Mathematics Interdisciplinary Research **6** (2021) 151 − 158

Original Scientific Paper

Scaling Symmetry and a New Conservation Law of the Harry Dym Equation

Mehdi Jafari ⋆, *Amirhesam Zaeim and Somayesadat Mahdion*

Abstract

In this paper, we obtain a new conservation law for the Harry Dym equation by using the scaling method. This method is algorithmic and based on variational calculus and linear algebra. In this method, the density of the conservation law is constructed by considering the scaling symmetry of equation and the associated flux is obtained by the homotopy operator. This density-flux pair gives a conservation law for the equation. A conservation law of rank 7 is constructed for the Harry Dym equation.

Keywords: Harry Dym equation, conversation laws, scaling symmetry, homotopy operator.

2010 Mathematics Subject Classification: 70S10, 58J70, 68W30.

How to cite this article
M. Jafari, A. Zaeim and S. Mahdion, Scaling Symmetry and a New Conservation Law of the Harry Dym Equation, *Math. Interdisc. Res.* **6** (2021) 151 − 158.

1. Introduction

Some partial differential equations (PDEs) which appear in applied sciences like physical chemistry, fluid mechanics, quantum physics and etc., admit conservation laws. In mathematical sense, conservation laws are divergence expressions that vanish on the solution of the PDE. Conservation laws are fundamental laws in physics and state that specific quantities of a system will remain unchanged during the time. There are some methods for obtaining the conservation laws of a

⋆Corresponding author (E-mail: m.jafarii@pnu.ac.ir)
Academic Editor: Hikmet Koyunbakan
Received 20 January 2020, Accepted 25 April 2020
DOI:10.22052/mir.2021.240441.1270

system [1, 4, 5, 6]. Noether's theorem that relates the variational symmetry and conservation laws of a PDE, were used in the common methods [12, 13]. There is another method that apply calculus of variation and linear algebra. This method is called scaling method and briefly works as follows [14]. A primitive density with arbitrary coefficients which is invariant under the scaling symmetry of the PDE is considered in the first step. Then, the time derivative of primitive density is calculated and is combined with the PDE. Using the Euler operator, a linear system is obtained and by solving this system, the actual density will be constructed. Finally, the associated flux is obtained by using the inverse divergence operator i.e, homotopy operator. In this work, we obtain the conservation laws of the Harry Dym equation,

$$u_t = -\frac{1}{2}u^3 u_{3x}. \tag{1}$$

This equation was obtained by Harry Dym and Martin Kruskal as a solvable evolution equation by a spectral problem based on the string equation instead of the Schrödinger equation [2, 8, 9, 10, 11].

The main purpose of this study is to find the conservation law of the Equation (1), by applying the scaling method. This paper is organized as follows. In Section 2, we have referred to some definitions and previous results that are used in the later sections. We will show that the Harry Dym equation is uniform in rank and admits a scaling symmetry in Section 3. In Section 4, the primitive density of rank 7 is constructed and actual density is obtained by removing the divergence and divergence-equivalent terms. Finally, we calculate the corresponding fluxes using the homotopy operator.

2. Preliminaries

Let $\Delta(x, u^{(n)}) = 0$ be a system of PDEs, where $x = (x^1, \ldots, x^p)$ and $u = (u^1, \ldots, u^q)$ are independent and dependent variables respectively and $u^{(n)}$ is all derivatives of u up to the n-th order. A conversation law is the following divergence expression

$$\mathrm{Div} Q = 0,$$

which is vanished for all solutions $u = f(x)$ of the given system. In dynamical problems, the time variable t and the spatial variables $x = (x^1, \ldots, x^p)$ are specified separately. So the conservation law takes the form

$$D_t \rho + \mathrm{Div}\mathbf{J} = 0, \tag{2}$$

where ρ is the conserved density, $\mathbf{J}$ is the corresponding flux, D_t is the total time derivative which is defined as following and Div is the total divergence of $\mathbf{J} = (\mathbf{J}_1, \ldots, \mathbf{J}_p)$ with respect to the spatial variables [13].

