(G) = \bigcup_{\frac{C_G(x)}{\sim} \in A(G)} K_{n_{\frac{C_G(x)}{\sim}}}, \tag{1}$$

where $n_{\frac{C_G(x)}{\sim}} = \frac{|C_G(x)|-|Z|}{[N_G(C_G(x)):C_G(x)]}$ and $A(G) = \frac{Cent(G)\setminus\{G\}}{\sim}$ such that $\sim$ is an equivalence relation on $Cent(G) \setminus \{G\}$ defined by $C_G(x) \sim C_G(y)$ if and only if $C_G(x)$ and $C_G(y)$ are conjugate in G. We need the structure of the centralizers in G. Since $\frac{G}{Z} \cong D_{2n}$, there are $a, b \in G \setminus Z$ such that

$$\frac{G}{Z} = \langle aZ, bZ \mid (aZ)^n = (bZ)^2 = Z,\ (bZ)^{-1}(aZ)(bZ) = (aZ)^{-1}\rangle.$$

For every $g \in G$, $gZ \in \frac{G}{Z}$. Hence, there are $0 \leq i \leq n-1$ and $0 \leq j \leq 1$ such that

$$gZ = (aZ)^i(bZ)^j = (a^iZ)(b^jZ) = a^ib^jZ.$$

Therefore,

$$G = \left\{a^ib^jz \mid 0 \leq i \leq n-1,\ 0 \leq j \leq 1,\ b^{-1}ab = a^{-1}z_r,\ a^n, b^2, z, z_r \in Z\right\}.$$

Assume $Z = \{1, z_1, z_2, \dots, z_{|Z|-1}\}$ and the elements of G are as follows:

$\boxed{1}$	a	$\cdots$	a^{n-1}	b	ab	$\cdots$	$a^{n-1}b$
$\boxed{z_1}$	az_1	$\cdots$	$a^{n-1}z_1$	bz_1	abz_1	$\cdots$	$a^{n-1}bz_1$
$\boxed{z_2}$	az_2	$\cdots$	$a^{n-1}z_2$	bz_1	abz_2	$\cdots$	$a^{n-1}bz_2$
$\vdots$	$\vdots$	$\cdots$	$\vdots$	$\vdots$	$\vdots$	$\cdots$	$\vdots$
$\boxed{z_{\vert Z\vert-1}}$	$az_{\vert Z\vert-1}$	$\cdots$	$a^{n-1}z_{\vert Z\vert-1}$	$bz_{\vert Z\vert-1}$	$abz_{\vert Z\vert-1}$	$\cdots$	$a^{n-1}bz_{\vert Z\vert-1}$

By Theorem 1.1(1.1) and the fact that $\frac{G}{Z} \cong D_{2n}$, $|Cent(G)| = n + 2$ and

$$Cent(G) = \{G, C_G(a), C_G(a^s b) : \ 0 \leq s \leq n-1\}.$$

Also, G is a CA-group and by Theorem 1.1(1.1), two non-central conjugacy classes of G are adjacent in $\Gamma(G)$ if and only if their centralizers are conjugate in G. Hence, we investigate the conjugation of centralizers of G. By definition of G, it is easy to see that $C_G(a) = \{a^i z \ \mid \ 0 \leq i \leq n-1, \ z \in Z\}$. Therefore, $|C_G(a)| = n|Z|$. On the other hand, $|G| = 2n|Z|$. Hence $[G : C_G(a)] = 2$ and $C_G(a) \lhd G$. Also, by definition $b^{-1}ab = a^{-1}z_r$ and so $ba = a^{-1}bz_r$. Thus, for each i

$$ba^i = a^{-i}bz_r^i \qquad \mathit{and} \qquad b^{-1}a^i = a^{-i}b^{-1}z_r^i. \tag{2}$$

