Mathematics Interdisciplinary Research **x** (202x) xx – yy

Big Finitistic Dimensions for Categories of Quiver Representations

Roghayeh Bagherian and Esmaeil Hosseini *

Abstract

Assume that $\mathcal{A}$ is a Grothendieck category and $\mathcal{R}$ is the category of all $\mathcal{A}$-representations of a given quiver $\mathcal{Q}$. If $\mathcal{Q}$ is left rooted and $\mathcal{A}$ has a projective generator, we prove that the big finitistic flat (resp. projective) dimension $\mathrm{FFD}(\mathcal{A})$ (resp. $\mathrm{FPD}(\mathcal{A})$) of $\mathcal{A}$ is finite if and only if the big finitistic flat (resp. projective) dimension of $\mathcal{R}$ is finite. When $\mathcal{A}$ is the Grothendieck category of left modules over a unitary ring R, we prove that if $\mathrm{FPD}(\mathcal{R}) < +\infty$ then any representation of $\mathcal{Q}$ of finite flat dimension has finite projective dimension. Moreover, if R is n-perfect then we show that $\mathrm{FFD}(\mathcal{R}) < +\infty$ if and only if $\mathrm{FPD}(\mathcal{R}) < +\infty$.

Keywords: quiver, representation of quiver, Grothendieck category, finitistic dimension.

2010 Mathematics Subject Classification: 18E30, 18G35, 16E05.

How to cite this article
R. Bagherian and E. Hosseini, Big finitistic dimensions for categories of quiver representations, *Math. Interdisc. Res.* **x** (202x) xx–yy.

1. Introduction

Assume that R is a ring and $\mathfrak{P}(R)$ (resp. $\mathfrak{p}(R)$) be the class of all (resp. finitely generated) left R-modules of finite projective dimension. The *big* (resp. *little*) *finitistic projective dimension* of R is defined by $\mathrm{FPD}(R) := \mathrm{supp}_{M\in\mathfrak{P}(R)}\mathrm{pd}M$ (resp. $\mathrm{fPD}(R) := \mathrm{supp}_{M\in\mathfrak{p}(R)}\mathrm{pd}M$). The finiteness of $\mathrm{fPD}(R)$ is a celebrated conjecture, called the finitistic dimension conjecture, which remains unsolved for more than 60 years. One of the reasons for the importance of $\mathrm{FPD}(R)$ lies in its

*Corresponding author (E-mail: e.hosseini@scu.ac.ir)
Academic Editor: Ali Reza Ashrafi
Received 31 January 2021, Accepted 3 June 2021
DOI: 10.22052/MIR.2021.240439.1273

relation to the fPD(R). It is clear that the inequality fPD(R) $\leq$ FPD(R) holds and hence, if FPD(R) $< +\infty$ then fPD(R) $< +\infty$. In the introduction of [2], Bass reminded that FPD(R) was introduced by Kaplansky. Furthermore, if A is a commutative noetherian ring, he proved in [3] that, the inequality FPD(A) $\geq$ dim$A$ holds over A where dim$A$ is the Krull dimension of A. The reverse inequality FPD(A) $\leq$ dim$A$ has been proved by Gruson and Raynaud in [12]. Therefore, FPD(A) is finite if and only if A is of finite Krull dimension. Therefore, FPD(A) may be infinite since Nagata's example shows that there exists a commutative noetherian ring of infinite Krull dimensions (see [17]).

Historically, finitistic dimension conjectures have been raised by Bass in [2]. In fact he presented the following two conjectures

(1) fPD(R) = FPD(R),

(2) fPD(R) $< \infty$.

They are called the first and the second Finitistic Dimension Conjecture, respectively. In 1992, Huisgen-Zimmermann showed that the first Finitistic Dimension Conjecture fails. He showed that there exists a ring R such that fPD(R) $\neq$ FPD(R) ([27, Section 3]). Later, in [25], Smalø gave an example and showed that the difference between fPD(R) and FPD(R) could be very large. However, the conjecture (2) is still an open problem and efforts are underway to find the answer, see [19].

Assume that R is a unitary ring, $\mathcal{Q}$ be an arbitrary quiver and $\mathcal{R} = \mathrm{Rep}(\mathcal{Q}, R)$ be the category of all R-representations of $\mathcal{Q}$, see [1]. The big finitistic projective (resp. injective) dimension FPD($\mathcal{R}$) (resp. FID($\mathcal{R}$)) of $\mathcal{R}$ was first studied in [7]. It was shown in [7, Proposition 3.3.1.] that FPD($\mathcal{R}$) $\leq$ FPD(R) + 1. Moreover, they proved if $\mathcal{Q}$ is not discrete then FPD($\mathcal{R}$) = FPD(R) + 1. In the present work, we study the big finitistic flat dimension in the category of quiver representations. In fact, we show that if $\mathcal{Q}$ is a left rooted quiver and $\mathcal{A}$ is a Grothendieck category with a projective (resp. flat) generator then the big finitistic projective (resp. flat) dimension of $\mathcal{A}$ is finite if and only if the big finitistic projective (resp. flat) dimension of $\mathcal{R}$ is finite, where $\mathcal{R}$ is the category of representations of $\mathcal{Q}$ by objects in $\mathcal{A}$. Furthermore, we obtain a generalization of [15, Theorem 6] in the category of quiver representations.