Definition 2.1. *The total time derivative D_t of the function $g = g(x, t, u^{(M)}(x, t))$ is defined as follows,*

$$D_t g = \frac{\partial g}{\partial t} + \sum_{\alpha=1}^{q} \sum_{J} u^{\alpha}_{J,t} \frac{\partial g}{\partial u^{\alpha}_{J}},$$

where $J = (j_1, \ldots, j_k)$ is a multi-index with $0 \le k \le M$ and

$$u^{\alpha}_{J,t} = \frac{\partial u^{\alpha}_{J}}{\partial t} = \frac{\partial^{k+1} u^{\alpha}}{\partial t \partial x^{j_1} \ldots \partial x^{j_k}}.$$

Definition 2.2. [15] *The zeroth-Euler operator acting on a scalar differential function g is defined as*

$$\mathcal{L}_{u(x)} g = (\mathcal{L}_{u^1(x)} g, \mathcal{L}_{u^2(x)} g, \ldots, \mathcal{L}_{u^q(x)} g),$$

where

$$\mathcal{L}_{u^{\alpha}(x)} g = \sum_{k=0}^{M_1^{\alpha}} (-D_x)^k \frac{\partial g}{\partial u^{\alpha}_{kx}}, \qquad \alpha = 1, \ldots, q, \tag{3}$$

where M_1^{α}'s are orders of g for the component u^{α} with respect to x.

Definition 2.3. *Let $g = g(x, u^{(M)}(x))$ of order M be given. g is called exact if a differential vector function $G(x, u^{(M-1)}(x))$ exists such that $g = \mathrm{Div} G$.*

The following theorem, called Exactness theorem, states the condition for exactness of a differential function using the zeroth-Euler operator. This theorem is so important for calculating the conservation laws.

Theorem 2.4. [14] *Exactness of a differential function $g = g(x, u^{(M)}(x))$ is equivalent to the condition $\mathcal{L}_{u(x)} g = 0$.*

Another operator whish is used to calculate the flux of conservation laws is called the homotopy operator and is defined as follows.

Definition 2.5. *The homotopy operator of a differential function $g = g(x, u^{(M)}(x))$ of one variable x, is the following operator,*

$$\mathcal{H}_{u(x)} g = \int_0^1 \left(\sum_{\alpha=1}^{q} \mathcal{I}_{u^{\alpha}(x)} g\right)[\lambda u] \frac{d\lambda}{\lambda}, \tag{4}$$

where the integrand is defined as

$$\mathcal{I}_{u^{\alpha}(x)} g = \sum_{k=1}^{M_1^{\alpha}} \left(\sum_{j=0}^{k-1} u^{\alpha}_{jx} (-D_x)^{k-(j+1)}\right) \frac{\partial g}{\partial u^{\alpha}_{kx}}. \tag{5}$$

Theorem 2.6. [14] *Assume that an exact differential function $g = g(x, u^{(M)}(x))$ be given. That is, there exists a function $G = G(x, u^{(M-1)}(x))$ such that $g = \mathrm{Div} G$. In this case,*

$$G = \mathrm{Div}^{-1} g = \mathcal{H}_{u(x)} g.$$

3. Scaling Symmetry of the Harry Dym Equation

To find the conservation laws of a system using the scaling method, we use the scaling (dilation) symmetry of that system to find the primitive density. We claim that the scaling symmetry of Equation (1) is the transformation

$$(x, t, u) \to (\lambda^{-1}x, \lambda^{-6}t, \lambda u),$$

where λ is an arbitrarily scaling constant. There are several algorithmic methods for finding the scaling symmetries [3, 4, 13], but here, we use the concept of the weight of variables to find it [7].

Definition 3.1. *Consider the scaling symmetry $x \to \lambda^{-p}x$. The weight of the variable x is denoted by $W(x)$ and is equal to $-p$. If $W(x) = -p$, the weight of D_x is defined as p.*

Definition 3.2. *If a monomial has more than one variable and each variable has a weight, then the sum of the weights of its variables is called the rank of the monomial. If all the monomials in a differential function have the same rank, it is called uniform in rank.*

An equation that admits a scaling symmetry is uniform in rank, so we can obtain the scaling symmetry of Harry Dym equation by assuming that (1) is uniform in rank. With this assumption, a system of weight-balance equations is obtained, then by solving this system, scaling symmetry will be obtained. For (1), the weight-balance equation is

$$W(u) + W(D_t) = 4W(u) + 3W(D_x). \tag{6}$$

Solving the Equation (6) gives $W(u) = 1$, $W(D_x) = 1$ and then $W(D_t) = 6$. Since (2) must vanish on all the solutions of the PDE, density and flux of conservation law must follow its scaling symmetries. So, clearly the conservation law must also be uniform in rank. In addition, according to the symmetry of the Harry Dym equation, we can construct the initial density, which is a linear combination of monomials with the selected rank (See [14] for more details).

