Mathematics Interdisciplinary Research **X** (202x) $xx - yy$

Conference Paper

Three Constructions on Graphs and Distance-Based Invariants

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Abstract

Many graphs are constructed from simpler ones by the use of operations on graphs, and as a consequence, the properties of the resulting constructions are strongly related to the properties of their constituents. This paper is concerned with computing some distance-based graph invariants for three constructions on graphs namely double graph, extended double cover, and strong double graph.

Keywords: distance in graph, graph invariant, composite graph.

2010 Mathematics Subject Classification: 05C12, 05C76.

How to cite this article
M. Azari, Three constructions on graphs and distance-based invariants, *Math. Interdisc. Res.* **x** (202x) xx-yy.

1. Introduction

Topological indices are numerical parameters of a graph that characterize its topology and are invariant under graph isomorphism. They are used in the development of quantitative structure-activity relationships (QSARs) and quantitative structure-property relationships (QSPRs) in which the biological activity or other properties of molecules are correlated with their chemical structure [10]. A large number of topological indices have been introduced so far with different levels of success in QSAR/QSPR researches. They are divided into different categories,

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Academic Editor: Tomislav Došlić
Received 5 June 2021, Accepted 21 June 2021
DOI: 10.22052/MIR.2021.242881.1292

two of the most famous of which are distance-based and degree-based indices. One of the important subcategory of distance-based topological indices is introduced based on the graph theoretical notion of eccentricity, most of which have been recognized as effective tools for predicting pharmaceutical and biological properties of diverse nature.

Many graphs are formed from simpler ones which serve as their fundamental building blocks, and as a consequence, the properties of these composite graphs are strongly related to the properties of their components. Three of them which have attracted much attention in recent years are double graph, extended double cover, and strong double graph (see [1, 2, 4, 5, 11, 13, 14, 17]). The purpose of this paper is to describe the connections between some distance-based graph invariants of these constructions with the corresponding invariants of their constituents.

2. Definitions and Preliminaries

All over the paper, G is considered to be a simple connected graph with n vertices and m edges, the vertex and edge sets of which are denoted by $V(G)$ and $E(G)$, respectively. The notations $d_G(u)$, $\varepsilon_G(u)$, and $\delta_G(u)$ related to a vertex $u \in V(G)$, represent the degree of u, eccentricity of u, and sum of degrees of all neighbors of u in G, respectively. For the edge $e = uv \in E(G)$, $n_u(e|G)$ represents the number of vertices of G which are closer to u than to v and similarly for $n_v(e|G)$.

Here, we give the definitions and some mathematical properties of double graph, extended double cover, and strong double graph. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$.

Definition 2.1. The *double graph* $D[G]$ of G is made from two distinct copies $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of G by keeping the primary edge set of each copy and adding the edges $x_i y_j$ and $x_j y_i$ for every edge $v_i v_j \in E(G)$.

Evidently, $D[G]$ has $2n$ vertices and $4m$ edges. The following results follow easily from definition 2.1.

Lemma 2.2. *The following relations hold.*

(i) $d_{D[G]}(x_i) = d_{D[G]}(y_i) = 2d_G(v_i)$.

(ii) $\varepsilon_{D[G]}(x_i) = \varepsilon_{D[G]}(y_i) = \begin{cases} 2 & \text{if } \varepsilon_G(v_i) = 1, \\ \varepsilon_G(v_i) & \text{if } \varepsilon_G(v_i) \geq 2. \end{cases}$

(iii) $\delta_{D[G]}(x_i) = \delta_{D[G]}(y_i) = 4\delta_G(v_i)$.

(iv) *If* $v_i v_j \in E(G)$, *then*
$n_{x_i}(x_i x_j|D[G]) = n_{y_i}(y_i y_j|D[G]) = n_{x_i}(x_i y_j|D[G]) = n_{y_i}(x_j y_i|D[G]) = 2n_{v_i}(v_i v_j|G)$,
$n_{x_j}(x_i x_j|D[G]) = n_{y_j}(y_i y_j|D[G]) = n_{y_j}(x_i y_j|D[G]) = n_{x_j}(x_j y_i|D[G]) = 2n_{v_j}(v_i v_j|G)$.

Remark 1. From part (ii) of Lemma 2.2, it can be easily seen that, if G contains a vertex of eccentricity 1, then each vertex of $D[G]$ has eccentricity 2.