Before starting, let us fix some notations and definitions. A *quiver* is a directed graph $\mathcal{Q}$ whose the set of *vertices* is denoted by $V_{\mathcal{Q}}$ and the set of *arrows* is denoted by $E_{\mathcal{Q}}$. An *arrow* from $v \in V_{\mathcal{Q}}$ to $w \in V_{\mathcal{Q}}$ is denoted by $v \overset{a}{\to} w$. The *initial* (resp. *terminal*) vertex of an arrow a in $\mathcal{Q}$ is denoted by $i(a)$ (resp. $t(a)$). A sequence $a_n \cdots a_2 a_1$ of arrows in $\mathcal{Q}$ is called a path if for each $1 \leq i \leq n-1$, $t(a_i) = i(a_{i+1})$. So, a quiver $\mathcal{Q}$ can be considered as a category in which $\mathrm{V}_{\mathcal{Q}}$ is the set of all objects and for each pair $v, w \in \mathrm{V}_{\mathcal{Q}}$, $\mathrm{Hom}_{\mathcal{Q}}(v.w)$ is the set of all path from v to w. If $\mathcal{K}$ is a category, a covariant functor from $\mathcal{Q}$ to $\mathcal{K}$ is called a $\mathcal{K}$-representation of $\mathcal{Q}$, i.e. if $\mathcal{T}$ is a $\mathcal{K}$-representation of $\mathcal{Q}$ then for each $v \in V_{\mathcal{Q}}$, $\mathcal{T}(v)$ is an object of $\mathcal{K}$ and for each arrow $a : v \longrightarrow w \in E_{\mathcal{Q}}$, $\mathcal{T}(a) : \mathcal{T}(v) \longrightarrow \mathcal{T}(w)$ is a morphism in $\mathcal{K}$. In addition, morphisms between $\mathcal{K}$-representations are precisely natural transformations. Therefore, $\mathcal{K}$-representations of $\mathcal{Q}$ form a category, denoted by $\mathrm{Rep}(\mathcal{Q},\mathcal{K})$. If $\mathcal{K}$ has limits (e.g. products, pullbacks etc.), then so is $\mathrm{Rep}(\mathcal{Q},\mathcal{K})$ which are computed vertex-wise in $\mathcal{K}$. Colimits, Cokernels, Kernels, and Images in $\mathrm{Rep}(\mathcal{Q},\mathcal{K})$ are computed vertex-wise in $\mathcal{K}$. A sequence $\mathcal{T}' \to \mathcal{T} \to \mathcal{T}''$ in $\mathrm{Rep}(\mathcal{Q},\mathcal{K})$ is called exact if for each $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathcal{T}'(v) \to \mathcal{T}(v) \to \mathcal{T}''(v)$ is an exact sequence in $\mathcal{K}$. Thus, if $\mathcal{K}$ is Grothendieck category then so is $\mathrm{Rep}(\mathcal{Q},\mathcal{K})$, (see [13]). If $\mathcal{K}$ admits both products and coproducts then for each $v \in \mathrm{V}_{\mathcal{Q}}$ and each $\mathcal{K}$-representation $\mathcal{T}$ of $\mathcal{Q}$ we have the following canonical morphisms

$$\bigoplus_{t(a)=v} \mathcal{T}(i(a)) \xrightarrow{\varphi_v^{\mathcal{T}}} \mathcal{T}(v) \quad (resp. \quad \mathcal{T}(v) \xrightarrow{\psi_v^{\mathcal{T}}} \prod_{i(a)=v} \mathcal{T}(t(a))),$$

where the coproduct (resp. product) is taken over all $a \in E_{\mathcal{Q}}$ where $t(a) = v$ (resp. $i(a) = v$). Recall from [9] that a quiver $\mathcal{Q}$ is called *left rooted* if there is no path of the form $\cdots \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$ in $\mathcal{Q}$.

Let R be an associative ring with identity and $\mathcal{A}$ be the category R-Mod of all left R-modules. The study of special objects in $\mathcal{R} = \mathrm{Rep}(\mathcal{Q},\mathcal{A})$ has been interested in the literature. If $\mathcal{Q}$ is sufficiently nice, the projective (resp. flat, Gorenstein projective) R-representations of $\mathcal{Q}$ has been characterized in [10] (resp. [9], [6]).

Set Up: Throughout this work $\mathcal{A}$ is a Grothendieck category, $\mathcal{Q}$ is a left rooted quiver and $\mathcal{R} = \mathrm{Rep}(\mathcal{Q},\mathcal{A})$ is the category of all $\mathcal{A}$-representations of $\mathcal{Q}$, R is an associative ring with identity and all modules are left R-modules unless otherwise specified.

2. On the Big Finitistic Projective Dimension

This section is devoted to $\mathrm{FPD}(\mathcal{R})$ in the category $\mathcal{R}$. First, let us recall some notations and definitions from [9, Section 3]. Any morphism $h : \mathcal{Q} \longrightarrow \mathcal{Q}'$ of quivers induces the following functor

$$h^\star : \mathcal{R}' \longrightarrow \mathcal{R}$$

where $\mathcal{R}'$ is the category of all representations of $\mathcal{Q}'$ by objects in $\mathcal{A}$ and $h^\star(\mathcal{T}) = \mathcal{T} \circ h$.