4. Computing Conservation Laws of the Harry Dym Equation

In this section, using the scaling method, we obtain the conservation laws of the Harry Dym equation. To calculate the conservation law with this method, we firstly construct the density ρ and then we will obtain the corresponding flux $\mathbf{J}$. To calculate the density, we will firstly consider an initial density, which is a linear combination of differential terms with arbitrary coefficients. This combination must be chosen from a fixed rank and must be invariant under the scaling symmetry of the Harry Dym equation. The total time derivative of the initial density is calculated in the next step and all time derivatives in the expressions are replaced by their equivalent expressions using (1). According to (2), the resulting expression must be exact. Therefore, the arbitrary coefficients are calculated by solving the linear system which is achieved by applying the exactness Theorem 2.4; i.e.,

$$\mathcal{L}_{u(x)}(D_t\rho) = 0.$$

By substituting the coefficients obtained from the above system with the initial density ρ, the actual density is obtained. Finally, the corresponding flux is calculated using the homotopy operator,

$$\mathbf{J} = -\mathrm{Div}^{-1}(D_t\rho).$$

4.1 Constructing the Candidate Density

As mentioned earlier, the first step to find conservation laws, is to obtain the density. We firstly choose an arbitrary rank for the initial density. Then, we construct the terms of ρ by combining monomials with a specified rank that include dependent variables and their partial derivatives. In the following, we obtain the initial density ρ of rank 7 for the Harry Dym Equation (1). Consider the list $\mathcal{P}$, which contains dependent variables up to rank 7. According to (6), $\mathcal{P} = \{u^7, u^6, u^5, u^4, u^3, u^2, u\}$. Then, we apply the total derivative operator with respect to the spatial variables on $\mathcal{P}$ to increase the rank of the terms in the list up to 7. We call this new list as $\mathcal{Q}$.

$$\begin{aligned}\mathcal{Q} = \{&u^7, u^5u_x, u^3u_x^2, u^4u_{2x}, uu_x^3, u^2u_xu_{2x}, u^3u_{3x}, u_x^2u_{2x},\\ &u_{2x}^2u, uu_xu_{3x}, u^2u_{4x}, u_{2x}u_{3x}, u_xu_{4x}, uu_{5x}, u_{6x}\}.\end{aligned} \tag{7}$$

In order to the density to be nontrivial, we have to delete the divergence terms and also keep one of the divergence-equivalent terms in the list and to remove the rest. By applying (3) over (7), we have

$$\begin{aligned}\mathcal{L}_{u(x)}\mathcal{Q} = \{&7u^6, 0, -3u^2u_x^2 - 2u^3u_{2x}, 8u^3u_{2x} + 12u^2u_x^2, -2u_x^3 - 6uu_xu_{2x},\\ &4uu_xu_{2x} + (2u_x^2 + 2uu_{2x})u_x, -(6u_x^2 + 6uu_{2x})u_x - 12uu_xu_{2x}, 0,\\ &3u_{2x}^2 + 4u_xu_{3x} + 2uu_{4x}, -2uu_{4x}, -4u_xu_{3x} - 3u_{2x}^2,\\ &4uu_{4x} + 8u_xu_{3x} + 6u_{2x}^2, 0, 0, 0, 0\}.\end{aligned} \tag{8}$$