Since $b^2 \in Z$, by Equation 2,

$$(a^s b)^2 = (a^s b)(a^s b) = a^s(ba^s)b = a^s(a^{-s}bz_r^s)b = b^2 z_r^s \in Z,$$

where $0 \leq s \leq n-1$. Therefore,

$$C_G(a^s b) = \{(a^s b)^k z \ \mid \ 0 \leq k \leq 1, z \in Z\} = Z \cup a^s bZ \tag{3}$$

and $|C_G(a^s b)| = 2|Z|$. Also, by Equation 2,

$$(a^i z)^{-1}(a^s b)(a^i z) = a^{-i}(a^s b)a^i = a^{-i}a^s(ba^i) = a^{s-2i}bz_r^i, \tag{4}$$

$0 \leq i \leq n-1$, and

$$\begin{aligned} (a^i bz)^{-1}(a^s b)(a^i bz) &= b^{-1}[a^{-i}(a^s b)a^i]b = b^{-1}(a^{s-2i}bz_r^i)b \\ &= (b^{-1}a^{s-2i})b^2 z_r^i = a^{2i-s}bz_r^{s-i}. \end{aligned} \tag{5}$$

We now assume that $C_G(a^s b)$ and $C_G(a^t b)$ are conjugate in G. There exists $x \in G$ such that $x^{-1}C_G(a^s b)x = C_G(a^t b)$. By Equations 4 and 5, $a^{s-2i}bz_r^i \in C_G(a^t b)$ or $a^{2i-s}bz_r^{s-i} \in C_G(a^t b)$. Without loss of generality, we consider the case that $a^{2i-s}bz_r^{s-i} \in C_G(a^t b)$. By Equation 3, there exists $z_m \in Z$ such that $a^{2i-s}bz_r^{s-i} = a^t bz_m$. So $a^{2i-s-t} \in Z$. Since $o(aZ) = n$, $n \mid 2i - s - t$. Hence there exists $k \in \mathbb{Z}$ such that $s + t = nk + 2i$.

Suppose n is even. Then $s + t$ is even. Therefore, $C_G(a^s b)$ and $C_G(a^t b)$ are conjugate in G if and only if s and t are both even or both odd. Then $\frac{C_G(b)}{\sim}$ and

$\frac{C_G(ab)}{\sim}$ are two distinct classes in $A(G)$. Thus $A(G) = \left\{\frac{C_G(a)}{\sim}, \frac{C_G(b)}{\sim}, \frac{C_G(ab)}{\sim}\right\}$. We now assume that n is odd. In this case, we show all $C_G(a^s b)$ are conjugate with $C_G(b)$ for $1 \le s \le n-1$. If s is even, then set $x = a^{\frac{s}{2}}$. By Equations 3 and 4, $xC_G(b)x^{-1} = C_G(a^s b)$. But s is odd, then set $x = a^{\frac{s+n}{2}}$. Because n is odd number, by Equations 3 and 4, $xC_G(b)x^{-1} = C_G(a^s b)$. Accordingly, for $0 \le s \le n-1$, all $C_G(a^s b)$ are conjugate with $C_G(b)$. Thus $A(G) = \left\{\frac{C_G(a)}{\sim}, \frac{C_G(b)}{\sim}\right\}$. Therefore,

$$A(G) = \begin{cases} \left\{\frac{C_G(a)}{\sim}, \frac{C_G(b)}{\sim}, \frac{C_G(ab)}{\sim}\right\}, & n \text{ is even,} \\ \left\{\frac{C_G(a)}{\sim}, \frac{C_G(b)}{\sim}\right\}, & n \text{ is odd.} \end{cases} \tag{6}$$

Since $C_G(a) \lhd G$, $N_G(C_G(a)) = G$ and $[N_G(C_G(a)) : C_G(a)] = 2$. Then

$$n_{\frac{C_G(a)}{\sim}} = \frac{|C_G(a)| - |Z|}{[N_G(C_G(a)) : C_G(a)]} = \frac{n|Z| - |Z|}{2} = \frac{(n-1)|Z|}{2}. \tag{7}$$