The forest $P(\mathcal{Q})$ associated to $\mathcal{Q}$ is defined as follows. A vertex of $P(\mathcal{Q})$ is a path of $\mathcal{Q}$, and an arrow in $P(\mathcal{Q})$ is of the form $(p, ap) : p \longrightarrow ap$ where $a \in E_{\mathcal{Q}}$ such that $t(p) = i(a)$. Any connected component of $P(\mathcal{Q})$ is a tree whose root is $v \in V_{\mathcal{Q}}$. This component is denoted by $P(\mathcal{Q})_v$.

For a given $v \in V_{\mathcal{Q}}$, assume that $f_v : \{v\} \longrightarrow \mathcal{Q}$, $g_v : \{v\} \longrightarrow P(\mathcal{Q})_v$ are embedding morphisms and $t_v : P(\mathcal{Q})_v \longrightarrow \mathcal{Q}$ is the morphism defined by

(i) For each $p \in \mathrm{V}_{P(\mathcal{Q})_v}$, $t_v(p) = t(p)$.

(ii) For each $(p, ap) \in E_{P(\mathcal{Q})_v}$, $t_v(p, ap) = a$.

So we have the following factorization

$$v \xrightarrow{g_v} P(\mathcal{Q})_v \xrightarrow{t_v} \mathcal{Q}, \tag{1}$$

for f_v, i.e., $f_v = t_v \circ g_v$ and Equation (1) induces the following functors

$$f_v^\star : \mathrm{Rep}(\mathcal{Q}, \mathcal{A}) \longrightarrow \mathrm{Rep}(\{v\}, \mathcal{A}),$$

$$g_v^\star : \mathrm{Rep}(P(\mathcal{Q})_v, \mathcal{A}) \longrightarrow \mathrm{Rep}(\{v\}, \mathcal{A}),$$

and

$$t_v^\star : \mathrm{Rep}(\mathcal{Q}, \mathcal{A}) \longrightarrow \mathrm{Rep}(P(\mathcal{Q})_v, \mathcal{A}),$$

such that $f_v^\star = (t_v \circ g_v)^\star = g_v^\star \circ t_v^\star$.

By the same arguments that are used in [9, Section 3], one can show that $g_v^\star$ and $t_v^\star$ are exact and admit the exact left adjoints

$$g'_v : \mathrm{Rep}(\{v\}, \mathcal{A}) \longrightarrow \mathrm{Rep}(P(\mathcal{Q})_v, \mathcal{A}),$$

and

$$t'_v : \mathrm{Rep}(P(\mathcal{Q})_v, \mathcal{A}) \longrightarrow \mathrm{Rep}(\mathcal{Q}, \mathcal{A}),$$

respectively. So, for any $v \in V_\mathcal{Q}$, the functor $f'_v = t'_v \circ g'_v$ is exact and it is the left adjoint of $f_v^\star = (t_v \circ g_v)^\star = g_v^\star \circ t_v^\star$. Therefore, for each pair $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ and $\mathcal{Y} \in \mathrm{Rep}(\{v\}, \mathcal{A})$, the adjoint pair (f', f^*) of exact functors induces the following isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}(\mathcal{Q},\mathcal{A})}(f'_v(\mathcal{Y}), \mathcal{X}) \cong \mathrm{Hom}_{\mathrm{Rep}(\{v\},\mathcal{A})}(\mathcal{Y}, f_v^\star(\mathcal{X})),$$

of Abelian groups. By the same argument that are used in the proof of [9, Theorem 3.3], we deduce the following result.

Proposition 2.1. *If $\mathcal{A}$ admits a projective generator then so is $\mathcal{R}$.*

Proof. Let $\mathcal{X}$ be a representation of $\mathcal{Q}$. Since, $\mathrm{Rep}(\{v\}, \mathcal{A})$ and $\mathcal{A}$ are obviously isomorphic and $f_v^\star(\mathcal{X})$ is a representation of $\{v\}$ then there is a projective object $P \in \mathcal{A}$ and an epimorphism $P \longrightarrow f_v^\star(\mathcal{X})$. Therefore, by the adjoint property of $f'_v \mapsto f^\star$, we have a unique epimorphism $f'_v(P) \longrightarrow \mathcal{X}$ in $\mathcal{R}$. So, for each $v \in V_\mathcal{Q}$, we have an epimorphism $\alpha_v : f'_v(P_v) \longrightarrow \mathcal{X}$ where P_v is a projective representation of $\{v\}$. Since $f_v^\star$ is an exact functor, then $f'_v(P_v)$ is projective in $\mathcal{R}$ and so is $\bigoplus_{v \in V_\mathcal{Q}} f'_v(P_v)$. In addition, $\bigoplus \alpha_v : \bigoplus f'_v(P_v) \longrightarrow \mathcal{X}$ is an epimorphism, since the restriction on each α_v is. Therefore, if P is a projective generator in $\mathcal{A}$, the set $\{f'_v(P) : v \in V_\mathcal{Q}\}$ of projective $\mathcal{A}$-representations generates $\mathcal{R}$. □

Proposition 2.1 tells us that $\mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ has enough projective objects. These objects can be characterized by the same arguments that are used in [10, Theorem 3.1].