Definition 2.3. The *extended double cover* $ED[G]$ of G is a bipartite graph with bipartition $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$, in which x_i is adjacent with y_j if and only if $i = j$ or $v_i v_j \in E(G)$.

It is obvious that, $ED[G]$ has $2n$ vertices and $n + 2m$ edges. The following results follow directly from Definition 2.3.

Lemma 2.4. *The following relations hold.*

(i) $d_{ED[G]}(x_i) = d_{ED[G]}(y_i) = d_G(v_i) + 1$.

(ii) $\varepsilon_{ED[G]}(x_i) = \varepsilon_{ED[G]}(y_i) = \varepsilon_G(v_i) + 1$.

(iii) $\delta_{ED[G]}(x_i) = \delta_{ED[G]}(y_i) = \delta_G(v_i) + 2d_G(v_i) + 1$.

Definition 2.5. The *strong double graph* $SD[G]$ of G is made by taking two distinct copies $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$ of G by keeping the primary edge set of each copy and by joining with an edge the vertex x_i with the closed neighborhood of the vertex y_i for all $1 \leq i \leq n$.

It is clear that, $SD[G]$ has $2n$ vertices and $4m + n$ edges. The following results follow directly from definition 2.5.

Lemma 2.6. *The following relations hold.*

(i) $d_{SD[G]}(x_i) = d_{SD[G]}(y_i) = 2d_G(v_i) + 1$.

(ii) $\varepsilon_{SD[G]}(x_i) = \varepsilon_{SD[G]}(y_i) = \varepsilon_G(v_i)$.

(iii) $\delta_{SD[G]}(x_i) = \delta_{SD[G]}(y_i) = 4\delta_G(v_i) + 4d_G(v_i) + 1$.

(iv) *If* $v_i v_j \in E(G)$, *then*
$n_{x_i}(x_i x_j|SD[G]) = n_{y_i}(y_i y_j|SD[G]) = n_{x_i}(x_i y_j|SD[G]) = n_{y_i}(x_j y_i|SD[G])$
$= 2n_{v_i}(v_i v_j|G) - 1$,
$n_{x_j}(x_i x_j|SD[G]) = n_{y_j}(y_i y_j|SD[G]) = n_{y_j}(x_i y_j|SD[G]) = n_{x_j}(x_j y_i|SD[G])$
$= 2n_{v_j}(v_i v_j|G) - 1$.

(v) *For each* $1 \leq i \leq n$, $n_{x_i}(x_i y_i|SD[G]) = n_{y_i}(x_i y_i|SD[G]) = 1$.

We end this section with the following simple lemma.

Lemma 2.7. *For positive real numbers* p *and* q,

$$\frac{1}{p+q} \leq \frac{1}{4}\left(\frac{1}{p} + \frac{1}{q}\right),$$

with equality if and only if $p = q$.

3. Main Results

In this section, we introduce some distance-based invariants of graph G and study them for double graph, extended double cover, and strong double graph of G. Following the notations introduced in the previous section, let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(D[G]) = V(ED[G]) = V(SD[G]) = \{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n\}$. We denote by n' the number of vertices with eccentricity 1 in G.

3.1 Zagreb Eccentricity Indices

Vukičević and Graovac [19] introduced the *first and second Zagreb eccentricity indices* of G as

$$E_1(G) = \sum_{i=1}^{n} \varepsilon_G(v_i)^2, \quad E_2(G) = \sum_{v_iv_j \in E(G)} \varepsilon_G(v_i)\varepsilon_G(v_j),$$

and Xu *et al.* [20] proposed the *third Zagreb eccentricity index* of G as

$$E_3(G) = \sum_{v_iv_j \in E(G)} \big|\varepsilon_G(v_i) - \varepsilon_G(v_j)\big|.$$