According to the Theorem 2.4, $u^5u_x, u_x^2u_{2x}, u_{2x}u_{3x}, u_xu_{4x}, uu_{5x}$, and u_{6x} are divergences and so they should be removed from $\mathcal{Q}$. The third and fourth terms of the list (8) are multiples of each other, so the third and fourth terms of the list (7) are divergence-equivalent and one of them must be removed from $\mathcal{Q}$. From the equivalent terms, select the one with the lowest rank and delete the rest. Therefore, u^4u_{2x} is removed from the list. In a similar way, the sixth and seventh terms, as well as the ninth, tenth, and eleventh terms, are equivalent. So $\mathcal{Q}$ can be summarized as

$$\mathcal{Q} = \{u^7, u^3u_x^2, uu_x^3, u^2u_{2x}u_x, u_{2x}^2u\}.$$

Now, consider a linear combination of the members of the above list with arbitrary coefficients to form the initial density of rank 7 for the Harry Dym equation

$$\rho = c_1u^7 + c_2u^3u_x^2 + c_3uu_x^3 + c_4u^2u_xu_{2x} + c_5u_{2x}^2u. \tag{9}$$

4.2 Determining the Actual Density

To determine the unspecified coefficients in (9), we calculate $D_t\rho$,

$$\begin{aligned}D_t\rho = &(7c_1u^6 + 3c_2u^2u_x^2 + c_3u_x^3 + 2c_4uu_xu_{2x} + c_5u_{2x}^2)u_t\\ &+(2c_2u^3u_x + 3c_3uu_x^2 + c_4u^2u_{2x})u_{xt}\\ &+(c_4u^2u_x + 2c_5u_{2x}u)u_{2xt}.\end{aligned}$$

Then, by using (1), we replace u_t and its derivatives with their equivalent values. Let $E = -D_t\rho$. So we have

$$\begin{aligned}E = &(7c_1u^6 + 3c_2u^2u_x^2 + c_3u_x^3 + 2c_4uu_xu_{2x} + c_5u_{2x}^2)(\tfrac{1}{2}u^3u_{3x})\\ &+(2c_2u^3u_x + 3c_3uu_x^2 + c_4u^2u_{2x})(\tfrac{3}{2}u_xu^2u_{3x} + \tfrac{1}{2}u^3u_{4x})\\ &+(c_4u^2u_x + 2c_5u_{2x}u)(\tfrac{3}{2}u_{2x}u^2u_{3x} + 3u_x^2uu_{3x} + \tfrac{3}{2}u_xu^2u_{4x} + \tfrac{1}{2}u^3u_{5x}).\end{aligned}$$

According to (2), E must be exact. Therefore, using Theorem 2.4, we have $\mathcal{L}_{u(x)}E = 0$. This equation forms a system of linear equations, by solving this system, unspecified coefficients are determined as follows

$$c_1 = c_2 = c_5 = 0, \qquad c_3 = c_4, \tag{10}$$

and c_4 is arbitrary. Assuming $c_4 = 1$, then $c_3 = 1$. Substituting (10) in (9), the actual density is obtained as follows

$$\rho = uu_x^3 + u^2u_xu_{2x}.$$

4.3 Computing the Flux

After the density has been determined, the corresponding flux can be calculated using the fact that $\mathbf{J} = \mathrm{Div}^{-1}(E)$. We use Theorem 2.6 and the homotopy operator to obtain the flux. Substituting (10) in E, we have

$$E = 8u^3u_{3x}u_x^3 + 4u^4u_{3x}u_xu_{2x} + \tfrac{9}{2}u^4u_x^2u_{4x} + \tfrac{1}{2}u^5u_{2x}u_4 + \frac{1}{2}u^5u_xu_{5x}.$$

The integrand $\mathcal{I}_{u(x)}E$ is calculated by relation (5) as follows

$$\mathcal{I}_{u(x)}E = \textstyle\sum_{k=1}^{3}\left(\sum_{j=0}^{k-1}u_{jx}(-D_x)^{k-(j+1)}\right)\frac{\partial f}{\partial u_{kx}} = 14u^4u_x^2u_{3x} + \tfrac{7}{2}u^5u_xu_{4x}.$$

Using (4), we obtain the flux as follows

$$\mathbf{J} = \mathcal{H}_{u(x)}E = \int_0^1 (\mathcal{I}_{u(x)}E)[\lambda u]\frac{d\lambda}{\lambda} = \frac{1}{2}u^4u_x(4u_xu_{3x} + uu_{4x}).$$

Therefore, the conservation law of rank 7 is obtained as follows

$$D_t\rho + \mathrm{Div}\mathbf{J} = D_t(uu_x^3 + u^2u_xu_{2x}) + \mathrm{Div}\big(2u^4u_x^2u_{3x} + \frac{1}{2}u^5u_xu_{4x}) = 0.$$

5. Conclusions

In this paper, the Harry Dym equation, which is one of the most widely used equations in various sciences, was examined. Since this equation is uniform in rank, its scaling symmetries were obtained and using them, the density of the conservation law was constructed for the equation. The corresponding flux was also obtained using the homotopy operator. Eventually, a conservation law of rank 7 was obtained for the Harry Dym equation.