Proposition 2.2. *Let $\mathcal{Q}$ be a left rooted quiver. If $\mathcal{A}$ admits a projective generator then an $\mathcal{A}$-representation $\mathcal{P}$ of $\mathcal{Q}$ is projective if and only if the following two conditions are satisfied:*

(i) *For each $v \in V_{\mathcal{Q}}$, $\mathcal{P}(v)$ is projective in $\mathcal{A}$.*

(ii) *For each $v \in V_{\mathcal{Q}}$, the morphism $\varphi_v^{\mathcal{P}}$ is a splitting monomorphism.*

Now by the previous results, the projective dimension of $\mathcal{A}$-representations of $\mathcal{Q}$ can be defined in the usual sense and so the big finitistic projective dimension in $\mathcal{R}$ is defined as follows.

Definition 2.3. Let $\mathcal{A}$ be a Grothendieck category, the big finitistic projective dimension of $\mathcal{A}$ is defined by

$$\mathrm{FPD}(\mathcal{A}) := \sup\{\mathrm{pd}(M) | M \in \mathcal{A} \ \text{ with } \ \mathrm{pd}(M) < \infty\}.$$

The following Lemma is playing a significant role in this section. In the case $\mathcal{A}$ is the category of R-representations (R is a ring) of an arbitrary quiver, the result has been proved in [7, Lemma 3.1.5].

Lemma 2.4. *Let $\mathcal{Q}$ be a left rooted quiver and $\mathcal{P}$ be a vertex-wise projective representation of $\mathcal{Q}$. Then,* $\mathrm{pd}(\mathcal{P}) \leq 1$.

Proof. We show that $\mathcal{P}$ admits a projective resolution of length 1. Consider the following short exact sequence in $\mathcal{R}$

$$0 \longrightarrow \mathcal{K} \xrightarrow{g} \mathcal{M} \xrightarrow{f} \mathcal{P} \longrightarrow 0,$$

where $\mathcal{M}$ is projective. By Proposition 2.2, for each $v \in V_{\mathcal{Q}}$, $\mathcal{M}(v_i)$ is a projective object in $\mathcal{A}$ and for each vertex $v \in V_{\mathcal{Q}}$, $\bigoplus_{t(a)=v} \mathcal{M}(i(a)) \xrightarrow{\varphi_v^{\mathcal{M}}} \mathcal{M}(v)$ is a splitting monomorphism of projective objects in $\mathcal{A}$. Now, consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \bigoplus_{t(a)=v} \mathcal{K}(i(a)) & \xrightarrow{i_1} & \bigoplus_{t(a)=v} \mathcal{M}(i(a)) & \xrightarrow{f_1} & \bigoplus_{t(a)=v} (\mathcal{P}(i(a)) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_v^{\mathcal{K}}} & & \downarrow{\scriptstyle \varphi_v^{\mathcal{M}}} & & \downarrow{\scriptstyle \varphi_v^{\mathcal{P}}} & & \\
0 & \longrightarrow & \mathcal{K}(v) & \xrightarrow{i_2} & \mathcal{M}(v) & \xrightarrow{f_2} & \mathcal{P}(v) & \longrightarrow & 0
\end{array}$$

where $i_1 = \bigoplus_{t(a)=v} g(i(a)), i_2 = g(v), f_1 = \bigoplus_{t(a)=v} f(i(a))$ and $f_2 = f(v)$. We will show that $\bigoplus_{t(a)=v} \mathcal{K}(i(a)) \to \mathcal{K}(v)$ is a splitting monomorphism of projective objects in $\mathcal{A}$. Because $\mathcal{Q}$ is left rooted, by Proposition 2.2, it is enough to show that each $\mathcal{K}(v_i)$ is a projective object in $\mathcal{A}$ and the morphism $\bigoplus_{t(a)=v} \mathcal{K}(i(a)) \xrightarrow{\varphi_v^{\mathcal{K}}} \mathcal{K}(v)$ is a splitting monomorphism. Clearly, for each $v \in V_{\mathcal{Q}}$, $\mathcal{K}(v)$ is projective, since $\mathcal{M}(v) \simeq \mathcal{K}(v) \oplus \mathcal{P}(v)$ and $\mathcal{M}(v)$ is projective. On the other hand since $\varphi_v^{\mathcal{M}}$ and i_1 are splitting monomorphisms, then they have sections β and s respectively. Therefor, $\varphi_v^{\mathcal{K}}$ is the section of $\alpha = s \circ \beta \circ i_2$ and so we are done. $\square$

Now, we can prove the main result of this section. In the case $\mathcal{A}$ is the category of R-representations (R is a ring) of an arbitrary quiver, it was shown in [7, Proposition 3.1.5] that if $\mathrm{FPD}(\mathcal{A})$ is finite then $\mathrm{FPD}(\mathcal{R}) \leq \mathrm{FPD}(\mathcal{A})$ is finite.