3.1.1 First Zagreb Eccentricity Index

Theorem 3.1. *The first Zagreb eccentricity index of $D[G]$ is given by*

$$E_1(D[G]) = 2\big(E_1(G) + 3n'\big). \tag{1}$$

Proof. By Definition 2.1 and Lemma 2.2,

$$\begin{aligned} E_1(D[G]) =& \sum_{i=1}^{n} \varepsilon_{D[G]}(x_i)^2 + \sum_{i=1}^{n} \varepsilon_{D[G]}(y_i)^2 = 2\Big[\sum_{\varepsilon_G(v_i)=1} 2^2 + \sum_{\varepsilon_G(v_i)\geq 2} \varepsilon_G(v_i)^2\Big] \\ =& 2\Big[\sum_{\varepsilon_G(v_i)=1} (2^2 - 1^2) + \sum_{i=1}^{n} \varepsilon_G(v_i)^2\Big] = 2\big(3n' + E_1(G)\big), \end{aligned}$$

from which Equation (1) follows. □

From Remark 1 and Theorem 3.1, we conclude that:

Corollary 3.2.

$$E_1(D[G]) = \begin{cases} 8n & \text{if } n' \neq 0, \\ 2E_1(G) & \text{if } n' = 0. \end{cases}$$

The invariant $\zeta(G) = \sum_{u \in V(G)} \varepsilon_G(u)$ is known as the *total eccentricity* of G.

Theorem 3.3. *The first Zagreb eccentricity index of $ED[G]$ is given by*

$$E_1(ED[G]) = 2\big(E_1(G) + 2\zeta(G) + n\big). \tag{2}$$

Proof. By Definition 2.3 and Lemma 2.4,

$$\begin{aligned} E_1(ED[G]) =& \sum_{i=1}^{n} \varepsilon_{ED[G]}(x_i)^2 + \sum_{i=1}^{n} \varepsilon_{ED[G]}(y_i)^2 = 2\sum_{i=1}^{n} \big(\varepsilon_G(v_i)+1\big)^2 \\ =& 2\sum_{i=1}^{n} \big(\varepsilon_G(v_i)^2 + 2\varepsilon_G(v_i) + 1\big) = 2\big(E_1(G) + 2\zeta(G) + n\big), \end{aligned}$$

from which Equation (2) follows. □

Theorem 3.4. *The first Zagreb eccentricity index of $SD[G]$ is given by*

$$E_1(SD[G]) = 2E_1(G). \tag{3}$$

Proof. By Definition 2.5 and Lemma 2.6,

$$E_1(SD[G]) = \sum_{i=1}^{n} \varepsilon_{SD[G]}(x_i)^2 + \sum_{i=1}^{n} \varepsilon_{SD[G]}(y_i)^2 = 2\sum_{i=1}^{n} \varepsilon_G(v_i)^2 = 2E_1(G),$$

and Equation (3) follows. □

3.1.2 Second Zagreb Eccentricity Index

Theorem 3.5. *The second Zagreb eccentricity index of $D[G]$ is given by*

$$E_2(D[G]) = 4\Big(E_2(G) + 3\binom{n'}{2} + 2n'(n-n')\Big). \tag{4}$$

Proof. By Definition 2.1 and Lemma 2.2,

$$\begin{aligned} E_2(D[G]) =& \sum_{x_i x_j \in E(D[G])} \varepsilon_{D[G]}(x_i)\varepsilon_{D[G]}(x_j) + \sum_{y_i y_j \in E(D[G])} \varepsilon_{D[G]}(y_i)\varepsilon_{D[G]}(y_j) \\ &+ \sum_{x_i y_j \in E(D[G])} \varepsilon_{D[G]}(x_i)\varepsilon_{D[G]}(y_j) + \sum_{x_j y_i \in E(D[G])} \varepsilon_{D[G]}(x_j)\varepsilon_{D[G]}(y_i) \\ =& 4\Big[\sum_{\substack{v_i v_j \in E(G): \\ \varepsilon_G(v_i)=\varepsilon_G(v_j)=1}} (2\times 2) + \sum_{\substack{v_i v_j \in E(G): \\ \varepsilon_G(v_i)=1, \varepsilon_G(v_j)=2}} (2\times 2) \\ &+ \sum_{\substack{v_i v_j \in E(G): \\ \varepsilon_G(v_i), \varepsilon_G(v_j)\geq 2}} \varepsilon_G(v_i)\varepsilon_G(v_j)\Big] \end{aligned}$$

$$=4\Big[\sum_{\substack{v_iv_j\in E(G):\\ \varepsilon_G(v_i)=\varepsilon_G(v_j)=1}}\big(4-(1\times 1)\big)+\sum_{\substack{v_iv_j\in E(G):\\ \varepsilon_G(v_i)=1,\varepsilon_G(v_j)=2}}\big(4-(1\times 2)\big)$$

$$+\sum_{v_iv_j\in E(G)}\varepsilon_G(v_i)\varepsilon_G(v_j)\Big]=4\Big[3\binom{n'}{2}+2n'(n-n')+E_2(G)\Big],$$

hence Equation (4) holds. □

From Remark 1 and Theorem 3.5, it can be verified that:

Corollary 3.6.

$$E_2(D[G])=\begin{cases}16m & \text{if } n'\neq 0,\\ 4E_2(G) & \text{if } n'=0.\end{cases}$$

The *eccentric connectivity index* [18] is defined for G as

$$\xi^c(G)=\sum_{i=1}^{n}d_G(v_i)\varepsilon_G(v_i)=\sum_{v_iv_j\in E(G)}(\varepsilon_G(v_i)+\varepsilon_G(v_j)).$$

Theorem 3.7. *The second Zagreb eccentricity index of $ED[G]$ is given by*

$$E_2(ED[G])=2E_2(G)+E_1(G)+2\xi^c(G)+2\zeta(G)+n+2m. \tag{5}$$

Proof. From Definition 2.3 and Lemma 2.4,

$$E_2(ED[G])=\sum_{x_iy_j\in E(ED[G])}\varepsilon_{ED[G]}(x_i)\varepsilon_{ED[G]}(y_j)$$

$$+\sum_{x_jy_i\in E(ED[G])}\varepsilon_{ED[G]}(x_j)\varepsilon_{ED[G]}(y_i)+\sum_{i=1}^{n}\varepsilon_{ED[G]}(x_i)\varepsilon_{ED[G]}(y_i)$$

$$=2\sum_{v_iv_j\in E(G)}\big(\varepsilon_G(v_i)+1\big)\big(\varepsilon_G(v_j)+1\big)+\sum_{i=1}^{n}\big(\varepsilon_G(v_i)+1\big)^2$$

$$=2\sum_{v_iv_j\in E(G)}\Big(\varepsilon_G(v_i)\varepsilon_G(v_j)+\big(\varepsilon_G(v_i)+\varepsilon_G(v_j)\big)+1\Big)$$

$$+\sum_{i=1}^{n}\big(\varepsilon_G(v_i)^2+2\varepsilon_G(v_i)+1\big)$$

$$=2\big(E_2(G)+\xi^c(G)+m\big)+E_1(G)+2\zeta(G)+n,$$

that can be simplified to Equation (5). □

Theorem 3.8. *The second Zagreb eccentricity index of $SD[G]$ is given by*

$$E_2(SD[G])=4E_2(G)+E_1(G). \tag{6}$$

Proof. From Definition 2.3 and Lemma 2.4,

$$\begin{aligned} E_2(SD[G]) = & \sum_{x_i x_j \in E(SD[G])} \varepsilon_{SD[G]}(x_i)\varepsilon_{SD[G]}(x_j) \\ & + \sum_{y_i y_j \in E(SD[G])} \varepsilon_{SD[G]}(y_i)\varepsilon_{SD[G]}(y_j) \\ & + \sum_{x_i y_j \in E(SD[G])} \varepsilon_{SD[G]}(x_i)\varepsilon_{SD[G]}(y_j) \\ & + \sum_{x_j y_i \in E(SD[G])} \varepsilon_{SD[G]}(x_j)\varepsilon_{SD[G]}(y_i) + \sum_{i=1}^{n} \varepsilon_{SD[G]}(x_i)\varepsilon_{SD[G]}(y_i) \\ = & 4 \sum_{v_i v_j \in E(G)} \varepsilon_G(v_i)\varepsilon_G(v_j) + \sum_{i=1}^{n} \varepsilon_G(v_i)^2 = 4E_2(G) + E_1(G), \end{aligned}$$

from which Equation (6) follows. □

3.1.3 Third Zagreb Eccentricity Index

In what follows, the third Zagreb eccentricity index of $D[G]$, $ED[G]$, and $SD[G]$ are computed. The proofs are similar to those given in Theorems 3.5, 3.7, and 3.8 and are not presented here.

Theorem 3.9. *The third Zagreb eccentricity index of $D[G]$, $ED[G]$, and $SD[G]$ is given by*

(i) $E_3(D[G]) = 4\big(E_3(G) - n'(n - n')\big)$.

(ii) $E_3(ED[G]) = 2E_3(G)$.