Conflicts of Interest. The authors declare that there are no conflicts of interest regarding the publication of this article.

References

[1] S. Anco and G. Bluman, Direct construction method for conservation laws of partial differential equations Part II: general treatment, *Euro. J. Appl. Math.* **13** (5) (2002) 567–585.

[2] M. Antonowicz and A. P. Fordy, Coupled Harry Dym equations with multi-Hamiltonian structures, *J. Phys. A: Math. Gen.* **21** (5) (1988) L269–L275.

[3] G. W. Bluman, S. C. Anco, *Symmetry and Integration Methods for Differential Equations*, Springer-Verlag, New York, 2004.

[4] G. W. Bluman, A. F. Cheviakov and S. C. Anco, *Applications of Symmetry Methods to Partial Differential Equations,* Appl. Math. Sci., vol. 168, Springer-Verlag, New York, 2010.

[5] A. F. Cheviakov, Computation of fluxes of conservation laws, *J. Eng. Math.* **66** (1-3) (2010) 153–173.

[6] A. F. Cheviakov, GeM software package for computation of symmetries and conservation laws of differential equations, *Comp. Phys. Comm.* **176** (1) (2007) 18–61.

[7] W. Hereman, P. J. Adams, H. L. Eklund, M. S Hickman and B. M. Herbst, Direct methods and symbolic software for conservation laws of nonlinear equations, *Advances in nonlinear waves and symbolic computation,* $19-78$*, loose errata, Nova Sci. Publ., New York*, 2009.

[8] W. Hereman, P. P. Banerjee and M. R. Chatterjee, Derivation and implicit solution of the Harry Dym equation and its connections with the Korteweg–de Vries equation, *J. Phys. A* **22** (1989) 241–255.

[9] M. D. Kruskal, Nonlinear wave equations, *Lecture Notes in Physics*, vol. 38, pp. 310–354, Berlin-Heidelberg-New York: Springer, 1975.

[10] F. Magri, A simple model of the integrable Hamiltonian equation, *J. Math. Phys.* **19** (5) (1978) 1156–1162.

[11] M. Nadjafikhah and P. Kabi-Nejad, On the changes of variables associated with the Hamiltonian structure of the Harry-Dym equation, *Glob. J. Adv. Res. Cl. Mod. Geom.* **6** (2) (2017) 83–90.

[12] E. Noether, Invariante variations probleme, Nachr. Akad. Wiss. Gött. Math. Phys. Kl., 2 (1918), 235–257. (English translation in Transp. Theory Stat. Phys. **1** (3) (1971) 186–207.

[13] P. J. Olver, *Applications of Lie Groups to Differential Equations,* Springer, New York, 1986.

[14] D. Poole, *Symbolic Computation of Conservation Laws of Nonlinear Partial Differential Equations Using Homotopy Operators*, Ph.D. Dissertation, Colorado School of Mines, Golden, Colorado, 2009.

[15] D. Poole and W. Hereman, The homotopy operator method for symbolic integration by parts and inversion of divergences with applications, *Appl. Anal.* **87** (2010) 433–455.

Mehdi Jafari
Department of Mathematics,
Payame Noor University,
Tehran, Iran
e-mail: m.jafarii@pnu.ac.ir

Amirhesam Zaeim
Department of Mathematics,
Payame Noor University,
P. O. BOX 19395-3697,
Tehran, Iran
e-mail: saadatmandi@kashanu.ac.ir

Somayesadat Mahdion
Department of Mathematics,
Payame Noor University,
P. O. BOX 19395-3697,
Tehran, Iran
e-mail: ssm.mahdion@gmail.com