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On Finding a Relative Interior Point of a Polyhedral Set

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Abstract

This paper proposes a new linear program for finding a relative interior point of a polyhedral set. Based on characterizing the relative interior of a polyhedral set through its polyhedral *representing sets*, two main contributions are made. First, we complete the existing results in the literature that require the non-negativity of the given polyhedral set. Then, we deal with the general case where this requirement may not be met.

Keywords: polyhedral set, representing set, relative interior point, maximal element, linear optimization.

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1. Introduction

Any convex subset of the non-negative orthant of Euclidean space is called a non-negative convex set. Any element of a non-negative convex set is said to be maximal, if the number of its positive components is maximum. If a closed convex set can be described by a finite number of linear equality and inequality constraints, it is called a (general) polyhedral set. By the *defining* constraints of such a polyhedral set, we refer to the constraints apart from the non-negativity constraints.

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The relative interior of a convex set is its interior relative to its affine hull. As known from convex analysis, the relative interior of a non-empty convex set is non-empty [11]. This paper proposes a linear optimization based approach to deal effectively with the problem of finding a relative interior point of a polyhedral set. The author's motivation comes from four facts. First, the information of a relative interior point of the optimal set of a linear program is useful for parametric analysis [1, 9]. Second, each pair of relative interior points of the primal and dual optimal sets of a linear program yields a strict complementary solution [10]. Third, a relative interior point of a polyhedral set can be used to recognize the presence of implicit equalities – inequality constraints that are satisfied as equalities for all feasible solutions (see, e.g., [4, 12, 13, 14], among others). Fourth, the problem under consideration has found applications in other fields such as linear fractional optimization, geometric optimization, vector optimization, compressed sensing and data envelopment analysis (see, e.g., [5, 7] and references therein).

Mehdiloozad et al. [8] develop a convex program for finding a maximal element of a non-negative convex set. They prove that the set of all maximal elements of a non-negative polyhedral set with equality defining constraints coincides with its relative interior. Exploiting this result, they derive a linear program from their convex program to find a relative interior point of such a polyhedral set. Mirdehghan and Mehdiloo [7] develop a linear program to find a relative interior point of a non-negative polyhedral set having both equality and inequality defining constraints. As an extension to their program, a new linear program is developed in this paper that is able to find a relative interior point of any polyhedral set which is not required to be *non-negative*.

For a subset of Euclidean space, we define a *representing set* as a higher-dimensional set that its projection onto the linear span of the given set is equal to that set. We use two nice results of Rockafellar [11] to characterize the relative interior of any convex representing set of a set. Restating one of his results, we show that *projecting the relative interior of each convex representing set of a set (if any exists) onto its linear span*[1] *results its relative interior.* To find a relative interior point of a non-negative polyhedral set having both equality and inequality defining constraints, we represent it equivalently by a higher-dimensional non-negative polyhedral set that its defining constraints are all equalities. Then, we demonstrate that *projecting the set of all maximal elements of this representing set onto the linear span of the given set results its relative interior.* This completes a result presented by Mirdehghan and Mehdiloo [7], and helps us derive their linear program from the convex program of Mehdiloozad et al. [8].

As a consequence of the Rockafellar's results, we characterize the relative interior of a convex set without using the relative interior of its representing sets. We show that this characterization generalizes the already-stated result of Mehdiloozad et al. [8], according to which the relative interior of a non-negative

[1] The linear span, also called *linear hull*, of a set is the smallest subspace containing that set.

polyhedral set with equality defining constraints consists of its maximal elements. Based on our new characterization, a new linear program is developed for finding a relative interior of a general polyhedral set. This original program is derived from the convex program of Mehdiloozad et al. [8].

The organization of the paper is as follows. Section 2 introduces the used notations and develops a linear program for finding a maximal element of a non-negative polyhedral set. It also presents an equivalent representation of any subset of Euclidean space. Section 3 deals with finding a relative interior point of a non-negative polyhedral set, and Section 4 considers the extension of this problem to general polyhedral sets. Section 5 contains some concluding remarks. Appendix A contains the computer program written in GAMS (General Algebraic Modeling System) that can be used for solving the proposed linear program.

2. Background

2.1. Notation

Throughout the paper we use the following notations. Let $\mathbb{R}^d$ denote the d-dimensional Euclidean space, and let $\mathbb{R}^d_+$ denote its non-negative orthant. We denote sets by uppercase calligraphic letters, vectors by boldface lowercase letters, and matrices by boldface uppercase letters. By convention, all vectors are column vectors. Superscript $\top$ denotes the transpose of a vector or matrix.

Vectors $\mathbf{0}$ and $\mathbf{1}$ are vectors all components of which are equal to 0 and 1, respectively. The dimensions of these vectors are clear from the context in which they are used. For simplicity, notation $(\mathbf{a};\mathbf{b}) \in \mathbb{R}^{d+d'}$ is used to show the column vector obtained by adding vector $\mathbf{b} \in \mathbb{R}^{d'}$ below vector $\mathbf{a} \in \mathbb{R}^d$. For two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d$, the inequality $\mathbf{a} \geq \mathbf{b}$ (resp., $\mathbf{a} > \mathbf{b}$) denotes $a_i \geq b_i$ (resp., $a_i > b_i$), for all $i = 1, \ldots, d$. Matrix $\mathbf{0}$ is the matrix all components of which are equal to 0, and matrix $\mathbf{I}$ is the identity matrix. The dimensions of these matrices are clear from the context in which they are used.

Let $\mathcal{S}$ be a set in $\mathbb{R}^d$. The relative interior of $\mathcal{S}$, denoted $\mathrm{ri}\,(\mathcal{S})$, is defined as interior relative to its affine hull. Formally,

$$\mathrm{ri}\,(\mathcal{S}) = \left\{\mathbf{s}^{\mathrm{o}} \in \mathcal{S} \colon \mathcal{N}_\varepsilon\,(\mathbf{s}^{\mathrm{o}}) \cap \mathrm{aff}\,(\mathcal{S}) \subseteq \mathcal{S} \text{ for some } \varepsilon > 0\right\},$$

where $\mathrm{aff}\,(\mathcal{S})$ denotes the affine hull of $\mathcal{S}$, and $\mathcal{N}_\varepsilon\,(\mathbf{s}^{\mathrm{o}}) = \left\{\mathbf{s} \in \mathbb{R}^d \colon \|\mathbf{s} - \mathbf{s}^{\mathrm{o}}\| < \varepsilon\right\}$. If the set $\mathcal{S}$ is non-empty and convex, then $\mathrm{ri}\,(\mathcal{S}) \neq \varnothing$ (see, e.g., [11]). If the set $\mathcal{S}$ is non-negative, we use the notation $\mathrm{me}\,(\mathcal{S})$ to denote the set of all its maximal elements:

$$\mathrm{me}\,(\mathcal{S}) = \{\mathbf{s} \in \mathcal{S} \colon \text{the number of positive components of } \mathbf{s} \text{ is maximum}\}\,.$$