(iii) $E_3(SD[G]) = 4E_3(G)$.

From Remark 1 and part (i) of Theorem 3.9, we reach the following corollary.

Corollary 3.10.

$$E_3(D[G]) = \begin{cases} 0 & \text{if } n' \neq 0, \\ 4E_3(G) & \text{if } n' = 0. \end{cases}$$

3.2 Eccentric Adjacency Index

The *eccentric adjacency index* of G was put forward by Gupta *et al.* [8] as

$$\xi^{ad}(G) = \sum_{i=1}^{n} \frac{\delta_G(v_i)}{\varepsilon_G(v_i)}.$$

Theorem 3.11. *The eccentric adjacency index of $D[G]$ is given by*

$$\xi^{ad}(D[G]) = 4\big(2\xi^{ad}(G) - \sum_{\varepsilon_G(v_i)=1} \delta_G(v_i)\big). \tag{7}$$

Proof. By Definition 2.1 and Lemma 2.2,

$$\begin{aligned}
\xi^{ad}(D[G]) =& \sum_{i=1}^{n} \frac{\delta_{D[G]}(x_i)}{\varepsilon_{D[G]}(x_i)} + \sum_{i=1}^{n} \frac{\delta_{D[G]}(y_i)}{\varepsilon_{D[G]}(y_i)} \\
=& 2\Big[\sum_{\varepsilon_G(v_i)=1} \frac{4\delta_G(v_i)}{2} + \sum_{\varepsilon_G(v_i)\geq 2} \frac{4\delta_G(v_i)}{\varepsilon_G(v_i)}\Big] \\
=& 2\Big[\sum_{\varepsilon_G(v_i)=1} \big(\frac{4\delta_G(v_i)}{2} - \frac{4\delta_G(v_i)}{1}\big) + \sum_{i=1}^{n} \frac{4\delta_G(v_i)}{\varepsilon_G(v_i)}\Big] \\
=& 2\Big(-2\sum_{\varepsilon_G(v_i)=1} \delta_G(v_i) + 4\xi^{ad}(G)\Big),
\end{aligned}$$

from which Equation (7) follows. □

The *first Zagreb index* [9] is defined for G as

$$M_1(G) = \sum_{i=1}^{n} d_G(v_i)^2 = \sum_{i=1}^{n} \delta_G(v_i) = \sum_{v_iv_j\in E(G)} \big(d_G(v_i) + d_G(v_i)\big).$$

Application of Remark 1 and Theorem 3.11 yields:

Corollary 3.12.

$$\xi^{ad}(D[G]) = \begin{cases} 4M_1(G) & \text{if } n' \neq 0, \\ 8\xi^{ad}(G) & \text{if } n' = 0. \end{cases}$$

The *inverse total eccentricity index* [16] and *connective eccentricity index* [7] of G are respectively defined as

$$\zeta^{-1}(G) = \sum_{i=1}^{n} \frac{1}{\varepsilon_G(v_i)}, \quad \xi^{ce}(G) = \sum_{i=1}^{n} \frac{d_G(v_i)}{\varepsilon_G(v_i)}.$$

Theorem 3.13. *The eccentric adjacency index of $ED[G]$ satisfies the following inequality:*

$$\xi^{ad}(ED[G]) \leq \frac{1}{2}\big(\xi^{ad}(G) + 2\xi^{ce}(G) + \zeta^{-1}(G) + M_1(G) + n + 4m\big), \tag{8}$$

with equality if and only if $G \cong K_n$.

Proof. By Definition 2.3 and Lemma 2.4,

$$\xi^{ad}(ED[G]) = \sum_{i=1}^{n} \frac{\delta_{ED[G]}(x_i)}{\varepsilon_{ED[G]}(x_i)} + \sum_{i=1}^{n} \frac{\delta_{ED[G]}(y_i)}{\varepsilon_{ED[G]}(y_i)} = 2\sum_{i=1}^{n} \frac{\delta_G(v_i) + 2d_G(v_i) + 1}{\varepsilon_G(v_i) + 1}.$$

