Mathematics Interdisciplinary Research **x** (202x) $xx - yy$

Weakly Compatible Maps and Fixed Points

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Abstract

Here, the existence of fixed points for weakly compatible maps is studied. The results are new generalization of the results of [5]. Finally, we study the new common fixed point theorems.

Keywords: weakly compatible, cone metric space, common fixed point.

2010 Mathematics Subject Classification: 47H10.

How to cite this article
M. Shahsavari, A. Razani and Gh. Abbasi, Weakly compatible maps and fixed points, *Math. Interdisc. Res.* **x** (202x) xx-yy.

1. Introduction

Huang et. al [5] extended the concept of the metric space (see [2, 10, 11, 12, 13] and References therein). They introduced the cone metric space. After then many authors studied some fixed point theorems in this setting (see [3, 4, 7, 8] and references therein).

Here, we extend the recent results [1, 5]. In order to do this, we recall some facts (see [1, 2, 4, 5]).

Definition 1.1. [5] Assume H is a real Bancah space. A subset $C \subset H$ is called cone if

(I) $C \neq \phi$, $C \neq 0$ and C is closed.

(II) If $u, -u \in C$, then $u = 0$.

(III) For every real positive α, β and $u, v \in C$, then $\alpha u + \beta v \in C$.

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Academic Editor: Hikmet Koyunbakan
Received 15 January 2021, Accepted 20 June 2021
DOI: 10.22052/MIR.2021.240433.1267

A partial order $\leq$ on c can be defined by

$$u \leq v \text{ iff } v - u \in C.$$

One can write

$$\begin{array}{rl} u < v & \text{if } u \leq v \text{ and } u \neq v \\ u \ll v & \text{if } v - u \in intC, \end{array}$$

where int$C$ is the interior of C.

Definition 1.2. [6] The cone C is said to be normal if

$$\exists\ M > 0 \text{ such that } 0 \leq u \leq v \ \Rightarrow\ \|u\| \leq M\|v\| \tag{1}$$

for all $u, v \in H$.

Definition 1.3. [6] One says the cone C is regular, if every bounded (from below) decreasing sequence $\{z_n\}$ (in C) is convergent.

The above definition implies that if C is a regular cone, then it is a normal cone.

Definition 1.4. [6] A cone metric on a set Y, is a function $\rho : Y \times Y \to H$, satisfies

- for all $u, v \in Y,\ \rho(u, v) \geq 0$ and $\rho(u, v) = 0$ iff $u = v$.
- for all $u, v \in Y\ \rho(u, v) = \rho(v, u)$.
- for all $u, v, w \in Y\ \rho(u, v) \leq \rho(u, w) + \rho(v, w)$.

then (Y, ρ) is called a cone metric space.

Next we introduce the concept of the convergence of a sequence and then one can write about Cauchy sequence.

Definition 1.5. [2] The sequence $\{u_n\}$ is called a convergent sequence,

$$\forall \alpha \gg 0 \in H\ \exists\ M \in \mathbb{N} \text{ such that } \forall\ n > M\ \rho(u_n, u) \ll \alpha,$$

for some fixed $u \in Y$. The sequence $\{u_n\}$ is called a Cauchy sequence, if

$$\forall \alpha \gg 0 \in H\ \exists\ M \in \mathbb{N} \text{ such that } \forall\ m, n > M\ \rho(u_n, u_m) \ll \alpha.$$

Notice that a complete cone metric space is a space where every Cauchy sequence is convergent. Also, It is necessary to mention that in a normal cone,

$$\{u_n\} \text{ is a Cauchy sequence } \iff \rho(u_n, u_m) \to 0,$$

when $(n, m \to \infty)$ (see [1]).

About the uniqueness of the limit, the following Remark from [6] is recalled.

Remark 1.

- Let C be a normal cone, the limit of a convergent sequence is unique.
- If $u \leq v$, where $u, v \in Y$, and $\alpha \geq 0$, then $\alpha u \leq \alpha v$.
- If for each $n \in N$, $u_n \leq v_n$, where $\{u_n\}, \{v_n\}$ are two sequences in Y and $\lim u_n = u, \lim v_n = v$, then $u \leq v$.
- If any sequence $u_n \to u_0$ implies $T(u_n) \to T(u_0)$, the function $T : Y \to Y$ is continuous at $u_0 \in Y$.