Theorem 2.5. *Let $\mathcal{Q}$ be a left rooted quiver. Then* $\mathrm{FPD}(\mathcal{A})$ *is finite if and only if* $\mathrm{FPD}(\mathcal{R})$ *is finite.*

Proof. Assume that $\mathrm{FPD}(\mathcal{A}) < +\infty$ and $\mathcal{X}$ is an object in $\mathcal{R}$ of finite projective dimension. Then, for any $v \in V_{\mathcal{Q}}$, $\mathcal{X}(v)$ is an object in $\mathcal{A}$ of finite projective dimension and so, for each $v \in V_{\mathcal{Q}}$, $\mathrm{pd}\mathcal{X}(v) \leq \mathrm{FPD}(\mathcal{A})$. Then, by Lemma 2.4, $\mathrm{pd}\mathcal{X} \leq \mathrm{FPD}(\mathcal{A})+1$. This shows that $\mathrm{FPD}(\mathcal{R}) < \mathrm{FPD}(\mathcal{A})+1$. Conversely, assume that $\mathrm{FPD}(\mathcal{R}) < +\infty$ and M is an object in $\mathcal{A}$ of finite projective dimension. Let $v \in V_{\mathcal{Q}}$ be an arbitrary vertex and $f_v : \{v\} \longrightarrow \mathcal{Q}$ be the inclusion. Then $f'_v(M)$ is a representation of $\mathcal{Q}$ of finite projective dimension. Therefore, $\mathrm{pd}f'_v(M) \leq \mathrm{FPD}(\mathcal{R})$. So by Proposition 2.2, $\mathrm{pd}M \leq \mathrm{FPD}(\mathcal{R})$. This implies that $\mathrm{FPD}(\mathcal{A}) \leq \mathrm{FPD}(\mathcal{R})$. □

Remark 1. Let A be a commutative noetherian ring, $\mathcal{A}$ be the category of all A-modules and $\mathcal{R} = \mathrm{Rep}(\mathcal{Q}, \mathcal{A})$. The Bass-Gruson-Raynaud Theorem states that $\mathrm{FPD}(A) = \mathrm{dim}A$ where $\mathrm{dim}A$ is the Krull dimension of A. So, if $\mathcal{Q}$ is a left rooted quiver, then, Theorem 2.5 yields:

(i) If A has finite Krull dimension then $\mathrm{FPD}(\mathcal{R}) \leq \mathrm{dim}A + 1$.

(ii) If A has infinite Krull dimension, then, $\mathrm{FPD}(\mathcal{R}) = \infty$.

Remark 2. Let R be a unitary ring and $\mathcal{A}$ be the category of all unital left R-modules, (i.e. R-modules ${}_RM$ such that $RM = M$). Suppose that X is the set of all paths in $\mathcal{Q}$. The path ring $R\mathcal{Q}$ of $\mathcal{Q}$ is defined as the free left R-module over X where the composition between two paths defines a multiplication in $R\mathcal{Q}$. So $R\mathcal{Q}$ is a ring with enough idempotents (see [26, Ch.10, §. 49]). Let $R\mathcal{Q}$-Mod be the category of unital left $R\mathcal{Q}$-modules. Clearly, $R\mathcal{Q}$ is a generator of $R\mathcal{Q}$-Mod. Since $R\mathcal{Q}$-Mod and $\mathcal{R} = \mathrm{Rep}(\mathcal{Q}, R)$ are equivalent categories then $R\mathcal{Q}$ is a projective generator for $\mathcal{R}$. It was shown in [11] that if $\mathrm{FPD}(R) = n$ then $\mathrm{FPD}(R\mathcal{Q}) \leq n+1$. Indeed, if $\mathcal{Q}$ is not discrete then $\mathrm{FPD}(R\mathcal{Q}) \leq n+1$ by [11, Proposition 3.5]. Notice that, a stronger statement has been proved in [7]. It was shown in [7, Proposition 3.3.1] that if $\mathcal{Q}$ is an arbitrary quiver, then, $\mathrm{FPD}(\mathcal{R}) \leq \mathrm{FPD}(R)+1$. Furthermore, if $\mathcal{Q}$ is not discrete then $\mathrm{FPD}(\mathcal{R}) = \mathrm{FPD}(R)+1$. Furthermore, the same statement holds for the finitistic injective dimension.

3. On the Big Finitistic Flat Dimension

Thoughout this section, $\mathcal{A}$ is the category of all R-modules. By the previous section, the category $\mathcal{R} = \mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ of all $\mathcal{A}$-representations of $\mathcal{Q}$ is a Grothendieck categories with a projective generator. Recall that an object $\mathcal{F}$ in $\mathcal{R}$ is said to be *flat* if it is a directed limit of a directed system of projective objects in $\mathcal{R}$. The flat representations of $\mathcal{Q}$ has been characterized in [9, Theorem 3.7] as follows.

Proposition 3.1. *An object $\mathcal{F} \in \mathcal{R}$ is flat if and only if*

(i) *For each $v \in \mathrm{V}_{\mathcal{Q}}$, $\mathcal{F}(v)$ is a flat object in $\mathcal{A}$.*

(ii) *For any $v \in \mathrm{V}_{\mathcal{Q}}$, the canonical morphism $\varphi_v^{\mathcal{F}}$ is a pure monomorphism in $\mathcal{A}$.*

Let $\mathrm{Flat}(\mathcal{R})$ be the class of all flat objects in $\mathcal{R}$ and $\mathrm{Flat}(\mathcal{R})^{\perp}$, be the class of all objects $\mathcal{C}$ in $\mathcal{R}$ such that $\mathrm{E}xt^1_{\mathcal{R}}(\mathcal{F},\mathcal{C}) = 0$ for every $\mathcal{F} \in \mathrm{Flat}(\mathcal{R})$. It was shown in [9] that the pair $(\mathrm{Flat}(\mathcal{R}),\mathrm{Flat}(\mathcal{R})^{\perp})$ is a complete hereditary cotorsion theory (for definitions and more details see [8]). So, any object $\mathcal{X}$ in $\mathcal{R}$ admits a flat cover and hence it has a minimal flat resolution. It follows that the flat dimension $\mathrm{fd}\mathcal{X}$ of $\mathcal{X}$ can be defined in the usual sense. To study more results in this direction the reader is referred to [6, 13]. Moreover, recently, the homotopy category of flat R-representations of $\mathcal{Q}$ has been studied by Eshraghi in [5].