Application of Lemma 2.7 yields:

$$\begin{aligned}\xi^{ad}(ED[G]) \le & 2 \times \frac{1}{4}\sum_{i=1}^{n} \Big(\frac{\delta_G(v_i) + 2d_G(v_i) + 1}{\varepsilon_G(v_i)} + \frac{\delta_G(v_i) + 2d_G(v_i) + 1}{1}\Big) \\ = & \frac{1}{2}\big(\xi^{ad}(G) + 2\xi^{ce}(G) + \zeta^{-1}(G) + M_1(G) + 4m + n\big)\end{aligned}$$

from which the inequality in Equation (8) holds. By Lemma 2.7, the equality holds in Equation (8) if and only if for each $1 \le i \le n$, $\varepsilon_G(v_i) = 1$, which implies that $G \cong K_n$. □

Theorem 3.14. *The eccentric adjacency index of $SD[G]$ is given by*

$$\xi^{ad}(SD[G]) = 2\big(4\xi^{ad}(G) + 4\xi^{ce}(G) + \zeta^{-1}(G)\big). \tag{9}$$

Proof. By Definition 2.5 and Lemma 2.6,

$$\begin{aligned}\xi^{ad}(SD[G]) = & \sum_{i=1}^{n} \frac{\delta_{SD[G]}(x_i)}{\varepsilon_{SD[G]}(x_i)} + \sum_{i=1}^{n} \frac{\delta_{SD[G]}(y_i)}{\varepsilon_{SD[G]}(y_i)} = 2\sum_{i=1}^{n} \frac{4\delta_G(v_i) + 4d_G(v_i) + 1}{\varepsilon_G(v_i)} \\ = & 2\big(4\xi^{ad}(G) + 4\xi^{ce}(G) + \zeta^{-1}(G)\big),\end{aligned}$$

from which Equation (9) holds. □

3.3 Modified Eccentric Connectivity Index

The *modified eccentric connectivity index* of G was proposed by Ashrafi and Ghorbani [3] as

$$\xi_c(G) = \sum_{i=1}^{n} \delta_G(v_i)\varepsilon_G(v_i).$$

In the following theorem, exact expressions for the modified eccentric connectivity index of $D[G]$, $ED[G]$, and $SD[G]$ are presented. The proofs are analogous to those presented in Subsection 3.2 and are therefore omitted.

Theorem 3.15. *The modified eccentric connectivity index of $D[G]$, $ED[G]$, and $SD[G]$ is given by*

(i) $\xi_c(D[G]) = 8\Big(\xi_c(G) + \sum_{\varepsilon_G(v_i)=1} \delta_G(v_i)\Big)$.

(ii) $\xi_c(ED[G]) = 2\big(\xi_c(G) + 2\xi^c(G) + \zeta(G) + M_1(G) + n + 4m\big)$.

(iii) $\xi_c(SD[G]) = 2\big(4\xi_c(G) + 4\xi^c(G) + \zeta(G)\big)$.

As a direct consequence of Remark 1 and part (i) of Theorem 3.15, we get the following corollary.

Corollary 3.16.

$$\xi_c(D[G]) = \begin{cases} 16M_1(G) & \text{if } n' \neq 0, \\ 8\xi_c(G) & \text{if } n' = 0. \end{cases}$$

3.4 Eccentric Harmonic Index

The *eccentric version of harmonic index* (also called *eccentric harmonic index*) of G was introduced by Ediz *et al.* [6] as

$$H_e(G) = \sum_{v_iv_j \in E(G)} \frac{2}{\varepsilon_G(v_i) + \varepsilon_G(v_j)}.$$

Theorem 3.17. *The eccentric harmonic index of $D[G]$ is given by*

$$H_e(D[G]) = 2\Big(2H_e(G) - \binom{n'}{2} - \frac{n'(n-n')}{3}\Big). \tag{10}$$