In the next definition we recall the concept of R-weakly commuting mapps.

Definition 1.6. Assume $T, S : Y \to Y$, if

$$\exists\ R \in \mathbb{R}^+ \text{ such that } \forall\ u \in Y,\ \rho(TS(u), ST(u)) \leq R\rho(T(u), S(u))$$

the mappings T and S are called R-weakly commuting maps on metric space (Y, ρ).

There is another concept which is called compatible and it can be as follows.

Definition 1.7. Assume $T, S : Y \to Y$, if $lim_{n\to\infty}\rho(TS(u_n), ST(u_n)) = 0$, where $\{u_n\} \in Y$ and there exists u in Y such that

$$lim_{n\to\infty}T(u_n) = lim_{n\to\infty}S(u_n) = u,$$

the mappings T and S are called compatible maps on (Y, ρ).

Remark 2. Notice that compatible maps aren't weakly commuting mappings (see [8]).

Jungck et. al. [9] in 1998 introduced the coincidentally commuting mappings (this concept is recalled as weakly compatible mappings, too).

Definition 1.8. Suppose $T, S : Y \to Y$, if

$$\exists\ u \in Y,\ T(u) = S(u) \text{ then } TS(u) = ST(u),$$

then the mappings T and S are called weakly compatible.

Notice that

$$\text{weakly commuting} \to \text{compatible} \to \text{weakly compatible}$$

but the reverse of the above fact is not true.

Now, we recall a Theorem of [1], which will be used in the next section.

Theorem 1.9. *Suppose $T, S : Y \to Y$ are weakly compatible maps and $z = S(u) = T(u)$ (i.e. T and S have a unique coincidence point). Then T and S have a unique common fixed point z.*

2. Weakly Compatible Maps

Here, two new generalization results of [5] will be presented.

Theorem 2.1. *Assume (Y, ρ) is a complete cone metric space, $T : Y \to Y$, C is a normal cone and $\chi : (0, \infty) \to (0, 1)$ is a monotonically decreasing function. If*

$$\rho(T(u), T(v)) \leq \chi(\|\rho(u, v)\|)\rho(u, v), \ \forall \ u, v \in Y,$$

then T has a unique fixed point in Y.

Proof. Due to prove the theorem, fixed $u_0 \in Y$ and define a sequence by

$$u_{n+1} = T^n(u_0).$$

This implies

$$\begin{aligned} \rho(u_{n+1}, u_n) \leq & \ \chi(\|\rho(u_n, u_{n-1})\|)\rho(u_n, u_{n-1}) \\ \leq & \ \chi(\|\rho(u_n, u_{n-1})\|)\chi(\|\rho(u_{n-1}, u_{n-2})\|)\rho(u_{n-1}, u_{n-2}) \\ \leq & \ \dots \\ \leq & \ \Pi_{k=0}^{k=n-1}\chi(\|\rho(u_k, u_{k+1})\|)\rho(u_0, u_1). \end{aligned}$$

By $\chi(s) < 1$, for $r > t$ we can conclude that

$$\begin{aligned} \|\rho(u_r, u_t)\| \leq & \ K\|\rho(u_r, u_{r-1}) + \rho(u_{r-1}, u_{r-2}) + \cdots + \rho(u_{t+1}, u_t)\| \\ \leq & \ M(\|\rho(u_r, u_{r-1})\| + \|\rho(u_{r-1}, u_{r-2})\| + \cdots + \|\rho(u_{t+1}, u_t)\|) \\ \leq & \ M^2\|\rho(u_0, u_1)\| = N, \end{aligned} \tag{2}$$

and

$$\|\rho(u_n, u_{n+p})\| \leq M\Pi_{k=0}^{k=n-1}\chi(\|\rho(u_k, u_{k+p})\|)\|\rho(u_0, u_p)\| \text{ for all } p > 0. \tag{3}$$

Now we show that $\{u_n\}$ is a Cauchy sequence or

$$\forall \epsilon > 0, \exists N \text{ such that } \|\rho(u_N, u_{N+p})\| < \frac{\epsilon}{2M},$$

for every $p > 0$.