The finitistic flat dimension was first introduced by Bass in [2] as follows

$$\mathrm{FFD}(R) = \sup\{\mathrm{fdM} | M \text{ is a left } R\text{-module with fdM} < \infty\}.$$

He compared several finitistic dimensions of R and proved in [2, pp. 487(8.3)] that if R is a left perfect ring, then there is the inequality, $\mathrm{FPD}(R) = \mathrm{FFD}(R) \geq \mathrm{fPD}(R)$.

The big *finitistic flat dimension* $\mathrm{FFD}(\mathcal{R})$ of $\mathcal{R} = \mathrm{Rep}(\mathcal{Q},\mathcal{A})$ is defines analogous to $\mathrm{FFD}(R)$. In this section we generalize [2, pp. 487(8.3)] and prove that if $\mathcal{R} = \mathrm{Rep}(\mathcal{Q},\mathcal{A})$ is n-perfect then $\mathrm{FFD}(\mathcal{R})$ is finite if and only if $\mathrm{FPD}(\mathcal{R})$ is finite. We recall from [14], for a given non-negative integer n, a category $\mathcal{A}$ (resp $\mathcal{R}$) is called n-perfect if for any flat object F (resp. $\mathcal{F}$) in $\mathcal{A}$ (resp. $\mathcal{R}$), we have $\mathrm{pd}F \leq n$ (resp. $\mathrm{pd}\mathcal{F} \leq n$).

Lemma 3.2. *Let $\mathcal{A}$ be an n-perfect category. Then the following conditions hold.*

(i) *If M is an R-module then,* fdM *is finite if and only if* $\mathrm{pd}M$ *is finite.*

(ii) *If $\mathcal{M}$ is an R-representation then,* $\mathrm{fd}\mathcal{M}$ *is finite if and only if* $\mathrm{pd}\mathcal{M}$ *is finite.*

Proof. The proof is straightforward. □

For more results related to Lemma 3.2 the reader is referred to [14, 16, 20, 21, 22, 23, 24].

Assume that $v \in V_{\mathcal{Q}}$ and $f_v : \{v\} \longrightarrow \mathcal{Q}$ is the embedding morphism. As stated in the previous section, we have the adjoint pair (f'_v, f^*_v) of exact functors, i.e. we have the following exact functors

$$f_v^{\star} : \mathrm{Rep}(\mathcal{Q},\mathcal{A}) \longrightarrow \mathrm{Rep}(\{v\},\mathcal{A}),$$

$$f'_v : \mathrm{Rep}(\{v\}, \mathcal{A}) \longrightarrow \mathrm{Rep}(\mathcal{Q}, \mathcal{A}),$$

such that for each $\mathcal{X} \in \mathrm{Rep}(\mathcal{Q}, \mathcal{A})$ and $\mathcal{Y} \in \mathrm{Rep}(\{v\}, \mathcal{A})$ we have the following isomorphism

$$\mathrm{Hom}_{\mathrm{Rep}(\mathcal{Q},\mathcal{A})}(f'_v(\mathcal{Y}), \mathcal{X}) \cong \mathrm{Hom}_{\mathrm{Rep}(\{v\},\mathcal{A})}(\mathcal{Y}, f^{\star}_v(\mathcal{X})),$$

of Abelian groups. The following result shows that f'_v preserves flatness in the sense that it maps flat modules to flat representations.

Lemma 3.3. *Let (f'_v, f^*_v) be as above. Then, f'_v preserves flatness in the sense that it maps flat modules to flat representations.*

Proof. Let $\mathcal{F}$ be a flat object in $\mathcal{A}$. By definition, there exists a directed system $\{\mathcal{P}_i : f_{ij}\}_{i\in I}$ of projective left R-modules such that $\mathcal{F} = \varinjlim_{i\in\mathbb{I}} \mathcal{P}_i$. Sine f'_v is a left adjoint, it preserves directed limits. Hence, $f'_v(\mathcal{F}) = f'_v(\varinjlim_{i\in\mathbb{I}} \mathcal{P}_i) = \varinjlim_{i\in\mathbb{I}} f'_v(\mathcal{P}_i)$. So we are done by Proposition 2.2. □

Lemma 3.4. *Let $\mathcal{F}$ be a vertex-wise flat $\mathcal{A}$-representation of $\mathcal{Q}$. Then,* $\mathrm{fd}(\mathcal{F}) \leq 1$.