Proof. By Definition 2.1 and Lemma 2.2,

$$\begin{aligned} H_e(D[G]) =& 4\Big[\sum_{\substack{v_iv_j \in E(G): \\ \varepsilon_G(v_i)=\varepsilon_G(v_j)=1}} \frac{2}{2+2} + \sum_{\substack{v_iv_j \in E(G): \\ \varepsilon_G(v_i)=1, \varepsilon_G(v_j)=2}} \frac{2}{2+2} \\ &+ \sum_{\substack{v_iv_j \in E(G): \\ \varepsilon_G(v_i), \varepsilon_G(v_j) \geq 2}} \frac{2}{\varepsilon_G(v_i) + \varepsilon_G(v_j)}\Big] \\ =& 4\Big[\sum_{\substack{v_iv_j \in E(G): \\ \varepsilon_G(v_i)=\varepsilon_G(v_j)=1}} \big(\frac{2}{2+2} - \frac{2}{1+1}\big) + \sum_{\substack{v_iv_j \in E(G): \\ \varepsilon_G(v_i)=1, \varepsilon_G(v_j)=2}} \big(\frac{2}{2+2} - \frac{2}{1+2}\big) \\ &+ \sum_{v_iv_j \in E(G)} \frac{2}{\varepsilon_G(v_i) + \varepsilon_G(v_j)}\Big] = 4\big(-\frac{1}{2}\binom{n'}{2} - \frac{n'(n-n')}{6} + H_e(G)\big), \end{aligned}$$

from which Equation (10) follows. □

As a direct consequence of Remark 1 and Theorem 3.17, we arrive at:

Corollary 3.18.

$$H_e(D[G]) = \begin{cases} 2m & \text{if } n' \neq 0, \\ 4H_e(G) & \text{if } n' = 0. \end{cases}$$

Theorem 3.19. *The eccentric harmonic index of $ED[G]$ satisfies the following inequality:*

$$H_e(ED[G]) \leq \frac{1}{4}\big(2H_e(G) + \zeta^{-1}(G) + n + 2m\big), \tag{11}$$

with equality if and only if $G \cong K_n$.

Proof. By Definition 2.3 and Lemma 2.4,

$$H_e(ED[G]) = 2 \sum_{v_iv_j \in E(G)} \frac{2}{\big(\varepsilon_G(v_i)+1\big) + \big(\varepsilon_G(v_j)+1\big)} + \sum_{i=1}^{n} \frac{2}{2\big(\varepsilon_G(v_i)+1\big)}.$$

Now from Lemma 2.7, we get

$$\begin{aligned} H_e(ED[G]) \leq& 2 \times \frac{1}{4} \sum_{v_iv_j \in E(G)} \Big(\frac{2}{\varepsilon_G(v_i)+\varepsilon_G(v_j)} + \frac{2}{2}\Big) + \frac{1}{4}\sum_{i=1}^{n}\big(\frac{1}{\varepsilon_G(v_i)} + \frac{1}{1}\big) \\ =& \frac{1}{2}\big(H_e(G) + m\big) + \frac{1}{4}\big(\zeta^{-1}(G) + n\big), \end{aligned}$$

hence the inequality in Equation (11) holds. The equality occurs in Equation (11) if and only if for each $v_iv_j \in E(G)$, $\varepsilon_G(v_i) + \varepsilon_G(v_j) = 2$ and for each $1 \leq i \leq n$, $\varepsilon_G(v_i) = 1$, which implies $G \cong K_n$. □

Theorem 3.20. *The eccentric harmonic index of $SD[G]$ is given by*

$$H_e(SD[G]) = 4H_e(G) + \zeta^{-1}(G). \tag{12}$$

Proof. From Definition 2.5 and Lemma 2.6,

$$H_e(SD[G]) = 4 \sum_{v_iv_j \in E(G)} \frac{2}{\varepsilon_G(v_i)+\varepsilon_G(v_j)} + \sum_{i=1}^{n} \frac{2}{2\varepsilon_G(v_i)} = 4H_e(G) + \zeta^{-1}(G),$$

and Equation (12) follows. □

3.5 Vertex PI index

The *vertex PI index* of G was introduced by Khadikar [15] as

$$PI_v(G) = \sum_{e=v_iv_j \in E(G)} \big(n_{v_i}(e|G) + n_{v_j}(e|G)\big).$$

Theorem 3.21. *The vertex PI index of $D[G]$ is given by*

$$PI_v(D[G]) = 8PI_v(G). \tag{13}$$

Proof. From Definition 2.1 and Lemma 2.2,

$$PI_v(D[G]) = 4 \sum_{e=v_iv_j \in E(G)} \big(2n_{v_i}(e|G) + 2n_{v_j}(e|G)\big) = 8PI_v(G),$$

and Equation (13) holds. □

Theorem 3.22. *The vertex PI index of $ED[G]$ is given by*

$$PI_v(ED[G]) = 2n(n + 2m). \tag{14}$$

Proof. From bipartiteness of $ED[G]$, for each edge $e = uv \in E(ED[G])$, we have $n_u(e|ED[G]) + n_v(e|ED[G]) = 2n$. Hence