If $\|\rho(u_k, u_{k+p})\| \geq \epsilon$ for $k = 0, 1, \dots, n-1$, then from monotonicity of $\chi(s)$ we have $\chi(\|\rho(u_k, u_{k+p})\|) \leq \chi(\epsilon)$, and (2) and (3) will imply

$$\|\rho(u_n, u_{n+p})\| \leq MN\,(\chi(\epsilon))^n.$$

We have $\chi(\epsilon) < 1$, thus

$$lim_{n\to\infty}\chi^n(\epsilon) = 0,$$

so there exists an integer N (independent of p) such that $\|\rho(x_N, x_{N+p})\| < \epsilon/2M$ for every $p > 0$.

Thus by triangle property for $n = N + p$ and $m = N + q$

$$\begin{aligned}\|\rho(u_n, u_m)\| &\leq M(\|\rho(u_N, u_{N+p})\| + \|\rho(u_N, u_{N+q})\|)\\ &< M(\tfrac{\epsilon}{2M} + \tfrac{\epsilon}{2M})\\ &= \epsilon.\end{aligned}$$

thus there exists $u \in Y$ such that

$$\lim_{n\to\infty} u_n \to u.$$

Also u is a fixed point of T, because

$$\|\rho(T(u), u)\| \leq M^2\|\rho(u_n, u)\| + M\|\rho(u_{n+1}, u)\|.$$

Suppose $v \neq u$ are distinct fixed points of T. Since $\chi(u, v) = k < 1$, then

$$\rho(u, v) \leq \chi(\|\rho(u, v)\|)\rho(u, v) \leq k\rho(u, v).$$

Thus the uniqueness is proved. □

In the next theorem, we study the existence of a fixed point in the sequentially compact cone metric space.

Theorem 2.2. *Suppose (Y, ρ) is a cone metric space and sequentially compact, $T : Y \to Y$, C is a regular cone, $\vartheta : H \to H$ is a continuous, $\vartheta(t) < t$ for all $t \neq 0$ and $\vartheta(0) = 0$. Suppose*

$$\rho(Tu, Tv) \leq \vartheta(\rho(u, v)) \text{ for all } u \neq v \in Y,$$

then T has a unique fixed point in Y.

Proof. Fixed $u_0 \in Y$ and define a sequence by

$$u_{n+1} := T^n(u_0).$$

The assumption $\vartheta(t) < t$ shows

$$\rho(T(u_{n-1}), T(u_n)) \leq \rho(u_{n-1}, u_n).$$

And since $\{\rho(u_{n+1}, u_n)\}$ is bounded from below, there is $a \in H$ such that

$$lim_{n\to\infty}\rho(u_{n+1}, u_n) = a.$$

Notice that the condition $\vartheta(t) < t$ implies that $a = 0$.

On the other hand, there exists a subsequence $\{u_{r_i}\}$ of $\{u_n\}$ (Y is sequentially compactness) such that

$$u_{r_i} \to u.$$

The contractive condition implies $\rho(T(u_{r_i}), T(u)) \leq \vartheta(\rho(u_{r_i}, u))$, so

$$\|\rho(T(u_{r_i}), T(u))\| \leq M\|\vartheta(\rho(u_{r_i}, u))\|.$$

By the continuity of ϑ, when $i \to \infty$ $\|\rho(T(u_{r_i}), T(u))\| \to 0$. Hence $T(x_{r_i}), \to T(u)$.

Now [5, Lemma 5] implies $\rho(T(u_{r_i}), u_{r_i}) \to \rho(T(u), u)$ when $i \to \infty$, therefore $\rho(u_{r_i+1}, u_{r_i}) \to \rho(T(u), u)$. Hence $\|\rho(T(u), u)\| = 0$ and u is a fixed point of T. The uniqueness of u is obvious. □

The two next Theorem 2.3 and Theorem 2.4 are new generalizations of the results of [1].