Let $\mathcal{S}$ be a set in $\mathbb{R}^{d+d'}$, and let each vector $\mathbf{s} \in \mathcal{S}$ be partitioned into two subvectors $\mathbf{x} \in \mathbb{R}^d$ and $\mathbf{y} \in \mathbb{R}^{d'}$ as $\mathbf{s} = (\mathbf{x};\mathbf{y})$. We define the $\mathbf{x}$-*space* $\Pi_1\,[\mathcal{S}] \subset \mathbb{R}^d$

of $\mathcal{S}$ as the projection of $\mathcal{S}$ onto the space of $\mathbf{x}$-variables. Formally,

$$\Pi_1[\mathcal{S}] = \{\mathbf{x} \in \mathbb{R}^d \colon (\mathbf{x};\mathbf{y}) \in \mathcal{S} \text{ for some } \mathbf{y} \in \mathbb{R}^{d'}\}.$$

For any $\mathbf{x} \in \Pi_1[\mathcal{S}]$, we define the $\mathbf{x}$-*section* $\mathcal{S}_2(\mathbf{x}) \subseteq \mathbb{R}^{d'}$ of $\mathcal{S}$ as the set of all subvectors $\mathbf{y} \in \mathbb{R}^{d'}$ such that $(\mathbf{x};\mathbf{y}) \in \mathcal{S}$. That is, $\mathcal{S}_2(\mathbf{x}) = \{\mathbf{y} \in \mathbb{R}^{d'} \colon (\mathbf{x};\mathbf{y}) \in \mathcal{S}\}$. Similarly, we define the $\mathbf{y}$-*space* $\Pi_2[\mathcal{S}] \subset \mathbb{R}^{d'}$ of $\mathcal{S}$ as follows:

$$\Pi_2[\mathcal{S}] = \{\mathbf{y} \in \mathbb{R}^{d'} \colon (\mathbf{x};\mathbf{y}) \in \mathcal{S} \text{ for some } \mathbf{x} \in \mathbb{R}^{d}\}.$$

For any $\mathbf{y} \in \Pi_2[\mathcal{S}]$, the $\mathbf{y}$-*section* $\mathcal{S}_1(\mathbf{y}) \subseteq \mathbb{R}^d$ of $\mathcal{S}$ is defined as $\mathcal{S}_1(\mathbf{y}) = \{\mathbf{x} \in \mathbb{R}^d \colon (\mathbf{x};\mathbf{y}) \in \mathcal{S}\}$.

2.2. Finding a Maximal Element of a Non-Negative Polyhedral Set

Consider the non-empty set $\mathcal{K} \subset \mathbb{R}^d$ defined as

$$\mathcal{K} = \left\{\mathbf{x} \in \mathbb{R}^d \colon \mathbf{P}\mathbf{x} + \mathbf{Q}\mathbf{y} = \mathbf{r},\, \mathbf{x} \geq \mathbf{0},\, \mathbf{y} \in \mathbb{R}^{d'}\right\},$$

where $\mathbf{x}$ and $\mathbf{y}$ are the vectors of variables that are non-negative and unrestricted in sign, respectively. Furthermore, $\mathbf{P}$ and $\mathbf{Q}$ are matrices of coefficients of dimensions $c \times d$ and $c \times d'$, respectively, and $\mathbf{r} \in \mathbb{R}^c$ is a constant vector.

By the *projection lemma* (see, e.g., Corollary 2.4 in [3]), the projection of any polyhedral set onto the space of any subset of its characterizing variables is a polyhedral set. Therefore, $\mathcal{K}$ is a non-negative polyhedral set in $\mathbb{R}^d_+$.

Mehdiloozad et al. [8] develop a general convex program for finding a maximal element of any non-negative convex set. As a consequence to their Theorem 3.2, the following result develops a linear program for finding a maximal element of $\mathcal{K}$.

Theorem 2.1. *Let $(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{y}^*, w^*)$ be an optimal solution to the following linear program:*

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{x}^1 \\
&\text{subject to} \\
&\mathbf{P}\left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{Q}\mathbf{y} = \mathbf{r}w, \\
&\mathbf{1} \geq \mathbf{x}^1 \geq \mathbf{0},\ \mathbf{x}^2 \geq \mathbf{0},\ \mathbf{y} \text{ sign free},\ w \geq 1.
\end{aligned} \tag{1}$$

Then, $\frac{1}{w^*}\left(\mathbf{x}^{1*} + \mathbf{x}^{2*}\right) \in \mathrm{me}(\mathcal{K})$.

Proof. Consider the following linear program:

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{x}^1 + w^1 \\
&\text{subject to} \\
&\mathbf{P}\left(\mathbf{x}^1 + \mathbf{x}^2\right) + \mathbf{Q}\mathbf{y} = \mathbf{r}\left(w^1 + w^2\right), \\
&\mathbf{1} \geq \mathbf{x}^1 \geq \mathbf{0},\ \mathbf{x}^2 \geq \mathbf{0},\ \mathbf{y} \text{ sign free},\ 1 \geq w^1 \geq 0,\ w^2 \geq 0.
\end{aligned} \tag{2}$$

The maximization program (2) is feasible and its objective function is upper bounded by $d+1$, where d is the dimension of vector $\mathbf{x}^1$. Consequently, program (2) has a finite optimal solution, namely $\big(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{y}^*, w^{1*}, w^{2*}\big)$. By assumption, we have $\mathcal{K} \neq \emptyset$. Hence, it follows from Theorem 3.2 in [8] that $w^{1*} = 1$. Therefore, program (1) can be derived from program (2) by replacing w^1 with its optimal value and using the variable substitution $w = 1 + w^2$. This indicates that $\big(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{y}^*, 1 + w^{2*}\big)$ is an optimal solution to program (1), and that any optimal solution of program (1) gives an optimal solution of program (2). Namely, if we define $\mathbf{x}^{1\prime} = \mathbf{x}^{1*}$, $\mathbf{x}^{2\prime} = \mathbf{x}^{2*}$, $\mathbf{y}' = \mathbf{y}^*$, $w^{1\prime} = 1$ and $w^{2\prime} = w^* - 1$, then $\big(\mathbf{x}^{1\prime}, \mathbf{x}^{2\prime}, \mathbf{y}', w^{1\prime}, w^{2\prime}\big)$ is an optimal solution to program (2). Therefore, the statement of the theorem follows from Theorem 3.2 in [8]. $\square$

Remark 1. In Theorem 2.1, there is no need for a prior knowledge about the nonemptiness of the set $\mathcal{K}$. This is due to the fact that the infeasibility of the linear program (1) implies $\mathcal{K} = \varnothing$.