Proof. By [9, Theorem 4.3], there is a short exact sequence

$$0 \longrightarrow \mathcal{K} \stackrel{g}{\longrightarrow} \mathcal{M} \stackrel{f}{\longrightarrow} \mathcal{F} \longrightarrow 0,$$

of $\mathcal{A}$-representations of $\mathcal{Q}$ such that $\mathcal{M}$ is flat. By Proposition 3.1, for each $v \in V_{\mathcal{Q}}$, $\mathcal{M}(v_i)$ is a flat R-module and for each $v \in V_{\mathcal{Q}}$, $\bigoplus_{t(a)=v} \mathcal{M}(i(a)) \stackrel{\varphi^{\mathcal{M}}_v}{\longrightarrow} \mathcal{M}(v)$ is a pure monomorphism of flat R-modules. Now consider the following commutative diagram

$$\begin{array}{ccccccccc}
0 \rightarrow & \bigoplus_{t(a)=v} \mathcal{K}(i(a)) & \stackrel{i_1}{\rightarrow} & \bigoplus_{t(a)=v} \mathcal{M}(i(a)) & \stackrel{f_1}{\rightarrow} & \bigoplus_{t(a)=v} (\mathcal{F}(i(a)) & \rightarrow 0 \\
 & \downarrow \varphi^{\mathcal{K}}_v & & \downarrow \varphi^{\mathcal{M}}_v & & \downarrow \varphi^{\mathcal{F}}_v & \\
0 \longrightarrow & \mathcal{K}(v) & \stackrel{i_2}{\longrightarrow} & \mathcal{M}(v) & \stackrel{f_2}{\longrightarrow} & \mathcal{F}(v) & \longrightarrow 0
\end{array}$$

where $i_1 = \bigoplus_{t(a)=v} g(i(a)), i_2 = g(v), f_1 = \bigoplus_{t(a)=v} f(i(a))$ and $f_2 = f(v)$. Now we show that $\bigoplus_{t(a)=v} \mathcal{K}(i(a)) \to \mathcal{K}(v)$ is a pure monomorphism of flat R-modules. It is known that the class of all flat left R-modules is closed under pure submodules and pure extensions, so the purity of $\varphi^{\mathcal{M}}_v$, i_1 and i_2 imply the purity $\varphi^{\mathcal{K}}_v$. Therefore, by Proposition 3.1, $\mathcal{K}$ is a flat $\mathcal{A}$-representation of $\mathcal{Q}$. □

Theorem 3.5. $\mathrm{FFD}(\mathcal{R}) < \infty$ *if and only if* $\mathrm{FFD}(R) < \infty$

Proof. Let $\mathrm{FFD}(R) < +\infty$ and $\mathcal{X}$ be an R-representation of $\mathcal{Q}$ of finite flat dimension. Then, for any $v \in V_{\mathcal{Q}}$, $\mathcal{X}(v)$ is an R-module of finite flat dimension. So, for each $v \in V_{\mathcal{Q}}$, $\mathrm{fd}\mathcal{X}(v) \leq \mathrm{FFD}(R)$. Consequently, by Lemma 3.4, $\mathrm{fd}\mathcal{X} \leq \mathrm{FFD(R)}+1$. Therefore, $\mathrm{FFD}(\mathcal{R}) \leq \mathrm{FFD}(R) + 1$.

Conversely, assume that $\mathrm{FFD}(\mathcal{R}) < +\infty$ and M is a R-module of finite flat dimension. Let $v \in V_{\mathcal{Q}}$ be an arbitrary vertex and $f_v : \{v\} \longrightarrow \mathcal{Q}$ be the inclusion. Then, by Lemma 3.3, $f'_v(M)$ is a representation of $\mathcal{Q}$ of finite flat dimension. Therefore, $\mathrm{fd} f'_v(M) \leq \mathrm{FFD}(\mathcal{R})$. So $\mathrm{fd} M \leq \mathrm{FFD}(\mathcal{R})$. This implies that $\mathrm{FFD}(R) \leq \mathrm{FFD}(\mathcal{R})$ and finished the proof. □

Proposition 3.6. *Assume that* $\mathcal{R} = \mathrm{Rep}(\mathcal{Q}, R\text{-Mod})$. *Then, there are positive integers* n *and* m *such that* $\mathcal{R}$ *is* m*-perfect if and only if* R *is* n*-perfect.*

Proof. Assume that R is an n-perfect ring and $\mathcal{X}$ is a flat R-representation of $\mathcal{Q}$. Then, for each $v \in V_{\mathcal{Q}}$, $\mathrm{pd}\mathcal{X}(v) \leq n$. Then, by Lemma 2.4, $\mathrm{pd}\mathcal{X} \leq n+1$ and hence $m = n+1$. Conversely, assume that $\mathcal{R}$ is m-perfect and F is a flat R-module. Then by Lemma 3.3, for an arbitrary $v \in V_{\mathcal{Q}}$, $f'_v(F)$ is a flat representation of $\mathcal{Q}$ and hence $\mathrm{pd} f'_v(F) \leq m$. Consequently, $\mathrm{pd} F \leq m$ and hence R is $m = n$-perfect. □

Proposition 3.7. *Let* $\mathcal{A}$ *be an* n*-perfect category. Then* $\mathrm{FPD}(\mathcal{R}) < \infty$ *if and only if* $\mathrm{FFD}(\mathcal{R}) < \infty$.

Proof. Let $\mathrm{FPD}(\mathcal{R}) < \infty$ and $\mathcal{X}$ be an object in $\mathcal{R}$ of finite flat dimension s. Then, by Lemma 3.2 and Proposition 3.6, $\mathrm{pd}\mathcal{X}$ is finite and hence $\mathrm{fd}\mathcal{X} \leq \mathrm{pd}\mathcal{X} \leq \mathrm{FPD}(\mathcal{R})$. Therefore, $\mathrm{FFD}(\mathcal{R}) \leq \mathrm{FPD}(\mathcal{R})$. The converse is trivial. □

Here, we show that a generalization of [15, Proposition 6] holds in the category $\mathcal{R}$.