$$PI_v(ED[G]) = \sum_{uv \in E(ED[G])} \big(n_u(e|ED[G]) + n_v(e|ED[G])\big) = 2n(n + 2m),$$

and Equation (14) follows. □

Theorem 3.23. *The vertex PI index of $SD[G]$ is given by*

$$PI_v(SD[G]) = 2\big(4PI_v(G) + n - 4m\big). \tag{15}$$

Proof. By Definition 2.5 and Lemma 2.6,

$$\begin{aligned} PI_v(SD[G]) =& 4 \sum_{e=v_iv_j \in E(G)} \Big(\big(2n_{v_i}(e|G) - 1\big) + \big(2n_{v_j}(e|G) - 1\big)\Big) + \sum_{i=1}^{n} \big(1 + 1\big) \\ =& 4\big(2PI_v(G) - 2m\big) + 2n, \end{aligned}$$

from which Equation (15) holds. □

3.6 Weighted Vertex PI Index

The *weighted vertex PI index* of G was introduced by Ilić and Milosavljelić [12] as

$$PI_w(G) = \sum_{e=v_iv_j \in E(G)} \big(d_G(v_i) + d_G(v_j)\big)\big(n_{v_i}(e|G) + n_{v_j}(e|G)\big).$$

Theorem 3.24. *The weighted vertex PI index of $D[G]$ is given by*

$$PI_w(D[G]) = 16PI_w(G). \tag{16}$$

Proof. From Definition 2.1 and Lemma 2.2,

$$\begin{aligned} PI_w(D[G]) =& 4 \sum_{e=v_iv_j \in E(G)} \big(2d_G(v_i) + 2d_G(v_j)\big)\big(2n_{v_i}(e|G) + 2n_{v_j}(e|G)\big) \\ =& 16PI_w(G), \end{aligned}$$

and Equation (16) follows. □

Theorem 3.25. *The weighted vertex PI index of $ED[G]$ is given by*

$$PI_w(ED[G]) = 4n\big(M_1(G) + n + 4m\big). \tag{17}$$

Proof. By Lemma 2.4 and bipartiteness of $ED[G]$,

$$\begin{aligned} PI_w(ED[G]) =& 2n \sum_{uv \in E(ED[G])} \big(d_{ED[G]}(u) + d_{ED[G]}(v)\big) \\ =& 2n\Big(2 \sum_{v_i v_j \in E(G)} \big((d_G(v_i) + 1) + (d_G(v_j) + 1)\big) + 2\sum_{i=1}^{n}(d_G(v_i) + 1)\Big) \\ =& 4n\big(M_1(G) + 4m + n\big), \end{aligned}$$

hence Equation (17) holds. □

Theorem 3.26. *The weighted vertex PI index of $SD[G]$ is given by*

$$PI_w(SD[G]) = 4\big(4PI_w(G) + 4PI_v(G) - 4M_1(G) + n\big). \tag{18}$$

Proof. From Definition 2.5 and Lemma 2.6,

$$\begin{aligned} PI_w(SD[G]) =& 4 \sum_{e = v_i v_j \in E(G)} \Big(\big(2d_G(v_i) + 1\big) + \big(2d_G(v_j) + 1\big)\Big) \\ & \times \Big(\big(2n_{v_i}(e|G) - 1\big) + \big(2n_{v_j}(e|G) - 1\big)\Big) \\ & + \sum_{i=1}^{n} \Big(\big(2d_G(v_i) + 1\big) + \big(2d_G(v_i) + 1\big)\Big)(1 + 1) \\ =& 4 \sum_{e = v_i v_j \in E(G)} \Big(2\big(d_G(v_i) + d_G(v_j)\big) + 2\Big) \\ & \times \Big(2\big(n_{v_i}(e|G) + n_{v_j}(e|G)\big) - 2\Big) + 4\sum_{i=1}^{n} \big(2d_G(v_i) + 1\big) \\ =& 4\big(4PI_w(G) - 4M_1(G) + 4PI_v(G) - 4m\big) + 4(4m + n), \end{aligned}$$

which can be simplified to Equation (18). □

Acknowledgement. The author would like to thank the editor and referee for their insightful comments.

Confilcts of Interest. The author declares that there is no conflict of interest regarding the publication of this article.

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