2.3. An Equivalent Representation of a Set

We begin this section with the following definition (see [2]).

Definition 2.2. Let $\mathcal{X} \subset \mathbb{R}^d$. A set $\mathcal{X}^+ \subset \mathbb{R}^{d+d'}$ is a *representing set* for $\mathcal{X}$, if $\Pi_1\left[\mathcal{X}^+\right] = \mathcal{X}$.

By Definition 2.2, the set $\mathcal{X}^+$ can be regarded as an equivalent representation of the set $\mathcal{X}$. That is, $\mathbf{x} \in \mathcal{X}$ if and only if there exists some $\mathbf{s} \in \mathbb{R}^{d'}$ such that $(\mathbf{x}; \mathbf{s}) \in \mathcal{X}^+$. We use superscripts "$+c$" and "$+n$" for a representing set of a set to denote its convexity and non-convexity, respectively.

It is straightforward to verify that the convexity of any representing set $\mathcal{X}^{+c}$ implies the convexity of the set $\mathcal{X}$ itself. In particular, as follows from the projection lemma, the projection of any polyhedral set onto the space of any subset of its variables is a polyhedral set. However, it should be noted that any representing set of a polyhedral, and more broadly convex, set is not generally convex. We illustrate this fact with the following example.

Example 2.3. Figure 1 shows the two-dimensional polyhedral set $\mathcal{L} = \big\{(x; y) \in \mathbb{R}^2 \colon 0 \le x \le 1,\, y = x\big\}$ as the segment OA. It further depicts the non-convex set $\mathcal{L}^{+n} = \big\{((x; y); z) \in \mathbb{R}^3 \colon 0 \le x \le 1,\, y = x,\, z = x^2\big\}$ as the curve OB. It is clear that the projection of the curve OB onto the $(x; y)$-space is the segment OA. Therefore, $\Pi_1\left[\mathcal{L}^{+n}\right] = \mathcal{L}$, and $\mathcal{L}^{+n}$ is a non-convex representing set for the polyhedral set $\mathcal{L}$.

Restating Theorem 6.8 in [11] as the next theorem, we characterize the relative interior of any convex representing set of a set.

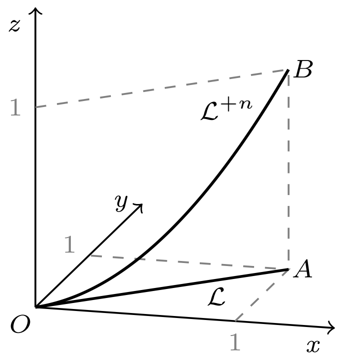

Figure 1: The sets $\mathcal{L}$ and $\mathcal{L}^{+n}$ in Example 2.3.

Theorem 2.4. *Let $\mathcal{X}^{+c} \subset \mathbb{R}^{d+d'}$ be a convex representing set for $\mathcal{X} \subset \mathbb{R}^d$. Then, the following equality is true:*

$$\mathrm{ri}\left(\mathcal{X}^{+c}\right) = \bigcup_{\mathbf{x} \in \mathrm{ri}(\mathcal{X})} \{\mathbf{x}\} \times \mathrm{ri}\left(\mathcal{X}_2^{+c}(\mathbf{x})\right). \tag{3}$$

Figure 2 draws the three-dimensional convex set $\mathcal{X}^{+c}$ as a representing set for the two-dimensional convex set $\mathcal{X}$. All points inside the half-sphere $\mathcal{X}^{+c}$ are its relative interior points. Furthermore, all points inside the ball $\mathcal{X}$ are its relative interior points. It is clear that the relative interior of $\mathcal{X}^{+c}$ can be stated as the union of all vertical open segments parallel to axis z each of which is projected onto a relative interior point of $\mathcal{X}$ in the (x, y)-space. This is a graphical illustration of Theorem 2.4.

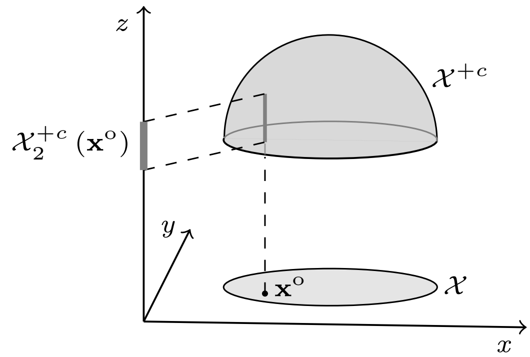

Figure 2: The relative interior of a convex representing set.

By the next example, we show that the convexity assumption made in Theorem 2.4 is not necessary for (3) to hold, though it is sufficient.

Example 2.5. Consider the polyhedral set $\mathcal{L}$ as defined in Example 2.3 and modify its representing set as $\mathcal{L}^{+n} = \big\{((x;y);z) \in \mathbb{R}^3 \colon 0 \le x \le 1,\, y = x,\, 0 \le z \le x^2\big\}$. Figure 3 shows this representing set as the shaded area OAB.

It is observed that the relative interior of $\mathcal{L}$ is the open segment OA and the relative interior of $\mathcal{L}^{+n}$ is the inside of the area OAB. Formally, $\mathrm{ri}\,(\mathcal{L}) = \big\{(x;y) \in \mathbb{R}^2 \colon 0 < x < 1,\, y = x\big\}$ and $\mathrm{ri}\,(\mathcal{L}^{+n}) = \big\{((x;y);z) \in \mathbb{R}^3 \colon 0 < x < 1,\, y = x,\, 0 < z < x^2\big\}$. Furthermore, for any $(x;y) \in \mathcal{L}$, we have $\mathcal{L}_2^{+n}\big((x;y)\big) = \big\{z \in \mathbb{R} \colon 0 \le z \le x^2\big\}$ and $\mathrm{ri}\big(\mathcal{L}_2^{+n}\big((x;y)\big)\big) = \big\{z \in \mathbb{R} \colon 0 < z < x^2\big\}$. Therefore, while the representing set $\mathcal{L}^{+n}$ is non-convex, it satisfies the equality (3).

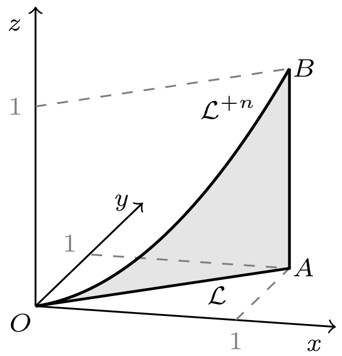

Figure 3: The sets $\mathcal{L}$ and $\mathcal{L}^{+n}$ in Example 2.5.