Suppose $0 \le s \le n-1$ is constant. We know that $C_G(a^s b) \trianglelefteq N(C_G(a^s b))$. Hence for every $x \in N(C_G(a^s b))$, $x^{-1}C_G(a^s b)x = C_G(a^s b)$. By Equations 4 and 5, $a^{s-2i}bz_r^i \in C_G(a^s b)$ or $a^{2i-s}bz_r^{s-i} \in C_G(a^s b)$. If $a^{s-2i}bz_r^i \in C_G(a^s b)$, then by Equation 3, there exists $z_m \in Z$ such that $a^{s-2i}bz_r^i = a^s b z_m$ and so $a^{-2i} \in Z$. Since $o(aZ) = n$, $n \mid -2i$. Hence there exists $k \in \mathbb{Z}$ such that $2i = -nk$. Since $0 \le i \le n-1$,

$$i = \begin{cases} 0 \quad \text{or} \quad \frac{n}{2}, & n \text{ is even,} \\ 0, & n \text{ is odd.} \end{cases} \tag{8}$$

If $a^{2i-s}bz_r^{s-i} \in C_G(a^s b)$, then by Equation 3, there exists $z_m \in Z$ such that $a^{2i-s}bz_r^{s-i} = a^s b z_m$ and hence $a^{2i-2s} \in Z$. Since $o(aZ) = n$, $n \mid 2i - 2s$. Hence there exists $k \in \mathbb{Z}$ such that $2i = nk + 2s$. Since $0 \le i, s \le n-1$,

$$i = \begin{cases} s \quad \text{or} \quad \frac{n}{2} + s, & n \text{ is even,} \\ s, & n \text{ is odd.} \end{cases} \tag{9}$$

By Equations 8 and 9,

$$N(C_G(a^s b)) = \begin{cases} C_G(a^s b) \cup a^{\frac{n}{2}} C_G(a^s b), & n \text{ is even,} \\ C_G(a^s b), & n \text{ is odd,} \end{cases}$$

and

$$|N(C_G(a^s b))| = \begin{cases} 2|C_G(a^s b)|, & n \text{ is even,} \\ |C_G(a^s b)|, & n \text{ is odd.} \end{cases}$$

Since $|C_G(a^s b)| = 2|Z|$,

$$n_{\frac{C_G(ab)}{\sim}} = n_{\frac{C_G(b)}{\sim}} = \frac{|C_G(b)| - |Z|}{[N_G(C_G(b)) : C_G(b)]}$$
$$= \begin{cases} \frac{2|Z|-|Z|}{2} = \frac{|Z|}{2}, & n \text{ is even,} \\ \frac{2|Z|-|Z|}{1} = |Z|, & n \text{ is odd.} \end{cases} \quad (10)$$

We now apply Equations 1, 6, 7 and 10 to prove that

$$\Gamma(G) = \begin{cases} K_{\frac{(n-1)|Z|}{2}} \cup 2K_{\frac{|Z|}{2}}, & n \text{ is even,} \\ K_{\frac{(n-1)|Z|}{2}} \cup K_{|Z|}, & n \text{ is odd.} \end{cases}$$

This completes the proof of our main result.