Theorem 3.8. *Assume that* $\mathrm{FPD}(\mathcal{R})$ *is finite. Then any object in* $\mathcal{R}$ *of finite flat dimension has finite projective dimension.*

Proof. Let $\mathrm{FPD}(\mathcal{R})$ be finite. By Theorem 2.5, we deduce that $\mathrm{FPD}(\mathcal{A})$ is finite. Then by [15, Proposition 6], any flat module has finite projective dimension. Indeed, there exists an integer n such that $\mathcal{A}$ is n-perfect. So we are done by Proposition 3.6. □

Aknowledgements. The authors wish to thank the referees for the valuable comments and suggestions in shaping the paper into its present form.

Conflicts of Interest. The author declares that there are no conflicts of interest regarding the publication of this article.

References

[1] I. Assem, D. Simson and A. Skowronski, *Elements of the Representation Theory of Associative Algebras*, Vol. 1: Techniques of Representation Theory (London Mathematical Society Student Texts, Series Number 65), Cambridge University Press, Cambridge-New York, 2006.

[2] H. Bass, Finitistic dimension and a homological generalization of semiprimary rings, *Trans. Amer. Math. Soc.* **95** (1960) 390–405.

[3] H. Bass, Injective dimension in noetherian rings, *Trans. Amer. Math. Soc.* **102** (1962) 18–29.

[5] H. Eshraghi, Homotopy category of cotorsion flat representations of quivers, *Math. Interdisc. Res.* **5** (2020) 279–294.

[6] H. Eshraghi, R. Hafezi, E. Hosseini and Sh. Salarian, Cotorsion theory in the category of quiver representations, *J. Alg. App.* **12** (6) (2013) 1350005.

[7] H. Eshraghi, R. Hafezi and Sh. Salarian, Total acyclicity for complexes of representations of quivers, *Comm. Algebra* **41** (12) (2013).

[8] E. E. Enochs and O. M. G. Jenda, Relative homological algebra, Vol. 1: Second revised and extended edition, *de Gruyter Exp. Math.* 30. Walter de Gruyter GmbH & Co. KG, Berlin, 2011.

[9] E. Enochs, L. Oyonarte and B. Torrecillas, Flat covers and flat representations of quivers, *Comm. Algebra* **32** (4) (2004) 1319–1338.

[10] E. Enoches and S. Estrada, Projective representation of quivers, *Comm. Algebra* **33** (10) (2005) 3467–3478.

[11] S. Estrada and S. Ozdemir, Finitistic dimension conjectures for representations of quivers, *Turk. J. Math.* **13** (2013) 585–591.

[12] L. Gruson and L. Raynaund, Critères de platitude et de projectivé, Techniqus de "platification" d'un module, *Invent. Math.* **13** (1971) 1–89.

[13] H. Holm and P. Jørgensen Cotorsion pairs in categories of quiver representations, *Kyoto J. Math.* **59** (3) (2019) 575–606.

[14] E. Hosseini, Flat quasi–coherent sheaves of finite cotorsion dimension, *J. Alg. App.* **141** (2017) 753–762.

[15] C. U. Jensen, On the vanishing of lim, *J. Algebra* **15** (1970) 151–166.

[16] P. Jørgensen, Finite flat and projective dimension, *Comm. Algebra* **33** (2005) 2275–2279.

[17] M. Nagata, Local Rings, Interscience, John Wiley & Sons, New york, 1962.

[19] J. Rickard, Unbounded derived categories and the finitistic dimension conjecture, *Adv. Math.* **354** (2019) 1–21.

[20] D. Simson, A remark on projective dimension of flat modules, *Math. Ann.* **209** (1974) 181–182.

[21] D. Simson, On pure global dimension of locally finitely presented Grothendieck categories, *Fund. Math.* **96** (2) (1977) 91–116.

[22] D. Simson, $\aleph$–flat and $\aleph$–projective modules, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **20** (1972) 109–114.

[23] D. Simson, On the structure of flat modules, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **20** (1975) 115–120.

[24] D. Simson, Note on projective modules, *Bull. Acad. Pol. Sci. Ser. Sci. Math. Astron. Phys.* **17** (1969) 355–359.

[25] S. O. Smalø, Homological differences between finite and infinite dimensional representations of algebras, Infinite length modules (Bielefeld, 1998), 425–439, Trends Math., Birkhäuser, Basel, 2000.

[26] R. Wisbauer, Foundations of Module and Ring Theory, Reading: Gordon and Breach, 1991.

[27] B. Zimmerman-Huisgen. The finitistic dimension conjecture–a tale 3.5 decades Abelian groups and modules, *Math. Appl.* **343** (1995) 501–517.

R. Bagherian
Department of Mathematics,
Isfahan University of Technology,
Isfahan, I. R. Iran
e-mail: r.bagherian@math.iut.ac.ir

E. Hosseini
Department of Mathematics,
Shahid Chamran University of Ahvaz,
Ahvaz, I. R. Iran
e-mail: e.hosseini@scu.ac.ir