3. Finding a Relative Interior Point of a Non-Negative Polyhedral Set

Let $\mathcal{X}^+ \subset \mathbb{R}^{d+d'}$ be a representing set for $\mathcal{X} \subset \mathbb{R}^d$. Then, it is not difficult to verify that $\Pi_1\,[\mathrm{aff}\,(\mathcal{X}^+)] = \mathrm{aff}\,(\Pi_1\,[\mathcal{X}^+]) = \mathrm{aff}\,(\mathcal{X})$ (see, e.g., Page 8 of [11]). By this equality, we obtain the next result.

Theorem 3.1. *Let $\mathcal{X}^+ \subset \mathbb{R}^{d+d'}$ be a representing set for $\mathcal{X} \subset \mathbb{R}^d$. Then, the following embedding is true:*

$$\Pi_1\left[\mathrm{ri}\left(\mathcal{X}^+\right)\right] \subseteq \mathrm{ri}\,(\mathcal{X}). \tag{4}$$

Proof. Let $\mathbf{x}^{\mathrm{ri}} \in \Pi_1\,[\mathrm{ri}\,(\mathcal{X}^+)]$. Then, by Definition 2.2, there exists some $\mathbf{y}^{\mathrm{ri}} \in \mathbb{R}^{d'}$ such that $\big(\mathbf{x}^{\mathrm{ri}};\mathbf{y}^{\mathrm{ri}}\big) \in \mathrm{ri}\,(\mathcal{X}^+)$. This means that $\mathcal{N}_\varepsilon\big(\big(\mathbf{x}^{\mathrm{ri}};\mathbf{y}^{\mathrm{ri}}\big)\big) \cap \mathrm{aff}\,(\mathcal{X}^+) \subseteq \mathcal{X}^+$, for some $\varepsilon > 0$. As a consequence, the following relationship holds between the $\mathbf{x}$-spaces of the two sets in the above embedding:

$$\Pi_1\left[\mathcal{N}_\varepsilon\left(\left(\mathbf{x}^{\mathrm{ri}};\mathbf{y}^{\mathrm{ri}}\right)\right)\right] \cap \Pi_1\left[\operatorname{aff}\left(\mathcal{X}^{+}\right)\right] \subseteq \Pi_1\left[\mathcal{X}^{+}\right]. \tag{5}$$

It is straightforward to show that $\mathcal{N}_\varepsilon\left(\mathbf{x}^{\mathrm{ri}}\right) \subseteq \Pi_1\left[\mathcal{N}_\varepsilon\left(\left(\mathbf{x}^{\mathrm{ri}};\mathbf{y}^{\mathrm{ri}}\right)\right)\right]$, $\Pi_1\left[\operatorname{aff}\left(\mathcal{X}^{+}\right)\right] = \operatorname{aff}(\mathcal{X})$ and $\Pi_1\left[\mathcal{X}^{+}\right] = \mathcal{X}$. Then, it follows from (5) that $\mathcal{N}_\varepsilon\left(\mathbf{x}^{\mathrm{ri}}\right) \cap \operatorname{aff}(\mathcal{X}) \subseteq \mathcal{X}$, which means that $\mathbf{x}^{\mathrm{ri}} \in \operatorname{ri}(\mathcal{X})$. Therefore, the embedding (4) is true. □

It follows from (4) that the projection of any relative interior point of $\mathcal{X}^{+}$ onto the space of the original variables is a relative interior point of $\mathcal{X}$. It is important to note that this inclusion may be strict. For example, let $\mathcal{L}$ and $\mathcal{L}^{+n}$ be as considered in Example 2.3. Then, we have $\operatorname{ri}\left(\mathcal{L}^{+n}\right) = \emptyset$, whereas $\operatorname{ri}(\mathcal{L}) = \left\{(x;y) \in \mathbb{R}^2 \colon 0 < x < 1,\ y = x\right\}$ (see Figure 1). Therefore, $\emptyset \subset \operatorname{ri}(\mathcal{L})$.

By Theorem 6.6 in [11], the following theorem demonstrates that the embedding (4) holds as equality for the class of convex representing sets.

Theorem 3.2. *Let $\mathcal{X}^{+c} \subset \mathbb{R}^{d+d'}$ be a convex representing set for $\mathcal{X} \subset \mathbb{R}^d$. Then, the following equality is true:*

$$\operatorname{ri}(\mathcal{X}) = \Pi_1\left[\operatorname{ri}\left(\mathcal{X}^{+c}\right)\right]. \tag{6}$$

Proof. Define the linear function $A \colon \mathbb{R}^{d+d'} \to \mathbb{R}^d$ as $A(\mathbf{x};\mathbf{y}) = \mathbf{x}$. Then, (6) follows from Theorem 6.6 in [11]. □

By Theorem 3.2, the convexity of a representing set of a set ensures the converse of the embedding (4). However, this condition is not necessary. For example, consider the sets $\mathcal{P}$ and $\mathcal{P}^{+n}$ as in Example 2.5. Then, we have $\operatorname{ri}(\mathcal{P}) = \Pi_1\left[\operatorname{ri}\left(\mathcal{P}^{+n}\right)\right]$, while the representing set $\mathcal{P}^{+n}$ is non-convex.

From convex analysis, it is known that the relative interior of a convex set is convex [11]. Therefore, it follows from Theorem 3.2 that the relative interior of $\mathcal{X}^{+c}$ is a convex representing set for the relative interior of $\mathcal{X}$. This representation suggests an approach for finding a relative interior of $\mathcal{X}$. Namely, it states that a relative interior point of $\mathcal{X}$ can be obtained by projecting a relative interior point of $\mathcal{X}^{+c}$ onto the space of original variables.

Note that the above approach is most effective when $\mathcal{X}^{+c}$ is a non-negative polyhedral set that is described only with equality constraints, because a relative interior point of such a polyhedral set can be found through Theorem 2.1. In this section, we use this approach to find a relative interior point of the following non-empty polyhedral set in $\mathbb{R}^n$:

$$\mathcal{P} = \left\{\mathbf{x} \in \mathbb{R}^n \colon \mathbf{A}\mathbf{x} = \mathbf{b},\ \mathbf{C}\mathbf{x} \le \mathbf{d},\ \mathbf{x} \ge \mathbf{0}\right\},$$

where $\mathbf{x}$ is the vector of variables, $\mathbf{A}$ and $\mathbf{B}$ are respectively matrices of coefficients of dimensions $m \times n$ and $m' \times n$, and $\mathbf{b} \in \mathbb{R}^m$ and $\mathbf{d} \in \mathbb{R}^{m'}$ are respectively constant vectors of equality and inequality constraints.

By the next result, we show that the relative interior of any non-negative convex set is included in the set of its maximal elements.