Example 2.1. By GAP, $Z(D_{24}) \cong Z(T_{24}) \cong Z(V_{24}) \cong \mathbb{Z}_2$ and $\frac{D_{24}}{Z(D_{24})} \cong \frac{T_{24}}{Z(T_{24})} \cong \frac{V_{24}}{Z(V_{24})} \cong D_{12}$. By Theorem 1.2, $\Gamma(D_{24}) = \Gamma(T_{24}) = K_5 \cup 2K_1$. Also, $Z(D_{20}) \cong Z(T_{20}) \cong \mathbb{Z}_2$, $\frac{D_{20}}{Z(D_{20})} \cong \frac{T_{20}}{Z(T_{20})} \cong D_{10}$ and by Theorem 1.2, $\Gamma(D_{20}) = \Gamma(T_{20}) = K_4 \cup K_2$. On the other hand, By Theorems 1.1(1.1), 1.1(1.1) and 1.1(1.1), it is easy to see that $\Gamma(D_{24}) = \Gamma(T_{24}) = K_5 \cup 2K_1$ and $\Gamma(D_{20}) = \Gamma(T_{20}) = K_4 \cup K_2$. Therefore, this confirms the correctness of Theorem 1.2, see Figure 1. Furthermore, $Z(V_{48}) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $Z(\mathbb{Z}_{24} \rtimes \mathbb{Z}_2) \cong \mathbb{Z}_4$ and $\frac{V_{48}}{Z(V_{48})} \cong \frac{\mathbb{Z}_{24} \rtimes \mathbb{Z}_2}{Z(\mathbb{Z}_{24} \rtimes \mathbb{Z}_2)} \cong D_{12}$. By Theorem 1.2, $\Gamma(V_{48}) = \Gamma(\mathbb{Z}_{24} \rtimes \mathbb{Z}_2) = K_{10} \cup 2K_2$. See Figure 1.

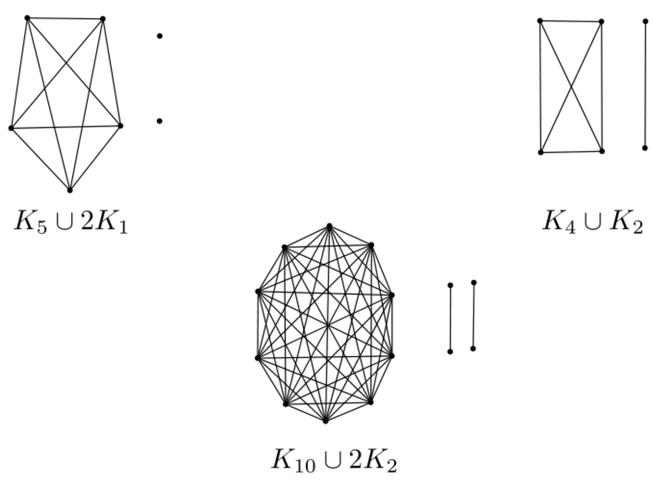

Figure 1: $\Gamma(D_{24})=\Gamma(T_{24})=\Gamma(V_{24})=K_5 \cup 2K_1$, $\Gamma(D_{20})=\Gamma(T_{20})=K_4 \cup K_2$ and $\Gamma(V_{48}) = \Gamma(\mathbb{Z}_{24} \rtimes \mathbb{Z}_2) = K_{10} \cup 2K_2$.

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References

[1] A. Abdollahi, S. M. Jafarian Amiri and A. Mohammadi Hassanabadi, Groups with specific number of centralizers, *Houston J. Math.* **33** (1) (2007) 43 − 57.

[2] S. J. Baishya, On finite groups with specific number of centralizers, *Int. Electron. J. Alg.* **13** (2013) 53 − 62.

[3] S. M. Belcastro and G. J. Sherman, Counting centralizers in finite groups, *Math. Mag.* **5** (1994) 111 − 114.

[4] F. Harary, *Graph Theory*, Addison-Wesley, Reading, MA, 1969.

[5] A. Mohammadian, A. Erfanian, M. Farrokhi D. G. and B. Wilkens, Triangle-free commuting conjugacy class graphs, *J. Group Theory* **19** (3) (2016) 1049−1061.

[6] D. J. Robinson, *A Course in the Theory of Groups*, Springer-Verlag, New York, 1996.

[7] M. A. Salahshour and A. R. Ashrafi, Commuting conjugacy class graph of finite CA-groups, *Khayyam J. Math.* **6** (1) (2020) 108–118.

[8] R. Schmidt, Zentralisatorverbände endlicher Gruppen (German), *Rend. Sem. Mat. Univ. Padova* **44** (1970) 97 − 131.

[9] The GAP Team, Group, GAP - Groups, Algorithms, and Programming, Version 4.5.5, 2012, http://www.gap-system.org.

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