Theorem 2.3. *Suppose (Y, ρ) is a cone metric space, $T, S : Y \to Y$, C is a normal cone, $\theta : \mathbb{R}^+ \to (0, 1)$ is a monotonically decreasing function. Assume $\rho(T(u), T(v)) \leq \theta(\|\rho(S(u), S(v))\|)\rho(S(u), S(v))$. Let $S(Y)$ be a sequentially compact subspace of Y, $T(Y) \subset S(Y)$, then T and S have a unique coincidence point in Y. In addition, S and T have a (unique) common fixed point if they are weakly compatible.*

Proof. First, we define a sequence in $S(Y)$. Fixed $u_0 \in Y$, since the range of S contains the range of T, one takes $u_1 \in Y$ such that $T(u_0) = S(u_1)$. By induction, assume $\{u_n\} \in Y$ is obtained. One chooses $u_{n+1} \in Y$ such that $T(u_n) = S(u_{n+1})$. Then

$$\begin{aligned}
\rho(S(u_{n+1}), S(u_n)) \leq\ & \theta(\|\rho(u_n, u_{n-1})\|)\rho(S(u_n), S(u_{n-1})) \\
\leq\ & \theta(\|\rho(u_n, u_{n-1})\|)\theta(\|\rho(u_{n-1}, u_{n-2})\|)\rho(S(u_{n-1}), S(u_{n-2})) \\
\leq\ & \cdots \\
\leq\ & \Pi_{k=0}^{k=n-1}\theta(\|\rho(u_k, u_{k+1})\|)\rho(S(u_0), S(u_1)).
\end{aligned}$$

Since $\theta(s) < 1$ for all $s \geq 0$, then for $r > t$

$$\begin{aligned}
\|\rho(S(u_r), S(u_t))\| \leq\ & M\|\rho(S(u_r), S(u_{r-1})) + \rho(S(u_{r-1}), S(u_{r-2})) \\
& \qquad + \cdots + \rho(S(u_{t+1}), S(u_t))\| \\
\leq\ & M(\|\rho(S(u_r), S(u_{r-1}))\| + \|\rho(S(u_{r-1}), S(u_{r-2}))\| \\
& \qquad + \cdots + \|\rho(S(u_{t+1}), S(u_t))\|) \\
\leq\ & M^2\|\rho(S(u_0), S(u_1))\| \\
=\ & N.
\end{aligned} \tag{4}$$

Then

$$\rho(S(u_n), S(u_{n+p})) \leq \Pi_{k=0}^{k=n-1}\theta(\|\rho(u_k, u_{k+p})\|)\rho(S(u_0), S(u_p)) \text{ for all } p > 0. \tag{5}$$

Notice that $\{S(u_n)\}$ is a Cauchy sequence. Due to prove this, for every $\epsilon > 0$ there exists number N, dependent on ϵ such that $\|\rho(S(u_N), S(u_{N+p}))\| < \frac{\epsilon}{2M}$ for every $p > 0$. If $\|\rho(S(u_k), S(u_{k+p}))\| \geq \epsilon$ for $k = 0, 1, \ldots, n-1$, then from monotonicity of $\theta(s)$ we have $\theta(\|\rho(u_k, u_{k+p}\|)) \leq \theta(\epsilon)$, and by (4) and (5), $\|\rho(S(u_n), S(u_{n+p}))\| \leq MN\,(\theta(\epsilon))^n$. Notice that $\theta^n(\epsilon) \to 0$, so there exists an integer N independent of p

such that $\|\rho(S(u_N), S(u_{N+p}))\| < \frac{\epsilon}{2M}$ for every $p > 0$, and for $n := N + p, m := N + q$

$$\begin{aligned} \|\rho(S(u_n), S(u_m))\| &\leq M(\|\rho(S(u_N), S(u_{N+p}))\| + \|\rho(S(u_N), S(u_{N+q})))\| \\ &< M(\tfrac{\epsilon}{2M} + \tfrac{\epsilon}{2M}) \\ &= \epsilon. \end{aligned}$$