Theorem 3.3. *Let $\mathcal{C}$ be a non-negative convex set in $\mathbb{R}^d_+$. Then, the following embedding is true:*

$$\mathrm{ri}\,(\mathcal{C}) \subseteq \mathrm{me}\,(\mathcal{C})\,. \tag{7}$$

Proof. If $\mathcal{C} = \emptyset$, there is nothing to prove. Otherwise, it follows from convex analysis [11] that $\mathrm{ri}\,(\mathcal{C}) \neq \emptyset$. Let $\mathbf{x}^{\mathrm{o}} \in \mathrm{ri}\,(\mathcal{C})$. By Lemma 3.1 in [8], we have $\mathrm{me}\,(\mathcal{C}) = \left\{\mathbf{x} \in \mathcal{C} \colon x_j > 0 \text{ for all } j \in \mathcal{J}^{\neq}\right\}$, where $\mathcal{J}^{\neq} = \left\{j \in \{1, \ldots, n\} : x_j > 0 \text{ for some } \mathbf{x} \in \mathcal{C}\right\}$. Therefore, proving the statement that $\mathbf{x}^{\mathrm{o}} \in \mathrm{me}\,(\mathcal{C})$ is equivalent to showing that $x^{\mathrm{o}}_j > 0$ for all $j \in \mathcal{J}^{\neq}$.

Let $\varepsilon > 0$ be an arbitrary positive constant. By contradiction, assume that $x^{\mathrm{o}}_{\hat{j}} = 0$ for some $\hat{j} \in \mathcal{J}^{\neq}$. By the definition of $\mathcal{J}^{\neq}$, there exists some $\bar{\mathbf{x}} \in \mathcal{C}$ such that $\bar{x}_{\hat{j}} > 0$. For any $\delta > 0$, define $\mathbf{x}^{\delta} = (1 + \delta)\,\mathbf{x}^{\mathrm{o}} - \delta\bar{\mathbf{x}}$. Clearly, $\mathbf{x}^{\delta} \in \mathrm{aff}\,(\mathcal{C})$ for all $\delta > 0$. Choose $\delta_\varepsilon > 0$ sufficiently small such that $\mathbf{x}^{\delta_\varepsilon} \in \mathcal{N}_\varepsilon\,(\mathbf{x}^{\mathrm{o}})$. Then, $\mathbf{x}^{\delta_\varepsilon} \in \mathcal{N}_\varepsilon\,(\mathbf{x}^{\mathrm{o}}) \cap \mathrm{aff}\,(\mathcal{C})$. At the same time, we have $x^{\delta_\varepsilon}_{\hat{j}} < 0$, because $x^{\mathrm{o}}_{\hat{j}} = 0$ and $\bar{x}_{\hat{j}} > 0$. It follows that $\mathbf{x}^{\delta_\varepsilon} \notin \mathcal{C}$, which contradicts the assumption that $\mathbf{x}^{\mathrm{o}}$ is in $\mathrm{ri}\,(\mathcal{C})$. This contradiction shows that $\mathbf{x}^{\mathrm{o}} \in \mathrm{me}\,(\mathcal{C})$. Therefore, $\mathrm{ri}\,(\mathcal{C}) \subseteq \mathrm{me}\,(\mathcal{C})$. □

Notice that, though the converse of the embedding in (7) is true for the special class of non-negative polyhedral sets with equality defining constraints (see Theorem 3.1 in [8]), it is not generally true for polyhedral and, therefore, convex sets. This is shown by the following counterexample.

Example 3.4. Consider the polyhedral set $\mathcal{L}$ as defined in Example 2.3. It is straightforward to see that $(1; 1)$ is a maximal element of $\mathcal{L}$, whereas it is not a relative interior point of $\mathcal{L}$. Geometrically speaking, this element corresponds to the extreme point A of the segment AO in Figure 1, which draws the set $\mathcal{L}$. Therefore, $\mathrm{me}\,(\mathcal{L}) \nsubseteq \mathrm{ri}\,(\mathcal{L})$.

The above counterexample shows that the problem of finding a relative interior point of $\mathcal{P}$ cannot be addressed by finding one of its maximal elements. To deal with this problem, the inequality constraints of $\mathcal{P}$ are converted to equalities by adding slack variables, and the following non-negative polyhedral set is defined:

$$\mathcal{P}^{+c} = \left\{(\mathbf{x}; \mathbf{s}) \in \mathbb{R}^{n+m'} : \mathbf{A}\mathbf{x} = \mathbf{b},\, \mathbf{C}\mathbf{x} + \mathbf{s} = \mathbf{d},\, \mathbf{x} \geq \mathbf{0},\, \mathbf{s} \geq \mathbf{0}\right\}.$$

Clearly, $\mathcal{P}^{+c}$ is a polyhedral representing set for $\mathcal{P}$. By Theorem 1 in [7], the projection of any maximal element of $\mathcal{P}^{+c}$ onto its $\mathbf{x}$-space is a relative interior of $\mathcal{P}$, i.e., $\Pi_1\,[\mathrm{me}\,(\mathcal{P}^{+c})] \subseteq \mathrm{ri}\,(\mathcal{P})$. The next theorem completes this result by showing that the inclusion holds as equality.

Theorem 3.5. *The following equality is true:*

$$\mathrm{ri}\,(\mathcal{P}) = \Pi_1\left[\mathrm{me}\left(\mathcal{P}^{+c}\right)\right]. \tag{8}$$

Proof. Because the non-negative polyhedral set $\mathcal{P}^{+c}$ is described only by equality defining constraints, it follows from Theorem 4.1 in [8] that $\operatorname{ri}(\mathcal{P}^{+c}) = \operatorname{me}(\mathcal{P}^{+c})$. Therefore, (8) is obtained by Theorem 3.2. □

We illustrate Theorem 3.5 with the following example.

Example 3.6. Consider the non-negative polyhedral set $\mathcal{L}$ as defined in Example 2.3. As already mentioned, we have $\operatorname{ri}(\mathcal{L}) = \big\{(x; y) \in \mathbb{R}^2 : 0 < x < 1,\, y = x\big\}$, which is shown as the segment OA in Figure 4. By adding the slack variable s to the inequality defining constraint $x \leq 1$, the representing set $\mathcal{L}^{+c}$ is obtained as follows:

$$\mathcal{L}^{+c} = \big\{\big((x; y); s\big) \in \mathbb{R}^3 : x - y = 0,\, x + s = 1,\, x, y, s \geq 0\big\}.$$

Figure 4 shows the set $\mathcal{L}^{+c}$ as the segment AC. It is clear that the points on the open segment AC are the maximal elements of $\mathcal{L}^{+c}$. Therefore, the projection of this segment onto the $(x; y)$-space is the open segment OA, which shows the relative interior of $\mathcal{L}$.

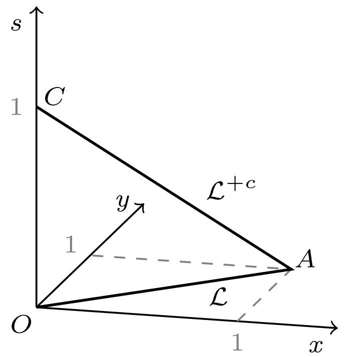

Figure 4: The sets $\mathcal{L}$ and $\mathcal{L}^{+c}$ in Example 3.6.

From Theorem 3.5, a relative interior point of $\mathcal{P}$ can be obtained from a maximal element of $\mathcal{P}^{+c}$. Using this fact, the following theorem develops a linear program (as in [7]) for finding a relative interior point of $\mathcal{P}$.

Theorem 3.7. *Let* $\big(\mathbf{x}^{\mathrm{ri}}, \mathbf{s}^{\mathrm{ri}}\big) = \frac{1}{w^*}\big(\mathbf{x}^{1*} + \mathbf{x}^{2*}, \mathbf{s}^{1*} + \mathbf{s}^{2*}\big)$*, where* $\big(\mathbf{x}^{1*}, \mathbf{x}^{2*}, \mathbf{s}^{1*}, \mathbf{s}^{2*}, w^*\big)$ *is an optimal solution to the following linear program:*

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{x}^1 + \mathbf{1}^\top \mathbf{s}^1 \\
&\text{subject to} \\
&\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{C} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{x}^1 + \mathbf{x}^2 \\ \mathbf{s}^1 + \mathbf{s}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{c} \\ \mathbf{d} \end{pmatrix} w, \\
&\mathbf{1} \geq \mathbf{x}^1, \mathbf{s}^1 \geq \mathbf{0},\, \mathbf{x}^2, \mathbf{s}^2 \geq \mathbf{0},\, w \geq 1.
\end{aligned} \tag{9}$$

Then, $\left(\mathbf{x}^{\mathrm{ri}}, \mathbf{s}^{\mathrm{ri}}\right) \in \mathrm{me}\left(\mathcal{P}^{+c}\right)$ *and, therefore,* $\mathbf{x}^{\mathrm{ri}} \in \mathrm{ri}\left(\mathcal{P}\right)$.[2]

Proof. The proof follows from Theorems 2.1 and 3.5. □

4. Finding a Relative Interior Point of a General Polyhedral Set

As already stated, finding a relative interior point of a set $\mathcal{X} \subset \mathbb{R}^d$ by (6) requires having a relative interior point of its convex representing set $\mathcal{X}^{+c}$. By the next result, we make an alternative characterization of the relative interior of $\mathcal{X}$ that is exempt from this requirement.

Theorem 4.1. *Let* $\mathcal{X}^{+c} \subset \mathbb{R}^{d+d'}$ *be a convex representing set for* $\mathcal{X} \subset \mathbb{R}^d$*, and let* $\mathcal{S} = \Pi_2\left[\mathcal{X}^{+c}\right]$*. Then, the following equality is true:*

$$\mathrm{ri}\left(\mathcal{X}\right) = \bigcup_{\mathbf{s} \in \mathrm{ri}(\mathcal{S})} \mathrm{ri}\left(\mathcal{X}_1^{+c}\left(\mathbf{s}\right)\right). \tag{10}$$

Proof. Let $\mathcal{S} = \Pi_2\left[\mathcal{X}^{+c}\right]$. Then, $\mathcal{X}^{+c}$ is regarded as a convex representing set for the set $\mathcal{S} \in \mathbb{R}^{d'}$. By Theorem 2.4, it follows that

$$\mathrm{ri}\left(\mathcal{X}^{+c}\right) = \bigcup_{\mathbf{s} \in \mathrm{ri}(\mathcal{S})} \mathrm{ri}\left(\mathcal{X}_1^{+c}\left(\mathbf{s}\right)\right) \times \{\mathbf{s}\}. \tag{11}$$

Therefore, the equality (10) follows from (11) by Theorem 3.2. □

Theorem 4.1 suggests an approach for finding a relative interior point of the set $\mathcal{X}$. Namely, it states that such a point can be obtained in two stages. First, a relative interior point of the projection of $\mathcal{X}^{+c}$ onto its $\mathbf{s}$-space, namely $\mathbf{s}^{\mathrm{ri}} \in \mathrm{ri}\left(\mathcal{S}\right)$, is found. Second, a relative interior point of the $\mathbf{s}^{\mathrm{ri}}$-section of $\mathcal{X}^{+c}$ is found as a relative interior point of $\mathcal{X}$.

The suggested approach is most effective when $\mathcal{S}$ is a non-negative polyhedral set with equality defining constraints and the affine hull of $\mathcal{X}_1^{+c}\left(\mathbf{s}^{\mathrm{ri}}\right)$ is equal to this set. This is due to two facts. First, the former condition allows for finding $\mathbf{s}^{\mathrm{ri}}$ through Theorem 2.1. Second, the latter condition requires finding an element of $\mathcal{X}_1^{+c}\left(\mathbf{s}^{\mathrm{ri}}\right)$. In this section, we use the above-described approach to deal with finding a relative interior point of the following non-empty polyhedral set in $\mathbb{R}^n$:

$$\mathcal{Q} = \left\{\mathbf{x} \in \mathbb{R}^n : \mathbf{E}\mathbf{x} = \mathbf{f},\, \mathbf{G}\mathbf{x} \leq \mathbf{h}\right\}, \tag{12}$$

where $\mathbf{x}$ is the vector of variables, $\mathbf{E}$ and $\mathbf{G}$ are respectively matrices of coefficients of dimensions $m \times n$ and $m'' \times n$, and $\mathbf{f} \in \mathbb{R}^m$ and $\mathbf{h} \in \mathbb{R}^{m''}$ are respectively constant vectors of equality and inequality constraints.

[2] It further follows by (3) that $\mathbf{s}^{\mathrm{ri}} \in \mathrm{ri}\left(\mathcal{P}_2^{+c}\left(\mathbf{x}^{\mathrm{ri}}\right)\right)$.