This shows $\{S(u_n)\}$ is Cauchy (see [1]) and there exists a u in $S(Y)$ such that

$$\lim_{n\to\infty} S(u_n) = u.$$

Thus one can find v in Y such that $S(v) = u$. Also

$$\rho(S(u_n), T(v)) = \rho(T(u_{n-1}), T(v)) \leq \theta(\|\rho((u_{n-1}, v))\|)\rho(S(u_{n-1}), S(v)).$$

The relation (1) implies

$$\|\rho(S(u_n), T(v))\| \leq M\theta(\|\rho(u_{n-1}, v)\|)\|\rho(S(u_{n-1}), u(v))\| \to 0.$$

Since $S(u_n)$ converges to $S(v)$ then $\rho(S(u_n), T(v)) \to 0$ as $n \to \infty$. In adition $\rho(S(u_n), S(v)) \to 0$ as $n \to \infty$. Thus $T(v) = S(v)$.

Notice that T and S have a unique point of coincidence. Suppose there exists a point $v'(\neq v)$ in Y such that $T(v') = S(v')$. Since $\theta(\|\rho(v', v)\|) = b < 1$, $\rho(S(v'), S(v)) = \rho(T(v'), T(v)) \leq \theta(\|\rho(v', v)\|)\rho(S(v'), S(v)) \leq b\rho(S(v'), S(v))$, which implies $\|\rho(S(v'), S(v))\| = 0$ and $S(v') = S(v)$. Thus T and S (by Theorem 1.9) have a unique common fixed point. □

Remark 3. Theorem 2.3 remains true if one consider R-weakly commuting maps.

Theorem 2.4. *Suppose (Y, ρ) is a cone metric space, $T, S : Y \to Y$, C is a regular cone, θ is a continuous selfmap on H, $\theta(t) < t$ for all $t \neq 0$ and $\theta(0) = 0$. Assume*

$$\rho(T(u), T(v)) \leq \theta(\rho(S(u), S(v))) \text{ for all } u, v \in Y, S(u) \neq S(v).$$

If $S(Y)$ is a sequentially compact subspace of Y and $T(Y) \subset S(Y)$, then T and S have a unique coincidence point in Y. In addition, S and T have a (unique) common fixed point if they are weakly compatible.

Proof. Fixed $u_0 \in Y$. Set $u_1 \in Y$ such that $T(u_0) = S(u_1)$. By induction, we have $\{u_n\}$ in Y. Define u_{n+1} in Y such that $T(u_n) = S(u_{n+1})$. Since $\theta(t) < t$,

$$\rho(T(u_n), T(u_{n+1})) \leq \rho(S(u_n), S(u_{n+1})).$$

Notice that $\{\rho(S(u_{n+1}), S(u_n))\}$ is bounded from below and is a decreasing sequence. Thus there exists $a \in E$ such that $\rho(S(u_{n+1}), S(u_n)) \to a$ as $n \to \infty$ (P is regular). The condition $\theta(t) < t$ implies $a = 0$.

On the other hand, there exists subsequence $\{S(u_{r_i})\}$ of $\{S(u_n)\}$ ($S(Y)$ is sequentially compactness) such that $S(u_{r_i}) \to v$ when $i \to \infty$ such that $v \in S(Y)$. Consequently, there is u in Y such that $S(u) = v$. Thus

$$\begin{aligned}\rho(S(u_{r_i}), T(u)) = & \ \rho(T(u_{r_i-1}), T(u)) \\ \leq & \ \theta(\rho(S(u_{r_i-1}), S(u))),\end{aligned}$$

so

$$\|\rho(S(u_{r_i}), T(u))\| \leq M\|\theta(\rho(S(u_{r_i-1}), S(u)))\|.$$

Then $S(u_{r_i}) \to S(u)$ and continuity of θ imply

$$\|\rho(S(u_{r_i}), T(u))\| \to 0(i \to \infty).$$

Hence $S(u_{r_i}) \to T(u)$. Also

$$\rho(S(u_{r_i}), S(u)) \to 0 \text{ as } i \to \infty.$$

Finally, $T(u) = S(u)$ (the uniqueness of the limit). The uniqueness of the coincidence point is obvious. In fact, T and S (by Theorem 1.9) have a common fixed point which is unique. □

Remark 4. Theorem 2.4 remains true if one consider R-weakly commuting maps.

Confilcts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

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