Observe that the set $\mathcal{Q}$ itself is not a non-negative polyhedral set. To find a relative interior point of this set, we build the following set by adding slack variables to the inequality defining constraints of $\mathcal{Q}$:

$$\mathcal{Q}^{+c} = \left\{ (\mathbf{x}; \mathbf{s}) \in \mathbb{R}^{n+m''} \colon \mathbf{E}\mathbf{x} = \mathbf{f},\, \mathbf{G}\mathbf{x} + \mathbf{s} = \mathbf{h},\, \mathbf{s} \geq \mathbf{0} \right\}.$$

Clearly, $\mathcal{Q}^{+c}$ is a polyhedral representing set for $\mathcal{Q}$ that is characterized only by equality defining constraints, but it is not non-negative. Denote $\mathcal{S}$ the projection of $\mathcal{Q}^{+c}$ onto its $\mathbf{s}$-space, i.e., $\mathcal{S} = \Pi_2\left[\mathcal{Q}^{+c}\right]$. Then, the set $\mathcal{S}$ is stated as follows:

$$\mathcal{S} = \left\{ \mathbf{s} \in \mathbb{R}^{m''} \colon \mathbf{E}\mathbf{x} = \mathbf{f},\, \mathbf{G}\mathbf{x} + \mathbf{s} = \mathbf{h},\, \mathbf{s} \geq \mathbf{0} \right\}. \tag{13}$$

Because the set $\mathcal{Q}^{+c}$ is polyhedral, it follows from the projection lemma that $\mathcal{S}$ is a non-negative polyhedral set in $\mathbb{R}^{m''}_{+}$. It is straightforward to prove that any maximal element of $\mathcal{S}$ is a relative interior point of this set. By Theorem 3.3, it follows that $\mathrm{ri}\,(\mathcal{S}) = \mathrm{me}\,(\mathcal{S})$. Using this equality in the next result, we characterize the relative interior of $\mathcal{Q}$ by the set of all vectors $\mathbf{x} \in \mathcal{Q}$ that their associated slack vectors are maximal elements of $\mathcal{S}$.

Theorem 4.2. *The following equality is true:*

$$\mathrm{ri}\,(\mathcal{Q}) = \bigcup_{\mathbf{s} \in \mathrm{me}(\mathcal{S})} \mathcal{Q}_1^{+c}\,(\mathbf{s}). \tag{14}$$

Proof. Take some arbitrary $\hat{\mathbf{s}} \in \mathcal{S}$. Then, we have

$$\mathcal{Q}_1^{+c}\,(\hat{\mathbf{s}}) = \left\{ \mathbf{x} \in \mathbb{R}^n \colon \mathbf{E}\mathbf{x} = \mathbf{f},\, \mathbf{G}\mathbf{x} = \mathbf{h} - \hat{\mathbf{s}} \right\}.$$

From the above equality, it is clear that the affine hull of $\mathcal{Q}_1^{+c}\,(\hat{\mathbf{s}})$ is equal to this set and, therefore, $\mathrm{ri}\big(\mathcal{Q}_1^{+c}\,(\hat{\mathbf{s}})\big) = \mathcal{Q}_1^{+c}\,(\hat{\mathbf{s}})$. Taking into account this equality, (14) follows from Theorem 4.1. □

Remark 2. It is worth noting that Theorem 4.2 can be viewed as an extension of Theorem 4.1 in [8]. To see this, let $\mathcal{M}$ denote the non-negative polyhedral set obtained from (12) by setting $\mathbf{G} = -\mathbf{I}$ and $\mathbf{h} = \mathbf{0}$. That is, $\mathcal{M} = \left\{ \mathbf{x} \in \mathbb{R}^d \colon \mathbf{E}\mathbf{x} = \mathbf{f},\, \mathbf{x} \geq \mathbf{0} \right\}$. Then, we have $\mathcal{M}^{+c} = \left\{ (\mathbf{x}; \mathbf{x}) \in \mathbb{R}^{2n} \colon \mathbf{x} \in \mathcal{M} \right\}$, and $\mathcal{M}_1^{+c}\,(\mathbf{s}) = \{\mathbf{s}\}$ for all $\mathbf{s}$ in $\mathcal{S} = \Pi_2\left[\mathcal{M}^{+c}\right]$. Consequently, $\bigcup_{\mathbf{s} \in \mathrm{me}(\mathcal{S})} \mathcal{M}_1^{+c}\,(\mathbf{s}) = \mathrm{me}\,(\mathcal{S})$. Because $\mathcal{S} = \mathcal{M}$, it thus follows from (14) that $\mathrm{ri}\,(\mathcal{M}) = \mathrm{me}\,(\mathcal{M})$. This is the same statement of Theorem 4.1 in [8].

Theorem 4.2 enables one to identify a relative interior point of $\mathcal{Q}$ without any use of the relative interiors of its representing sets. It shows that such a point can be obtained in two stages. First, a maximal element of the set $\mathcal{S}$ as defined in (13), namely $\mathbf{s}^{\mathrm{me}} \in \mathrm{me}\,(\mathcal{S})$, is found. Second, a relative interior point of the $\mathbf{s}^{\mathrm{ri}}$-section of $\mathcal{Q}^{+c}$, namely $\mathbf{x}^{\mathrm{ri}}$, is identifies as a relative interior point of $\mathcal{Q}$. The next result develops a linear program that accomplishes these two tasks simultaneously.

Theorem 4.3. *Suppose that* $\mathbf{x}^{\mathrm{ri}} = \frac{1}{w^*}\mathbf{x}^*$ *and* $\mathbf{s}^{\mathrm{me}} = \frac{1}{w^*}\left(\mathbf{s}^{1*} + \mathbf{s}^{2*}\right)$, *where* $\left(\mathbf{x}^*, \mathbf{s}^{1*}, \mathbf{s}^{2*}, w^*\right)$ *is an optimal solution to the following linear program:*

$$\begin{aligned}
&\max \quad \mathbf{1}^\top \mathbf{s}^1 \\
&\text{subject to} \\
&\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{G} & \mathbf{I} \end{bmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{s}^1 + \mathbf{s}^2 \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ \mathbf{h} \end{pmatrix} w, \\
&\mathbf{x} \text{ sign free}, \ \mathbf{1} \geq \mathbf{s}^1 \geq \mathbf{0}, \ \mathbf{s}^2 \geq \mathbf{0}, \ w \geq 1.
\end{aligned} \tag{15}$$

Then, $\mathbf{s}^{\mathrm{me}} \in \mathrm{me}\,(\mathcal{S})$ *and, therefore,* $\mathbf{x}^{\mathrm{ri}} \in \mathrm{ri}\big(\mathcal{Q}\big)$.

Proof. The proof follows from Theorems 2.1 and 4.2. □

Remark 3. Note that linear program (15) includes linear program (9) as a special case. Precisely, linear program (9) is derived from (15) by setting $\mathbf{E} = \mathbf{A}$, $\mathbf{f} = \mathbf{b}$, $\mathbf{G} = \begin{bmatrix} \mathbf{C} \\ -\mathbf{I} \end{bmatrix}$ and $\mathbf{h} = \begin{pmatrix} \mathbf{d} \\ \mathbf{0} \end{pmatrix}$.

Based on Theorem 4.3, we devise the following Algorithm 1 for finding a relative interior point of the polyhedral set $\mathcal{Q}$.

Algorithm 1 Finding a relative interior point of the polyhedral set $\mathcal{Q}$

Start
Read $\mathbf{E}$, $\mathbf{G}$, $\mathbf{f}$ and $\mathbf{h}$;
Solve the linear program (15);
If program (15) is not feasible, **then**
  Print "$\mathcal{Q} = \varnothing$";
 Else
  Set $\mathbf{x}^{\mathrm{ri}} = \frac{1}{w^*}\mathbf{x}^*$, where $\left(\mathbf{x}^*, \mathbf{s}^{1*}, \mathbf{s}^{2*}, w^*\right)$ is an optimal solution of (15);
  Return $\mathbf{x}^{\mathrm{ri}}$;
end if
Stop

We illustrate Algorithm 1 by the following example.

Example 4.4. Let the two-dimensional polyhedral set $\mathcal{Q} \subset \mathbb{R}^2$ be defined as follows:

$$\begin{aligned}
\mathcal{Q} = \big\{(x; y) \in \mathbb{R}^2 \colon \ & -2x + 3y \leq 6, \\
& -2x + \ \ y \leq 6, \\
& -2x - 3y \leq 14, \\
& \qquad\ \ -y \leq 4 \ \big\}.
\end{aligned}$$

Figure 5 shows the relative interior of the set $\mathcal{Q}$ as the interior of the shaded area $A'ABCC'$. To find a relative interior point of $\mathcal{Q}$, the linear program (15) was solved by using the GAMS code provided in Appendix A. The code was executed on a Laptop with Intel Core i7-3537U 2.06 GHz processor, 8GB of RAM and 64 bit Windows 7. Table 1 shows the six relative interior points of the set $\mathcal{Q}$ obtained by using different solvers of GAMS. The last column of this table provides the amount of CPU time (in seconds) taken by each solver. It is observed that using different solvers may lead to different relative interior points.

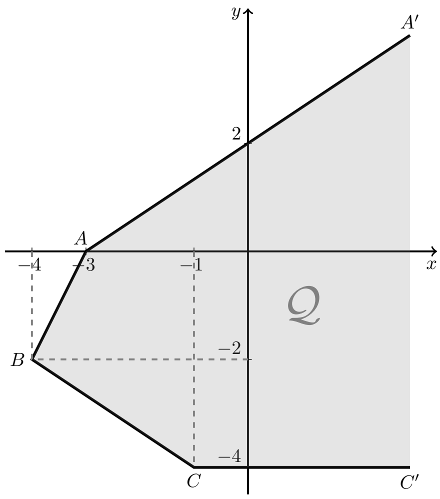

Figure 5: The relative interior of the set $\mathcal{Q}$ in Example 4.4.

Table 1: Six relative interior points of the set $\mathcal{Q}$ in Example 4.4.

Solver	w^*	x^*	y^*	x^{ri}	y^{ri}	CPU
BARON	1	−2	−3	−2	−3	0.030
BDMLP	1	−2.5	0	−2.5	0	0.062
CBC	1	−3.5	−2	−3.5	−2	0.002
CONOPT	1	−3.5	−2	−3.5	−2	0.015
CPLEX	1	−2	−3	−2	−3	0.078
GUROBI	1	−2.5	0	−2.5	0	0.177
IPOPT	1	−3.339	−1.809	−3.339	−1.809	0.796
KNITRO	1	−3.001	−1.525	−3.001	−1.525	0.624
MOSEK	1	0	0	0	0	0.109

5. Concluding Remarks

In this paper, we address the problem of finding a relative interior point of a polyhedral set through its representation in higher-dimensional space. For a given set, we say that a higher-dimensional set is representing, if its projection onto the linear span of the given set is the same as that set. Using two nice results of Rockafellar [11], we characterize the relative interior of any convex representing set of a set in two alternative ways. We then use these characterizations to deal with the problem under investigation.

We first reduce the statement of the problem under investigation to the class of non-negative polyhedral sets to complete the existing results in the literature. We represent the relative interior of a non-negative polyhedral set as the projection of the relative interior of a specific polyhedral representing set of the given set onto its linear span. Based on this result, we derive the linear program of Mirdehghan and Mehdiloo [7] from the convex program proposed in [8] that identifies a maximal element of any non-negative convex set.

We next consider the problem under investigation for general polyhedral sets. For such a polyhedral set, a polyhedral representing set results from adding slacks to the inequality constraints. By projecting this representing set onto the space of the slack variables, we make an alternative statement of the relative interior of the given polyhedral set.

By our consequence of the results of Rockafellar, we extend a result in [8] by which the relative interior of a non-negative polyhedral set described by equality defining constraints coincides with the set of its maximal elements. Based on our extended result, we develop a new linear program for finding a relative interior of a polyhedral set. This linear program is derived from the general convex program of Mehdiloozad et al. [8], and includes the linear program of Mirdehghan and Mehdiloo [7] as a special case.

Appendix A

The following computer program written in GAMS has been used for identifying a relative interior point of the polyhedral set $\mathcal{Q}$ in Example 4.4.

```
** The code has been developed by Mahmood Mehdiloo for finding a relative interior point
   of a polyhedral set.
** Making the code applicable for any polyhedral set just requires modifying 'Sets',
   'Table A(ma,n)' and 'Table B(me,n)'.

Sets
   ma      row number of matrix A         /ma1*ma1/
   me      row number of matrix E         /me1*me4/
   n    column number of matrices A and E / n1*n2 /;

Table A(ma,n)
        n1      n2
ma1    0       0;

```

```
Table E(me,n)
       n1     n2
me1   -2      3
me2   -2      1
me3   -2     -3
me4    0     -1;

Parameters
   c/ma1    0/
   f/me1    6
     me2    6
     me3    14
     me4    4/;

Free Variables
     Theta
     x(n);

Positive Variables
     s1(me)
     s2(me)
     w;
************************************
Equations
     Obj
       Con1
       Con2
       Con3
       Con4;

     Obj..       Theta =E= Sum(me, s1(me));
       Con1(ma)..     Sum(n, a(ma,n)*x(n))                   =E= c(ma)*w;
       Con2(me)..     Sum(n, e(me,n)*x(n)) + s1(me) + s2(me) =E= f(me)*w;
       Con3(me)..     s1(me) =L= 1;
       Con4..         w =G= 1;

Model ProposedLP /All/;
************************************
File Program /Results.txt/; Put Program;

Put 'Finding a relative interior point of Q';
Put /'--------------------------------------'/;
Solve ProposedLP using LP Maximizing Theta;
      Put ' w = ':5; Put w.L:<5:0; Put / /;
      Loop(me,
            Put 's1(':3; Put ord(me):<>3:0; Put ') = ':4;
            Put ( s1.L(me) ):<10:3;
            Put 's2(':3; Put ord(me):<>3:0; Put ') = ':4;
            Put ( s2.L(me) ):<10:3 /;
           );
      Put /;
      Loop(n,
           Put ' x(':3; Put ord(n):<>3:0; Put ') = ':4;
           Put ( x.L(n)/w.L ):<10:3/;
          );
Put '--------------------------------------'/;
```

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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